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Abstract. We study a nonlocal version of the one-phase Stefan problem which develops mushy regions, even if they were not present initially, a model which can be of interest at the mesoscopic scale. The equation involves a convolution with a compactly supported kernel. The created mushy regions have the size of the support of this kernel. If the kernel is suitably rescaled, such regions disappear and the solution converges to the solution of the usual local version of the one-phase Stefan problem. We prove that the model is well posed, and give several qualitative properties. In particular, the long-time behavior is identified by means of a nonlocal mesa solving an obstacle problem.

1. Introduction

The aim of this paper is to study the following nonlocal version of the one-phase Stefan problem posed in $\mathbb{R}^N \times (0,\infty)$,

$$\begin{cases} \partial_t u = J * v - v, \text{ where } v = (u-1)_+, \\ u(\cdot,0) = f \geqslant 0, \end{cases} \tag{1.1}$$

which presents interesting features from the physical point of view. The function J is assumed to be continuous, compactly supported, radially symmetric and with $\int_{\mathbb{R}^N} J = 1$. We denote by R_J the radius of the support of J. For some results we will also assume that J is nonincreasing in the radial variable.

The local model – The well-known usual local Stefan problem is a mathematical model that describes the phenomenon of phase transition, for example between water and ice, [21], [22]. Its history goes back to Lamé and Clapeyron [20] and, afterwards, Stefan [23]. The one-phase Stefan problem corresponds to the simplified case in which the temperature of the ice phase is supposed to be maintained at the value where the phase transition occurs, say 0°C. The thermodynamical state of the system is characterized by two state variables, *temperature* v and *enthalpy* u. Conservation of energy implies that they satisfy, in the absence of heat sources or sinks, the evolution equation

$$\rho \partial_t u = \nabla \cdot (\kappa \nabla v),$$

where the *density* $\rho > 0$ and the *thermal conductivity* $\kappa > 0$ are assumed to be constant.

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On the other hand, there is a constitutive equation relating v to u, given in an ideal situation by the formula

$$v = c^{-1}(u - L)_+, \tag{1.2}$$

where $c > 0$, which we assume to be constant, is called the *specific heat* (the amount of energy needed to increase in one unit the temperature of a mass unit of water) and $L > 0$ is the *latent heat* (the amount of energy needed to transform a mass unit of ice into water). All the parameters can be set to one with a change of units, and we arrive to

$$\partial_t u = \Delta v, \qquad v = (u - 1)_+. \tag{1.3}$$

In contrast with the standard heat equation, this problem has *finite speed of propagation*: if the initial data are compactly supported, the same is true for u (and hence for v) for any later time. This is one of the main features of the model, and gives rise to the existence of two *free boundaries*, one for u, the set $\partial\{u > 0\}$, and one for v, $\partial\{v > 0\}$.

On the ice we have $u = 0$, while $u \geqslant 1$ on the liquid phase. The points where $0 < u < 1$ correspond to the *mushy region*, where we have neither ice nor water, but something in an intermediate state between solid and liquid. The temperature in this zone is 0, but not its enthalpy.

There is a major drawback in the model: either $u(x, t) \geqslant 1$ or $u(x, t) = f(x)$ [15]. This means, on one hand, that a point can belong to the mushy region only if it belonged to it at the initial time. On the other hand, there is no evolution of the enthalpy inside the mushy region. Both things are unsatisfactory from the physical point of view. Indeed, once the structure of ice starts to break, it should take some time until it melts completely. The region where this occurs may be small, but it should be noticed at some intermediate (mesoscopic) scale between the microscopic and macroscopic ones.

THE NONLOCAL MODEL – Bearing the above discussion in mind, we consider the nonlocal diffusion version (1.1) of the Stefan problem. As we shall see, this model also has finite speed of propagation, and hence free boundaries. Moreover, in sharp contrast with the local problem, it develops mushy regions, even if they were not present initially. The size of these mushy regions is given by the size of the support of the convolution kernel J. Finally, if the initial data are continuous, the solution remains continuous for all times.

It is possible to rescale the kernel in such a way that solutions of the nonlocal model converge to solutions of the local version (1.3). In this scaling, the support of the kernel shrinks to a point, and hence the mushy regions disappear. Thus, the local model can be viewed as a limit problem when going from the intermediate scale to the macroscopic one.

What is the point in considering this particular diffusion operator, $\mathcal{L}v := J * v - v$? With this choice, the evolution of u at a certain point is governed by a balance between the value of the temperature at this point and a certain average of this physical magnitude in a fixed neighborhood. This is a way to take into account possible middle-range interactions between water and ice. In the local model, the region where the average is taken shrinks to a point, and the evolution of the enthalpy is governed by the Laplacian of the temperature.

Choice of the kernel – One could use other non-local operators, as fractional-Laplacian type operators of the form $\mathcal{K}v := -(-\Delta)^s v$. However, we will stick to operators $\mathcal{L}$ involving kernels which are both compactly supported and non-singular, though most (if not all) of our results should be valid for singular (yet compactly supported) kernels. Why are we being so restrictive? There are two main reasons:

(i) Using a non-compactly supported kernel means that "infinite-range" interactions are not considered to be negligible. This would imply for instance that the mushy regions which are automatically created are instantaneously spread throughout the space (see Section 3.5 with $R_J = \infty$), something which is not satisfactory.

(ii) Kernels which have a singularity at the origin have the big advantage of implying regularization properties ([6] and [11]). It is challenging from a mathematical point of view to try to handle situations where no regularizing effect can occur.

Another way to think about this absence of regularization is to understand it as a lack of compactness of the inverse of $\mathcal{L}$, which is the source of several problems. Our approach, which consists in controlling the supports of solutions, allows us to bypass such problems, up to a certain point.

About the initial data – We would like to consider typical situations in which initially there is ice with zero enthalpy everywhere except at some places where there is water at some positive temperature. In this case, we shall see that a mushy region will develop as ice melts from an initial configuration in which no mushy region was present. Thus, we have to consider initial data which are in principle not continuous, but only integrable. This will lead to solutions that are not continuous. Therefore, we have to handle the various qualitative properties concerning the supports and mushy regions not in the usual (continuous) sense, but in the sense of distributions, which requires some effort.

If the initial data are continuous and bounded, we can construct a solution with these two properties. Some of our techniques can be greatly simplified for such solutions, and can be later applied to more general data by approximation. Hence we will devote some time to discuss the basic theory for solutions in this class.

As expected, when the initial data are at the same time bounded, continuous and integrable, the two concepts of solution mentioned above coincide.

In the one-phase Stefan problem, the temperature, and hence the enthalpy, are assumed to be nonnegative. Therefore, in principle we should only consider nonnegative initial data f. Indeed, if the initial enthalpy had sign changes, we would be dealing with a two-phase Stefan problem, and the relation between the temperature and the enthalpy would not be given by (1.2). In particular, the temperature should be negative for negative values of the enthalpy. Therefore, the relevant model would be different.

Though the problem might not have any physical meaning for functions f that have sign changes, some of our results are still true for such general initial data. Whenever we know that this is the case, we will state the corresponding theorem for a general f, and will add the restriction of nonnegativity only if required.

Long-time behavior – Another important aspect of evolution equations is the long time behavior of solutions. For a certain class of initial data (which does not include all non-negative functions in $\mathrm{L}^1(\mathbb{R}^N)$ if $N = 1, 2$), we prove that the solution converges as time goes to infinity to a solution of a nonlocal mesa problem which is not the "classical" (local) mesa problem. The identification of the limit (the mesa) can be done by solving an obstacle problem. More precisely, the asymptotic behavior of the solution u is given by the projection

$$\mathcal{P}f := f + J * w - w$$

where w solves an obstacle problem detailed in Section 5.

The projection operator $\mathcal{P}$ is L^1-contractive in the set $\mathcal{S}$ of admissible functions for which the above convergence result is valid, Corollary 5.5. On the other hand, $\mathcal{S}$ is dense in the set of $\mathrm{L}^1(\mathbb{R}^N)$ functions which are non-negative a.e. Therefore, $\mathcal{P}$ can be extended by continuity to the latter class of functions. This will allow us to identify the large time limit of solutions of (1.1) for the wider class of integrable and nonnegative initial data.

Abstract setting – Equations like (1.1) and (1.3) can be embedded in the abstract setting of semi-group theory for equations of the form

$$\gamma(u)_t + Au \ni 0,$$

where γ is a maximal monotone graph, and A is a linear (bounded or unbounded) m-accretive operator, see [13] and also [2], [3]. This theory provides existence and uniqueness for our model, though not the main qualitative properties considered here. As for existence and uniqueness, in the special case that we are studying, γ is the inverse of the Lipschitz graph $\Gamma : s \mapsto (s-1)_+$, and many arguments can be written at a lower cost.

Notation – Given a set Ω we define:

- $\mathrm{BC}(\Omega) = \{\varphi \in \mathrm{C}(\Omega) : \varphi \text{ bounded in } \mathbb{R}^N\}$;
- $\mathrm{C_c}(\Omega) = \{\varphi \in \mathrm{C}(\Omega) : \varphi \text{ compactly supported}\}$;
- $\mathrm{C_c^\infty}(\Omega) = \{\varphi \in \mathrm{C}^\infty(\Omega) : \varphi \text{ compactly supported}\}$.

We also denote

- $\mathrm{L}^1_+(\mathbb{R}^N) = \{\varphi \in \mathrm{L}^1(\mathbb{R}^N) : \varphi \geqslant 0 \text{ a.e.}\}$;
- $\mathrm{BC}_+(\mathbb{R}^N) = \{\varphi \in \mathrm{BC}(\mathbb{R}^N) : \varphi \geqslant 0\}$;
- $\mathrm{C}_0(\mathbb{R}^N) = \{\varphi \in \mathrm{C}(\mathbb{R}^N) : \varphi \to 0 \text{ as } |x| \to \infty\}$.

The elements ψ belonging to $\mathrm{L}^1((0,T); \mathrm{L}^1(\mathbb{R}^N))$ will sometimes be viewed as elements of $\mathrm{L}^1(\mathbb{R}^N \times (0,T))$. In such cases we denote $\psi(x,t) = \psi(t)(x)$.

Organization of the paper – In Section 2 we derive the general theory of the model both for integrable initial data and for continuous and bounded initial data. Section 3 is devoted to the study of mushy regions and free boundaries. Convergence to the local Stefan problem and disappearance of the mushy regions in the macroscopic scale are done in Section 4. In Section 5 we study the large time behavior of solutions. Numerical experiments and illustrations of the qualitative properties of the model are collected in Section 6. We devote the last section of the paper to establish some conclusions and make some comments on the model. In order to prove some of our results, we have needed to improve slightly the existing results on the asymptotic behavior of solutions to the non-local heat equation. Such improvements are proved in an appendix.

2. Basic theory of the model

We will develop here the basic theory for the two concepts of solution mentioned in the introduction.

2.1. L^1 **theory.** We start with the theory for integrable initial data. In this case the solution is regarded as a continuous curve in $\mathrm{L}^1(\mathbb{R}^N)$.

Definition 2.1. Let $f \in \mathrm{L}^1(\mathbb{R}^N)$. An L^1-solution of (1.1) is a function $u \in \mathrm{C}([0,\infty);\mathrm{L}^1(\mathbb{R}^N))$ such that (1.1) holds in the sense of distributions, or equivalently, if for every $t>0$, $u(t) \in \mathrm{L}^1(\mathbb{R}^N)$ and

$$u(t) = f + \int_0^t \big(J * v(s) - v(s)\big)\,\mathrm{d}s, \qquad v = (u-1)_+ \quad \text{a.e.} \tag{2.1}$$

Remark. If u is an L^1-solution, then $u \in \mathrm{L}^1(\mathbb{R}^N \times [0,T])$ for all $T>0$. Hence, (1.1) holds, not only in the sense of distributions, but also a.e., and u is said to be a *strong* solution. Moreover, since $v=(u-1)_+ \in \mathrm{C}([0,\infty);\mathrm{L}^1(\mathbb{R}^N))$, we also have $u \in \mathrm{C}^1([0,\infty);\mathrm{L}^1(\mathbb{R}^N))$, and the equation holds a.e. in x for all $t \geqslant 0$.

Theorem 2.2. *For any $f \in \mathrm{L}^1(\mathbb{R}^N)$, there exists a unique L^1-solution of* (1.1).

Proof. Let $\mathcal{B}_{t_0}$ be the Banach space consisting of the functions $u \in \mathrm{C}([0,t_0];\mathrm{L}^1(\mathbb{R}^N))$ endowed with the norm

$$|||u||| = \max_{0\leqslant t\leqslant t_0} \|u(t)\|_{\mathrm{L}^1(\mathbb{R}^N)}.$$

We define the operator $\mathcal{T}:\mathcal{B}_{t_0}\to\mathcal{B}_{t_0}$ through

$$(\mathcal{T}_f u)(t) = f + \int_0^t (J*(u-1)_+(s) - (u-1)_+(s))\,\mathrm{d}s.$$

This operator turns out to be contractive if t_0 is small enough. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\mathcal{T}_f\varphi - \mathcal{T}_f\psi|(t)\\
&\quad\leqslant \int_{\mathbb{R}^N}\int_0^t \Big(|J*((\varphi-1)_+ - (\psi-1)_+)(s)| + |(\varphi-1)_+ - (\psi-1)_+|(s)\Big)\,\mathrm{d}s\\
&\quad\leqslant \int_0^t \Big(\|J\|_{\mathrm{L}^1(\mathbb{R}^N)}+1\Big)\|((\varphi-1)_+ - (\psi-1)_+)(s)\|_{\mathrm{L}^1(\mathbb{R}^N)}\,\mathrm{d}s.
\end{aligned}$$

Hence

$$\begin{aligned}
|||\mathcal{T}_f\varphi - \mathcal{T}_f\psi||| &\leqslant 2t_0 \max_{0\leqslant t\leqslant t_0} \|((\varphi-1)_+ - (\psi-1)_+)(t)\|_{\mathrm{L}^1(\mathbb{R}^N)}\\
&\leqslant 2t_0|||\varphi-\psi|||.
\end{aligned}$$

Thus, $\mathcal{T}$ is a contraction if $t_0<1/2$. Existence and uniqueness in the time interval $[0,t_0]$ now follow easily, using Banach's fixed point theorem. Since the length of the existence and uniqueness time interval does not depend on the initial data, we may iterate the argument to extend the result to all positive times. □

The energy of the L^1-solutions is constant in time.

