

BLACKBOX IDENTITY TESTING FOR BOUNDED TOP FANIN DEPTH-3 CIRCUITS: THE FIELD DOESN'T MATTER

Nitin Saxena
Hausdorff Center for Mathematics
Bonn, Germany
ns@hcm.uni-bonn.de

C. Seshadhri
Sandia National Laboratories*
Livermore, USA
scomand@sandia.gov

Abstract

Let C be a depth-3 circuit with n variables, degree d and top fanin k (called $\Sigma\Pi\Sigma(k, d, n)$ circuits) over base field \mathbb{F} . It is a major open problem to design a deterministic polynomial time blackbox algorithm that tests if C is identically zero. Klivans & Spielman (STOC 2001) observed that the problem is open even when k is a constant. This case has been subjected to a serious study over the past few years, starting from the work of Dvir & Shpilka (STOC 2005).

We give the first polynomial time blackbox algorithm for this problem. Our algorithm runs in time $\text{poly}(n)d^k$, regardless of the base field. The *only* field for which polynomial time algorithms were previously known is $\mathbb{F} = \mathbb{Q}$ (Kayal & Saraf, FOCS 2009, and Saxena & Seshadhri, FOCS 2010). This is the first blackbox algorithm for depth-3 circuits that does not use the rank based approaches of Karnin & Shpilka (CCC 2008).

We prove an important tool for the study of depth-3 identities. We design a blackbox polynomial time transformation that reduces the number of variables in a $\Sigma\Pi\Sigma(k, d, n)$ circuit to k variables, but preserves the identity structure.

*Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

1 Introduction

Polynomial identity testing (PIT) is a major open problem in theoretical computer science. The input is an arithmetic circuit that computes a polynomial $p(x_1, x_2, \dots, x_n)$ over a base field \mathbb{F} . We wish to check if p is the zero polynomial, or in other words, is identically zero. We may be provided with an explicit circuit, or may only have blackbox access. In the latter case, we can only evaluate the polynomial p at various domain points. The main goal is to devise a *deterministic* blackbox polynomial time algorithm for PIT. One of the main reasons for interest in this problem is the connection between PIT algorithms and circuit lower bounds (Heintz & Schnorr [HS80], Kabanets & Impagliazzo [KI03] and Agrawal [Agr05, Agr06]). Refer to surveys for a detailed treatment of PIT [Sax09, AS09].

Since the problem of PIT is very hard, restricted versions of it have been studied. One common and natural variant is that of the *bounded depth circuits*. Results of Agrawal & Vinay [AV08] justify this restriction. They essentially show that an efficient blackbox identity test for depth-4 circuits leads to (almost) the complete resolution of PIT and also provides exponential lower bounds. Raz [Raz10] showed that even lower bounds for depth-3 circuits imply super-polynomial lower bounds for general arithmetic formulas. Not surprisingly, the problem of PIT is still wide open for the special case of depth-3 circuits.

A depth-3 circuit C over a field \mathbb{F} is of the form $C(x_1, \dots, x_n) = \sum_{i=1}^k T_i$, where T_i (a *multiplication term*) is a product of at most d linear polynomials with coefficients in \mathbb{F} . The size of the circuit C can be expressed in three parameters: the number of variables n , the degree d , and the *top fanin* k . Such a circuit is referred to as a $\Sigma\Pi\Sigma(k, d, n)$ circuit. Even when the top fanin k is constant, blackbox polynomial time algorithms were not known¹.

The study of PIT algorithms for depth-3 circuits was initiated by Dvir & Shpilka [DS05], who gave a quasi-polynomial time non-blackbox algorithm. The first non-trivial *blackbox* algorithm was given by Karnin & Shpilka [KS08]. There have been many recent results in this area by Kayal & Saxena [KS06], Saxena & Seshadhri [SS09, SS10a], and Kayal & Saraf [KS09b]. Our main result is the first polynomial time blackbox tester for bounded top fanin depth-3 circuits over *any* field.

Theorem 1 *There exists a deterministic blackbox $\text{poly}(nd^k)$ time algorithm for PIT on $\Sigma\Pi\Sigma(k, d, n)$ circuits, regardless of the base field \mathbb{F} .*

Table 1 details the time complexities² of previous algorithms. For convenience, we do not give the list of all algorithms, but only the important milestones for the case of arbitrary fields. We stress that the time complexities bound the number of bit operations.

Table 1: Depth-3 blackbox PIT algorithms over any field

Paper	Time complexity
[KS08]	$nd^{(2^{k^2} \log^{k-2} d)}$
[SS09, SS10a]	$nd^{k^2 \log d}$
This paper	nd^k

The only field for which such polynomial time algorithms were previously known was \mathbb{Q} . This was a breakthrough result of Kayal & Saraf [KS09b], which was followed by improvements in [SS10a]. These used beautiful incidence geometry theorems for the reals, but analogues of these results are either unknown or false for other fields. Since the best running time of these algorithms is $\text{poly}(nd^{k^2})$, we get an improved algorithm for this case as well.

