

NONREPETITIVE CHOICE NUMBER OF TREES

JAKUB KOZIK AND PIOTR MICEK

ABSTRACT. A nonrepetitive coloring of a path is a coloring of its vertices such that the sequence of colors along the path does not contain two identical, consecutive blocks. The remarkable construction of Thue asserts that 3 colors are enough to color nonrepetitively paths of any length. A nonrepetitive coloring of a graph is a coloring of its vertices such that all simple paths are nonrepetitively colored. Assume that each vertex v of a graph G has assigned a set (list) of colors L_v . A coloring is chosen from $\{L_v\}_{v \in V(G)}$ if the color of each v belongs to L_v . The Thue choice number of G , denoted by $\pi_l(G)$, is the minimum k such that for any list assignment $\{L_v\}$ of G with each $|L_v| \geq k$ there is a nonrepetitive coloring of G chosen from $\{L_v\}$. Alon et al. (2002) proved that $\pi_l(G) = O(\Delta^2)$ for every graph G with maximum degree at most Δ . We propose an almost linear bound in Δ for trees, namely for any $\varepsilon > 0$ there is a constant c such that $\pi_l(T) \leq c\Delta^{1+\varepsilon}$ for every tree T with maximum degree Δ . The only lower bound for trees is given by a recent result of Fiorenzi et al. (2011) that for any Δ there is a tree T such that $\pi_l(T) = \Omega(\frac{\log \Delta}{\log \log \Delta})$. We also show that if one allows repetitions in a coloring but still forbid 3 identical consecutive blocks of colors on any simple path, then a constant size of the lists allows to color any tree.

1. INTRODUCTION

A *repetition* of length h ($h \geq 1$) in a sequence is a subsequence of consecutive terms of the form: $x_1 \dots x_h x_1 \dots x_h$. A sequence is *nonrepetitive* if it does not contain a repetition of any length.

In 1906 Thue proved that there exist arbitrarily long nonrepetitive sequences over only 3 different symbols (see [2, 10]). The method discovered by Thue is constructive and uses substitutions over a given set of symbols. Recently a completely different approach to creating long nonrepetitive sequences emerged (see [7]). Consider the following naive procedure: generate consecutive terms of a sequence by choosing symbols at random and every time a repetition occurs, erase the repeated block and continue. For instance, if the generated sequence is *abcbc*, we must cancel the last two symbols, which brings us back to *abc*. By a simple counting one can prove that with positive probability the length of a constructed sequence exceeds any finite bound, provided the number of symbols is at least 4. This is slightly weaker than Thue's result, but the argument seems to be more flexible for adaptations to other settings. This approach leads e.g. to a very short proof (see [7]) that for every $n \geq 1$ and every sequence of sets L_1, \dots, L_n , each of size at least 4, there is a nonrepetitive sequence s_1, \dots, s_n where $s_i \in L_i$ (first proved with an enhanced Local Lemma in [8]). The analogous statement for lists of size 3 remains an exciting open problem. In this paper we make use of the above-mentioned approach to nonrepetitive colorings of trees.

For a given graph G we denote by $V(G)$ the set of vertices of G . A coloring function $f : V(G) \rightarrow \mathbb{N}$ is a *nonrepetitive coloring* of G if there is no repetition on

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the color sequence of any simple path in G . The minimum number of colors used in a nonrepetitive coloring of G is called the *Thue number* of G and denoted by $\pi(G)$. The dependence between the Thue number and maximum degree of graphs is already quite well understood.

Theorem 1 (Alon et al. [1]). *For any graph G with maximum degree Δ there is a nonrepetitive coloring of G using at most $16\Delta^2$ colors. Moreover, for every $\Delta > 1$ there is a graph with maximum degree Δ which needs $\Omega\left(\frac{\Delta^2}{\log \Delta}\right)$ colors in any nonrepetitive coloring.*

The Thue number of any tree is at most 4 (see [1]). Kündgen and Pelsmajer [9] proved that $\pi(G) \leq 12$ for all outerplanar G , and $\pi(G) \leq 4^k$ for all graphs G with tree-width at most k . Probably the most intriguing question in the area concerns planar graphs.

Conjecture 2 (Grytczuk 2007 [6]). *There is a constant such that $\pi(G) \leq c$, for all planar graphs G .*

Very recently Dujmović et al. [3] showed $\pi(G) = O(\log n)$ for all planar G on n vertices.

Now, we turn to the list-version of nonrepetitive colorings of graphs. This is an analog of the classical graph choosability introduced by Vizing [11] and independently by Erdős, Rubin and Taylor [4]. Given a graph G suppose that each $v \in V(G)$ has a preassigned set of colors L_v . We call $\{L_v\}_{v \in V(G)}$ a *list assignment* of G , or just *lists* of G . A coloring f is *chosen from $\{L_v\}$* if $f(v) \in L_v$ for all $v \in V(G)$. The *Thue choice number* of G , denoted by $\pi_l(G)$, is the minimum k such that for any list assignment $\{L_v\}$ of G with each $|L_v| \geq k$ there is a nonrepetitive coloring of G chosen from $\{L_v\}$. The upper bound from Theorem 1 works also in the list-setting, i.e., $\pi_l(G) \leq 16\Delta^2$ for all G with maximum degree Δ . As we mentioned $\pi_l(P_n) \leq 4$ for all paths P_n and the problem whether 3 or 4 is the right bound remains open. The first significant difference between the Thue number and the Thue choice number has been proved recently for trees.

Theorem 3 (Fiorenzi et al. [5]). *For any constant c there is a tree T such that $\pi_l(T) \geq c$.*

In fact one can extract from [5] that for any $\Delta > 1$ there is a tree T with $\pi_l(T) = \Omega\left(\frac{\log \Delta}{\log \log \Delta}\right)$. We propose two results complementary to Theorem 3. First is an improved upper bound for the Thue choice number of trees.