Theorem 2.3. *Let $f \in \mathrm{L}^1(\mathbb{R}^N)$. The L^1-solution u to* (1.1) *satisfies*

$$\int_{\mathbb{R}^N} u(t) = \int_{\mathbb{R}^N} f \qquad \text{for every } t > 0.$$

Proof. Since $u(t) \in \mathrm{L}^1(\mathbb{R}^N)$ for any t, integration of the equation (2.1) in space yields, thanks to Fubini's Theorem:

$$\int_{\mathbb{R}^N} u(t) = \int_{\mathbb{R}^N} f + \int_0^t \Big(\int_{\mathbb{R}^N} J * v(s) - \int_{\mathbb{R}^N} v(s) \Big) \,\mathrm{d}s = \int_{\mathbb{R}^N} f.$$

□

Our next aim is to derive an L^1-contraction property for L^1-solutions. In order to obtain it, we need first to approximate the graph $\Gamma(s) = (s-1)_+$ by a sequence of strictly monotone graphs $\Gamma_n(s)$ such that:

(i) there is a constant L independent of n such that $|\Gamma_n(s) - \Gamma_n(t)| \leqslant L|s-t|$, for all $n \in \mathbb{N}$ and $s, t \geqslant 0$;
(ii) for all $n \in \mathbb{N}$, $\Gamma_n(0) = 0$ and Γ_n is strictly increasing on $[0,\infty)$;
(iii) $\Gamma_n(s) \leqslant s$ for all $n \in \mathbb{N}$ and $s \geqslant 0$;
(iv) $\Gamma_n \to \Gamma$ as $n \to \infty$ uniformly in $[0,\infty)$;

(take for instance $\Gamma_n(s) = s/(n+1)$ for $0 \leqslant s \leqslant (n+1)/n$, and $\Gamma_n(s) = s-1$ for $s > (n+1)/n$).

Since Γ_n is Lipschitz, for any $f \in \mathrm{L}^1(\mathbb{R}^N)$ and any $n \in \mathbb{N}$ there exists a unique L^1-solution $u_n \in \mathrm{C}([0,\infty); \mathrm{L}^1(\mathbb{R}^N))$ of the approximate problem

$$\partial_t u_n = J * \Gamma_n(u_n) - \Gamma_n(u_n) \tag{2.2}$$

with initial data $u_n(0) = f$. The proof is just like that of Theorem 2.2. Moreover, $\Gamma_n(u_n) \in \mathrm{C}([0,\infty); \mathrm{L}^1(\mathbb{R}^N))$, and, hence, $u_n \in \mathrm{C}^1([0,\infty); \mathrm{L}^1(\mathbb{R}^N))$. Conservation of energy also holds.

The L^1-contraction for our original problem will follow from an analogous result for the approximate problems.

Lemma 2.4. *Let $u_{n,1}$ and $u_{n,2}$ be two L^1-solutions of* (2.2) *with initial data $f_1, f_2 \in \mathrm{L}^1(\mathbb{R}^N)$. Then,*

$$\int_{\mathbb{R}^N} \big(u_{n,1} - u_{n,2}\big)_+(t) \leqslant \int_{\mathbb{R}^N} \big(f_1 - f_2\big)_+ \qquad \text{for every } t \geqslant 0. \tag{2.3}$$

Proof. We subtract the equations for $u_{n,1}$ and $u_{n,2}$ and multiply by $\mathbb{1}_{\{u_{n,1} > u_{n,2}\}}$. Since $u_{n,1} - u_{n,2} \in \mathrm{C}^1([0,\infty); \mathrm{L}^1(\mathbb{R}^N))$, then

$$\partial_t(u_{n,1} - u_{n,2})\mathbb{1}_{\{u_{n,1} > u_{n,2}\}} = \partial_t(u_{n,1} - u_{n,2})_+.$$

On the other hand, since $0 \leqslant \mathbb{1}_{\{u_{n,1} > u_{n,2}\}} \leqslant 1$, we have

$$J * (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))\mathbb{1}_{\{u_{n,1} > u_{n,2}\}} \leqslant J * (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+.$$

Finally, since Γ_n is strictly monotone, $\mathbb{1}_{\{u_{n,1} > u_{n,2}\}} = \mathbb{1}_{\{\Gamma_n(u_{n,1}) > \Gamma_n(u_{n,2})\}}$. Thus,

$$(\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))\mathbb{1}_{\{u_{n,1} > u_{n,2}\}} = (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+.$$

We end up with

$$\partial_t(u_{n,1} - u_{n,2})_+ \leqslant J * (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+ - (\Gamma_n(u_{n,1}) - \Gamma_n(u_{n,2}))_+.$$

Integrating in space, and using Fubini's Theorem, which can be applied, since $(\Gamma_n(u_{n,1}(t)) - \Gamma_n(u_{n,2}(t)))_+ \in \mathrm{L}^1(\mathbb{R}^N)$, we get

$$\partial_t \int_{\mathbb{R}^N} (u_{n,1} - u_{n,2})_+(t) \leqslant 0.$$

□

Remark. It follows immediately from (2.3) that

$$\|(u_{n,1} - u_{n,2})(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} \leqslant \|f_1 - f_2\|_{\mathrm{L}^1(\mathbb{R}^N)}.$$

Corollary 2.5. *Let u_1 and u_2 be two L^1-solutions of* (1.1) *with initial data $f_1, f_2 \in \mathrm{L}^1(\mathbb{R}^N)$. Then, for every $t \geqslant 0$,*

$$\int_{\mathbb{R}^N} (u_1 - u_2)_+(t) \leqslant \int_{\mathbb{R}^N} (f_1 - f_2)_+. \tag{2.4}$$

Proof. The idea is to pass to the limit in n in the contraction property (2.3) for the approximate problems (2.2). Hence, the first step is to prove that any solution u of (1.1) is the limit of solutions u_n to (2.2).

Let ω be an open set whose closure is contained in $\mathbb{R}^N \times (0,\infty)$, $\omega \subset\subset \mathbb{R}^n \times (0,\infty)$. By the conservation of energy, $\|u_n(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} = \|f\|_{\mathrm{L}^1(\mathbb{R}^N)}$. Hence $\{u_n\}$ is uniformly bounded in $\mathrm{L}^1(\omega)$. Therefore, in order to apply Fréchet-Kolmogorov's compactness criterium, it is enough to control

$$I = \iint_\omega |u_n(x+h, t+s) - u_n(x,t)| \,\mathrm{d}x \,\mathrm{d}t$$

for h and s small enough (how small not depending on n).

On one hand, thanks to the L^1-contraction property,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} |u_n(x+h, t+s) - u_n(x, t+s)| \,\mathrm{d}x \,\mathrm{d}t \\ &\qquad \leqslant \int_0^T \int_{\mathbb{R}^N} |f(x+h) - f(x)| \,\mathrm{d}x \,\mathrm{d}t \leqslant T o_h(1) \end{aligned} \tag{2.5}$$

as $h \to 0$ uniformly in s and n. On the other hand, using the regularity in time, then Fubini's Theorem, and finally the sublinearity of Γ_n and the L^1-contraction property, we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} |u_n(x,t+s)-u_n(x,t)|\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\leqslant \int_0^T \int_{\mathbb{R}^N} \int_t^{t+s} |\partial_t u_n|(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad= \int_0^T \int_t^{t+s} \int_{\mathbb{R}^N} |J*\Gamma_n(u_n)-\Gamma_n(u_n)|(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t \\
&\qquad\leqslant \int_0^T \int_t^{t+s} (\|J\|_{\mathrm{L}^\infty(\mathbb{R}^N)}+1)\|\Gamma_n(u_n(\tau))\|_{\mathrm{L}^1(\mathbb{R}^N)}\,\mathrm{d}\tau\,\mathrm{d}t \\
&\qquad\leqslant (\|J\|_{\mathrm{L}^\infty(\mathbb{R}^N)}+1)\|f\|_{\mathrm{L}^1(\mathbb{R}^N)} sT.
\end{aligned} \tag{2.6}$$

Taking T such that $\omega\subset\mathbb{R}^N\times(0,T)$, and using the estimates (2.5) and (2.6) we get the required control.

Summarizing, along a subsequence (still noted u_n), $u_n\to\bar u$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N\times(0,\infty))$ for some function $\bar u$. Moreover: (i) since the sequence $\{u_n(t)\}$ is uniformly bounded in $\mathrm{L}^1(\mathbb{R}^N)$, we deduce from Fatou's lemma that for almost every $t>0$, $\bar u(t)\in\mathrm{L}^1(\mathbb{R}^N)$; (ii) using that the nonlinearities Γ_n are uniformly Lipschitz, and their uniform convergence, we get that $\Gamma_n(u_n)\to\Gamma(\bar u)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N\times(0,\infty))$; (iii) as a consequence, since J is compactly supported, $J*\Gamma_n(u_n)\to J*\Gamma(\bar u)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N\times(0,\infty))$. All this is enough to pass to the limit in the integrated version of (2.2),

$$u_n(t)=f+\int_0^t \big(J*\Gamma_n(u_n(s))-\Gamma_n(u_n(s))\big)\,\mathrm{d}s,$$

for almost every $t>0$. If we extend $\bar u(t)$ to all $t>0$ by continuity, so that it belongs to the space $\mathrm{C}^1([0,\infty);\mathrm{L}^1(\mathbb{R}^N))$, we get that $\bar u$ is the L^1-solution to (1.1) with initial data f, i.e., $\bar u=u$. As a consequence, convergence is not restricted to a subsequence.

Now we turn to the contraction property. Let u_1, u_2 be the L^1-solutions with initial data f_1 and f_2 respectively. We approximate them by the above procedure, which yields sequences $\{u_{n,i}\}$, $i=1,2$, such that $u_{n,i}\to u_i$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N\times(0,\infty))$ (and hence a.e.). The approximations satisfy (2.3). Using Fatou's lemma to pass to the limit in this last inequality we get that (2.4) holds for almost every $t\geqslant 0$. Finally, since the solutions are in $\mathrm{C}([0,\infty);\mathrm{L}^1(\mathbb{R}^N))$, we deduce that this inequality holds for any $t\geqslant 0$. □

Remark. Equation (2.4) implies a comparison principle. In particular, if $f\geqslant 0$, we conclude that $u\geqslant 0$. Hence, u is truly a *one-phase* solution.

The temperature turns out to be subcaloric.

Lemma 2.6. *Let $f\in\mathrm{L}^1(\mathbb{R}^N)$ and u the corresponding L^1-solution. Then the temperature $v=(u-1)_+$ satisfies $v_t\leqslant J*v-v$ in the sense of distributions and a.e. in $\mathbb{R}^N\times(0,\infty)$.*

Proof. Since $u\in\mathrm{C}^1([0,\infty);\mathrm{L}^1(\mathbb{R}^N))$,

$$v_t=u_t\mathbb{1}_{\{u>1\}}\qquad\text{a.e.}$$

In the set $\{u \leqslant 1\}$ we have $u_t = J * (u-1)_+ \geqslant 0$ and $v_t = 0$, whereas in the set $\{u > 1\}$ we have $v_t = u_t$ a.e.. In both cases we obtain $v_t \leqslant u_t = J * v - v$ a.e., and in the sense of distributions since these are locally integrable functions. $\square$

This property allows to estimate the size of the solution in terms of the L^∞-norm of the initial data.

Lemma 2.7. *Let $f \in \mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$. Then the L^1-solution u of* (1.1) *satisfies $\|u(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant \|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$ for any $t > 0$. Moreover, $\limsup_{t\to\infty} u(t) \leqslant 1$ a.e. in $\mathbb{R}^N$.*

Proof. The result is obvious if $\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant 1$, since in this case $u(t) = f$ for any $t > 0$. So let us assume that $\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)} > 1$. Since v is subcaloric and locally integrable we may use [8, Proposition 3.1] (with a compactly supported kernel), to obtain that

$$0 \leqslant \|v(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant \|(f-1)_+\|_{\mathrm{L}^\infty(\mathbb{R}^N)} = \|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)} - 1.$$

Therefore, $\|u(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant 1 + \|v(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant \|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$.

We can also compare v with the solution w of the following problem:

$$w_t = J * w - w, \quad w(0) = (f-1)_+ \in \mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N).$$

Using Theorem A.1 in the Appendix, we obtain that the solution v goes to zero asymptotically like $ct^{-N/2}$, so that $(u-1)_+ \to 0$ almost everywhere, which implies the result. $\square$

2.2. BC **theory.** We now develop a theory in the class of continuous and bounded functions whenever the initial data f belong to that class.