As Table 1 shows, *even for the simple case of $k = 3$ and $\mathbb{F} = \mathbb{F}_2$, no deterministic polynomial-time blackbox PIT algorithm was known*. Kayal & Saxena [KS06] gave a non-blackbox algorithm

¹Until this work.

²Technically, the running times are polynomial in the stated times.

(over all fields), which runs in $\text{poly}(nd^k)$ time [KS06]. Theorem 1 closes the gap between blackbox and non-blackbox algorithms.

Throughout the following discussion, we will think of k as a constant. Hence, when we refer to polynomial time, the dependence on k will be ignored.

1.1 Variable reduction

Dvir & Shpilka [DS05] introduced a powerful idea. They defined the notion of the *rank* of a $\Sigma\Pi\Sigma(k, d, n)$ circuit. We will not explain this precisely here but merely say that this is the number of “free variables” in a $\Sigma\Pi\Sigma$ circuit. They proved the remarkable fact that the rank of every identity³ is small. This led to the reduction of PIT for general $\Sigma\Pi\Sigma(k, d, n)$ circuits to PIT on $\Sigma\Pi\Sigma$ circuits over few variables. They developed a non-blackbox quasi-polynomial time algorithm through their rank bounds. Karnin & Shpilka [KS08] used the idea of rank to devise *blackbox* algorithms for $\Sigma\Pi\Sigma$ circuits. Their algorithms had a running time that depended exponentially in the rank. Hence, constant rank bounds would lead to polynomial time algorithms. Unfortunately, Kayal & Saxena [KS06] gave constructions (extended in [SS09]) showing that for $\Sigma\Pi\Sigma$ identities over finite fields, the rank could be unbounded (as large as $k \log d$). This means that the best running time one could hope for over finite fields via this approach was $d^{k \log d}$. Tighter rank bounds from [SS09, SS10a] gave algorithms that almost match this running time. For the special case when the field \mathbb{F} is \mathbb{Q} , Kayal & Saraf [KS09b] proved a constant rank bound, establishing the first polynomial time blackbox algorithm for this case. Refer to [SS10a] for a more detailed treatment of rank bounds.

Until this work, all blackbox algorithms relied solely on the rank approach of Karnin & Shpilka [KS08]. As the examples of [KS06] show, even for the case of \mathbb{F}_2 , a new idea is required to get polynomial time algorithms. We provide the first blackbox algorithm that circumvents the problem of large rank identities. Interestingly, one of the main ideas has roots in the non-blackbox polynomial time algorithm of Kayal & Saxena [KS06]. This algorithm had a completely different idea and used generalizations of the *Chinese Remainder Theorem*. These algebraic ideas were further developed in a previous work of the authors [SS10a]. Karnin & Shpilka [KS09a] used extractors of Gabizon & Raz [GR05] to construct their blackbox algorithm. We combine these extractor ideas with the algebraic framework to develop a very useful algorithmic tool. Any $\Sigma\Pi\Sigma(k, d, n)$ circuit can be converted into a small family of $\Sigma\Pi\Sigma$ circuits over just k variables. The original circuit is an identity iff all circuits in the family are identities. This transformation requires no knowledge of the circuit and has a running time of $\text{poly}(kdn)$.

Theorem 2 *Let \mathbb{F} be an arbitrary field such that $|\mathbb{F}| > dnk^2$. There is a deterministic algorithm that takes as input a triple (k, d, n) of natural numbers and in time $\text{poly}(kdn)$, outputs a set of linear maps $\Psi_i : \mathbb{F}[x_1, \dots, x_n] \rightarrow \mathbb{F}[y_1, \dots, y_k]$ ($1 \leq i \leq \text{poly}(kdn)$). A $\Sigma\Pi\Sigma(k, d, n)$ circuit C is identically zero iff $\forall i, \Psi_i(C) = 0$.*

Remark: The size restriction made in the theorem is really no loss of generality. In the blackbox model, it is standard to assume that we can query the given circuit on points in an extension field. If \mathbb{F} is small, then we just need to move to a large enough extension field \mathbb{F}' . Such an extension can be found by a deterministic $\text{poly}(\log |\mathbb{F}'|)$ time method of Adleman & Lenstra [AL86], or even by a slower brute-force method of finding a suitable irreducible polynomial over \mathbb{F} . We do all our computations in this extension field \mathbb{F}' . Henceforth, for convenience, we will assume that the field has size $> dnk^2$.

Observe the power of this theorem. Regardless of n, d or \mathbb{F} , every $\Sigma\Pi\Sigma(3, d, n)$ non-identity can be converted to an “equivalent” non-identity involving just 3 variables. The time required to generate this transformation is truly polynomial in the *size* of the circuit. We believe that this theorem will be useful in reaching the holy grail of a truly polynomial time PIT algorithm for $\Sigma\Pi\Sigma$ circuits.

