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ON SEARCH GAMES THAT INCLUDE AMBUSH*

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Abstract. We present a stochastic game that models ambush/search in a finite region Q which has area but no other structure. The searcher can search a unit area of Q in unit time or adopt an "ambush" mode for a certain period. The searcher "captures" the hider when the searched region contains the hider's location or if the hider moves while the searcher is in ambush mode. The payoff in this zero sum game is the capture time. Our game is motivated by the (still unsolved) princess and monster game on a star graph with a large number of leaves.

Key words. noisy search game, ambush strategy, Poisson process

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Isaacs [22] introduced the princess and monster game on a graph in his book on differential games. This game on a graph has remained largely unsolved, contrary to the game in a domain, which was solved by Gal [17]. In the 1980s Alpern and Asic [3, 4] solved the game on a graph with three unit length arcs connecting two nodes. This is the only nontrivial graph for which the game has been solved up to now. Gal [18] already noticed that the game on a graph can be solved if ambush strategies are not allowed. Progress on the princess and monster game on a graph has been slow, and that is why we propose to approach the game from a stochastic point of view, ignoring the geometry of the space, in order to focus on ambush strategies.

In the original princess and monster game, the players have no visibility. There are versions of the game in which either one or both players have limited or full visibility, so there is an increased probability of detection. However, even if there is no visibility, there may be central locations in which the monster has a higher probability of detecting the princess, simply because she has a high probability of crossing that location once she moves to a new hiding place. In the princess and monster game on a graph, this would apply if all nodes are connected to just a few central nodes. Instead of searching the entire space as quickly as possible, the Monster may want to spend extra time in these central locations, waiting in ambush. In our paper, we study the trade-off between ambush and search in the princess and monster game. A similar type of trade-off occurs in the search of heterogeneous environments, when the searcher has a sensor with a sensitivity that depends on the location, such as studied in [11].

There is a considerable literature on games that involve search, and the names of the games vary. They can be games between cops and robbers, or pursuers and evaders, or searchers and hiders, or hunters and rabbits. For instance, the game that we consider here is similar to the hunter and rabbit game on a graph that is studied in [1]. In our paper, we study the expected capture time, or search value [3], which

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is equivalent to the escape length in [1]. Our work is motivated by Isaacs' game. Related games are studied in computer science. These games are based on Parson's pursuit-evasion game [28], in which a party of searchers is looking for a lost man in a dark cave. In Isaacs' game the searcher minimizes the expected time of capture, while in Parsons' game, the party of searchers minimizes their total number, which is an NP-complete problem [24].

1. Introduction. In his 1965 book *Differential Games*, Isaacs included a final chapter on pursuit-evasion games of incomplete information and presented the following game which he named "princess and monster" [22, p. 349]:

> *The monster P searches for the princess E, the time required being the payoff. They are both in a totally dark room $\mathcal{R}$ (of any shape) but they are each cognizant of its boundary (possibly through small light admitting perforations high in the walls). Capture means that the distance $PE \leq l$, a quantity small in comparison with the dimension of $\mathcal{R}$. The monster, supposed highly intelligent, moves at a known speed. We permit the princess full freedom of locomotion.*

The princess and monster game has developed over the years into the theory of search games [7, 18, 19, 21]. Isaacs' game was eventually solved by Gal [17] under the restriction that the search space $\mathcal{R}$ is convex; see also [16, 20]. It was subsequently shown that this restriction can be removed by some minor modifications on the proof in [17], so the solution extends to any connected $\mathcal{R}$; see [18, p. 88]. The value of the game is equal to the time it takes the monster to exhaustively search the whole space. Later, Lalley and Robbins [23] introduced a solution of the game that is stochastic and involves point processes, enabling the monster to randomize his search strategy so that it remains optimal even if the princess has some information on his whereabouts. The solution of [23] applies to convex $\mathcal{R}$ only.

Isaacs already suggested simplifying the search space by considering search games on one-dimensional spaces. The princess is captured if the monster is in the same place at the same time. It is customary to normalize the game by choosing the maximum speed of the searcher such that an exhaustive search takes a unit amount of time, i.e., the total length of the space is one and the monster's maximum speed is one as well. The search game on a circle was solved by Zelikin [32] and Alpern [2], who showed that it was optimal for both players to adopt the so-called coin-half-tour strategy: start at a random location and oscillate between it and its antipode, choosing the direction by a coin toss every time. The value of the game on the circle with a fixed starting point for the searcher is equal to 1, and it is $\frac{3}{4}$ if the starting point is random.