Definition 2.8. Let $f \in \mathrm{BC}(\mathbb{R}^N)$. The function u is a BC-solution of (1.1) if $u \in \mathrm{BC}(\mathbb{R}^N \times [0,T])$ for all $T \in (0,\infty)$ and

$$u(x,t) = f(x) + \int_0^t \big(J * v(x,s) - v(x,s)\big)\,\mathrm{d}s, \quad v = (u-1)_+$$

for all $x \in \mathbb{R}^N$ and $t \in [0,\infty)$.

Notice that if u is a BC-solution, then u_t is continuous. Hence equation (1.1) is satisfied for all x and t, and u is a *classical* solution.

Theorem 2.9. *For any $f \in \mathrm{BC}(\mathbb{R}^N)$ there exists a unique* BC*-solution of* (1.1).

Proof. As in the case of L^1-solutions, existence and uniqueness follow from a fixed-point argument.

We start by proving existence and uniqueness in a small time interval $[0, t_0]$. We define the operator $\mathcal{T} : \mathrm{BC}(\mathbb{R}^N \times [0,t_0]) \to \mathrm{BC}(\mathbb{R}^N \times [0,t_0])$ through

$$(\mathcal{T}_f u)(x,t) = f(x) + \int_0^t \left(J * (u-1)_+(x,s) - (u-1)_+(x,s)\right)\,\mathrm{d}s.$$

This operator is contractive if $t_0 < 1/2$, which implies the local existence and uniqueness result. Indeed, a similar computation to that of the proof of Theorem 2.2 yields

$$|\mathcal{T}_f\varphi - \mathcal{T}_f\psi|(x,t) \leqslant 2t_0 \max_{\mathbb{R}^N\times[0,t_0]} |\varphi-\psi|, \qquad t\in[0,t_0].$$

By iteration, taking as initial data $u(x,t_0)\in \mathrm{BC}(\mathbb{R}^N)$, we obtain existence and uniqueness for $[0,2t_0]$ and hence for all times. □

The BC-solutions depend continuously on the initial data.

Lemma 2.10. *Let u_1 and u_2 be the* BC*-solutions with initial data f_1, $f_2 \in \mathrm{BC}(\mathbb{R}^N)$ respectively . Then, for all $T\in(0,\infty)$ there exists a constant $C=C(T)$ such that*

$$\max_{x\in\mathbb{R}^N} |u_1-u_2|(x,t) \leqslant C(T)\max_{\mathbb{R}^N}|f_1-f_2|, \qquad t\in[0,T].$$

Proof. Since u_i is a fixed point of the operator $\mathcal{T}_{f_i}$, we have (see the proof of Theorem 2.9)

$$|u_1-u_2|(x,t)\leqslant |f_1-f_2|(x) + 2t_0\max_{\mathbb{R}^N\times[0,t_0]}|u_1-u_2|, \quad x\in\mathbb{R}^N, t\in[0,t_0].$$

Taking $t_0=1/4$, we get

$$\max_{\mathbb{R}^N\times[0,1/4]}|u_1-u_2|\leqslant 2\max_{\mathbb{R}^N}|f_1-f_2|,$$

from where the result follows by iteration, with a constant $C(T)=2^{4T}$. □

We also have a control of the size of the solutions in terms of the initial data. The proof is identical to the one for L^1-solutions.

Lemma 2.11. *Let u be the* BC*-solution u of* (1.1) *with initial data $f\in\mathrm{BC}(\mathbb{R}^N)$, and let v be the corresponding temperature. Then, v is subcaloric, and $\|u(\cdot,t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant \|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$ for any $t>0$.*

3. Free boundaries and mushy regions

In the sequel, unless we say explicitly something different, we will be dealing with L^1-solutions. Since the functions we are handling are in general not continuous in the space variable, their positivity sets have to be considered in the distributional sense. To be precise, for any locally integrable and nonnegative function g in $\mathbb{R}^N$, we can consider the distribution T_g associated to the function g. Then the distributional support of g, $\mathrm{supp}_{\mathcal{D}'}(g)$ is defined as the support of T_g:

$$\mathrm{supp}_{\mathcal{D}'}(g) := \mathbb{R}^N\setminus\mathcal{O}, \quad \text{where } \mathcal{O}\subset\mathbb{R}^N \text{ is the biggest open set such that } T_g|_{\mathcal{O}}\equiv 0.$$

In the case of nonnegative functions g, this means that $x\in\mathrm{supp}_{\mathcal{D}'}(g)$ if and only if

$$\forall\varphi\in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N),\ \varphi\geqslant 0 \text{ and } \varphi(x)>0 \Longrightarrow \int_{\mathbb{R}^N} g(y)\varphi(y)\,\mathrm{d}y>0.$$

If g is continuous, then the support of g is nothing but the usual closure of the positivity set, $\mathrm{supp}_{\mathcal{D}'}(g)=\overline{\{g>0\}}$.

3.1. **Existence of free boundaries.** We first prove that the solution does not move far away from the support of v.

Lemma 3.1. *Let $f \in \mathrm{L}^1_+(\mathbb{R}^N)$. Then,*

$$\mathrm{supp}_{\mathcal{D}'}(u_t(t)) \subset \mathrm{supp}_{\mathcal{D}'}(v(t)) + B_{R_J} \quad \text{for any } t \geqslant 0.$$

Proof. Recall first that the equation holds down to $t = 0$ so that we may consider here $t \geqslant 0$ (and not only $t > 0$). Let $\varphi \in \mathrm{C}^\infty_\mathrm{c}(A^c)$, where $A = \mathrm{supp}_{\mathcal{D}'}(v(t)) + B_{R_J}$. Notice that the support of $J * v$ (which is a continuous function) lies inside A, so that

$$\int_{\mathbb{R}^N} (J * v)\varphi = 0.$$

Similarly, the supports of v and φ do not intersect, so that

$$\int_{\mathbb{R}^N} u_t\varphi = \int_{\mathbb{R}^N} (J * v)\varphi - \int_{\mathbb{R}^N} v\varphi = 0,$$

which means that the support of u_t is contained in A. $\square$

As a direct consequence, we get the finite speed of propagation property.

Theorem 3.2. *Let $f \in \mathrm{L}^1_+(\mathbb{R}^N)$ and compactly supported. Then, for any $t > 0$, the solution $u(t)$ and the corresponding temperature $v(t)$ are compactly supported.*

Proof. Estimate of the support of v. Notice first that

$$\big(J * (u-1)_+\big)(x,t) \leqslant \|J\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \|(u-1)_+\|_{\mathrm{L}^1(\mathbb{R}^N)} \leqslant \|J\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \|f\|_{\mathrm{L}^1(\mathbb{R}^N)} := c_0,$$

where we have used the L^1-contraction property of the equation for the last estimate. Multiplying (2.1) by a nonnegative test function $\varphi \in \mathrm{C}^\infty_\mathrm{c}\left((\mathrm{supp}_{\mathcal{D}'} f)^c\right)$ and integrating in space and time we have

$$\int_{\mathbb{R}^N} u(t)\varphi \leqslant \int_0^t \int_{\mathbb{R}^N} \big(J * (u(t)-1)_+\big)\varphi \leqslant c_0\, t \int_{\mathbb{R}^N} \varphi\,.$$

Taking $t_0 := 1/c_0$, we get $\int_{\mathbb{R}^N}(u(t)-1)\varphi \leqslant 0$ for all $t \in [0, t_0]$. Using an approximation $\varphi\chi_n$ where $\chi_n \to \mathrm{sign}_+(u-1)$, we deduce that $\int_{\mathbb{R}^N}(u(t)-1)_+\varphi = 0$, so that

$$\mathrm{supp}_{\mathcal{D}'}(v(t)) \subset \mathrm{supp}_{\mathcal{D}'}(f), \qquad t \in [0, t_0]. \tag{3.1}$$

Estimate of the support of u. Lemma 3.1 implies then that

$$\mathrm{supp}_{\mathcal{D}'}(u_t(t)) \subset \mathrm{supp}_{\mathcal{D}'}(f) + B_{R_J}, \qquad t \in [0, t_0].$$

This means that for any $\varphi \in \mathrm{C}^\infty_\mathrm{c}\left((\mathrm{supp}_{\mathcal{D}'}(f) + B_{R_J})^c\right)$ we have

$$\int_{\mathbb{R}^N} u(t)\varphi = \int_0^t \int_{\mathbb{R}^N} u_t(t)\varphi = 0, \qquad t \in [0, t_0],$$

that is,

$$\mathrm{supp}_{\mathcal{D}'}(u(t)) \subset \mathrm{supp}_{\mathcal{D}'}(f) + B_{R_J}, \qquad t \in [0, t_0]. \tag{3.2}$$

Iteration. Notice that t_0 depends on the initial data f only through its L^1 norm. Hence, since the L^1 norm of the enthalpy is time invariant, the arguments can be iterated to obtain the result for all times. $\square$

Remark. The same argument can be used for initial data which are not compactly supported, to show that some positive time will pass before the temperature becomes positive at any given point in the complement of the support of $(f-1)_+$.

The last two results have counterparts for BC-solutions.

Theorem 3.3. *Let $f \in \mathrm{BC}_+(\mathbb{R}^N)$, and let u be the corresponding* BC*-solution. Then:*

(i) $u_t(x,t) = 0$ *for any* $x \in (\mathrm{supp}(v(\cdot,t)) + B_{R_J})^c$, $t \geqslant 0$.
(ii) *If* $\sup_{|x|\geqslant R} f(x) < 1$ *for some* $R > 0$, *then* $v(\cdot,t)$ *is compactly supported for all* $t > 0$. *If moreover* $f \in \mathrm{C_c}(\mathbb{R}^N)$, *then* $u(\cdot,t)$ *is also compactly supported for all* $t > 0$.

Proof. (i) The proof is similar (though even easier, since the supports are understood in the classical sense) to the one for L^1-solutions.

(ii) Using Lemma 2.11 we get

$$\big(J * (u-1)_+\big)(x,t) \leqslant \|J\|_{\mathrm{L}^1(\mathbb{R}^N)} \|(u-1)_+\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant \|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}.$$

Therefore, from (2.1) we have

$$u(x,t) \leqslant f(x) + t\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant \sup_{|x|\geqslant R} f(x) + t\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}, \qquad |x| \geqslant R. \tag{3.3}$$

Thus, for all $|x| \geqslant R$ and $t \leqslant (1-\sup_{|x|\geqslant R} f(x))/(2\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)})$ we have $u(x,t) < 1$, and hence $v(x,t) = 0$. Then, by (i), $u(x,t) = f(x)$ for all $|x| \geqslant R + R_J$ and $t = (1-\sup_{|x|\geqslant R} f(x))/(2\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)})$. We now proceed by iteration. □

3.2. **Equivalence of formulations.** As a corollary of the control of the supports, we will prove that if the initial data are in $\mathrm{L}^1_+(\mathbb{R}^N) \cap \mathrm{C}_0(\mathbb{R}^N)$, then the L^1-solution is in fact continuous.

Proposition 3.4. *Let $f \in \mathrm{L}^1_+(\mathbb{R}^N) \cap \mathrm{C}_0(\mathbb{R}^N)$. The corresponding* L^1*-solution is continuous.*

Proof. We start by considering the case where f is continuous, nonnegative and compactly supported. Since a BC-solution with a continuous and compactly supported initial data stays compactly supported in space for all times, it is also integrable in space for all times. Moreover, $u \in \mathrm{C}\big([0,T]; \mathrm{L}^1(\mathbb{R}^N)\big)$. Hence, it coincides with the L^1-solution with the same initial data.

We now turn to the general case, that will be dealt with by approximation: let $\{f_n\}$ be a sequence of continuous and compactly supported functions such that

$$\|f_n - f\|_{\mathrm{L}^\infty(\mathbb{R}^N)} < \frac{1}{n}, \qquad \|f_n - f\|_{\mathrm{L}^1(\mathbb{R}^N)} < \frac{1}{n}.$$

Let u_n^1, u^1 be the L^1-solutions with initial data respectively f_n and f, and u_n^c, u^c the corresponding BC-solutions. We know that $u_n^1 = u_n^c$. Now, using the L^1-contraction property for L^1-solutions, we have that $\|u_n^1 - u^1\|_{\mathrm{L}^1(\mathbb{R}^N\times[0,T])} \to 0$ for any $T \in [0,\infty)$. Moreover, by Lemma 2.10, $\|u_n^1 - u^c\|_{\mathrm{L}^\infty(\mathbb{R}^N\times[0,T])} \to 0$. Hence the result. □

We can now use an approximation argument to prove a comparison principle for general initial data in $\mathrm{BC}_+(\mathbb{R}^N)$. This can in turn be used to show that initial data in that class yield *one-phase* solutions (satisfying $u \geqslant 0$).