³A small caveat: there are some technical restrictions of simplicity and minimality.

This theorem has a uniform treatment of all fields, and is hence stronger than rank bounds. The circuits $\Psi_i(C)$ only involve k variables. Using some standard PIT techniques, we can construct the following hitting set. This proves Theorem 1.

Theorem 3 *Given the triple of natural numbers (k, d, n) , a hitting set $\mathcal{H} \subseteq \mathbb{F}^n$ for $\Sigma\Pi\Sigma(k, d, n)$ circuits can be constructed in deterministic $\text{poly}(nd^k)$ time. In other words, for every non-zero $\Sigma\Pi\Sigma(k, d, n)$ circuit C over \mathbb{F} , there is some vector $(\alpha_1, \dots, \alpha_n) \in \mathcal{H}$ such that $C(\alpha_1, \dots, \alpha_n) \neq 0$.*

We make a somewhat philosophical remark. By Schwartz-Zippel we know that PIT for a depth-3 circuit $C(x_1, \dots, x_n)$ can be done by feeding an n -wise independent random distribution. The proof of Theorem 2 shows that for $\Sigma\Pi\Sigma(k, d, n)$ circuits, k -wise independent random distribution suffices.

1.2 History

The first randomized polynomial time PIT algorithm was given (independently) by Schwartz [Sch80] and Zippel [Zip79]. Algorithms using less randomness were devised by Chen & Kao [CK97], Lewin & Vadhan [LV98], and Agrawal & Biswas [AB99]. For depth-2 circuits, there has been a long line of work studying blackbox PIT algorithms [BOT88, CDGK91, Wer94, KS96, SS96, GKS90, KS01]. Raz & Shpilka studied non-blackbox algorithms for non-commutative formulas [RS05].

Klivans & Spielman [KS01] first observed that deterministic PIT was open even for depth-3 circuits with bounded top fanin. Progress towards this was first made by the quasi-polynomial time algorithm of Dvir & Shpilka [DS05]. The problem was resolved (in the non-blackbox setting) by a polynomial time algorithm given by Kayal & Saxena [KS06], with a running time exponential in the top fanin.

The remaining history of depth-3 PIT has been explained quite a bit in the previous sections. Identity tests are known only for very special depth-4 circuits [AM07, Sax08, SV09, KMSV10]. Agrawal & Vinay [AV08] showed that an efficient blackbox identity test for depth-4 circuits will actually give a quasi-polynomial blackbox test, and exponential lower bounds, for circuits of *all depths* that compute *low degree* polynomials. Thus, understanding depth-3 identities seems to be a natural first step towards the goal of PIT and circuit lower bounds.

At the end, we would just like to indicate how all the depth-3 PIT results are interrelated and how they collectively influenced progress in this problem. The rank notion of Dvir & Shpilka [DS05], the Chinese Remaindering of Kayal & Saxena [KS06], the rank preserving subspaces of Karnin & Shpilka [KS09a], the series of improved rank bounds by the authors and Kayal & Saraf [SS09, KS09b, SS10a]: each paper built of previous results and provided enough food for thought for subsequent papers. This paper also builds on the edifice constructed so far.

1.3 Organization

In Section 2, we give some basic definitions and give an intuitive overview of our approach. Section 3 gives some of the tools that were developed in previous works. In Section 4, we give our main analysis and prove Theorems 2 and 3.

2 Preliminaries and Intuition

We will denote the set $\{1, \dots, n\}$ by $[n]$. We fix the base field to be \mathbb{F} , so the circuits compute multivariate polynomials in the *polynomial ring* $\mathcal{R} := \mathbb{F}[x_1, \dots, x_n]$. We use \mathbb{F}^* to denote $\mathbb{F} \setminus \{0\}$. For any subset $S \subseteq [k]$, the *sub-circuit* C_S is $\sum_{s \in S} T_s$. For an $i \in \{0, \dots, k-1\}$, define $[i]'$:= $[k] \setminus [i]$. Conventionally, $[0] := \emptyset$ and $C_\emptyset := 0$.

A *linear form* is a linear polynomial in \mathcal{R} with zero constant term. We will denote the set of all linear forms by $L(\mathcal{R}) := \{\sum_{i=1}^n a_i x_i \mid a_1, \dots, a_n \in \mathbb{F}\}$. Clearly, $L(\mathcal{R})$ is a vector (or linear) space over \mathbb{F} and that will be quite useful. Much of what we do shall deal with *multi-sets* of linear forms (also

product of linear forms) and equivalence classes inside them. A *list* of linear forms is a multi-set of forms with an arbitrary order associated with them.