Search games with a mobile hider on general networks remain unsolved, and even the game on the interval has not been completely settled [6]. For some specific networks the analysis becomes manageable if the searcher is restricted to *nonloitering* strategies, i.e., he is not allowed to sit in *ambush* at a node during any positive time interval (see [18]). The value of the game on these networks under the nonloitering restriction is equal to 1. Not much is known if the nonloitering restriction is dropped. Alpern and Asic [4] studied the game on the graph I_k of two nodes that are connected by k arcs of length $1/k$. They solved the game for $k = 3$ and this remains the only search game on a network containing nodes with degree > 2 that is solved with ambush allowed. The value of the game on I_3 is equal to 1. Originally Gal conjectured in [17] that the value of the search game on any network with a fixed starting point is at least 1. This was refuted in [3], where it is shown that the search value of the spectacles network, i.e., two circles connected by a short line segment, is less than 1.

Let C_k be the star with k edges, i.e., a single node with k edges of length 1. In [4] it is conjectured that the minimal value of the game on a network is obtained by the game on C_k by letting k go to infinity.

The same conjecture applies to the k-leafed clover, because the search game on the clover is asymptotically equivalent to the star. Some evidence for this conjecture is supported in [1], where the clover is used to obtain a lower bound on the escape length in terms of the diameter of a graph.

In this paper we present a new stochastic approach to the princess and monster game that is motivated by the game on C_k. A unit speed searcher can search the star exhaustively in time one and will find an *immobile* hider in mean time $\frac{1}{2}$ if he does so. To avoid making the searcher's task so simple, a mobile hider can "run" at intermittent times and switch to a random end node. To counter this, the searcher can wait in ambush at the central node, in which case he catches the hider if she runs, but he wastes time if she does not. If the searcher *searches* when the hider runs, then he does not catch the hider, but if he *ambushes*, then the hider is caught with full probability. The princess and monster game has never been solved on the star, or almost any other nontrivial network, because the analysis of the ambush strategy is difficult. The main idea behind our paper is that the analysis of the ambush strategy can be simplified by ignoring the geometry. We define the search space only by its size (length or area or measure), which we normalize to one. The searcher has two abstract modes, *searching*, corresponding to moving along the edges of the star, and *ambushing*, corresponding to waiting at the central node. We call this an *ambush game*. Our results are relevant for graphs in which all nodes are connected to a small number of central nodes, and we will discuss this at greater length at the end of our paper.

To simplify the analysis we impose the restriction that both players are "noisy": the hider knows how much of the search region the searcher has cleared (since she last moved) and the searcher knows how long ago the hider last moved. In terms of Isaacs' game, the princess hears the monster moving and the monster hears the princess run. Thus the "state variable" s that is known to both players [9] is the fraction of the region that has been searched since the last move of the hider. A small time delay prevents the players from knowing each other's current mode (ambush or search, run or hide). The hider cannot run knowing that the searcher is not ambushing, nor can the searcher ambush knowing that the hider is running. We call this the *(noisy) ambush game*. It is related to the search-and-ambush game in [31] and the infiltration-ambush game in [10]. Our ambush game is a stochastic analogue of the search game of a moving fugitive on a network that is described in [27]. In that game, which is motivated by counterterrorism studies, both the searcher and the hider are noisy. Traditionally, search games are motivated by military applications or counterterrorism studies, but more recently biologists have proposed that search games may be used to understand animal behavior, in particular, the behavior of predators and prey that have no direct visual contact [15].

In the animal behavior literature, the tactics of predator and prey have been studied mostly separately. One of the few integrated studies so far is on the tactics between a spider and a cricket [14]. Crickets are particularly challenging prey to catch because their air-sensing system enables them to detect the movement of approaching predators. It has been observed that spiders adopt two strategies when they try to catch a cricket: they either remain motionless while the cricket moves around, and when they try to catch the cricket they either move slowly or move suddenly at maximum speed. Crickets, on the other hand, can move around slowly, or suddenly jump to a different location. This type of run-or-hide tactic of the prey animal has also been observed in lizards [25]. So far, this run-or-hide behavior has only been explained by means of pursuit-evasion games, in which predator and prey can see each other [12]. In our paper, the predator and prey have no direct visual contact.

In an initial attempt that was addressed to the animal behavior literature on predators searching for prey [5], we considered a search game in which the hider (prey animal) knows the current ambush probability and hence the probability of being caught if the decision is to run. This assumption was made to simplify the mathematical analysis. In the simplified game, a pure searcher strategy is a differentiable function $s(t)$ that represents the fraction of the space that has been searched by time t. The hider knows the three state variables $t, s(t), s'(t)$, respectively, time, the amount of space that has been searched, and the probability of search. This simplified game was solved in [5], but the optimal hider maximin strategy is not very satisfactory: the hider moves as soon as the searcher deviates from his optimal minimax strategy $s(t) = t - \frac{1}{4}t^2$. The assumption that the hider knows $s'(t)$, and therefore the probability of ambush in the next infinitesimal time interval, is unrealistic. In this paper we remove this unrealistic assumption, so now the hider only knows the state variables t and $s(t)$. This also brings the ambush game closer to the original princess and monster game, in which the hider only knows the state variable t. It turns out that the value of the game and the optimal searcher strategy remain the same if the hider no longer knows $s'(t)$, but the optimal hider strategy changes: it is described by a nonhomogeneous Poisson process, which is the mathematical equivalent of run-or-hide.