Corollary 3.5. *Let $f_1, f_2 \in \mathrm{BC}_+(\mathbb{R}^N)$, and u_1, u_2 the corresponding* BC*-solutions. If $f_1 \leqslant f_2$ then $u_1 \leqslant u_2$.*

Proof. Let $\{f_{1,n}\}$, $\{f_{2,n}\}$ be sequences of nonnegative, continuous and compactly supported functions such that $f_{1,n} \to f_1$, $f_{n,2} \to f_2$ uniformly, and $f_{1,n} \leqslant f_{2,n}$. Since $f_{1,n}, f_{2,n} \in \mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{C}_0(\mathbb{R}^N)$, the corresponding BC-solutions $u_{1,n}$, $u_{2,n}$, are also L^1-solutions. Hence, the comparison principle for L^1-solutions, Corollary 2.5, yields $u_{1,n} \leqslant u_{2,n}$. Passing to the limit in BC, and using the continuous dependence of BC-solutions on the initial data, Lemma 2.10, we conclude that $u_1 \leqslant u_2$. □

3.3. **Retention for u and v.** We next prove that the supports of both u and v are nondecreasing. We denote this property as retention.

We start by considering the case of BC-solutions.

Proposition 3.6. *Let $f \in \mathrm{BC}_+(\mathbb{R}^N)$, and let u be the* BC*-solution to problem* (1.1)*. Then $\partial_t u \geqslant -u$ and $\partial_t v \geqslant -v$ for all $t \geqslant 0$. In particular, u and v have the retention property.*

Proof. We have

$$\partial_t u = J * (u-1)_+ - (u-1)_+ \geqslant -u,$$

which, after integration, yields

$$u(x,t) \geqslant u(x,s)\mathrm{e}^{-(t-s)}, \qquad t \geqslant s. \tag{3.4}$$

This implies retention for u.

Concerning $v = (u-1)_+$, we have

$$\partial_t (u-1)_+ = \partial_t u \cdot \mathbb{1}_{\{u>1\}} \geqslant -(u-1)_+,$$

that is, $\partial_t v \geqslant -v$, from where retention follows. □

For L^1-solutions we also have retention for both u and v. In this case the supports have to be understood in the distributional sense.

Proposition 3.7. *Let $f \in \mathrm{L}^1_+(\mathbb{R}^N)$, and let u be the L^1-solution to problem* (1.1)*. Then u and the corresponding v have the retention property.*

Proof. Let $\{f_n\}$ be a sequence of functions in $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^N)$ such that $\|f_n - f\|_{\mathrm{L}^1(\mathbb{R}^N)} \to 0$. Let u_n be the L^1-solution (which coincides with the BC-solution) with initial data f_n. Thanks to the L^1-contraction property, $\|u_n(t) - u(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} \to 0$ for all $t > 0$. Moreover, since the temperature v is a Lipschitz function of u, we also have $\|v_n(t) - v(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} \to 0$.

Let φ be any nonnegative function, $\varphi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^N)$. Multiplying (3.4) (with u_n instead of u) by φ, integrating and letting $n \to \infty$, we get

$$\int_{\mathbb{R}^N} u(t)\varphi \geqslant \mathrm{e}^{-(t-s)} \int_{\mathbb{R}^N} u(s)\varphi, \qquad t \geqslant s,$$

from where retention for u in the distributional sense is immediate. The argument for v is identical. □

3.4. **Localization of the supports.** Our next aim is to prove, in the case of a nonincreasing kernel J, that for a wide class of initial data the supports of both u and v are *localized*: they are contained in a ball of fixed radius for all times.

The result will follow from comparison with solutions with initial data in $\mathrm{C}_0(\mathbb{R}^N)$ that are radial and strictly decreasing in the radial variable. Such solutions are continuous and radial (the latter fact comes from the uniqueness of the solutions and invariancy under rotations of the equation). Moreover, for any time $t \geqslant 0$, the water zone $\{v(t) > 0\}$ is compactly supported (see Theorem 3.3). The main technical difficulty stems from the fact that we are not able to prove that these solutions are decreasing in the radial variable for all times.

Lemma 3.8. *Let J be nonincreasing in the radial variable and $f \in \mathrm{C}_0(\mathbb{R}^N)$ nonnegative, radial, and strictly decreasing in the radial variable. Then the support of $v(t)$ is a ball of radius $r(t)$ for every $t \geqslant 0$ and the function r is continuous on $[0,\infty)$.*

Proof. For $t > 0$, let $r(t) := \inf\{r > 0 : \operatorname{supp}(v(\cdot,t)) \subset B_{r(t)}\}$. Thanks to Theorem 3.3 this quantity is well defined. By the retention property for v (Proposition 3.7), the function r is nondecreasing. Notice that a priori $\operatorname{supp}(v(t))$ could be strictly contained in $B_{r(t)}$, though we will prove that this is not the case.

Continuity of r. Assume for contradiction that $r(t_0^-) < r(t_0^+)$ at some time $t_0 > 0$. For any x such that $|x| > r(t_0^-)$ we have $v(x,t) = 0$ for all $t \leqslant t_0$. If moreover $|x| = r(t_0^+)$, the continuity of u yields $u(x,t_0) = 1$.

Let $x_a = (a,0,\dots,0)$, $x_b = (b,0,\dots,0)$, with $r(t_0^-) < a < b = r(t_0^+)$. We consider $w(t) := u(x_a,t) - u(x_b,t)$. For any $t \leqslant t_0$, we have

$$\begin{aligned} w'(t) &= ((J*v)(x_a,t) - v(x_a,t)) - ((J*v)(x_b,t) - v(x_b,t)) \\ &= (J*v)(x_a,t) - (J*v)(x_b,t) \\ &= \int_{|y|<r(t_0^-)} v(y,t)\big(J(x_a-y) - J(x_b-y)\big)\,\mathrm{d}y. \end{aligned}$$

Since $|x_a - y| < |x_b - y|$, for all $|y| \leqslant r(t_0^-)$, then $J(x_a-y) \geqslant J(x_b-y)$ in this region, because J is radially nonincreasing. Thus we obtain $w' \geqslant 0$. Using that $w(0) = f(x_a) - f(x_b) > 0$, we obtain $w(t_0) > 0$, so that $u(x_a,t_0) > u(x_b,t_0) = 1$. This is a contradiction, since $v(x_a,t_0) = 0$.

Continuity at $t = 0$ is easier. On one hand, from the retention property we have $r(0^+) \geqslant r(0)$. On the other hand, since f is strictly decreasing, we have $f(x) < 1$ if and only if $|x| > r(0)$. But, by the continuity of u, $f(x) = 1$ if $|x| = r(0^+)$. Therefore, $r(0^+)$ cannot be strictly greater that $r(0)$. We end up with $r(0) = r(0^+)$.

Conectedness. We now prove that $\operatorname{supp}(v(\cdot,t))$ is connected for all positive times. Assume, on the contrary, that there exists some $t_* > 0$ such that $\operatorname{supp}(v(\cdot,t_*))$ is disconnected. Hence, there are values $r(0) < a < b < r(t_*)$ such that $v(x,t_*)=0$ if $a\leqslant |x|\leqslant b$ and $v(x,t_*)>0$ if $b<|x|<b+\delta$ for some $\delta>0$. The retention property for v implies that $v(x,t)=0$ for $a\leqslant |x|\leqslant b$, $0\leqslant t<t_*$.

Let $t_d\in(0,t_*)$ be the time when the disconnected region outside the ball B_b appears,

$$t_d:=\sup\{t>0: v(x,t)=0 \text{ for } |x|\geqslant b\}=\sup\{t>0: v(x,t)=0 \text{ for } |x|\geqslant a\}.$$

Obviously, $r(t_d)\leqslant a$ and, on the other hand, $r(t_d^+)\geqslant b>a$, a contradiction with the continuity of r. □

Lemma 3.9. *Let J be nonincreasing in the radial variable. If $0\leqslant f\leqslant g$ a.e. for some $g\in \mathrm{L}^1_+(\mathbb{R}^N)\cap \mathrm{C}_0(\mathbb{R}^N)$ radial and strictly decreasing in the radial variable, then there exists some R, depending only on g, such that $\operatorname{supp}(v(t))\subset B_R$ for all $t\geqslant 0$.*

Proof. Let u_g be the L^1-solution with initial datum g and $v_g=(u_g-1)_+$. By comparison, $\operatorname{supp}(v(t))\subset \operatorname{supp}(v_g(t))$. Lemma 3.8 implies that $\operatorname{supp}(v_g(t))=B_{r_g(t)}$. The radius $r_g(t)$ can be estimated using the conservation of mass,

$$\int_{\mathbb{R}^N} g=\int_{\mathbb{R}^N} u_g(t)\geqslant \int_{\{u_g(t)>1\}} 1=\big|\operatorname{supp}(v_g(t))\big|=\omega_N r_g(t)^N,$$

where ω_N is the volume of the unit sphere in $\mathbb{R}^N$. This implies that

$$\operatorname{supp}(v(t))\subset \operatorname{supp}(v_g(t))\subset B_R,\qquad R=\left(\int_{\mathbb{R}^N} g/\omega_N\right)^{1/N}.$$

□

As a corollary of the localization of the support of the temperature, we obtain that $\|v(t)\|_{\mathrm{L}^1(\mathbb{R}^N)}$ tends to zero as $t\to\infty$ with an exponential rate.

Corollary 3.10. *Let J and f satisfy the hypotheses of Lemma 3.9. Then there are constants $C,k>0$ such that $\|v(t)\|_{\mathrm{L}^1(\mathbb{R}^N)}\leqslant C\mathrm{e}^{-kt}$ for all $t\geqslant 0$.*

Proof. Let R be such that $\operatorname{supp}(v(t))\subset B_R$ for all times, and V the L^1-solution to the nonlocal heat equation in B_R,

$$\begin{cases} V_t(x,t)=J*V(x,t)-V(x,t), & x\in B_R\ t>0,\\ V(x,t)=0, & x\notin B_R,\ t>0,\\ V(x,0)=(f-1)_+(x), & x\in B_R.\end{cases}$$

Since f is bounded, $(f-1)_+\in \mathrm{L}^2(B_R)$. Hence [12, Theorem 2], $\|V(t)\|_{\mathrm{L}^2(B_R)}$ decays exponentially in time.

As v is subcaloric, Lemma 2.6, $v\leqslant V$ in $B_R\times(0,\infty)$. This implies

$$\int_{\mathbb{R}^N} v(t)\leqslant (\omega_N R^N)^{1/2}\|V(t)\|_{\mathrm{L}^2(B_R)}\leqslant C\mathrm{e}^{-kt}.$$

□

For general integrable data we are only able to obtain a power-like decay rate.

Corollary 3.11. *Let $f\in \mathrm{L}^1(\mathbb{R}^N)$. Then $\|v(t)\|_{\mathrm{L}^1(\mathbb{R}^N)}=O(t^{-N/2})$.*

Proof. Since v is subcaloric and nonnegative, it is enough to compare it from above with the solution, V, to the non-local heat equation (A.1) with the same initial data. Thus, using the representation formula (A.2) for solutions to (A.1) (see the appendix), we get

$$\int_{\mathbb{R}^N} v(t) = \int_{\{v(t)>0\}} V(t) \leqslant e^{-t}\|(f-1)_+\|_{\mathrm{L}^1(\mathbb{R}^N)} + \int_{\{v(t)>0\}} \omega(t) * (f-1)_+,$$

where ω is the regular part of the fundamental solution to (A.1). Then we notice that the measure of the support of $v(t)$ is uniformly controlled,

$$|\{v(t)>0\}| = |\{u(t) \geqslant 1\}| \leqslant \int_{\{u(t)\geqslant 1\}} u(t) \leqslant \|f\|_{\mathrm{L}^1(\mathbb{R}^N)}.$$

Thus, using that $\|\omega(t)\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant Ct^{-N/2}$ (see [18]), we obtain

$$\int_{\mathbb{R}^N} v(t) \leqslant \mathrm{e}^{-t}\|(f-1)_+\|_{\mathrm{L}^1(\mathbb{R}^N)} + Ct^{-N/2}\|f\|_{\mathrm{L}^1(\mathbb{R}^N)}\|(f-1)_+\|_{\mathrm{L}^1(\mathbb{R}^N)} = O(t^{-N/2}).$$

□

If the initial data are bounded and compactly supported, we can obtain quantitative estimates for the supports of u and v. These estimates are sharp, as can be checked by considering indicator initial data.

Lemma 3.12. *Let J be nonincreasing in the radial variable and f nonnegative, bounded and compactly supported, contained in the ball of radius R_f. Then*

$$\operatorname{supp}(v(t)) \subset B_{R_v}, \quad \operatorname{supp}(u(t)) \subset B_{\max\{R_f, R_v+R_J\}}, \qquad R_v = \|f\|^{1/N}_{\mathrm{L}^\infty(\mathbb{R}^N)} R_f.$$

Proof. To obtain the estimate for the support of v we use Lemma 3.9 with functions g_n approximating $\mathbb{1}_{B_{R_f}}\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$ from above, and then pass to the limit in n. The estimate for the support of u then follows from Lemma 3.1. □

Remark. A better result should hold with a radius R_v depending on the mass of the initial data above level one, instead of the L^∞-norm of f.

3.5. **Creation of mushy regions.** Since the L^1-solutions to our problem are not necessarily continuous, we need a distributional definition of the mushy region.

Definition 3.13. The mushy region at time $t \geqslant 0$ of a nonnegative (L^1- or BC-) solution u to (1.1) is

$$\mathcal{M}(t) := \operatorname{Int}\Big(\operatorname{supp}_{\mathcal{D}'}(u(t)) \setminus \operatorname{supp}_{\mathcal{D}'}((u(t)-1)_+)\Big).$$

Remark. When u is continuous, $\mathcal{M}(t) = \{0 < u(x,t) < 1\}$.