Theorem 4. *For every $\varepsilon > 0$ there is a constant c such that $\pi_l(T) \leq c\Delta^{1+\varepsilon}$ for all trees T with maximum degree Δ .*

A sequence is *of the form x^r* for real $r \geq 1$ if it can be divided into $\lceil r \rceil$ blocks where all the blocks but the last are the same, say $x_1 \dots x_n$ for some $n \geq 1$, and the last block is the prefix of $x_1 \dots x_n$ of size $\lceil \text{frac}(r) \cdot n \rceil$, where $\text{frac}(r)$ is the fractional part of r . The sequence $x_1 \dots x_n$ repeated in those blocks is also called *the base* of the given sequence. For example any repetition is a sequence of the form x^2 and $abcdabcdab$ is of the form $x^{2.5}$ with the base $abcd$. A coloring of a graph G is x^r -free for real $r > 1$ if there is no sequence of the form x^r among the color sequences of simple paths in G . Thus, an x^2 -free coloring is simply a nonrepetitive coloring while an x^3 -free coloring satisfies a weaker condition, in particular it allows a coloring to have a repetitions. A consequence of our second result is that for any tree T and lists $\{L_v\}$ each of size 8 there is an x^3 -free coloring of T chosen from $\{L_v\}$. This explains somehow the tightness of Theorem 3.

Theorem 5. *For every $\varepsilon > 0$ there is a constant c such that for every tree T and lists $\{L_v\}_{v \in V(T)}$ each of size c there is $x^{2+\varepsilon}$ -free coloring of T chosen from $\{L_v\}$.*

2. PROOFS

In both proofs given a tree T we are going to fix an arbitrary vertex for a *root* and denote it by $\text{root}(T)$. For $u, v \in V(T)$ we say that u is a *descendant* of v if the unique simple path from u to $\text{root}(T)$ contains v . The set of all descendants of v , including v , is denoted by $v\downarrow$. The $\text{depth}(v)$ is the number of vertices on a simple path from v to $\text{root}(T)$. A vertex u is a *child* of v if u is a descendant of v and they are adjacent in T . We also pick an arbitrary planar embedding of T . This means we fix an ordering of children of every vertex in T . If v has a child, the first child of v in a determined order is $\text{first-child}(v)$. If u is a child of v , but not the last child, then $\text{next-child}(v, u)$ is the child of v that is next to u .

A *vertical* path in a rooted tree is a simple path whose first vertex is a descendant of the last or vice versa. A coloring of a rooted tree T is *vertically x^r -free* for real $r > 1$ if there is no sequence of the form x^r among the color sequences of vertical paths in T .

For any planar embedding of a given rooted tree T and list assignment $\{L_v\}_{v \in V(T)}$, a pair (f, u) is a *partial coloring* if $u \in V(T)$ and f is a partial function from $V(T)$ to \mathbb{N} defined only for the vertices of T up to u in the preorder traversal of T and $f(v) \in L(v)$, whenever $f(v)$ defined. The set of all partial colorings of a tree T with fixed $\{L_v\}_{v \in V(T)}$ is denoted by PCOL.

Following usual convention we define $[n]$ to be $\{1, \dots, n\}$. For a set of integers A we use A^+ to denote the set of finite sequences over A of length at least 1. For $s \in A^+$ and $n \in \mathbb{N}$ we write $s \cdot n$ to denote the sequence s with appended element n . For a sequence $s = (s_1, \dots, s_n)$ we put $s_{1..i} = (s_1, \dots, s_i)$.

Consider a coloring of a rooted tree with an $x^{2+\varepsilon}$ -block on some simple path. Clearly, at least half of the vertices of this path forms a vertical path whose color sequence is of the form $x^{1+\varepsilon/2}$. Thus, Theorem 5 is an immediate consequence of the following lemma.

Lemma 6. *For every $\varepsilon > 0$ there is a constant $c = 4 \cdot \lceil \frac{1}{\varepsilon} \rceil$ such that every rooted tree is vertically $x^{1+\varepsilon}$ -free colorable from any lists of size c .*

Proof. For a given $\varepsilon > 0$ put $c = 4 \cdot \lceil \frac{1}{\varepsilon} \rceil$. Let T be a rooted tree and $\{L_v\}_{v \in V(T)}$ be the list assignment with each $|L(v)| = c$. In order to get a contradiction, suppose that there is no vertically $x^{1+\varepsilon}$ -free coloring of T chosen from $\{L_v\}$. Fix an arbitrary planar embedding of T .

We propose a very naive procedure struggling to build a proper coloring of T from $\{L_v\}$. The procedure maintains (f, v) , a partial coloring of T from $\{L_v\}$, with no color sequence of the form $x^{1+\varepsilon}$ on any vertical paths other than paths going upwards from v . To start the procedure we just pick a color for $\text{root}(T)$ from $L(\text{root}(T))$ and all other vertices are uncolored. Every consecutive step of the procedure tries to correct and/or extend the current partial coloring. This is encapsulated by the call of $\text{nextV}((f, v), n)$ function (see Algorithm 1), where (f, v) is the current partial coloring and n is the hint for the next decision to be made. The call of nextV checks first whether (f, v) is vertically $x^{1+\varepsilon}$ -free. If not then the colors from vertices in the repeated ε -part of $x^{1+\varepsilon}$ occurrence starting from v are erased (as well as colors of all descendants of erased vertices) and the color for the top-most vertex with erased color is set again to be the n -th color from its list. If (f, v) is vertically $x^{1+\varepsilon}$ -free, $\text{nextV}((f, v), n)$ tries to extend the partial coloring (f, v) onto the consecutive subtrees of v . We will keep an invariant that any extension of an input partial coloring (f, v) onto all descendants of v contains a vertical $x^{1+\varepsilon}$ -block. We will extend (f, v) onto $u\downarrow$ for u being consecutive childs of v and if u is the first child of v whose subtree cannot be colored in this way then nextV sets the color of u to be the n -th color from $L(u)$.