Definition 4 *We collect some important definitions from [SS09]:*

[Multiplication term and operators $L(\cdot)$ & $M(\cdot)$] A multiplication term f is an expression in \mathcal{R} given as (the product may have repeated ℓ 's), $f := c \cdot \prod_{\ell \in S} \ell$, where $c \in \mathbb{F}^*$ and S is a list of nonzero linear forms. The list of linear forms in f , $L(f)$, is just the list S of forms occurring in the product above. For a list S of linear forms we define the multiplication term of S , $M(S)$, as $\prod_{\ell \in S} \ell$ or 1 if $S = \phi$.

[$\Sigma\Pi\Sigma$ circuits] An $\Sigma\Pi\Sigma(k, d)$ circuit C is a sum of k multiplication terms of degree d , $C = \sum_{i=1}^k T_i$. The list of linear forms occurring in C is $L(C) := \bigcup_{i \in [k]} L(T_i)$. Note that $L(C)$ is a list of size exactly kd . (For the purposes of this paper T_i 's are given in circuit representation and thus the list $L(T_i)$ is unambiguously defined from C .)

[Span $\text{sp}(\cdot)$ and Rank $\text{rk}(\cdot)$] For any $S \subseteq L(\mathcal{R})$ we let $\text{sp}(S) \subseteq L(\mathcal{R})$ be the linear span of the linear forms in S over the field \mathbb{F} . (Conventionally, $\text{sp}(\emptyset) = \{0\}$.) We use $\text{rk}(S)$ to denote the rank of S , considered as vectors in \mathbb{F}^n .

2.1 Intuition and main ideas

We give a high-level description of the main ideas used in this paper. Some notions are deliberately left vague, and others may even be formally incorrect. Nonetheless, this sketch is “morally” correct and, at some level, shows how the authors arrived at their conclusions.

How do we convert a high variate $\Sigma\Pi\Sigma(k, d, n)$ circuit C into a low variate one and still preserve the structure of C ? We wish to do this by a linear transformation $\Psi : \mathbb{F}[x_1, \dots, x_n] \rightarrow \mathbb{F}[y_1, \dots, y_\ell]$, where ℓ is comparable to k . When the rank of the linear forms in C is itself comparable to k , this can be done quite directly. We will get a circuit $\Psi(C)$ that is essentially *isomorphic* to C . For identities of large rank, such a transformation seems impossible. Any linear transformation will necessarily destroy some of the structure of C . This is because forms that were independent in C are now dependent in $\Psi(C)$. But maybe we are trying too hard to preserve the dependencies in C ? After all, we want a transformation Ψ that sends identities to identities. (And, of course, non-identities to non-identities. Surely, satisfying only the former condition is not too hard.) We are not particularly bothered about how well $\Psi(C)$ preserves the exact structure of C .

This is where the Chinese Remaindering techniques of [KS06] and the ideal framework of [SS10a] enters the picture. For any non-identity C , it was shown that there exists an ideal I generated by products of forms in $L(C)$ that “certifies” that C is non-zero. Essentially, the polynomial C is not in ideal I and hence, must be non-zero. In reality, it is much more complicated than that, but for the sake of explanation, it captures the main idea. The forms involved in generating I have rank at most k . This gives a low-dimensional certificate of the non-zerosness of C .

We argue that if Ψ can selectively preserve the forms generating I , then $\Psi(I)$ remains a certificate for $\Psi(C)$. In other words, $\Psi(C)$ will not be in $\Psi(I)$. All that is needed is, to find such a Ψ that is independent of C (since we are interested in blackbox algorithms). Enter [KS09a]. They develop the notion of *rank-preserving subspaces*. These can be viewed as linear transformations from a large-dimensional vector space \mathcal{S}_1 to a smaller dimensional one \mathcal{S}_2 . These preserve the structure (in terms of linear independence) of specific low dimensional subspaces of \mathcal{S}_1 . Furthermore, they can be constructed in a blackbox manner using extractors of [GR05]. To make this work, we will actually need a *set* of transformations, and one Ψ will not suffice.

The circuit $\Psi(C)$ is of the form $\Sigma\Pi\Sigma(k, d, k)$. This has only a constant number of variables, and the Schwartz-Zippel lemma [Sch80, Zip79] gives a simple blackbox algorithm for such circuits. This is combined with the transformation Ψ to construct the final hitting set.

3 Necessary tools

In this section, we list out the basic tools that we need. In the first part, we explain the low-rank ideal certificates for the non-zerosness of C . This requires some technical definitions, before we can state the exact theorem. In the second part, we give the key lemma of the Vandermonde matrix transformation used in [KS09a]. Once these tools are set in place, we will explain how the variable reduction works.

3.1 Low rank certificates

This framework and the following theorems were developed in [SS10a]. The definitions are extremely technical, and may appear to be somewhat unmotivated. These are needed to precisely formalize the notion of the low-rank certificate for non-zerosness. We reproduce many of the definitions and details for convenience. For more details, the interested reader should see the full version of [SS10a] ([SS10b]).