Here comes one of the main features of our model: it allows the creation of mushy regions.

Theorem 3.14. *Let $f \in \mathrm{L}^1_+(\mathbb{R}^N)$ be a nontrivial initial data such that* $\operatorname{supp}_{\mathcal{D}'}(f) = \operatorname{supp}_{\mathcal{D}'}((f-1)_+)$. *Then,*

$$\mathcal{M}(t) = \big\{0 < \operatorname{dist}\big(x, \operatorname{supp}_{\mathcal{D}'}(f)\big) < R_J\big\}, \quad t \in [0, t_0], \quad t_0 = \frac{1}{\|J\|_{\mathrm{L}^\infty(\mathbb{R}^N)}\|f\|_{\mathrm{L}^1(\mathbb{R}^N)}}.$$

Proof. We first observe that the assumptions on the initial data imply that there are no mushy regions initially, $\mathcal{M}(0) = \emptyset$. Moreover,

$$\mathrm{supp}_{\mathcal{D}'}(v(t)) = \mathrm{supp}_{\mathcal{D}'}(f), \qquad t \in [0, t_0]. \tag{3.5}$$

The upper inclusion is just (3.1), and the lower one follows from the retention property and the equality of the supports of f and $(f-1)_+$.

The inclusion $\mathcal{M}(t) \subset \big\{0 < \mathrm{dist}\big(x, \mathrm{supp}_{\mathcal{D}'}(f)\big) < R_J\big\}$ is an immediate consequence of equations (3.2) and (3.5).

Let us turn then to the other inclusion. Let φ be a nonnegative and nontrivial test function compactly supported in $\{0 < \mathrm{dist}(x, \mathrm{supp}_{\mathcal{D}'}(f)\big) < R_J\}$. Using (3.1), we have

$$\begin{aligned}\int_{\mathbb{R}^N} u(t)\varphi &= \int_{\mathbb{R}^N} f\varphi + \int_0^t \int_{\mathbb{R}^N} J * v(t)\varphi - \int_0^t \int_{\mathbb{R}^N} v(t)\varphi \\ &= \int_0^t \int_{\mathbb{R}^N} J * v(t)\varphi, \qquad t \in [0, t_0].\end{aligned}$$

Since $\mathrm{supp}(J * v(t)) = \mathrm{supp}_{\mathcal{D}'}((v(t)) + B_{R_J} = \mathrm{supp}_{\mathcal{D}'}(f) + B_{R_J}$, we conclude that

$$\int_{\mathbb{R}^N} u(t)\varphi = \int_{\mathbb{R}^N} J * v(t)\varphi > 0, \quad t \in [0, t_0].$$

In other words,

$$\{0 < \mathrm{dist}(x, \mathrm{supp}_{\mathcal{D}'}(f))\big) < R_J\} \subset \mathrm{supp}_{\mathcal{D}'}(u(t)), \qquad t \in [0, t_0],$$

which combined with (3.5) gives the required inclusion. □

3.6. **Emergence of disconnected water regions.** Another interesting feature of our model is that disconnected components of water may appear suddenly at a positive distance from the already existing water components. This is another example of a phenomenon that occurs for the non-local model but not for the local one. The reason is that, contrary to the local model, problem (1.1) allows middle-range interactions, (up to a distance R_J).

Let us now construct examples exhibiting this phenomenon. We will keep things simple, so that the underlying mechanism is better understood. But the result can be easily generalized to more complex situations.

Let us assume that f is a bounded and continuous initial data with three well differentiated zones:

$$\begin{cases} \text{a "warm" water zone, } \mathcal{W}, \text{ where the enthalpy is above 1;} \\ \text{a low-enthalpy ice zone, } \mathcal{I}_L, \text{ where } f \text{ is clearly below 1;} \\ \text{a "high"-enthalpy ice zone, } \mathcal{I}_H, \text{ where } f \text{ is close to, but below level 1.} \end{cases} \tag{3.6}$$

We shall see that if the enthalpy in $\mathcal{I}_H$ is close enough to 1 and this zone is not too far from $\mathcal{W}$, then $\mathcal{I}_H$ melts before the low-enthalpy zone does. Therefore, if $\mathcal{I}_L$ "separates" $\mathcal{W}$ and $\mathcal{I}_H$ initially, a new disconnected component will emerge in the water zone. It is enough to prove it for $\mathcal{I}_H = \{x\}$, the general case resulting from this.

Definition 3.15. A set $\mathcal{S} \subset \mathbb{R}^N$ separates $\mathcal{A}$ and $\mathcal{B}$ if:

(i) $\mathcal{S}^c$ has at least two open connected components;

(ii) $\mathcal{A}$ and $\mathcal{B}$ lie in two different connected components of $\mathcal{S}^c$.

Theorem 3.16. *Let $J > 0$ in B_{R_J}, $x \in \mathbb{R}^N$ and consider two non-empty sets $\mathcal{W}, \mathcal{I}_L \subset \mathbb{R}^N$ such that*

- *$\mathcal{W}$ is open;*
- *$\operatorname{dist}(x, \mathcal{W}) < R_J$;*
- *$\mathcal{I}_L$ separates $\overline{\mathcal{W}}$ and $\{x\}$.*

Let $f \in \mathrm{BC}_+(\mathbb{R}^N)$ such that

- *$f > 1$ in $\mathcal{W}$, $f < 1$ in $\overline{\mathcal{W}}^c$;*
- *$0 \leqslant f \leqslant 1 - \eta$ in $\mathcal{I}_L$ for some fixed $\eta \in (0,1)$;*

Then there exists $\varepsilon \in (0, \eta)$ such that if $1 - \varepsilon < f(x) < 1$, then in a finite time there appears in the water zone a new connected water component $\mathcal{C}_x$ containing x.

Figures 1 and 2 illustrate the phenomenon, the exact meaning of notations being defined within the proof that follows.

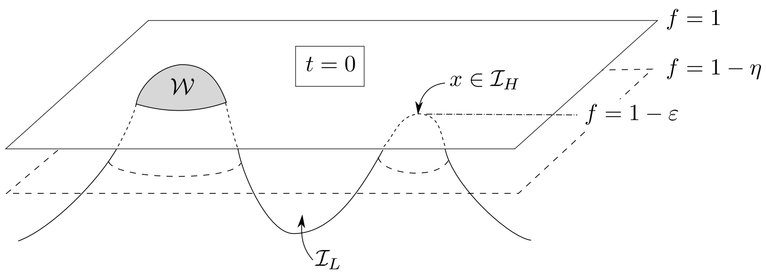

FIGURE 1. High-enthalpy zone near melting.

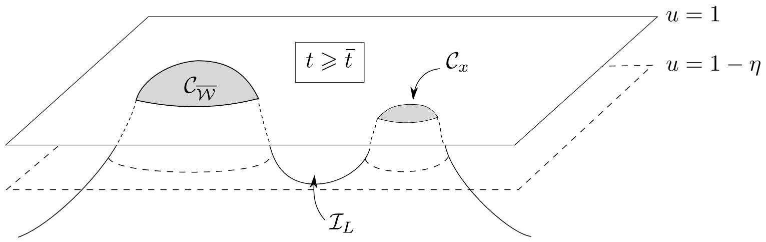

FIGURE 2. New disconnected component in water zone.

Proof. We proceed in three steps as follows.

STEP 1 – *For $0 \leqslant t < \eta/\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$, $\mathcal{I}_L \cap \{v(t) > 0\} = \emptyset$.*

Let $y \in \mathcal{I}_L$ and $t < \eta/\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$. Using (3.3), we have $u(y,t) < 1$. Thus, such an y remains in the ice zone.

STEP 2 – *Given $0 < \bar{t} < \eta/\|f\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, we have $x \in \{v(\bar{t}) > 0\}$.*

Assume on the contrary that $v(x, \bar{t}) = 0$ for all $\varepsilon > 0$. By our assumptions, since $\mathrm{dist}(x, \{f > 1\}) < R_J$, there exists $0 < \rho < R_J$ and $y_0 \in \{f > 1\}$ such that $\mathrm{dist}(x, y_0) < \rho$. Taking $\delta = (f(y_0) - 1)/2 > 0$, we have $f(y_0) > 1 + \delta$ so that the set

$$f^{(-1)}\big((1+\delta, +\infty)\big) = \{f > 1 + \delta\}$$

is open and contains y_0. Similarly, the ball $B_\rho(x)$ is also open and contains y_0 so that the intersection $\{f > 1+\delta\} \cap B_\rho(x)$ contains at least a ball $B_\tau(y_0)$ centered at y_0 with a positive radius $\tau > 0$. This has two consequences that we use below: first, $\{f > 1 + \delta\} \cap B_\rho(x)$ has a positive Lebesgue measure; second, $x - B_\tau(y_0) \subset B_{R_J}$, so that by assumption on J (up to taking a $\tau' < \tau$), $J(x - y)$ is uniformly bounded away from zero on $B_\tau(y_0)$.

Then, using the retention property for v, Proposition 3.6, we have

$$u(x, \bar{t}) > 1 - \varepsilon + \int_0^{\bar{t}} (J * v)(x, s)\,\mathrm{d}s \geqslant 1 - \varepsilon + \mathrm{e}^{-\bar{t}} \int_0^{\bar{t}} (J * v)(x, 0)\,\mathrm{d}s.$$

We estimate the integral as follows:

$$\begin{aligned} (J * v)(x, 0) &\geqslant \int_{\{f > 1+\delta\} \cap B_\rho(x)} J(x - y)\delta\,\mathrm{d}y \\ &\geqslant \delta \cdot \min_{B_\tau(y_0)} J(x - y) \cdot \big|\{f > 1 + \delta\} \cap B_\rho(x)\big| = C > 0, \end{aligned}$$

with C independent of ε. Hence, if $\varepsilon > 0$ is small enough we get

$$u(x, \bar{t}) > 1 - \varepsilon + \bar{t}\,\mathrm{e}^{-\bar{t}} C > 1,$$

which is a contradiction.

STEP 3 – *For $\bar{t}$ and ε as above, there is a new connected component containing x in the water zone.*

Since x and $\overline{\mathcal{W}}$ are separated by $\mathcal{I}_L$, there exist two open sets $\mathcal{O}_1, \mathcal{O}_2$ such that $\mathcal{O}_1 \cup \mathcal{O}_2 \subset (\mathcal{I}_L)^c$, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ and

$$\{x\} \in \mathcal{O}_1, \quad \overline{\mathcal{W}} \subset \mathcal{O}_2.$$

Thus, initially all the water zone is contained in $\mathcal{O}_2$ while x belongs to $\mathcal{O}_1$.

Now, by the retention property, we know that at time $\bar{t}$ the water zone still contains $\overline{\mathcal{W}}$. But at time $\bar{t}$ the water zone also contains x. Hence we are in the following situation:

$$\{v(\bar{t}) > 0\} \supset \{x\} \cup \overline{\mathcal{W}}, \quad \{v(\bar{t}) > 0\} \subset (\mathcal{I}_L)^c.$$

At time $\bar{t}$, let us denote by $\mathcal{C}_x$ the connected water component containing x and by $\mathcal{C}_{\overline{\mathcal{W}}}$ the one containing $\overline{\mathcal{W}}$, which are both non-empty. Since $\{v(\bar{t}) > 0\} \subset (\mathcal{I}_L)^c = \mathcal{O}_1 \cup \mathcal{O}_2$, the union being disjoint, it follows that necessarily $\mathcal{C}_x \subset \mathcal{O}_1$, $\mathcal{C}_{\overline{\mathcal{W}}} \subset \mathcal{O}_2$. Hence we deduce that

$$\mathcal{C}_x \cap \mathcal{C}_{\overline{\mathcal{W}}} = \emptyset.$$

In other words, at time $\bar{t}$, a new component has appeared in $\mathcal{O}_1$ which was not present initially, and it is even disconnected from all the water zones in $\mathcal{O}_2$. $\square$

Remark. A similar phenomenon takes place for non-continuous, integrable data. This can be proved either by using L^1-theory techniques, or by approximation from above and from below with continuous data.

4. The local Stefan problem as a limit in the macroscopic scale

If the support of the kernel is shrunk to a point through a suitable rescaling, we recover the local model. For the case of a bounded domain with Neumann boundary data, such convergence was already considered in [3] in the abstract setting of semi-group theory. We will give here an alternative, more direct proof, adapted to our problem. In addition, we will prove that mushy regions disappear in the limit.

Given a fixed initial datum $f \in \mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$, we consider the problem

$$\partial_t u^\varepsilon = \frac{1}{\varepsilon^2}(J_\varepsilon * v^\varepsilon - v^\varepsilon), \qquad v^\varepsilon = (u^\varepsilon - 1)_+, \qquad u^\varepsilon(\cdot, 0) = f, \tag{4.1}$$

where $J_\varepsilon = \varepsilon^{-N} J(\cdot/\varepsilon)$. Since J_ε is a unit mass kernel, compactly supported in the ball $B_{\varepsilon R_J}$, the various properties of solutions of (1.1) that we derived in the previous sections are still valid for solutions of (4.1). This latter problem admits a weak formulation, which will show to be quite convenient when passing to the limit: for any test function $\phi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N \times [0,\infty))$ we have

$$\int_{\mathbb{R}^N} u^\varepsilon(t)\phi(t) = \int_{\mathbb{R}^N} f\phi(0) + \int_0^t \int_{\mathbb{R}^N} (\partial_t \phi) u^\varepsilon + \frac{1}{\varepsilon^2}\int_0^t \int_{\mathbb{R}^N} (J_\varepsilon * \phi - \phi) v^\varepsilon. \tag{4.2}$$

This follows from Fubini's theorem, since J_ε is symmetric.