Algorithm 1: $\text{nextV}((f, v), n)$

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1  if  $x^{1+\varepsilon}$  occurs in  $(f, v)$  starting from  $v$  on the way to  $\text{root}(T)$  then
2     $l =$  the length of the base of  $x^{1+\varepsilon}$  sequence
3     $m = \lceil l \cdot \varepsilon \rceil$ 
4     $(v_{l+m}, \dots, v_1) =$  the path starting from  $v_{l+m} = v$  going upwards in  $T$ 
5      with  $f(v_i) = f(v_{l+i})$  for  $1 \leq i \leq m$ 
6     $u \leftarrow v_{l+1}$ 
7    erase all values of  $f$  in  $u \downarrow$ 
8  else
9     $u =$  first-child( $v$ )
10   while  $f$  has a vertically  $x^{1+\varepsilon}$ -free extension onto  $u \downarrow$  do
11     extend  $f$  onto  $u \downarrow$  in vertically  $x^{1+\varepsilon}$ -free manner
12      $u =$  next-child( $v, u$ )
13  extend  $f$  with  $\{u \rightarrow \alpha\}$ , where  $\alpha$  is the  $n$ -th element of  $L(u)$ 
14  return  $(f, u)$ 

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The partial function $\text{nextV} : \text{PCOL} \times [c] \rightarrow \text{PCOL}$ is defined by Algorithm 1. Note that $\text{nextV}((f, v), n)$ is well-defined for partial colorings (f, v) with

- (i) no color sequence of the form $x^{1+\varepsilon}$ on a vertical path other than paths going upwards from v , and
- (ii) no $x^{1+\varepsilon}$ -free extension of (f, v) onto $v \downarrow$.

Moreover, if $(f', u) = \text{nextV}((f, v), n)$ then this new partial coloring also satisfies (i) and (ii). This allows us to iterate the calls of nextV . Note also that vertex u is determined only by (f, v) , i.e. the first argument of nextV , while $f'(u)$ is simply the n -th color in $L(u)$.

Now, we define recursively a function $h : [c]^+ \rightarrow \text{PCOL}$ which captures the idea of our naive procedure trying to color T from $\{L_v\}$. For $s \in [c]^+$, $1 \leq n \leq c$ and α being the n -th color in $L(\text{root}(T))$ put

$$h(n) = (\{\text{root}(T) \rightarrow \alpha\}, \text{root}(T)),$$

$$h(s \cdot n) = \text{nextV}(h(s), n).$$

First of all note that $h(s)$ is well-defined for all $s \in [c]^+$. Indeed, $h(s)$ is explicitly constructed for all s of length 1 and it trivially satisfies (i), while (ii) holds as we supposed that there is no vertically $x^{1+\varepsilon}$ -free coloring of T from $\{L_v\}$. Now $h(s \cdot n)$ is well-defined as nextV is well-defined for partial colorings satisfying (i)-(ii) and a new partial coloring also satisfies (i)-(ii).

It is convenient to see $s \in [c]^+$ as a seed driving to a sequence of partial colorings of T : $h(s_1), h(s_{1..2}), h(s_{1..3}), \dots, h(s)$. Now, we aim to get a concise description of this sequence. Let $(f_i, v_i) = h(s_{1..i})$ for $1 \leq i \leq |s|$. We define $\text{chosen}(s) = (f_1(v_1), \dots, f_{|s|}(v_{|s|}))$. In other words, $\text{chosen}(s)$ is a sequence of colors set by instruction 13 of Algorithm 1 in consecutive calls of nextV on a way to build $h(s)$.

Claim. *The function chosen is injective.*

Proof of the Claim. Note that the length of $\text{chosen}(s)$ is equal the length of s . To get a contradiction let $s \neq s'$ be the shortest sequences for which $\text{chosen}(s) = \text{chosen}(s')$. Let $n = |s| = |s'|$. By minimality of s, s' we have $s_{1..(n-1)} = s'_{1..(n-1)}$. The first $n - 1$ values of $\text{chosen}(s)$ depend only on $s_{1..(n-1)}$, therefore they are the same for both sequences. Moreover, the last values of $\text{chosen}(s)$ and $\text{chosen}(s')$ are picked from the same list. By the construction of the procedure, the list is

determined by $h(s_{1..(n-1)}) = h(s'_{1..(n-1)})$ or it is just $L(\text{root}(t))$ in case when $n = 1$. Since $s \neq s'$ they must differ on the last coordinate. It means that indices of the last colors in $\text{chosen}(s)$ and $\text{chosen}(s')$ on the list are different, and hence the colors are different. \square

Let $s \in [c]^+$, $(f_i, v_i) = h(s_{1..i})$ for all $1 \leq i \leq |s|$. For $2 \leq i \leq |s|$ we denote by l_i, m_i the evaluations of variables l, m in the $(i-1)$ -th call to the procedure `nextV` (for some calls, they may be undefined). Then $W(s) = (\text{depth}(v_1), \dots, \text{depth}(v_{|s|}))$ is a *supporting walk* of s . The walk contains two kind of steps: positive, when $W(s)_i = W(s)_{i-1} + 1$, and negative, when $W(s)_i \leq W(s)_{i-1}$. Positive steps occur when procedure `nextV` descends into a subtree, i.e. evaluates the case from line 9 to 12. Negative steps correspond to the calls in which repeated part of an $x^{1+\varepsilon}$ -block in the partial assignment is erased (lines 2 to 7). Let us suppose that the i -th step was negative. Note that just from $W(s)$ we can decode the length of the erased block, i.e. the value of m_i . This is exactly $W(s)_{i-1} - W(s)_i + 1$. However, to decode the corresponding value l_i we need some additional information. All we know is that $m_i = \lfloor l_i \cdot \varepsilon \rfloor$, which leaves $\lceil 1/\varepsilon \rceil$ possible values for l_i . Therefore we annotate every step of $W(s)$ with a number from $\{0, \dots, \lceil 1/\varepsilon \rceil - 1\}$. The number is meaningful only for negative steps. Formally the annotation function $A : [c]^+ \rightarrow \{0, \dots, \lceil 1/\varepsilon \rceil - 1\}^+$ is defined as follows. For $1 \leq i \leq |s|$,