We assume that the reader is well aware of the basic notions of game theory, which can be found in [9, 26]. The paper is organized as follows. In section 2 we define the noisy ambush game. In section 3 we first discuss the value of the game and derive the optimal strategies of the players by assuming that the value of the game exists and is differentiable. We then prove in section 4 that the value indeed exists and is differentiable, based on the optimal strategies that we found in section 3. In section 5 we indicate some lines of further research. Our assumption that an ambushing searcher catches a moving hider with full probability is mainly for convenience. This assumption can be relaxed, as we discuss in the conclusion of our paper. The main work that has to be done is the removal of the restriction that the players are noisy. There has been some recent progress in this direction; see [8].

2. The noisy ambush game Γ. We study a zero-sum search game in which a hider H has taken cover in a confined area. A searcher S tries to find H, either by searching a part of the confined area or by laying an ambush, waiting for H to run and take cover in a new hiding place within that area. The hider's payoff is the time T when S finds H, either by encountering H while she is hiding or by ambushing her when she is running. The time scale is chosen such that it takes S one unit to search the entire space. At the start of the game, H is uniformly distributed in a search space of unit measure. At time t since the start of the game both players decide what to do in the subsequent infinitesimal time interval $[t, t + dt)$: S may *search* or *ambush* and H may *run* (move to a new random location) or *hide* (remain where she is at time t). If H runs successfully at time t, then the same game is played again from the start, but the hider's payoff is raised by t. This game Γ is a recursive game which depends on the time variable t and the space variable s, which represents the fraction of the search space that has been searched since H made her last run. Both players know the value of s and t (but t turns out to be irrelevant). The *stage game* $\Gamma(t, s)$

occurs if the hider has not yet been found at time t and if the searched region since the last run has size s. The stage game takes place over an infinitesimal time interval $[t, t+dt)$. Each player has two strategies: H can *run* or *hide*; S can *search* or *ambush*. Depending on the strategy pair played, either the game will end or a new stage game is played. These two outcomes may both occur with positive probability. In case H makes a successful run to a new random location within that time interval, we could say that the game ends and the payoff is $t + \mathcal{V}$, where $\mathcal{V}$ is the value of the game Γ. This is shorthand for saying that the game restarts with t and s both zero, and the hider has remained uncaught to a time t. In other words, the hider gets t and $\Gamma(0, 0)$, which is equivalent to a payoff of $t + \mathcal{V}$. There are two combinations which may lead to the game ending. If H runs while S is ambushing, the game ends and the payoff is t. If H hides while S is searching, then she is found with probability $dt/(1-s)$, and the game ends on average with payoff $t + dt/2$.

The stage game can be written as a 2×2 matrix game where the payoffs are either numbers or games at a later stage. The maximizing hider as row player and the minimizing searcher as column player:

$\Gamma(t,s)$	search	ambush
run	$t + dt/2 + \mathcal{V}$	$t + dt/2$
hide	$t + dt/2$, probability $\frac{dt}{1-s}$ found $\Gamma(t+dt, s+dt)$, probability $1 - \frac{dt}{1-s}$ not found	$\Gamma(t+dt, s)$

Since $\Gamma(t,s) = t + \Gamma(0,s)$, in terms of the payoff t is a "sunk cost": it has gone toward the hider's payoff, but it is irrelevant to the future play of the game. Thus we can refer to the stage game simply as $\Gamma(s)$. A mixed strategy for the hider in the stage game $\Gamma(s)$ is specified by the probability $r = r(s)dt$ that she runs during a time interval of size dt when the state variable is s. For the searcher, the mixed strategy can similarly be specified by the probability $p = p(s)$ that he will search. In analyzing the game, we forget that p and r are probabilities and simply view them as control variables of the players in a differential game. Our aim is to find the optimal mixed strategies and the value for the stage game $\Gamma(s)$. In particular we are interested in $\mathcal{V}$, the value of the original game $\Gamma(0)$. From now on, we write $r(s)dt$ for the optimal mixed hider strategy and $p(s)$ for the optimal mixed searcher strategy.

The main result of this paper is that the noisy ambush game has a value of $\frac{2}{3}$. The optimal mixed strategy of the hider is to run with intensity $r(s) = \frac{3}{2\sqrt{1-s}}$. The optimal mixed strategy of the searcher is to search with probability $\sqrt{1-s}$.