Lemma 4.1. *Let $f \in \mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$. The family $\{u^\varepsilon\}$ is relatively compact in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times (0,\infty))$.*

Proof. We first prove the relative compactness of $\{u^\varepsilon(t)\}$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$ for all $t > 0$ by means of Frechet-Kolmogorov's compactness criterium. To this aim we use the L^1-contraction property to show that: (i) the functions $u^\varepsilon(t)$ are uniformly bounded in $\mathrm{L}^1(\mathbb{R}^N)$; and (ii) for any compact set $\omega \subset \mathbb{R}^N$

$$\int_\omega |u^\varepsilon(x+h,t) - u^\varepsilon(x,t)| \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |f(x+h) - f(x)| \,\mathrm{d}x = o_h(1),$$

where $o_h(1)$ tends to 0 as $h \to 0$ independently of ε. Hence, along a subsequence, $u^\varepsilon(t) \to u(t)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$, for some function $u(t) \in \mathrm{L}^1(\mathbb{R}^N)$. We infer that $u^\varepsilon(x,t)$ converges for almost every (x,t).

Since $\|u^\varepsilon(t)\|_\infty \leqslant \|f\|_\infty$, we may now use the dominated convergence theorem to prove that u^ε converges to u in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times [0,\infty))$. □

This compactness result gives convergence along subsequences. The possible limit functions turn out to be weak solutions to the local Stefan problem

$$\partial_t u = \frac{m_2}{2}\Delta(u-1)_+, \quad u(\cdot,0) = f, \tag{4.3}$$

where $m_2 := \int_{\mathbb{R}^N} |z|^2 J(z)\,\mathrm{d}z$ is the second-order momentum of the kernel J, which is finite, since J is compactly supported. Since this problem has a unique weak solution [1], convergence is not restricted to subsequences.

Theorem 4.2. *Let $f \in \mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$. The sequence $\{u^\varepsilon\}$ of solutions to* (4.1) *converges as $\varepsilon \to 0$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times [0,\infty))$ to the unique weak solution of* (4.3).

Proof. Along a subsequence, $\{u^\varepsilon\}$ converges strongly in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times [0,\infty))$ to some function u, see Lemma 4.1. We also have convergence for $\{v^\varepsilon\}$ along some subsequence to $v = (u-1)_+$. Since the L^1-norms of solutions to (4.1) do not increase with time, see Corollary 2.5, the sequence $\{u_\varepsilon\}$ is uniformly bounded in $\mathrm{L}^\infty((0,\infty);\mathrm{L}^1(\mathbb{R}^N))$.

If we perform a Taylor expansion and use the symmetry of J, we get

$$\begin{aligned}
\frac{1}{\varepsilon^2}\Big(J_\varepsilon * \phi - \phi\Big) &= \frac{1}{\varepsilon^{2+N}}\int_{\mathbb{R}^N} J\Big(\frac{x-y}{\varepsilon}\Big)\Big(\phi(x)-\phi(y)\Big)\,\mathrm{d}y\\
&= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J(z)\Big(\phi(x)-\phi(x-\varepsilon z)\Big)\,\mathrm{d}z\\
&= \frac{m_2}{2}\Delta\phi + o(1) \qquad \text{as } \varepsilon \to 0^+
\end{aligned}$$

uniformly in (x,t) (recall that ϕ is compactly supported and smooth). Hence, passing to the limit in (4.2), we get that u is the unique weak solution to the local problem: for any test-function $\phi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^N \times [0,\infty))$,

$$\int_{\mathbb{R}^N} u(t)\phi(t) = \int_{\mathbb{R}^N} f\phi(0) + \int_0^t\int_{\mathbb{R}^N} \phi_t u + \frac{m_2}{2}\int_0^t\int_{\mathbb{R}^N} (\Delta\phi)(u-1)_+.$$

□

We next study the limit $\varepsilon \to 0$ for the mushy region $\mathcal{M}^\varepsilon(t)$ associated to u^ε for initial data such that $\mathcal{M}^\varepsilon(0) = \emptyset$. We first estimate the size of the mushy region for fixed $\varepsilon > 0$.

Theorem 4.3. *Let $f \in \mathrm{L}^1_+(\mathbb{R}^N)$ be a nontrivial initial data such that* $\mathrm{supp}_{\mathcal{D}'}(f) = \mathrm{supp}_{\mathcal{D}'}((f-1)_+)$. *For any $t > 0$,*

$$\mathcal{M}^\varepsilon(t) \subset \{x \in \mathbb{R}^N : 0 < \mathrm{dist}(x, \mathrm{supp}_{\mathcal{D}'}(v^\varepsilon(\cdot,t)) < \varepsilon R_J\}. \tag{4.4}$$

Proof. Let $x \notin \mathrm{supp}_{\mathcal{D}'}(v^\varepsilon(\cdot,t)) + B_{\varepsilon R_J}$ for some fixed $t > 0$. Since the support of $v^\varepsilon(t)$ is nondecreasing, then for any $0 \leqslant s \leqslant t$, $x \notin \mathrm{supp}_{\mathcal{D}'}(v^\varepsilon(\cdot,s)) + B_{\varepsilon R_J}$. Then Lemma 3.1 implies that $u^\varepsilon(x,t) = f(x) = 0$, which implies (4.4). □

Unfortunately, since the convergence of the functions $\{u^\varepsilon\}$ is rather weak, this is not enough to prove the convergence of the mushy regions. However, we will be able to prove that their limsup,

$$\mathcal{M}^*(t) = \limsup_{\varepsilon\to 0}\mathcal{M}^\varepsilon(t) = \bigcap_{\eta>0}\bigcup_{\eta<\varepsilon}\mathcal{M}^\varepsilon(t),$$

consisting of all points x such that for any $\eta > 0$ there exists an $\varepsilon \in (0,\eta)$ such that $x \in \mathcal{M}^\varepsilon(t)$, is a negligible set.

Recall that for the local problem, under our assumptions on the initial data no mushy regions are created, so that the supports of $v(t)$ and $u(t)$ coincide for all $t \geqslant 0$. Moreover, v is continuous in $\mathbb{R}^N \times (0,\infty)$ [9], hence the distributional support of $v(t)$ can be understood as the closure of the set $\{v(t) > 0\}$.

Corollary 4.4. *Let $f \in \mathrm{L}^1_+(\mathbb{R}^N)$ such that $\mathrm{supp}_{\mathcal{D}'}(f) = \mathrm{supp}_{\mathcal{D}'}((f-1)_+)$. Then, for any $t > 0$, $\mathcal{M}^*(t)$ has zero N-dimensional Lebesgue measure.*

Proof. We know that $u^\varepsilon(\cdot, t)$ converges pointwise to $u(\cdot, t)$ except on a set F which has zero N-dimensional Lebesgue measure. Apart from the set F there are three possibilities:

(i) $x \in \partial\{v(t) > 0\}$. This set is known to have zero N-dimensional Lebesgue measure [10].

(ii) $x \in \{v(t) = 0\} \setminus \partial\{v(t) > 0\}$. Then since there are no mushy regions for the limit (local) equation here, necessarily, $u(x,t) = 0$. This implies that for ε small enough (say less than ε_0), we have $u^\varepsilon(x,t) < 1$, thus $v^\varepsilon(x,t) = 0$. Hence x cannot belong to $\mathcal{M}^*(t)$ because x belongs to no mushy region for $\varepsilon < \varepsilon_0$, see Theorem 4.3.

(iii) $x \in \{v(t) > 0\}$. Then for ε small enough, $u^\varepsilon(x,t) > 1$ because it converges to $u(x,t) > 1$. Thus, for ε small enough, such an x does not belong to any mushy region, hence it is not in the limsup

Thus we have proved that $\mathcal{M}^*(t)$ is included in $F \cup \partial\{v(t) > 0\}$ which is a negligible set. □

5. Asymptotic behavior

Our next aim is to describe the large time behavior of the solutions to our model. For the local Stefan problem it is given by a 'mesa'-type problem [17]. To be more precise, u converges to $\tilde{f} = f + \Delta w$, where w solves the elliptic obstacle-type problem

$$w \geqslant 0, \quad 0 \leqslant f + \Delta w \leqslant 1, \quad (f + \Delta w - 1)w = 0.$$

In our case the limit is also given by a 'mesa', but now of a *non-local* character, see below. This is to be contrasted with the large time behavior of the non-local heat equation in the whole space, which is given by the solution of the *local* heat equation with the same data, and hence by a multiple of the fundamental solution of the latter equation.

5.1. Formulation of the Stefan problem as a parabolic non-local obstacle problem (in complementarity form).

We consider here a nonnegative initial u giving rise to a nonnegative solution u. To identify the asymptotic limit for u, we define the *Baiocchi variable*

$$w(t) = \int_0^t v(s)\,\mathrm{d}s.$$

A variable of this kind was first used by Baiocchi in 1971 to deal with the dam problem [4], [5]. The enthalpy and the temperature can be recovered from w through the formulas

$$u = f + J * w - w, \qquad v = \partial_t w, \tag{5.1}$$

where the time derivative has to be understood in the sense of distributions. Moreover,

$$0 \leqslant u - v \leqslant 1, \quad (u - 1 - v)v = 0 \quad \text{a.e.}$$

The distributional supports of v and w coincide for all times.

Lemma 5.1. *For any $t > 0$, we have* $\operatorname{supp}_{\mathcal{D}'}(v(t)) = \operatorname{supp}_{\mathcal{D}'}(w(t))$.

Proof. Let $x \in \operatorname{supp}_{\mathcal{D}'}(w(t))$. Then, given $\varphi \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^N)$, $\varphi \geqslant 0$, $\varphi(x) > 0$, we have

$$0 < \int_{\mathbb{R}^N} w(t)\varphi = \int_0^t \left(\int_{\mathbb{R}^N} v(s)\varphi \right) \mathrm{d}s.$$

Hence, there exists $s < t$ such that $\int_{\mathbb{R}^N} v(s)\varphi > 0$, i.e., $x \in \operatorname{supp}_{\mathcal{D}'}(v(s))$. Using the retention property for v we finally get that $x \in \operatorname{supp}_{\mathcal{D}'}(v(t))$.

Conversely, assume that $x \in \operatorname{supp}_{\mathcal{D}'}(v(t))$. Then, given $\varphi \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^N)$, $\varphi \geqslant 0$, $\varphi(x) > 0$, we have $\int_{\mathbb{R}^N} v(t)\varphi > 0$. Since $v \in \mathrm{C}([0,\infty); \mathrm{L}^1(\mathbb{R}^N))$, we have that there exists a value $\delta > 0$ such that $\int_{\mathbb{R}^N} v(s)\varphi > 0$ for all $s \in (t-\delta, t)$. This implies that $\int_{\mathbb{R}^N} w(t)\varphi > 0$, hence $x \in \operatorname{supp}_{\mathcal{D}'}(w(t))$. □

Thanks to this lemma, $(u - 1 - v)w = 0$ a.e. Hence w solves a.e. the complementarity problem

$$w \geqslant 0, \quad 0 \leqslant f + J * w - w - \partial_t w \leqslant 1, \quad (f + J * w - w - 1 - \partial_t w)w = 0, \tag{5.2}$$

plus the initial condition $w(0) = 0$. An analogous formulation for the local Stefan problem was given in [14], see also [16].

5.2. **A non-local elliptic obstacle problem.** If $\int_0^\infty \|v(t)\|_{\mathrm{L}^1(\mathbb{R}^N)}\,\mathrm{d}t < \infty$, then $w(t)$ converges monotonically and in $\mathrm{L}^1(\mathbb{R}^N)$ as $t \to \infty$ to

$$w_\infty = \int_0^\infty v(s)\,\mathrm{d}s \in \mathrm{L}^1(\mathbb{R}^N).$$

Thus, see (5.1), $u(\cdot, t)$ converges point-wisely and in $\mathrm{L}^1(\mathbb{R}^N)$ to

$$\tilde{f} = f + J * w_\infty - w_\infty.$$

Passing to the limit as $t \to \infty$ in (5.2), we get that w_∞ is a solution with data f to the *nonlocal obstacle problem*:

$$\text{(OP)} \qquad \begin{cases} \text{Given a non-negative data } f \in \mathrm{L}^1(\mathbb{R}^N)\text{, find a non-negative} \\ \text{function } w \in \mathrm{L}^1(\mathbb{R}^N) \text{ such that} \\ 0 \leqslant f + J * w - w \leqslant 1, \qquad (f + J * w - w - 1)w = 0 \quad \text{a.e.} \end{cases}$$

This non-local obstacle problem has a unique solution. The proof is based on the following Liouville type lemma for J-subharmonic functions.