$$A(s)_i = \begin{cases} l_i - \lfloor m_i/\varepsilon \rfloor & \text{if } i\text{-th step is negative} \\ 0 & \text{otherwise.} \end{cases}$$

Let $s \in [c]^+$ and $(f, v) = h(s)$ then $\text{vcolors}(s)$ is the sequence of colors on the path from $\text{root}(T)$ to v in a partial coloring f . Thus, the last value in $\text{vcolors}(s)$ is $f(v)$.

Finally we define a total encoding function $\text{Log} : [c]^+ \rightarrow \mathbb{N}^+ \times [1/\varepsilon]^+ \times \mathbb{N}^+$ as $\text{Log}(s) = (W(s), A(s), \text{vcolors}(s))$.

Claim. *The function Log is injective.*

Proof. Let $s \in [c]^+$. First we show that $\text{Log}(s)$ uniquely determines sequences $\text{vcolors}(s_{1..i})$ for all $1 \leq i \leq |s|$. Recall that $\text{vcolors}(s)$ is written explicitly in $\text{Log}(s)$.

Suppose $\text{vcolors}(s_{1..i})$ is already known and now we reconstruct $\text{vcolors}(s_{1..(i-1)})$. If the i -th step of $W(s)$ is positive, i.e. $W(s)_i > W(s)_{i-1}$ then the length of the path from the root to the current vertex increased by 1 in step i . Thus, $\text{vcolors}(s_{1..(i-1)})$ is exactly the same as $\text{vcolors}(s_{1..i})$ but with the last color erased. If the i -th step of $W(s)$ is negative then $m_i = W(s)_i - W(s)_{i-1} + 1$ is the size of the repeated ε -block and $l_i = \lfloor m_i/\varepsilon \rfloor + A(s)_i$ is the size of the base of a $x^{1+\varepsilon}$ sequence fixed in this step. Clearly, the last color in $\text{vcolors}(s_{1..i})$ is introduced in the i -th step and l_i colors before form a base of the $x^{1+\varepsilon}$ sequence that was retracted. Let $(\alpha_1, \dots, \alpha_{l_i}, \beta)$ be the suffix of $\text{vcolors}(s_{1..i})$. Then $\text{vcolors}(s_{1..(i-1)})$ is just $\text{vcolors}(s_{1..i})$ with the last color, namely β , erased and sequence $(\alpha_1, \dots, \alpha_m)$ appended.

Once we have sequences $\text{vcolors}(s_{1..i})$ for all $1 \leq i \leq |s|$, we may simply read their last values to reconstruct $\text{chosen}(s)$. Now, the previous Claim assures that $\text{chosen}(s)$ uniquely determines s . \square

Let us fix $M \in \mathbb{N}$. We are going to give a bound for the number of distinct $\text{Log}(s)$ for s of length M based on the structure of $\text{Log}(s)$. For $s \in [c]^M$, supporting walk $W(s)$ is a sequence of M positive integers with $W(s)_i - W(s)_{i-1} \leq 1$. Now replace all negative steps $W(s)_i, W(s)_{i+1}$ with a sequence $W(s)_i, W(s)_i + 1, W(s)_i, W(s)_i - 1, W(s)_i - 2, \dots, W(s)_{i+1}$. It is easy to see that such an operation is reversible and it results in a sequence of positive integers of size at most $2M$ with all steps in $\{-1, 1\}$. The number of such sequences is well-known to be $o(2^{2M})$. The number

of possible annotation sequences $A(s)$ is bounded by $\lceil 1/\varepsilon \rceil^M$. Finally, $\text{vcolors}(s)$ is a sequence of colors which appear on some simple path starting from $\text{root}(T)$ in a final partial coloring $h(s)$. There are $|V(T)|$ simple paths starting from $\text{root}(T)$ and each of them has at most $c^{|V(T)|}$ possible color assignments.

By the last Claim the number of distinct $\text{Log}(s)$ for $s \in [c]^M$ is simply c^M . On the other hand the upper bounds from obtained just now we get the following inequality

$$c^M \leq o(4^M) \cdot \left\lceil \frac{1}{\varepsilon} \right\rceil^M \cdot (|V(T)| \cdot c^{|V(T)|}).$$

For $c = 4\lceil \frac{1}{\varepsilon} \rceil$ this gives a contradiction for sufficiently large M . \square

Proof of Theorem 4. Clearly, it suffices to prove the theorem for small values of ε . We fix any $\varepsilon \in (0, 1)$, and choose δ so that it satisfies $1 + \varepsilon = \frac{1+\delta}{1-\delta}$ (note that $\delta < \frac{1}{2}$). We are going to prove a bit stronger statement. There is a constant c such that for every rooted tree T with maximum degree at most Δ and lists $\{L_v\}_{v \in V(T)}$ each of size at least $c\Delta^{1+\varepsilon}$, there exists a coloring of T from $\{L_v\}$ with

- (1) no color sequences of the form x^2 on simple paths in T , and
- (2) no color sequences of the form $x^{1+\delta}$ on vertical paths in T .