3. Solution of the stage game, assuming its value exists. To solve the stage game $\Gamma(s)$ we assume for now that it has a well-defined value $\mathcal{V}(s)$ that depends smoothly on s. We will only prove in the next section that this assumption is justified. We can replace the stage games in the matrix above by their continuation values, regarding $t + dt$ as a sunk cost (it is equivalent to work from t, but the formulas are slightly easier from $t + dt$):

(3.1)

	search	ambush
run	$\mathcal{V}(0) + O(dt)$	$O(dt)$
hide	$(1 - \frac{dt}{1-s})\mathcal{V}(s+dt) + \frac{dt}{1-s} \cdot -\frac{1}{2}dt$	$\mathcal{V}(s)$

The four continuation values in this matrix are calculated as follows. The first row of the matrix contains the payoffs in case H runs. If S searches during $[t, t+dt)$, then

H escapes. The state variable s is reset to zero and the game starts again at the time of running. Since $t + dt$ is a sunk cost, we put the payoff at $\mathcal{V}(0) + O(dt)$, where the order dt term depends on the actual time of running. If H runs while S ambushes, then she is caught at the time of running and the payoff is $O(dt)$. The second row of the matrix contains the payoffs if H hides. In this case, if S searches during $[t, t+dt)$, then H is caught with probability $\frac{dt}{1-s}$ at expected time $t + \frac{1}{2}dt$. Since we calculate from a sunk cost of $t + dt$, the hider's payoff in the stage game is $-\frac{1}{2}dt$. If H hides but is not caught during $[t, t + dt)$ while S searches, then the game continues with state variable $s + dt$. Finally, if S ambushes while H hides, then the state variable s remains invariant while the time increases to $t + dt$. In other words, the continuation value is $\mathcal{V}(s)$. This explains all entries in the matrix.

If we ignore the contributions of order dt, then we get the following matrix:

(3.2)

	search	ambush
run	$\mathcal{V}(0)$	0
hide	$\mathcal{V}(s)$	$\mathcal{V}(s)$

This matrix has a saddle point at hide-ambush. If the hider hides and does not run, then the hider's payoff is $\mathcal{V}(s)$ regardless of the strategy of the searcher. Conversely, if the searcher searches with probability

$$p(s) = \frac{\mathcal{V}(s)}{\mathcal{V}(0)}, \tag{3.3}$$

and ambushes with probability $1 - p(s)$, then the hider's payoff is $\mathcal{V}(s)$ regardless of her strategy. So we have found strategies for both players that up to order dt make their opponents indifferent. Since we have considered the payoff matrix up to order dt, we know the optimal strategies up to order dt. The searcher strategy in the ambush game is given by $s(t)$, which is the integral over the probability of search up to time t. To determine an integral, one needs to know the function value up to first order only. So we can determine $s(t)$ from $p(s) = \frac{\mathcal{V}(s)}{\mathcal{V}(0)}$. The strategy of the hider, on the other hand, requires higher order terms. Since the hider runs with probability $O(dt)$ in the time interval $[t, t + dt)$, her strategy is a random running time τ that has an intensity $\mathrm{P}(\tau \in [t, t + dt) \mid \tau \geq t) = r(t)dt$. To determine this intensity $r(t)$ we need to include the first order terms in the game matrix:

(3.4)

	search	ambush
run	$\mathcal{V}(0) + O(dt)$	$O(dt)$
hide	$\mathcal{V}(s) + \left(\mathcal{V}'(s) - \frac{\mathcal{V}(s)}{1-s}\right) dt + O(dt^2)$	$\mathcal{V}(s)$

Since $\mathcal{V}'(s)$ is negative, the saddle point has disappeared and the matrix has mixed equilibrium strategies. To make S indifferent up to $O(dt)$, H hides with probability $1 - O(dt)$ and runs with probability $O(dt)$. We will calculate the value of the game $\mathcal{V}(s)$ and the intensity $r(s)$ from the continuation matrix in (3.4).

3.1. Derivation of the value $\mathcal{V}(s)$. The value of the game $\mathcal{V}(s)$ is equal to the value of the continuation matrix (3.4) plus dt, since the continuation values are calculated with a sunk cost of $t + dt$. The optimal searcher strategy is to search with probability $p(s)$ and to ambush with probability $1 - p(s)$. By evaluating the second row of the matrix we find that

$$\mathcal{V}(s) = dt + p(s)\left(\mathcal{V}(s) + \left(\mathcal{V}'(s) - \frac{\mathcal{V}(s)}{1-s}\right) dt + O(dt^2)\right) + (1 - p(s))\mathcal{V}(s).$$

The constant terms cancel and the first order term gives rise to

$$1 + p(s)\left(\mathcal{V}'(s) - \frac{\mathcal{V}(s)}{1-s}\right) = 0.$$

By taking limits and by using the fact that we have determined $p(s)$ up to order dt in (3.3), we arrive at the differential equation

$$\mathcal{V}(0) + \left(\mathcal{V}'(s)\mathcal{V}(s) - \frac{\mathcal{V}(s)^2}{1-s}\right) = 0. \tag{3.5}$$