Definition 5 (Ideal) *An ideal I of \mathcal{R} with generators $f_i, i \in [m]$, is the set $\{\sum_{i \in [m]} q_i f_i \mid q_i \text{ 's } \in \mathcal{R}\}$ and is denoted by the notation $\langle f_1, \dots, f_m \rangle$. For an $f \in \mathcal{R}$, the three notations $f \equiv 0 \pmod{I}$, $f \equiv 0 \pmod{f_1, \dots, f_m}$ and $f \in I$, mean the same.*

(Radical-span) *Let $S := \{f_1, \dots, f_m\}$ be multiplication terms generating an ideal I . We associate a linear space to S called the radical span, $\text{radsp}(S) := \text{sp}(L(f_1) \cup \dots \cup L(f_m))$.*

When the set of generators S are clear from the context we will also use the notation $\text{radsp}(I)$. Similarly, $\text{radsp}(I, f)$ would be a shorthand for $\text{radsp}(S \cup \{f\})$.

(Nodes) *Let f be a multiplication term and let I be an ideal generated by some multiplication terms. As the relation “similarity mod $\text{radsp}(I)$ ” is an equivalence relation on $L(\mathcal{R})$, it partitions the list $L(f)$ into equivalence classes.*

[rep $_I$ (f)] *For each such class, pick a representative ℓ_i and define their collection $\text{rep}_I(f) := \{\ell_1, \dots, \ell_r\}$. (Note that form 0 can also appear in this set, it represents the class $L(f) \cap \text{radsp}(I)$.)*

[nod $_I$ (f)] *For each $\ell_i \in \text{rep}_I(f)$, we multiply the forms in f that are similar to $\ell_i \pmod{\text{radsp}(I)}$. We define nodes of $f \pmod{I}$ as the set of polynomials $\text{nod}_I(f) := \{M(L(f) \cap (\mathbb{F}^* \ell + \text{radsp}(I))) \mid \ell \in \text{rep}_I(f)\}$. (Remark: When $I = \{0\}$, nodes of f are just the coprime powers-of-forms dividing f .)*

(Paths) *Let I be an ideal generated by some multiplication terms. Let $C = \sum_{i \in [k]} T_i$ be a $\Sigma\Pi\Sigma(k, d)$ circuit. Let v_i be a sub-term of T_i (i.e. $L(v_i) \subseteq L(T_i)$), for all $i \in [k]$. We call the tuple (I, v_1, \dots, v_k) a path of $C \pmod{I}$ if, for all $i \in [k]$, $v_i \in \text{nod}_{\langle I, v_1, \dots, v_{i-1} \rangle}(T_i)$. It is of length k . (Remark: We have defined path \bar{p} as a tuple but, for convenience, we will sometimes treat it as a set of multiplication terms, eg. when operated upon by $\text{sp}(\cdot)$, $\langle \cdot \rangle$, $\text{radsp}(\cdot)$, etc.)*

Conventionally, when $k = 0$ the circuit C has just “one” gate: 0. In that case, the only path $C \pmod{I}$ has is (I) , which is of length 0.

Observe that the product of polynomials in $\text{nod}_I(f)$ just gives f (upto a constant multiple). Also, modulo $\text{radsp}(I)$, each node is just a form-power ℓ^m . In other words, modulo $\text{radsp}(I)$, a node is a rank-one term. Figure 1 should clarify the definition. The oval bubbles represent the list of forms in a term, and the rectangles enclose forms in a node. The arrows show a path. Starting with I as the zero ideal, $v_1 := x_1^2$, $v_2 := x_2(x_2 + 2x_1)$, and $v_3 := (x_4 + x_2)(x_4 + 4x_2 - x_1)(x_4 + x_2 + x_1)(x_4 + x_2 - 2x_1)$ form a path. Initially the path is just the zero ideal, so x_1^2 is a node. Note how v_2 is a power of x_2 modulo $\text{radsp}(v_1)$ and v_3 is a power of x_4 modulo $\text{radsp}(v_1, v_2)$.

Theorem 25 of [SS10b] claims that in a non-zero depth-3 circuit C , there always exists a path “certifying” the non-zerosness of C . Essentially, there is a path \bar{p} such that modulo \bar{p} , the circuit C reduces to a single non-zero multiplication term. The theorem is stated for general settings, but we