Let c be sufficiently large integer ($c \geq 12 \cdot (\lceil \frac{1}{\delta} \rceil + 1)$ will do). Let T be a tree and $\{L_v\}_{v \in V(T)}$ be a lists assignment with each $|L(v)| = \hat{c} \geq c\Delta^{1+\varepsilon}$. In order to get a contradiction, suppose that there is no coloring of T chosen from $\{L_v\}$ with (1) and (2) satisfied. Fix an arbitrary planar embedding of T .

Like in the proof of Lemma 6 we propose a procedure struggling to accomplish an impossible mission that is to produce a coloring of T from $\{L_v\}$ satisfying (1) and (2). The procedure maintains (f, v) a partial coloring of T from $\{L_v\}$ with the only possible violations of (1) and (2) on paths starting at v . To start the procedure we just pick a color for $\text{root}(T)$ from $L(\text{root}(T))$ and all other vertices are uncolored. Every consecutive step of the procedure tries to correct and/or extend the current partial coloring. This is encapsulated by the call of $\text{next}((f, v), n)$ function (see Algorithm 2), where (f, v) is the current partial coloring and n is the hint for the next decision to be made. The call of next checks first whether (f, v) is vertically $x^{1+\delta}$ -free. If not then the colors from vertices in the repeated δ -part of $x^{1+\delta}$ occurrence starting from v are erased (as well as colors of all descendants of erased vertices) and the color for the top-most vertex with color cleared is set again to be the n -th color from its list. If (f, v) is vertically $x^{1+\delta}$ -free then next checks whether it is x^2 -free (see lines 9-16 of Algorithm 2). If there is a path P with a color sequence of the form x^2 then it must start at v and next clears the colors along P up to the last vertex which is a predecessor of v or up to the vertex which finishes the repeated block of x^2 occurrence. Again, the color of the top-most vertex with color cleared is set to be the n -th color from its list. Finally, if there is no violation of (1) and (2) then $\text{next}((f, v), n)$ tries to extend the partial coloring (f, v) onto subtrees rooted at consecutive childs of v . We will keep an invariant that such an extension of an input partial coloring (f, v) can not be done, and if u is the first child of v whose subtree cannot be colored in this way then next sets the color of u to be the n -th color from $L(u)$.

The partial function $\text{next} : \text{PCOL} \times [\hat{c}] \rightarrow \text{PCOL}$ is defined by Algorithm 2. Note that $\text{next}((f, v), n)$ is well-defined for partial colorings (f, v) with

- (i) no color sequence of the form $x^{1+\delta}$ on a vertical path other than paths going upwards from v , and
- (ii) no color sequence of the form x^2 on simple paths other than paths starting at v , and

Algorithm 2: $\text{next}((f, v), n)$

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1  if  $x^{1+\delta}$  occurs in  $(f, v)$  starting at  $v$  on the way to  $\text{root}(T)$  then
2       $l$  = the length of the base of  $x^{1+\delta}$  sequence
3       $m = \lceil l \cdot \delta \rceil$ 
4       $(v_{l+m}, \dots, v_1)$  = the path starting at  $v_{l+m} = v$  going upwards in  $T$ 
5          with  $f(v_i) = f(v_{l+i})$  for  $1 \leq i \leq m$ 
6       $u \leftarrow v_{l+1}$ 
7      erase all values of  $f$  in  $u \downarrow$ 
8  else if  $x^2$  occurs in  $(f, v)$  starting at  $v$  then
9       $(v_{2l}, \dots, v_1)$  = the path starting at  $v_{2l} = v$ 
10         with  $f(v_i) = f(v_{l+i})$  for  $1 \leq i \leq l$ 
11          $k$  = the least integer  $i$  such that  $v$  is a descendant of  $v_i$ 
12         if  $k \leq l$  then
13              $u \leftarrow v_{l+1}$ 
14         else
15              $u \leftarrow v_{k+1}$ 
16         erase all values of  $f$  in  $u \downarrow$ 
17  else
18       $u = \text{first-child}(v)$ 
19      while  $f$  has an extension onto  $u \downarrow$  satisfying (1) and (2) do
20          extend  $f$  onto  $u \downarrow$  and keep (1) and (2) satisfied
21           $u = \text{next-child}(v, u)$ 
22  extend  $f$  with  $\{u \rightarrow \alpha\}$ , where  $\alpha$  is the  $n$ -th element of  $L(u)$ 
23  return  $(f, u)$ 

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(iii) no extension of (f, v) onto $v \downarrow$ preserving (1) and (2).

Moreover, if $\text{next}((f, v), n)$ exists then this new partial coloring also satisfies (i)-(iii). This allows us to iterate the calls of **next**.

Now, we define recursively function $h : [\hat{c}]^+ \rightarrow \text{PCOL}$ which captures the idea of our naive procedure trying to color T from $\{L_v\}$. For $s \in [\hat{c}]^+$, $1 \leq n \leq c$ and α being the n -th color in $L(\text{root}(T))$ put

$$h(n) = (\{\text{root}(T) \rightarrow \alpha\}, \text{root}(T)),$$

$$h(s \cdot n) = \text{next}(h(s), n).$$

First of all note that $h(s)$ is well-defined for all $s \in [\hat{c}]^+$. Indeed, $h(s)$ is explicitly constructed for all s of length 1 and it trivially satisfies (i) and (ii), while (iii) holds as we supposed that there is no coloring of T from $\{L_v\}$ satisfying (1) and (2). Now $h(s \cdot n)$ is well-defined as **next** is well-defined for partial colorings satisfying (i)-(iii) and a new partial coloring also satisfies (i)-(iii).