We write $y = \mathcal{V}^2$ and we multiply by $1 - s$ in order to obtain a linear differential equation

$$\frac{1}{2}(1-s)y' - y + (1-s)\mathcal{V}(0) = 0$$

which has general solution

$$\frac{K}{(1-s)^2} + \frac{2}{3}(1-s)\mathcal{V}(0)$$

for an arbitrary constant K. By definition $\mathcal{V}(1) = 0$ and therefore $K = 0$. We conclude that $\mathcal{V}(s) = \sqrt{\frac{2}{3}\mathcal{V}(0)(1-s)}$ and evaluating this at $x = 0$ we get $\mathcal{V}(0) = \frac{2}{3}$. So we conclude that if the stage game has a value, then it is given by

$$\mathcal{V}(s) = \frac{2}{3}\sqrt{1-s}. \tag{3.6}$$

It is remarkable that $\frac{2}{3}\sqrt{1-s}$ is the only bounded solution to our differential equation. All other solutions have a value of $\pm\infty$ at $s = 1$.

3.2. Derivation of the optimal strategies. Now that we know the value of the game, we derive the optimal mixed strategies for the players. The optimal searcher strategy $p(s)$ follows immediately by substituting $\mathcal{V}(s)$ into (3.3). We need to determine the optimal hider strategy. The probability that H runs during the time interval $[t, t + dt)$ is given by $r(s)dt$. Multiply the continuation matrix A^T in (3.4) by the row-vector $v = (r(s)dt, 1 - r(s)dt)$ and equate the first order terms in the coordinates of $v \cdot A$, to make the searcher indifferent up to first order. We get the first order differential equation:

$$r(s)\mathcal{V}(0) + \mathcal{V}'(s) - \frac{\mathcal{V}(s)}{1-s} = 0. \tag{3.7}$$

By substituting the value that we found in (3.6) and solving this differential equation we find that the optimal hider strategy is

$$r(s) = \frac{3}{2\sqrt{1-s}}. \tag{3.8}$$

Interestingly, this differential equation for $r(s)$ gives another way to calculate the value of the game, as follows. If S decides on a perpetual ambush from time t on, then s remains constant and hence so does the hider's intensity to run $r(s)$. Indeed, the time that H runs is given by a stationary Poisson process with intensity $r(s)$.

It has expected capture time $\frac{1}{r(s)}$. Since S is indifferent, the expected capture time by perpetual ambush must be equal to $\mathcal{V}(s)$. We conclude that $\frac{\mathcal{V}(0)}{\mathcal{V}(s)}+\mathcal{V}'(s)-\frac{\mathcal{V}(s)}{1-s}=0$, which is equivalent to the differential equation for $\mathcal{V}(s)$ above. Summarizing these results we arrive at the following.

THEOREM 3.1. *If the value of the noisy ambush game* $\Gamma(s)$ *exists, then the value is* $\mathcal{V}(s)=\frac{2}{3}\sqrt{1-s}$. *The probability of search is given by* $p(s)=\sqrt{1-s}$ *and the intensity to run is given by* $r(s)=\frac{3}{2\sqrt{1-s}}$.

4. Proof that the value of the stage game exists.

We have determined the optimal strategies for the players on the assumption that the value of the game $\mathcal{V}(s)$ exists. We also assumed that the value is differentiable with respect to s, while a priori, even continuity at $s=0$ is not immediately obvious. To conclude the proof of our main result, we still need to prove that the value of the game exists and is differentiable. We now give this proof, using the probability $p(s)$ and the intensity $r(s)$ that we found in the previous section. To be precise, we prove that $p(s)$ guarantees an expected capture time of at most $\mathcal{V}(s)$ against any hider strategy and that $r(s)$ guarantees an expected capture time of at least $\mathcal{V}(s)$ against any searcher strategy.

4.1. Optimal searcher strategy.

If the state variable is equal to s, then the searcher searches with probability $p(s)$, independently of the game's history. In other words,

$$\frac{ds}{dt}=p(s).$$

Since $p(s)$ is known, we can solve this differential equation and express s as a function of t.

PROPOSITION 4.1. *If* $p(s)=\sqrt{1-s}$, *then* $s(t)=t-\frac{1}{4}t^2$.

Note that the probability of search in terms of the state variable is given by $p(t)=1-\frac{1}{2}t$. So the probability of search decays linearly until $t=2$, by which time the entire space has been searched.