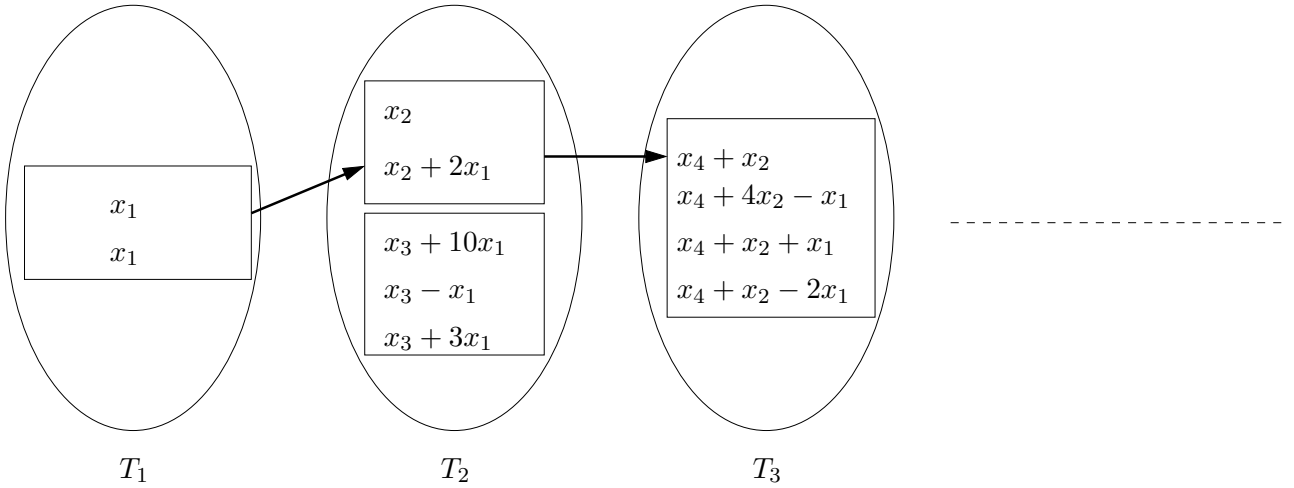


Figure 1: Nodes and paths in $C = T_1 + T_2 + T_3 + \dots$

will always assume that $I = \langle 0 \rangle$. Note that the rank of $\text{radsp}(\bar{p})$ is at most $(k - 1)$, since each node can increase the rank by at most 1 and since one term is to be left uncanceled.

Theorem 6 (Certificate for a Non-identity) *Let I be an ideal generated by some multiplication terms. Let $C = \sum_{i \in [k]} T_i$ be a $\Sigma\Pi\Sigma(k, d)$ circuit that is nonzero modulo I . Then $\exists i \in \{0, \dots, k - 1\}$ such that $C_{[i]} \bmod I$ has a path \bar{p} satisfying: $C_{[i]'} \equiv \alpha \cdot T_{i+1} \not\equiv 0 \pmod{\bar{p}}$ for some $\alpha \in \mathbb{F}^*$.*

3.2 The Vandermonde linear transformation

Linear transformations based on the Vandermonde matrix are used widely, for eg. in [GR05] to construct *linear seeded extractors for affine sources*. This was adapted in [KS08] to design depth-3 blackbox PIT algorithms. We will need ideas from Lemma 6.1 from [GR05] to construct linear transformations that reduce the dimension of a space but still preserve the structure of low rank subspaces.

We will design a linear transformation $\Psi_\beta : \mathbb{F}^n \rightarrow \mathbb{F}^k$ for any $\beta \in \mathbb{F}$ as follows. Let $V_{n,k,\beta}$ denote the $n \times k$ Vandermonde matrix. This is defined as $(V_{n,k,\beta})_{i,j} := \beta^{ij}$. We have

$$(b_1 \dots b_k) = (a_1 \dots a_n) \cdot V_{n,k,\beta} \quad (b_r = \sum_{i=1}^n a_i \beta^{ri}) \quad (1)$$

So far, Ψ_β has been seen as a linear transformation. But we wish to eventually understand its action on ideals. For that reason, it is necessary to view Ψ_β as a *linear homomorphism* from $\mathcal{R} = \mathbb{F}[x_1, \dots, x_n]$ to $\mathcal{R}' := \mathbb{F}[y_1, \dots, y_k]$. This means that Ψ_β maps \mathcal{R} to \mathcal{R}' and preserves the ring operations of addition and multiplication. We can equivalently define Ψ_β as

$$\forall i \in [n], \quad \Psi_\beta : x_i \mapsto \sum_{j=1}^k \beta^{ij} y_j \quad (2)$$

We define $\Psi_\beta(\alpha) = \alpha$ for all $\alpha \in \mathbb{F}$. This (naturally) defines the action of Ψ_β , on *all* the elements of \mathcal{R} , that preserves the ring operations of \mathcal{R} . We now state a key property of Ψ (Lemma 6.1 of [GR05]). For completeness, a proof is provided in Appendix A.

Lemma 7 (Ψ_β preserves k -rank) *Let $\Psi_\beta : \mathcal{R} \rightarrow \mathcal{R}'$ be the linear homomorphism defined by Equation (2). Let $S \subseteq L(\mathcal{R})$ be a subset of linear forms with $\text{rk}(S) \leq k$. For all but nk^2 values of β , $\text{rk}(\Psi_\beta(S)) = \text{rk}(S)$.*

The homomorphism Ψ_β depends solely on the triple of natural numbers (k, d, n) and is computable in $\text{poly}(kdn)$ time.

Remark: At first sight Ψ_β might seem independent of d but recall that we assume $|\mathbb{F}| > dnk^2$.