Now, for given $s \in [c]^+$ we aim to get a concise description of $h(s_1)$, $h(s_{1..2})$, $h(s_{1..3}), \dots, h(s)$. Let $(f_i, v_i) = h(s_{1..i})$ for $1 \leq i \leq |s|$. We define $\text{chosen}(s) = (f_1(v_1), \dots, f_{|s|}(v_{|s|}))$. In other words (and exactly as in the proof of Lemma 6), $\text{chosen}(s)$ is a sequence of colors set by instruction 22 of Algorithm 2 in consecutive calls of **next** on a way to build $h(s)$.

Claim. *The function chosen is injective.*

Note that if $(f', u) = \text{next}((f, v), n)$ is defined then vertex u is determined only by (f, v) , i.e. the first argument of **next**, while $f'(u)$ is simply the n -th color in $L(u)$. That is why the proof of the claim above follows exactly the same lines as the proof of the corresponding claim in the proof of Lemma 6.

For a partial coloring (f, v) let $\text{vcolors}((f, v))$ be the color sequence in f on vertices from $\text{root}(T)$ to v . In particular, the last color in $\text{vcolors}(s)$ is simply $f(v)$.

Claim. *The function vcolors is injective on partial colorings from the image of h .*

Proof. We are going to prove that for any two partial colorings $(f, v), (f', v')$ from the image of h , if $\text{vcolors}((f, v)) = \text{vcolors}((f', v'))$ then $(f, v) = (f', v')$. The proof goes by induction on the length of $\text{vcolors}((f, v))$.

When the length of $\text{vcolors}((f, v))$ and so $\text{vcolors}((f', v'))$ is 1 then $v = v' = \text{root}(T)$. Thus, $\text{root}(T)$ is the only vertex colored by f and f' , and the statement is trivial.

Suppose that $|\text{vcolors}((f, v))| = |\text{vcolors}((f', v'))| = n$ and the claim holds for all shorter sequences. Since (f, v) and (f', v') are in the image of h there exist $s, s' \in [\hat{c}]^+$ such that $h(s) = (f, v)$ and $h(s') = (f', v')$. Let $(f_i, v_i) = h(s_{1\dots i})$ for $1 \leq i \leq |s|$ and $(f'_i, v'_i) = h(s'_{1\dots i})$ for $1 \leq i \leq |s'|$. Let j be the least index such that $\text{depth}(v_i) \geq n$ for $j < i \leq |s|$. Analogously, let j' be the least index such that $\text{depth}(v'_i) \geq n$ for $j' < i \leq |s'|$. Now, we need a basic property of Algorithm 2 that is, if $(g', u') = \text{next}(g, u)$ then the coloring of a path from $\text{root}(T)$ to u' , with excluded u' , is the same in g and g' . This implies that the color sequence from $\text{root}(T)$ to v_j is the same in partial colorings $h(s_{1\dots i})$ for all $j \leq i \leq |s|$, which is just the prefix of $\text{vcolors}((f, v))$ of length $n - 1$. Analogously, a color sequence from $\text{root}(T)$ to $v'_{j'}$ is the same in partial colorings $h(s'_{1\dots i})$ for all $j' \leq i \leq |s'|$, which is just the prefix of $\text{vcolors}((f', v')) = \text{vcolors}((f, v))$ of length $n - 1$. In particular this means that $\text{vcolors}((f_j, v_j)) = \text{vcolors}((f'_{j'}, v'_{j'}))$. By the induction hypothesis we get $(f_j, v_j) = (f'_{j'}, v'_{j'})$. Now we do know that partial colorings (f_{j+1}, v_{j+1}) and $(f'_{j'+1}, v'_{j'+1})$ are generated by the calls of next with the same first arguments. Note that Algorithm 2 is deterministic (in particular line 20) in a sense that for the same input it always generates the same output. Thus, we immediately get that $v_{j+1} = v'_{j'+1}$, say $w = v_{j+1}$, and two partial colorings $(f_{j+1}, v_{j+1}), (f'_{j'+1}, v'_{j'+1})$ differ at most with the color of w . By the definition of j and j' , in all the consecutively built partial colorings (f_i, v_i) for $j < i \leq |s|$, (f'_i, v'_i) for $j' < i \leq |s'|$ vertex w is on the path from $\text{root}(T)$ to the current vertex, i.e. v_i or v'_i , respectively. Moreover, all these partial colorings differ at most in the subtree of w . But the only vertex from $w \downarrow$ colored in the final colorings (i.e. (f, v) and (f', v')) is w itself. Finally, in both of these colorings a vertex w receives the same color which is at the end of $\text{vcolors}((f, v)) = \text{vcolors}((f', v'))$. Thus, $(f, v) = (f', v')$. \square

Again (as in the proof of Lemma 6) we aim to get a concise description of all these partial colorings and then apply a double counting argument. For $s \in [\hat{c}]^+$, let $(f_i, v_i) = h(s_{1\dots i})$ for all $1 \leq i \leq |s|$. For $2 \leq i \leq |s|$ we denote by l_i, k_i the valuations of variables l, k in the $(i - 1)$ -th call to the procedure next (for some calls, they may be undefined). Define $W(s) = (\text{depth}(v_1), \dots, \text{depth}(v_{|s|}))$ to be a supporting walk of s . We distinguish three kind of steps (differences) in $W(s)$

- (a) positive, when $W(s)_i = W(s)_{i-1} + 1$, i.e. no obstruction occurs in the i -th step and Algorithm 2 evaluates lines 18-21,
- (b) $x^{1+\delta}$ -negative, when $W(s)_i \leq W(s)_{i-1}$ and color sequence of the form $x^{1+\delta}$ is fixed in the i -th step; this corresponds to the evaluation of lines 2-7,
- (c) x^2 -negative, when $W(s)_i \leq W(s)_{i-1}$ and color sequence of the form x^2 is fixed in i -th step; this corresponds to the evaluation of lines 9-16.