PROPOSITION 4.2. *If S uses the strategy $p(s)$, then the expected capture time is finite.*

Proof. We prove that the expected capture time is finite even if S lets the hider get away if she is caught in ambush. In other words, we compute the expected time that on the assumption that S only catches H in hiding and show that it is finite. The probability that S finds H in the time interval $[t,t+dt)$ is equal to

$$\frac{p(s)dt}{1-s}=\frac{\left(1-\frac{1}{2}t\right)dt}{1-t-\frac{1}{4}t^2}=\frac{dt}{1-\frac{1}{2}t}.$$

So the probability that S finds H in the time interval $[t,t+dt)$ regardless of the state variable s is $\geq dt$, which is the intensity of an exponential random variable with parameter one. It follows that the expected capture time is stochastically dominated by an exponential random variable of expected value 1. So the expected capture time is ≤ 1. ∎

THEOREM 4.3. *If S adopts the strategy $p(s)$, then the expected capture time is $\frac{2}{3}$ against any pure hider strategy.*

This theorem is proved in [5] for the game in which the hider knows both the state variable s and its time derivative s'.

Proof. Suppose H adopts the pure strategy in which she runs at time T. If $T\geq 2$, then S has searched the entire interval before H runs, and S finds H in hiding in expected time $\frac{2}{3}$. Therefore we may suppose that $T < 2$. The probability that H is found in hiding is equal to $s(T) = T - \frac{1}{4}T^2$. If H is not found by time T, then the probability that he is caught in ambush is equal to $1 - p(T) = \frac{1}{2}T$. Therefore H gets away with probability $\left(1 - T + \frac{1}{4}T^2\right)\left(1 - \frac{1}{2}T\right)$ and the game renews. Summarizing, the expected payoff is equal to

$$\int_0^T tp(t)dt + \left(1 - T + \frac{1}{4}T^2\right)\frac{T^2}{2} + \left(1 - T + \frac{1}{4}T^2\right)\left(1 - \frac{1}{2}T\right)(T + \mathcal{V}),$$

where $\mathcal{V}$ denotes the continuation value of $\Gamma(0)$ in case the searcher uses the strategy $p(s)$. If we write $h(T) = T - \frac{1}{2}T^2 + \frac{1}{12}T^3$, then this expected payoff is equal to

$$h(T) + \left(1 - \frac{3}{2}h(T)\right)\mathcal{V}.$$

The hider maximizes the expected capture time by choosing an optimal T, so by recursion we get that $\mathcal{V} = h(T) + \left(1 - \frac{3}{2}h(T)\right)\mathcal{V}$, and it follows that $\mathcal{V} = \frac{2}{3}$, regardless of T, since we have already ruled out that $\mathcal{V}$ is infinite and it obviously is greater than zero. ∎

COROLLARY 4.4. *If S adopts the strategy $p(s)$ in the stage game $\Gamma(s_0)$, then the expected capture time is $\frac{2}{3}\sqrt{1 - s_0}$ against any pure hider strategy.*

Proof. In the stage game $\Gamma(s_0)$ the search starts with state variable $s(0) = s_0$. We compute the optimal searcher strategy $s(t)$ for the stage game by solving the differential equation $s' = \sqrt{1 - s}$ with initial condition $s(0) = s_0$:

$$s(t) = s_0 + t\sqrt{1 - s_0} - \frac{1}{4}t^2.$$

Now we repeat the computation of the previous proof. Let T be the running time of the hider. There are three possibilities: either H is caught hiding at time some $t < T$, or H is caught in ambush at time T, or the stage game renews at state variable zero at time T. Now H is caught hiding at time t with probability $\frac{p(t)}{1-s_0}$. At time T the hider has not been caught with probability $\frac{1-s(T)}{1-s_0}$. At that time, the game continues with probability $p(T)$ with an additional expected capture time of $\frac{2}{3}$. Putting this all together we find an expected capture time of

$$\int_0^T t\frac{p(t)}{1 - s_0}dt + \frac{1 - s(T)}{1 - s_0} \cdot \left(T + \frac{2}{3}p(T)\right).$$

To see that this is constant, differentiate with respect to T to find the derivative

$$T\frac{p(T)}{1 - s_0} - \frac{s'(T)}{1 - s_0} \cdot \left(T + \frac{2}{3}p(T)\right) + \frac{1 - s(T)}{1 - s_0} \cdot \left(1 + \frac{2}{3}p'(T)\right).$$

Since $p(t) = s'(t)$ this can be simplified to

$$\frac{1}{(1 - s_0)}\left(-\frac{2}{3}s'(T)^2 + (1 + \frac{2}{3}s''(T))(1 - s(T))\right).$$

Now $s''(T) = -\frac{1}{2}$ and $s'(T) = \sqrt{1 - s(T)}$ so the derivative is zero, from which we conclude that the hider is indifferent. ∎

4.2. Optimal hider strategy. We already determined that provided the value of the game is well-defined, the optimal hider strategy is to run at a random time T with intensity $r(s)$. We say briefly that H runs with intensity $r(s)$. Recall that the distribution of T is determined by its intensity.