Lemma 5.2. *Let $w \in \mathrm{L}^1(\mathbb{R}^N)$ such that $w \geqslant 0$, $w \leqslant J * w$ a.e. Then $w = 0$ a.e.*

Proof. Assume first that w is continuous, and fix $\varepsilon > 0$. Since w is integrable, there is a radius R such that

$$\int_{|x| \geqslant R} w \leqslant \frac{\varepsilon}{\|J\|_{\mathrm{L}^\infty(\mathbb{R}^N)}}.$$

Hence, for $|x| \geqslant R + R_J$

$$w(x) \leqslant (J * w)(x) \leqslant \|J\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \int_{B_{R_J}(x)} w \leqslant \|J\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \int_{|x| \geqslant R} w \leqslant \varepsilon. \tag{5.3}$$

So, let us assume that for some $x \in \mathbb{R}^N$, $w(x) > \varepsilon$. Then the maximum of w is attained at some point $\bar{x} \in B_{R+R_J}$ and

$$\max_{\mathbb{R}^N} w = w(\bar{x}) > \varepsilon.$$

Using that $w \leqslant J * w$, we first deduce that $w(x) = w(\bar{x})$ in $B_{R_J}(\bar{x})$ and then, spreading this property to all the space by adding each time the support of J, we conclude that $w = w(\bar{x}) > \varepsilon$ in all $\mathbb{R}^N$. But this is a contradiction with (5.3). So, we deduce that $0 \leqslant w \leqslant \varepsilon$ for any $\varepsilon > 0$, hence $w \equiv 0$.

If w is not continuous, we consider $w_n = w * \rho_n$, where ρ_n is an approximation of the identity. The continuous function w_n satisfies all the hypotheses of the lemma, hence $w_n = 0$. Letting $n \to \infty$ we obtain $w = 0$ a.e. □

We will also need the following non-local version of Kato's inequality (see [19] for the local inequality),

$$(J * w - w)\mathbb{1}_{\{w>0\}} \leqslant J * w_+ - w_+ \quad \text{a.e.,} \tag{5.4}$$

which is trivially valid for any function in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$.

Theorem 5.3. *Problem* (OP) *has at most one solution.*

Proof. The key point is that solutions to Problem (OP) satisfy

$$\tilde{f} = f + J * w - w, \quad \tilde{f} \in \beta(w) \quad \text{a.e.,}$$

where β is the sign graph, see [7] for the local case.

Let w_i, $i = 1, 2$, be two solutions to (OP) with initial data f, and $\tilde{f}_i$ be the corresponding projections. Since $\tilde{f}_i \in \beta(w_i)$, we have

$$0 \leqslant (\tilde{f}_1 - \tilde{f}_2)\mathbb{1}_{\{w_1>w_2\}} = \big(J * (w_1 - w_2) - (w_1 - w_2)\big)\mathbb{1}_{\{w_1>w_2\}} \quad \text{a.e.,}$$

from where we get, using Kato's inequality (5.4), that

$$(w_1 - w_2)_+ \leqslant J * (w_1 - w_2)_+.$$

Therefore, $(w_1 - w_2)_+$ satisfies the hypotheses of Lemma 5.2. We conclude that $w_1 \leqslant w_2$. Interchanging the roles of w_1 and w_2, we get the result. □

5.3. **The mesa problem.** At this point we have a precise characterization of the large time behavior of solutions to the non-local Stefan problem (1.1) whenever $\int_0^\infty \|v(t)\|_{\mathrm{L}^1(\mathbb{R}^N)}\,\mathrm{d}t$ is finite. This is the case, for instance, under the hypotheses of Lemma 3.9, see Corollary 3.10, or for general $f \in \mathrm{L}^1_+(\mathbb{R}^N)$ if $N \geqslant 3$, see Corollary 3.11.

Theorem 5.4. *Let $f \in \mathrm{L}^1_+(\mathbb{R}^N)$, and assume in addition, if $N = 1, 2$, the hypotheses of Lemma* 3.9 . *If u is the solution to problem* (1.1) *and w is the solution of problem* (OP), *then $u(t) \to f + J * w - w$ in $\mathrm{L}^1(\mathbb{R}^N)$ as $t \to \infty$.*

The map $\mathcal{P} : f \mapsto \tilde{f} = f + J * w - w$ projects the data f onto a 'mesa'-type profile: $0 \leqslant \tilde{f} \leqslant 1$, $\tilde{f} = 1$ in the *non-coincidence set* $\{w > 0\}$. Notice that $\|\tilde{f}\|_{\mathrm{L}^1(\mathbb{R}^N)} = \|f\|_{\mathrm{L}^1(\mathbb{R}^N)}$. However, in contrast with the local problem, the projection $\tilde{f}$ is not necessarily equal to f in the coincidence set $\{w = 0\}$.

Up to now we have been able to prove the existence of a solution of (OP) for any $f \in \mathrm{L}^1_+(\mathbb{R}^N)$ only if $N \geqslant 3$. For low dimensions, $N = 1, 2$, we have needed to add the hypotheses of Lemma 3.9. Hence, for low dimensions the projection operator $\mathcal{P}$ is in principle only defined under these extra assumptions. However, $\mathcal{P}$ is continuous, in the L^1-norm, in the subset of $\mathrm{L}^1_+(\mathbb{R}^N)$ of functions satisfying the hypotheses of Theorem 5.4.

Corollary 5.5. *Let f_i, $i = 1, 2$, satisfying the hypotheses of Theorem* 5.4. *Then*

$$\|\tilde{f}_1 - \tilde{f}_2\|_{\mathrm{L}^1(\mathbb{R}^N)} \leqslant \|f_1 - f_2\|_{\mathrm{L}^1(\mathbb{R}^N)}.$$

Proof. Since (OP) has uniqueness, any solution with initial data satisfying the hypotheses of Theorem 5.4 can be obtained as the limit as $t \to \infty$ of the solution with the same initial data of the non-local Stefan problem. Hence the result follows just passing to the limit as $t \to \infty$ in the contraction property for this latter problem. □

Since the class of functions satisfying the hypotheses of Theorem 5.4 is dense in $\mathrm{L}^1_+(\mathbb{R}^N)$, we can extend $\mathcal{P}$ by continuity to the whole of this bigger space. Thus, for any $f \in \mathrm{L}^1_+(\mathbb{R}^N)$, $\mathcal{P}f$ is the limit in L^1 of $\{\mathcal{P}f_n\}$, where $\{f_n\}$ is any sequence of non-negative, measurable, bounded, and compactly supported functions approximating f in $\mathrm{L}^1(\mathbb{R}^N)$.

Let us notice that, though for any sequence of functions $\{f_n\}$ converging to f in $\mathrm{L}^1(\mathbb{R}^N)$ we have convergence of $\{\mathcal{P}f_n\}$, we are not able to prove the convergence of the corresponding solutions $\{w_n\}$ to a solution of (OP), except under the hypotheses of Theorem 5.4. The main obstacle to prove this convergence is the lack of compactness of the inverse of $\mathcal{L}$ (recall $\mathcal{L} = J * v - v$).

5.4. **Asymptotic limit for general data.** A simple argument now leads to the following characterization of the asymptotic limit of the non-local Stefan problem for general integrable initial data.

Theorem 5.6. *Let $f \in \mathrm{L}^1_+(\mathbb{R}^N)$, and u the corresponding solution to problem* (1.1). *Let $\mathcal{P}f$ be the projection of f onto a non-local mesa. Then $u(\cdot, t) \to \mathcal{P}f$ in $\mathrm{L}^1(\mathbb{R}^N)$ as $t \to \infty$.*

Proof. Given f, let $\{f_n\} \subset \mathrm{L}^1(\mathbb{R}^N)$ be a sequence of functions satisfying the hypotheses of Theorem 5.4 which approximate f in $\mathrm{L}^1(\mathbb{R}^N)$. Let u_n be the corresponding solutions to the non-local Stefan problem. We have,

$$\|u(t) - \mathcal{P}f\|_{\mathrm{L}^1(\mathbb{R}^N)} \leqslant \|u(t) - u_n(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} + \|u_n(t) - \mathcal{P}f_n\|_{\mathrm{L}^1(\mathbb{R}^N)} + \|\mathcal{P}f_n - \mathcal{P}f\|_{\mathrm{L}^1(\mathbb{R}^N)}.$$

Using the contraction property for the non-local Stefan problem, and the large time behavior for bounded and compactly supported initial data,

$$\limsup_{t\to\infty} \|u(t) - \mathcal{P}f\|_{\mathrm{L}^1(\mathbb{R}^N)} \leqslant \|f - f_n\|_{\mathrm{L}^1(\mathbb{R}^N)} + \|\mathcal{P}f_n - \mathcal{P}f\|_{\mathrm{L}^1(\mathbb{R}^N)}.$$

Letting $n \to \infty$ we get the result. □

6. Numerical experiments

In order to illustrate some of the previous results concerning solutions of (1.1), we show some numerical experiments. We will take $f \in \mathrm{L}^1_+(\mathbb{R}^N)$ compactly supported, and $J(x) = 0.75(1 - x^2)_+$. The space discretization is implemented using the trapezoidal rule. For the integration in time we have used an ODE integrator provided by Matlab®.

6.1. **Creation of mushy regions.** In Figure 3 we illustrate the creation of mushy regions. As mentioned in the introduction, this phenomenon is absent in the local problem: if there are no mushy regions initially, this is also true for any later time. On the contrary, in our non-local model, regions with u between 0 and 1 appear in the neighborhood of the water region as time passes. This is one of the main qualitative features of the model. For this simulation we have taken $f(x) = 2 \cdot \mathbb{1}_{[-1,1]}$.

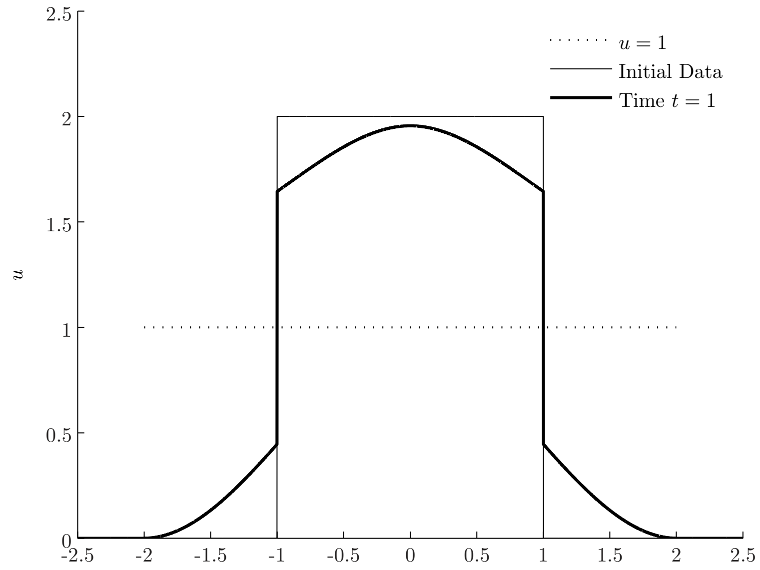

Figure 3. Creation of mushy regions.

6.2. **Appearance of disconnected water regions.** In the local case, it never happens that a new water region appears disconnected from the ones that were already present immediately before. On the contrary, this indeed happens sometimes in our non-local model, as shown in Section 3.6. We have exemplified this fact in figures 4 and 5, which correspond to an initial datum which is the sum of two characteristic functions, $f(x) = 2.5 \cdot \mathbb{1}_{[-1.25,-0.5]} + 0.99 \cdot \mathbb{1}_{[0.5,1]}$.

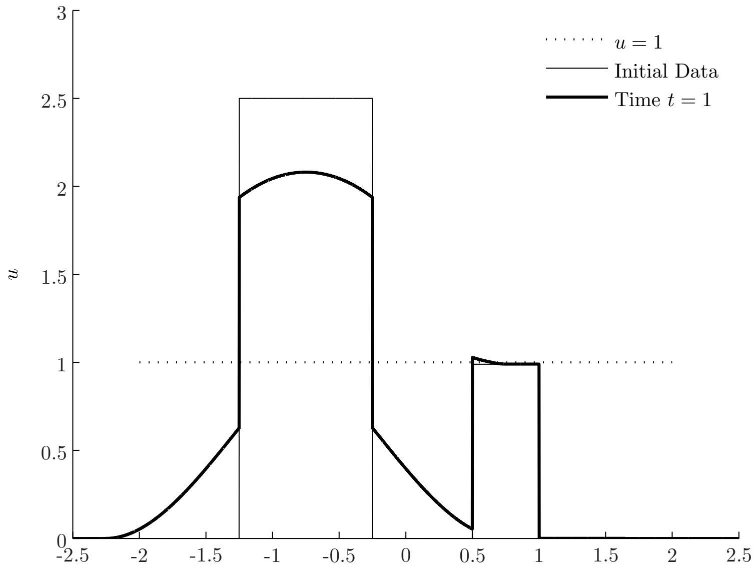

FIGURE 4. Appearance of disconnected water regions.

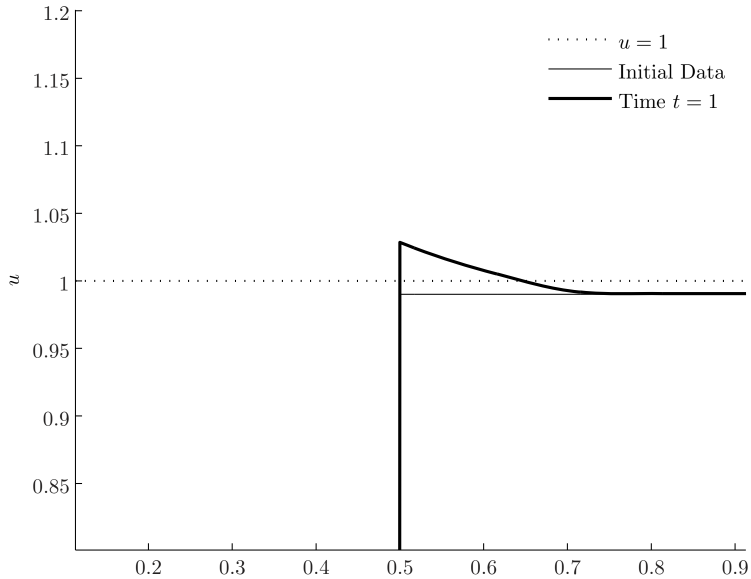

FIGURE 5. Appearance of disconnected water regions. Zoom.