Additionally we put $m_i = W(s)_i - W(s)_{i-1} + 1$. For $x^{1+\delta}$ -negative steps, m_i corresponds to the value of variable m in the corresponding call to the procedure next .

This time we need three kinds of annotations enriching the information given in $W(s)$. The first is analogous to the one in the proof of Lemma 6 and helps to recover lengths of the base of the $x^{1+\delta}$ sequence in $x^{1+\delta}$ -negative steps. Suppose that the i -th step was $x^{1+\delta}$ -negative. Note that just from $W(s)$ we can decode the length of the repeated block, i.e. the value of variable m_i . However, to decode a corresponding value l_i we need some additional information. All we know is that $m_i = \lceil l_i \cdot \delta \rceil$, which leaves $\lceil 1/\delta \rceil$ possible values for l_i . Therefore we annotate every negative step with a number from $\{0, \dots, \lceil 1/\delta \rceil - 1\}$ and use an extra value for all steps which are not $x^{1+\delta}$ -negative. The annotation function $A : [\hat{c}]^+ \rightarrow \{-1, 0, \dots, \lceil 1/\delta \rceil - 1\}^+$ is defined as follows. For $1 \leq i \leq |s|$,

$$A(s)_i = \begin{cases} l_i - \lfloor m_i/\delta \rfloor & \text{if } i\text{-th step is } x^{1+\delta}\text{-negative} \\ -1 & \text{otherwise.} \end{cases}$$

The second annotation function will serve to recover basic information concerning the paths whose part was retracted in x^2 -negative step. Suppose that the i -th step is x^2 -negative. We want to recover the values of l_i and k_i set in lines 9 and 11, which represents the half of length of the path forming a repetition and the position of the tip in this path. Note that $m_i = W(s)_{i-1} - W(s)_i + 1$ is equal to $\min(l_i, 2l_i - k_i)$. Hence, we need information what is the difference between l_i and k_i . For $1 \leq i \leq |s|$ let

$$B(s)_i = \begin{cases} l_i - k_i & \text{if } i\text{-th step is } x^2\text{-negative} \\ \text{whatever} & \text{otherwise.} \end{cases}$$

To get a more convenient description of function B , we make a list of important values of function B and encode it into a sequence over $\{-1, 0, 1\}$. If the i -th step is x^2 -negative then we convert $B(s)_i$ into a sequence of 0's of length $m_i = W(s)_i - W(s)_{i-1} + 1$ and if $B(s)_i \neq 0$ we put $\text{sgn}(B(s)_i)$ in $|B(s)_i|$ -th position. We need to argue here that $|B(s)_i| \leq m_i$. Indeed, as the partial coloring in the i -th step has no $x^{1+\delta}$ occurrence we get that $|l_i - k_i| \leq \delta l_i$ and $l_i - m_i \leq \delta l_i$, which give

$$|B(s)_i| = |l_i - k_i| \leq \delta l_i \leq \frac{\delta}{1 - \delta} m_i \leq m_i.$$

The last inequality holds as $\delta < \frac{1}{2}$. We define $B^*(s)$ to be the concatenation of the sequences produced for all x^2 -negative steps.

The third annotation contains the further description of the paths involved in x^2 -negative steps. Suppose that the i -th step is x^2 -negative and let $P = (v_{2l_i}, \dots, v_1)$ be the path whose color sequence forms a repetition. Already from $W(s)$ and $B^*(s)$ we will recover the size of the path and the value of k_i such that v_{k_i} is the tip of P . Now, we want to describe the way in which P goes down in T from v_{k_i} up to v_1 . Let n_j for $1 < j \leq k - 1$ be the position of v_{j-1} on the list of children of vertex v_j . Then put $C(i) = (n_1, \dots, n_{k_i-1})$ and $C^*(s)$ be the concatenation of $C(i)$'s for i being the indices of x^2 -negative steps.

A total encoding function is defined as $\text{Log}(s) = (W(s), A(s), B^*(s), C^*(s), h(s))$ for $s \in [\hat{c}]^+$. Length of a $\text{Log}(s) = (W(s), A(s), B^*(s), C^*(s), h(s))$ is defined to be the length of $W(s)$, hence $|\text{Log}(s)| = |s|$. Here comes the key property of Log function.

Claim. *The function Log is injective.*

Proof. Take any L from the image of Log . Suppose that $|L| = n$. Then, there exists $s \in [\hat{c}]^n$ such that $\text{Log}(s) = L$. We are going to show that there is only one such s . We prove that reconstructing the sequence $\text{chosen}(s)$ from L . This will prove the claim as we already know that $\text{chosen}(s)$ is injective.

Let s' be the prefix of s of size $n - 1$. In one step of reconstruction we decode from L the last chosen color α and the value of $\text{Log}(s')$. Then, by simple iteration of this process, we reconstruct the whole chosen (s) . The value of α may be simply read from $h(s)$, which is explicitly given in $\text{Log}(s)$. In order to get $\text{Log}(s')$ note that $W(s')$ and $A(s')$ are just the prefixes of $W(s)$ and $A(s)$ of length $|s| - 1$. It remains to reconstruct $h(s')$, $B^*(s')$ and $C^*(s')$. The way we proceed depends on the type of the last step in $W(s)$, which can be recognized from $W(s)$ itself and $A(s)$. Indeed, if $W(s)_n = W(s)_{n-1} + 1$, then the last step is positive. Otherwise the value of $A(s)$ indicates which type of negative step we deal with.

Cases 1 and 2. The last step in $W(s)$ is positive or $x^{1+\delta}$ -negative. Then $B^*(s') = B^*(s)$, $C^*(s') = C^*(s)$. The partial coloring $h(s')$ is reconstructed exactly as in the analogous cases in the proof of Lemma 6.