4 Analysis

With all the basic definitions in place, we are now ready to convert any $\Sigma\Pi\Sigma(k, d, n)$ non-identity C into a $\Sigma\Pi\Sigma(k, d, k)$ non-identity. This will be done through a two-step process. We will use the properties of Ψ_β to prove a major generalization of Lemma 7. Not only does Ψ_β preserve small subspaces, but it also maintains the structure of ideals having low radical-span. This leads to the second step. We deduce that Ψ_β must preserve all paths of C , since paths have a low radical span. Since C has a certifying path p , we show that $\Psi_\beta(p)$ must also be a certificate for $\Psi_\beta(C)$.

4.1 The moral nature of Ψ_β : it maintains ideals

We study the action of Ψ_β on ideals. Our main lemma is the following.

Lemma 8 (Ψ_β preserves ideals) *Let f_1, \dots, f_m, f be multiplication terms in \mathcal{R} . Define the ideal $I := \langle f_1, \dots, f_m \rangle$. Let the span $\text{sp}(L(f) \cup \text{radsp}(I))$, over \mathbb{F} , be of rank at most k . Then, for all but nk^2 values of β : $f \in I$ iff $\Psi_\beta(f) \in \langle \Psi_\beta(f_1), \dots, \Psi_\beta(f_m) \rangle$.*

To prove this, we need to show that the map Ψ_β is an isomorphism on small enough subrings of \mathcal{R} . This is a fairly direct consequence of Lemma 7. For $\ell_1, \dots, \ell_k \in L(\mathcal{R})$, $\mathbb{F}[\ell_1, \dots, \ell_k] \subset \mathcal{R}$ denotes the set of all polynomials $g(\ell_1, \dots, \ell_k)$, where $g \in \mathbb{F}[y_1, \dots, y_k]$. This is the *subalgebra* of \mathcal{R} generated by $\{\ell_1, \dots, \ell_k\}$.

Lemma 9 *Let $\ell_1, \dots, \ell_k \in L(\mathcal{R})$ be k linearly independent forms. Then, for all but nk^2 values of β , Ψ_β induces an isomorphism between $\mathbb{F}[\ell_1, \dots, \ell_k]$ and \mathcal{R}' .*

Proof. Let \mathcal{B} denote $\mathbb{F}[\ell_1, \dots, \ell_k]$ and set $S := \{\ell_1, \dots, \ell_k\} \subset L(\mathcal{R})$. It will be convenient to define Φ_β to be the map induced by Ψ_β on \mathcal{B} . We will show that $\Phi_\beta : \mathcal{B} \rightarrow \mathcal{R}'$ is an isomorphism. Since S is of rank k , \mathcal{B} is isomorphic to \mathcal{R}' (under an invertible linear transformation of variables). To show that the homomorphism Φ_β is an isomorphism, it suffices to prove that Φ_β is onto. We need to show that for any $g(y_1, \dots, y_k) \in \mathcal{R}'$, there exists $p \in \mathcal{B}$ such that $\Phi_\beta(p) = g$.

For the set S , choose a value of β other than the nk^2 values given by Lemma 7. We have $\text{rk}(\Phi_\beta(S)) = \text{rk}(S) = k$. Therefore, for each y_i , there exist constants $\alpha_j \in \mathbb{F}$ such that $y_i = \sum_j \alpha_j \Phi_\beta(\ell_j)$. By the linearity of Φ_β , $y_i = \Phi_\beta(\sum_j \alpha_j \ell_j)$. Hence, for each y_i , there is some linear form $t_i \in L(\mathcal{B})$, such that $\Phi_\beta(t_i) = y_i$. The polynomial $p := g(t_1, \dots, t_k)$ is certainly in \mathcal{B} . Since Φ_β is a homomorphism, $\Phi_\beta(p) = g$. ■

We can now complete the proof of Lemma 8.

Proof. If $f \in I$, then $f = \sum_{i \in [m]} g_i f_i$, where $g_i \in \mathcal{R}$. Since Ψ_β is a homomorphism, $\Psi_\beta(f) = \sum_{i \in [m]} \Psi_\beta(g_i) \Psi_\beta(f_i)$. So $\Psi_\beta(f) \in \langle \Psi_\beta(f_1), \dots, \Psi_\beta(f_m) \rangle$.

Let the span $\text{sp}(L(f) \cup \text{radsp}(I))$ over \mathbb{F} be generated by linear forms $\ell_1, \dots, \ell_r \in L(\mathcal{R})$. Since the rank of $\text{sp}(L(f) \cup \text{radsp}(I))$ is at most k , $r \leq k$. Choose arbitrary forms $\ell_{r+1}, \dots, \ell_k$ such that ℓ_1, \dots, ℓ_k are linearly independent. Define subring (of \mathcal{R}) $\mathcal{B} := \mathbb{F}[\ell_1, \dots, \ell_k]$. Applying Lemma 9, we get that for all but nk^2 values of β , Ψ_β induces an isomorphism $\Phi_\beta : \mathcal{B} \rightarrow \mathcal{R}'$. Choose any such β and fix the unique elements $t_1, \dots, t_k \in \mathcal{B}$ such that $\Phi_\beta(t_i) = y_i$ for all $i \in [k]$.