PROPOSITION 4.5.

$$\mathbb{P}(T \geq x) = \exp\left(-\int_0^x \frac{3}{2\sqrt{1-s(t)}}dt\right).$$

Proof. See [29, p 38]. □

Let τ be the time that S has searched the entire interval, so $s(\tau) = 1$ or $\tau = \infty$ if S decides on a perpetual ambush without searching the entire interval. If $T \geq \tau$, then H is certainly caught hiding. Remarkably, $\mathbb{P}(T \geq \tau)$ is positive if τ is finite. In other words, the hider decides not to run with positive probability. For instance, if $\tau = 1$, then S searches and never ambushes $s(t) = t$ and $P(T \geq 1) = e^{-3}$.

In a pure searcher strategy, S either ambushes or searches during a time interval $[t, t + dt)$. In other words, $s'(t)$ is either equal to zero (S ambushes) or equal to one (S searches).

PROPOSITION 4.6. *Consider the stage game $\Gamma(s_0)$ in which H runs with intensity $r(s)$. Suppose that it is optimal for the searcher to initially ambush. In other words, suppose that there exists an optimal pure searcher strategy such that $s'(0) = 0$. Then the expected capture time is $\frac{2}{3}\sqrt{1-s_0}$.*

Proof. Suppose that the searcher ambushes during $[0, t_0)$. At time t_0 either the game has ended or the players enter the same stage game. Therefore, it remains optimal to ambush and so we may assume that S ambushes perpetually. Since H runs with intensity $r(s_0)$ and since s_0 remains constant, the hider running time T is exponentially distributed with constant parameter. The expected capture time is $\frac{2}{3}\sqrt{1-s_0}$. □

The argument in the proof of this proposition extends to general pure strategies. Suppose that $s(t)$ is an optimal pure strategy in which some $s'(t) = 0$. Then ambush remains the optimal strategy since the state variable remains the same. Therefore, an optimal pure strategy has the property that there exists a time t_0 such that $s'(t) = 1$ if $t < t_0$ and $s'(t) = 0$ if $t \geq t_0$. Note that if $t_0 \geq 1$, then S has already searched the entire space. So the strategy in which S only searches can be included by taking $t_0 = 1$ and we may assume that $t_0 \leq 1$. We say that t_0 is the time of ambush.

THEOREM 4.7. *Suppose that H runs with intensity $r(s)$. Then the expected capture time is $\frac{2}{3}$ against any pure searcher strategy.*

Proof. Let $s(t)$ be an optimal pure searcher strategy against a running time T with intensity $r(s)$. At time T, the hider has been caught with probability $s(T)$. If not, then the hider is caught in ambush at that time with probability $1 - s'(T)$. Otherwise the game continues with state variable $s = 0$. Let $\mathcal{U}$ be the expected capture time in the stage game with parameter $s = 0$. Then the expected capture time is equal to

$$(1 - s(T))(T + s'(T)\mathcal{U}) + \int_0^T ts'(t)dt.$$

Let t_0 be the time of ambush of the optimal pure strategy $s(t)$. If $T < t_0$, then S only searchers up to time T, so $s(T) = T$. The hider escapes at time T and the game renews at state variable $s = 0$. The expected capture time in this case is equal to

$$(1 - T)(T + \mathcal{U}) + \frac{T^2}{2}.$$

If $T \geq t_0$, then the hider is caught in ambush at time T and the expected capture time is equal to

$$(1-t_0)T + \frac{t_0^2}{2}.$$

By introducing the random variable $C = \min\{t_0, T\}$ we can express the expected capture time by one single equation,

$$(1-C)(T + \mathcal{U}\ 1_{\{C<t_0\}}) + C^2/2.$$

Recall that $\mathcal{U}$ is the expected capture time in the stage game $\Gamma(0)$, so we have a recursive equation

$$\mathcal{U} = \mathbb{E}[(1-C)(T + \mathcal{U}\ 1_{\{C<t_0\}}) + C^2/2],$$

or equivalently

$$\mathcal{U} = \frac{\mathbb{E}[T - CT + \frac{C^2}{2}]}{\mathbb{E}[C] + (1-t_0)\mathbb{P}(C = t_0)}.$$