Case 3. The last step in $W(s)$ is x^2 -negative. Let $P = (v_{2l_n}, \dots, v_1)$ be the path whose color sequence forms a repetition and let v_{k_n} be the tip of P . The number of vertices in P with colors erased can be read from $W(s)$ and it is $m_n = W(s)_n - W(s)_{n-1} + 1$. By the construction of the Algorithm 2 (lines 9-16) we have

$$2l_n - m_n = \max(l_n, k_n).$$

From the last m_n values of sequence $B^*(s)$ we can extract the value of $l_n - k_n$. If all these values are zeros then $l_n - k_n = 0$. Otherwise exactly one of these m_n values is equal to 1 or -1 and the position of this non-zero value determines $|l_n - k_n|$ while the sign of $l_n - k_n$ is the same as the sign of this non-zero entry. Once we know $d = l_n - k_n$ we can deduce that

$$\begin{aligned} l_n = m_n \text{ and } k_n = m_n - d, & & \text{if } d = l_n - k_n \geq 0, \\ l_n = m_n - d \text{ and } k_n = m_n - 2d, & & \text{if } d = l_n - k_n < 0. \end{aligned}$$

Let $h(s) = (f, u)$, $h(s') = (f', u')$. As we supposed that call of `nextV` generating $h(s)$ from $h(s')$ retracts a repetition on path P , we get that $u' = v_{2l_n}$ and $u = v_{2l_n - m_n + 1}$. The color of $u = v_{2l_n - m_n + 1}$ in f' was erased by line 16 and replaced in line 22 of Algorithm 2. The colors of $v_{2l_n - m_n + 1}, \dots, v_{2l_n}$ were erased from f' and are not visible in f but the colors of $v_1, \dots, v_{2l_n - m_n}$ remain unchanged. The vertex $v_{2l_n - m_n}$ is clearly the parent of u . As we already reconstructed the value of k_n , i.e. the position of the tip of P , we know the vertices of P lying on a path from $v_{2l_n - m_n}$ to $\text{root}(T)$. In particular, we reconstructed the vertex v_{k_n} in T . Now, we make use of $C^*(s)$. The last $k_n - 1$ values of $C^*(s)$ indicates how the path P goes down in T from v_{k_n} up to v_1 . This way we reconstructed the position of $(v_{2l_n - m_n}, \dots, v_1)$ in T and we know that their colors are the same in f and f' . Once we know the colors of at least first half of the vertices of P (as $m_n \leq l_n$) and as the color sequence of vertices from P forms a repetition in f' we may deduce the colors of $v_{2l_n - m_n + 1}, \dots, v_{2l_n}$.

Putting all together we finally reconstruct $\text{vcolors}(h(s'))$ which is the sequence of colors in f' from $\text{root}(T)$ down to $u' = v_{2l_n}$. Indeed, the colors from $\text{root}(T)$ down to $v_{2l_n - m_n}$ are the same in f' and f , while the colors from $v_{2l_n - m_n + 1}$ to v_{2l_n} has just been reconstructed. Now, recall that the function `vcolors` is injective on the partial colorings from the image of h which means that we can reconstruct from $\text{vcolors}(h(s'))$ a partial coloring $h(s')$ itself. \square

We are going to bound the number of distinct $\text{Log}(s)$ for s of length M . For every $s \in [\hat{c}]^M$ we have $\text{Log}(s) = (W(s), A(s), B^*(s), C^*(s), h(s))$. Just like before, the number of integer walks $W(s)$ of length M is $o(4^M)$. The number of possible annotation sequences $A(s)$ is bounded by $(\lceil 1/\delta \rceil + 1)^M$. The annotation $B^*(s)$ is a sequence of numbers $\{-1, 0, 1\}$ of length $\sum_i m_i$, where i goes over all the indices of $x^{1+\delta}$ -negative steps. Clearly, $\sum_i m_i \leq M$ and so the number of distinct $B^*(s)$

is bounded by 3^M . The annotation $C^*(s)$ is the concatenation of sequences over $\{1, \dots, \Delta - 1\}$. The length of $C^*(s)$ is equal to $\sum_i k_i$, where the sum goes over the set I_{x^2} of all the indices of x^2 -negative steps. Clearly,

$$\sum_{i \in I_{x^2}} k_i \leq \sum_{i \in I_{x^2}} (1 + \delta) l_i \leq \sum_{i=1}^M \frac{1 + \delta}{1 - \delta} m_i \leq \frac{1 + \delta}{1 - \delta} M = (1 + \varepsilon)M,$$

By the last Claim the number of distinct $\text{Log}(s)$ for $s \in [\hat{c}]^M$ is simply $\hat{c}^M \geq (c\Delta)^{(1+\varepsilon)M}$. On the other hand we just obtained an independent upper bound and altogether we get the following inequality

$$(c\Delta)^{(1+\varepsilon)M} \leq o(4^M) \cdot \left(\left\lceil \frac{1}{\delta} \right\rceil + 1 \right)^M \cdot 3^M \cdot \Delta^{(1+\varepsilon)M} \cdot (|V(T)|c^{|V(T)|}).$$

For $c \geq 12 \cdot (\lceil \frac{1}{\delta} \rceil + 1)$ and sufficiently large M we get a contradiction. \square

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THEORETICAL COMPUTER SCIENCE DEPARTMENT, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, JAGIELLONIAN UNIVERSITY, 30-387 KRAKÓW, POLAND

E-mail address: jkozik@tcs.uj.edu.pl

THEORETICAL COMPUTER SCIENCE DEPARTMENT, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, JAGIELLONIAN UNIVERSITY, 30-387 KRAKÓW, POLAND

E-mail address: piotr.micek@tcs.uj.edu.pl