6.3. **Behavior as $\varepsilon \to 0$.** In Figure 6 we illustrate the effect described in Section 4. When $\varepsilon \to 0$, the solution of the non-local problem converges to the solution of the local Stefan problem with the same initial data. Moreover, the mushy regions that were created because of the non-local effect disappear. The initial datum is again the characteristic function $f(x) = 2 \cdot \mathbb{1}_{[-1,1]}$. We compute the solution for $\varepsilon = 1$, 0.5 and 0.2.

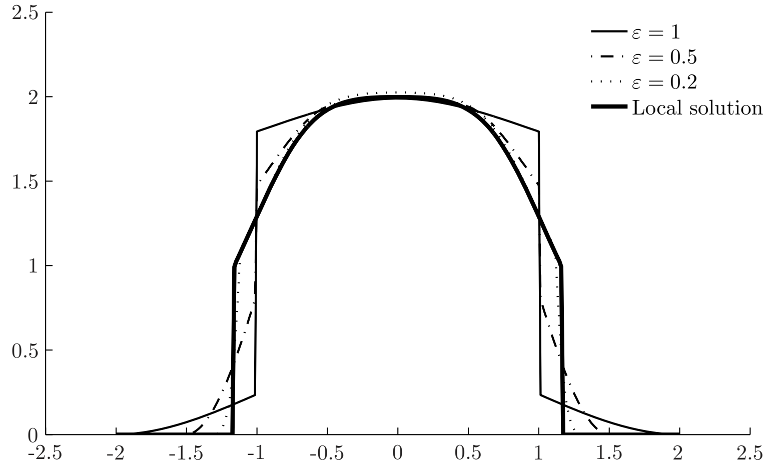

FIGURE 6. Convergence to the solution of the local problem as $\varepsilon \to 0$.

6.4. **Behavior as $t \to \infty$.** Finally we show an example of the asymptotic behavior of the solutions as $t \to \infty$, which is described in Section 5. We have taken here as initial datum

$$f(x) = \begin{cases} 0, & x < -4, \\ (\sin(5x))_+, & -4 \leqslant x \leqslant -1.5, \\ \sin(2x) + 3, & |x| < 1.5, \\ 0, & 1.5 \leqslant x \leqslant 6, \\ 0.3, & 6 < x < 6.5, \end{cases}$$

so that the convergence to a non-local mesa is more evident.

7. Conclusion

We have proposed a non-local model to describe the evolution of a mixture of ice and water in an intermediate mesoscopic scale, and derived some properties for the solutions which are interesting from the physical point of view, in particular the creation of mushy regions. We have also proved that the local Stefan problem is obtained in the macroscopic limit, and that mushy regions disappear in this latter scale if they were not present initially.

Theorem 3.14 does not only assert that mushy regions are indeed created. It has another interesting consequence reflected in (3.5): there is a waiting time $t_0 = t_0(f)$ until which the support of $v = (u-1)_+$ remains identical to that of $(f-1)_+$. In other words, the support of the water phase does not evolve until time t_0, though the temperature in this phase is decreasing because energy is consumed to break

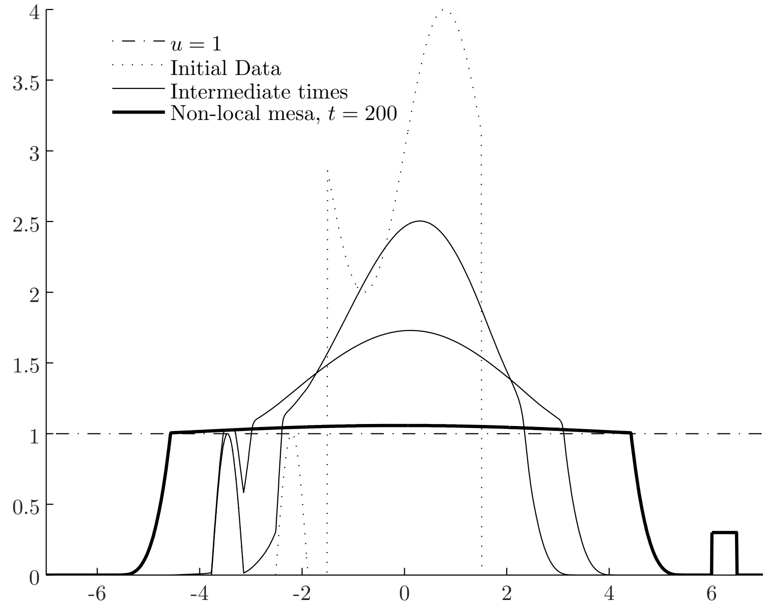

FIGURE 7. Convergence to the mesa as $t \to \infty$.

the ice. This waiting time can be interpreted as the time needed to break the nearby ice phase in order to convert it into water. We are facing a typical "latent heat" phenomenon which is not included in the usual local model. In the case of equation (1.3), the water phase begins to move instantaneously and ice is melted without any waiting time.

Summarizing, this nonlocal model seems meaningful and allows several natural phenomena at an intermediate scale which are not present in the local model.

Appendix

In the spirit of [12], we prove asymptotic estimates concerning the decay of solutions of the nonlocal heat equation

$$u_t = J * u - u, \qquad u(0) = f. \tag{A.1}$$

We only assume here that the kernel J is symmetric, nonnegative, with unit mass and a finite second order momentum denoted by m_2. We prove that the solution u is asymptotically similar to the solution of the local heat equation with the same initial data. The gain with respect to the paper [12] is that our results are valid for general initial data $f \in \mathrm{L}^1(\mathbb{R}^N)$, without assuming anything about their Fourier transform or their L^∞-norm.

Notice that the solutions will not eventually enter in the class of data considered in [12] (unless they were already there initially). Indeed, they can be written as

$$u(t) = \mathrm{e}^{-t} f + \big(\omega(t) * f\big) \tag{A.2}$$

with $\omega(t)$ smooth, integrable and bounded, [8]. Hence, u is not bounded if $f \notin \mathrm{L}^\infty(\mathbb{R}^N)$. However, by subtracting $\mathrm{e}^{-t} f$ to $u(t)$ we ensure that $u(t) - \mathrm{e}^{-t} f$ is bounded, since $\omega(t)$ is bounded and $f \in \mathrm{L}^1(\mathbb{R}^N)$. Moreover,

$$\|\hat{u}(t) - \mathrm{e}^{-t}\hat{f}\|_{\mathrm{L}^1(\mathbb{R}^N)} \leqslant \|\hat{\omega}(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} \|\hat{f}\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \leqslant \|\hat{\omega}(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} \|f\|_{\mathrm{L}^1(\mathbb{R}^N)} < \infty,$$

so that the Fourier transform of the difference is in $\mathrm{L}^1(\mathbb{R}^N)$. This is important to go back to the original variables after making the computations in Fourier variables.

Theorem A.1. *Let $f \in \mathrm{L}^1(\mathbb{R}^N)$ and u be the solution of* (A.1) *with $u(0) = f$. Let h be the solution of*

$$h_t = \frac{m_2}{2}\Delta h, \qquad h(0) = u(0) = f.$$

Then, as $t \to \infty$, there exists a function $\varepsilon(t) \to 0$ (depending only on J and N) such that

$$t^{N/2} \max_{\mathbb{R}^N} \left|u(t) - \mathrm{e}^{-t} f - h(t)\right| \leqslant \|f\|_{\mathrm{L}^1(\mathbb{R}^N)} \varepsilon(t). \tag{A.3}$$

Proof. Following [12, Theorem 2.2], we start from

$$\hat{u}(\xi, t) = \mathrm{e}^{(\hat{J}(\xi)-1)t} \hat{f}(\xi) \quad \text{and} \quad \hat{h}(\xi, t) = \mathrm{e}^{-c|\xi|^2 t} \hat{f}(\xi), \quad \text{with } c = \frac{m_2}{2},$$

but we make a different estimate using $u(t) - \mathrm{e}^{-t} f$ instead of $u(t)$,

$$\begin{aligned}
\int_{\mathbb{R}^N} |\hat{u} - \mathrm{e}^{-t}\hat{f} - \hat{f}|(\xi, t)\,\mathrm{d}\xi &= \int_{\mathbb{R}^N} \left|\left(\mathrm{e}^{t(\hat{J}(\xi)-1)} - \mathrm{e}^{-t} - \mathrm{e}^{-c|\xi|^2 t}\right)\hat{f}(\xi)\right| \mathrm{d}\xi \\
&= \int_{|\xi| \geqslant r(t)} \left|\left(\mathrm{e}^{t(\hat{J}(\xi)-1)} - \mathrm{e}^{-t} - \mathrm{e}^{-c|\xi|^2 t}\right)\hat{f}(\xi)\right| \mathrm{d}\xi \\
&\quad + \int_{|\xi| < r(t)} \left|\left(\mathrm{e}^{t(\hat{J}(\xi)-1)} - \mathrm{e}^{-t} - \mathrm{e}^{-c|\xi|^2 t}\right)\hat{f}(\xi)\right| \mathrm{d}\xi \\
&= I + II.
\end{aligned}$$

As in [12], we split I into two parts

$$I \leqslant \int_{|\xi| \geqslant r(t)} \left|\mathrm{e}^{-c|\xi|^2 t} \hat{u}_0(\xi)\right| \mathrm{d}\xi + \int_{|\xi| \geqslant r(t)} \left|\mathrm{e}^{t(\hat{J}(\xi)-1)} - \mathrm{e}^{-t}\right| |\hat{u}_0(\xi)|\,\mathrm{d}\xi = I_1 + I_2.$$

The only important modification with respect to the proof of [12, Theorem 2.2] concerns the term I_2. In [12] this term is estimated by using the L^1-norm of $\hat{f}$, which is something we do not want to do here.

In order to estimate I_2 we use that that $\hat{J}$ verifies

$$\hat{J}(\xi) \leqslant 1 - c|\xi|^2 + |\xi|^2 h(\xi), \quad \text{with } h \text{ bounded}, \quad h(\xi) \to 0 \text{ as } \xi \to 0.$$

Hence there exist $a, D, \delta > 0$ such that

$$\hat{J}(\xi) \leqslant 1 - D|\xi|^2, \text{ for } |\xi| \leqslant a \quad \text{and} \quad |\hat{J}(\xi)| \leqslant 1 - \delta, \text{ for } |\xi| \geqslant a.$$

We decompose I_2 by considering separately the sets $\{r(t) \leqslant |\xi| \leqslant a\}$ and $\{|\xi| \geqslant a\}$. The integration over $\{|\xi| \geqslant a\}$ is estimated here taking into account the term $\mathrm{e}^{-t} f$,

$$\int_{|\xi| \geqslant a} \left|\mathrm{e}^{t(\hat{J}(\xi)-1)} - \mathrm{e}^{-t}\right| |\hat{f}(\xi)|\,\mathrm{d}\xi = \mathrm{e}^{-t} \int_{|\xi| \geqslant a} \left|\mathrm{e}^{t\hat{J}(\xi)} - 1\right| |\hat{f}(\xi)|\,\mathrm{d}\xi.$$

Using that in this set $|\hat{J}| \leqslant 1 - \delta$, we get that

$$\begin{aligned}
\left|\mathrm{e}^{t\hat{J}(\xi)} - 1\right| |\hat{f}(\xi)| &\leqslant |\hat{f}(\xi)| \sum_{n=1}^{\infty} \frac{t^n |\hat{J}(\xi)|^n}{n!} \\
&\leqslant t|\hat{J}(\xi)||\hat{f}(\xi)| \sum_{n=0}^{\infty} \frac{t^n (1-\delta)^n}{(n+1)!} \\
&\leqslant t\mathrm{e}^{(1-\delta)t} |\hat{J}(\xi)| \|\hat{f}\|_\infty.
\end{aligned}$$

Hence

$$t^{N/2}\int_{|\xi|\geqslant a}\left|\mathrm{e}^{\,t(\hat{J}(\xi)-1)}-\mathrm{e}^{\,-t}\right||\hat{f}(\xi)|\,\mathrm{d}\xi\leqslant t^{N/2+1}\mathrm{e}^{\,-\delta t}\|\hat{f}\|_\infty\int_{|\xi|\geqslant a}|\hat{J}(\xi)|\,\mathrm{d}\xi\leqslant\|f\|_1\varepsilon_2(t),$$

where $\varepsilon_2(t)\to 0$ exponentially fast, and independently of the initial data.

Summing up (see [12, Theorem 2.2] for the estimates of the other terms of $I+II$)

$$t^{N/2}\|\hat{u}(t)-\mathrm{e}^{\,-t}\hat{f}-\hat{h}(t)\|_{\mathrm{L}^1(\mathbb{R}^N)}\leqslant\|f\|_{\mathrm{L}^1(\mathbb{R}^N)}\varepsilon(t)\to 0$$

as $t\to\infty$ for some function ε which only depends on J and N. This implies (A.3) by going back to the original variables. □

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