To compute the expected value in the numerator we condition on C:

$$\mathbb{E}[T - CT + C^2/2 \mid C] = \begin{cases} C - \frac{C^2}{2}, & C < t_0, \\ t_0 - \frac{t_0^2}{2} + \frac{2}{3}(1-t_0)^{3/2}, & C = t_0, \end{cases}$$

where we use that in the stage game $\Gamma(t_0)$ the hider runs at expected time $\frac{2}{3}\sqrt{1-t_0}$ if the searcher ambushes. We find that

$$\mathcal{U} = \frac{\mathbb{E}[C] - \mathbb{E}[C^2/2] + \frac{2}{3}(1-t_0)^{3/2} \cdot \mathbb{P}(C = t_0)}{\mathbb{E}[C] + (1-t_0)\mathbb{P}(C = t_0)},$$

which can be computed since we know the distribution $\mathbb{P}(C \geq x)$ from Proposition 4.5. More specifically, if we denote the tail distribution function by $G(x) = \mathbb{P}(C \geq x)$, then $G(x) = \exp\left(3\sqrt{1-x} - 3\right)$ for $x \leq t_0$ and $G(x) = 0$ if $x > t_0$. Now $\mathbb{E}[C] = \int_0^{t_0} G(x)dx$ and $\mathbb{E}[C - C^2/2] = \int_0^{t_0} (1-x)G(x)dx$ and $\mathbb{P}(C = t_0) = G(t_0)$. By a straightforward computation we obtain

$$\mathcal{U} = \frac{\frac{8}{27} + \left(\frac{2}{3}(1-t_0) - \frac{4}{9}\sqrt{1-t_0} + \frac{4}{27}\right)\mathbb{P}(C = t_0)}{\frac{4}{9} + \left((1-t_0) - \frac{2}{3}\sqrt{1-t_0} + \frac{2}{9}\right)\mathbb{P}(C = t_0)} = \frac{2}{3},$$

which is independent of t_0. ∎

COROLLARY 4.8. *If H adopts the strategy $r(s)$ in the stage game $\Gamma(s_0)$, then the expected capture time is $\frac{2}{3}\sqrt{1-s_0}$ against any pure searcher strategy.*

Proof. We know that the expected capture time is $\frac{2}{3}$ in the stage game $\Gamma(0)$ regardless of the searcher's strategy. So the searcher may as well start with a search $s(t) = t$ for $t \leq s_0$ and then ambush, which is an optimal pure strategy. In other words, $t_0 = s_0$. If the hider is not caught by time s_0 and still has not run, then the players arrive in the stage game $\Gamma(s_0)$ upon which the searcher ambushes. Since the searcher's strategy is optimal, ambush is optimal in the stage game $\Gamma(s_0)$. The corollary now follows from Proposition 4.6. ∎

5. Concluding remarks. The noisy ambush game is a search game in which we ignore the geometry of the search space. In the introduction we mentioned that our ambush game is motivated by a search game on a star with k rays (or a k-leafed clover) with k going to infinity, which is a game that was proposed in [3] and in [7, Remark 4.15] and that has been studied in [1]. The noisy search game on a star with k rays has been studied by Timmer in [30]. The numerical results in that report led to a conjectured asymptotic solution [30, Conjecture 3.7] that is exactly equal to our solution of the noisy ambush game.

More generally, we expect that our ambush game describes a noisy search game on a homogeneous network with a finite number of nodes if the number of arcs k between the nodes goes to infinity. Such a network with only two nodes and k arcs and silent players has been studied in [4]. On such a network, the searcher may ambush at two nodes instead of one, but if the searcher ambushes while the hider runs, then the game ends with probability $\frac{1}{2}$ only, instead of probability 1. Generalizing this, one may consider the noisy ambush game in which the probability of capture in ambush is $0 \leq c \leq 1$. This game is related to the analysis of the trade-off between speed and accuracy of predators that has been carried out in [13]. If the capture probability is c, then the value of the game satisfies the differential equation

$$1 + p(s)\left(\mathcal{V}'(s) - \frac{\mathcal{V}(s)}{1-s}\right) = 0$$

with $p(s) = \frac{\mathcal{V}(s)-(1-c)\mathcal{V}(0)}{c\mathcal{V}(0)}$ and with boundary condition $\mathcal{V}(1) = 0$. It is not possible to solve this differential equation analytically, but it is possible to solve it numerically. We find that the value of the game is approximately equal to 0.760 if $c = \frac{1}{2}$.

The next step toward a solution of the princess and monster game on a network, from a stochastic point of view, would be to consider the ambush game between silent players. This is a much more difficult game to study, since contrary to the noisy ambush game it is not open to recursion and a much more intricate analysis is required that is closer to the solution of the princess and monster game on domains. In that game, Gal found that the hider remains at the same place for a fixed amount of time before running to the next hiding place. So in terms of our ambush game, the hider uses a pure strategy with a fixed T. Gal found that this running time is relatively small in comparison to μ/r, where μ is the measure of the domain and r is the radius of detection of the searcher. His analysis, which is very intricate and is exhibited in detail in [7, section 4.5.6], depends on the fact that the probability of detection is negligible if the hider runs to the next hiding place. In the princess and monster game on a graph this probability is not negligible, and that is why it is important to solve the ambush game with silent players. Recently, Arculus [8] has made some progress in this direction. He conjectures optimal strategies and bounds on the value of the game that have been derived from numerical simulations.

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