Suppose $\Psi_\beta(f) \in \langle \Psi_\beta(f_1), \dots, \Psi_\beta(f_m) \rangle$. Then there exists $g_1, \dots, g_m \in \mathcal{R}'$ such that,

$$\Psi_\beta(f) = \sum_{i=1}^m g_i \cdot \Psi_\beta(f_i) \quad (3)$$

Each g_i is a polynomial in (y_1, \dots, y_k) over \mathbb{F} . So we can define the polynomial,

$$h := f - \sum_{i=1}^m g_i(t_1, \dots, t_k) \cdot f_i.$$

Note that f and f_i 's are multiplication terms generated by forms in the $\text{sp}(\ell_1, \dots, \ell_k)$. Hence all of these are in \mathcal{B} . The polynomials $g_i(t_1, \dots, t_k)$ are also in \mathcal{B} , so $h \in \mathcal{B}$. Since Φ_β is a homomorphism,

$$\Phi_\beta(h) = \Phi_\beta(f) - \sum_{i=1}^m g_i(\Phi_\beta(t_1), \dots, \Phi_\beta(t_k)) \cdot \Phi_\beta(f_i) = \Psi_\beta(f) - \sum_{i=1}^m g_i(y_1, \dots, y_k) \cdot \Psi_\beta(f_i) = 0$$

But Φ_β is an isomorphism, so $h = 0$. This implies $f = \sum_{i=1}^m g_i(t_1, \dots, t_k) \cdot f_i$. Note that the evaluations of the g_i 's are in \mathcal{R} . Thus, $f \in \langle f_1, \dots, f_m \rangle$. ■

4.2 Variable reduction

We come to the main part where we merge the path certificates with the properties of Ψ_β to prove the variable reduction. We will need a technical cancellation lemma from [SS10b], proven in Appendix B.

Lemma 10 *Let f_1, \dots, f_m be multiplication terms generating an ideal I , let $\ell \in L(\mathcal{R})$ and $g \in \mathcal{R}$. If $\ell \notin \text{radsp}(I)$ then: $\ell g \in I$ iff $g \in I$.*

We now state our main variable reduction lemma. This, combined with the polynomial time constructions of the Ψ_β 's, completes the proof of Theorem 2.

Lemma 11 *Let C be a $\Sigma\Pi\Sigma(k, d, n)$ circuit and $U \subseteq \mathbb{F}$ such that $|U| = dnk^2 + 1$. Then $C = 0$ iff $\forall \beta \in U, \Psi_\beta(C) = 0$.*

Proof. Since Ψ_β is a homomorphism, $C = 0$ implies $\forall \beta, \Psi_\beta(C) = 0$.

Suppose $C \neq 0$, but $\forall \beta \in U, \Psi_\beta(C) = 0$. Applying Theorem 6 on C (with $I := \langle 0 \rangle$) yields a certifying path \bar{p} . Thus, $\exists i \in \{0, \dots, k-1\}$ such that $C_{[i]}$ mod I has a path \bar{p} satisfying,

$$C_{[i]'} \equiv \alpha \cdot T_{i+1} \not\equiv 0 \pmod{\bar{p}}, \text{ for some } \alpha \in \mathbb{F}^*. \quad (4)$$

Note that \bar{p} is basically a sequence of multiplication terms such that $\text{rk}(\text{radsp}(\bar{p})) < k$. Let $g := M(L(T_{i+1}) \cap \text{radsp}(\bar{p}))$. This is just the product of all forms in T_{i+1} that are in $\text{radsp}(\bar{p})$. Note that T_{i+1}/g is a product of forms *not* in $\text{radsp}(\bar{p})$. By repeated applications of Lemma 10, since $T_{i+1} \notin \langle \bar{p} \rangle$, $g \notin \langle \bar{p} \rangle$. The rank of $\text{sp}(L(g) \cup \text{radsp}(\bar{p}))$ is less than k . Indeed, for any linear form $\ell \in L(T_{i+1})$, the rank of $\{\ell\} \cup \text{radsp}(\bar{p})$ is at most k .

We will now collect a set B of “bad” β values. By Lemma 7, for each $\ell \in L(T_{i+1})$, there are at most nk^2 values of β such that Ψ_β does not preserve $\text{rk}(\{\ell\} \cup \text{radsp}(\bar{p}))$. Add all of these values to B . The total number of all these bad β values is at most dnk^2 . Therefore, there exists a good β in U .

For any $\beta \in U \setminus B$, we know that Ψ_β preserves $\text{rk}(L(g) \cup \text{radsp}(\bar{p})) = \text{rk}(\text{radsp}(\bar{p}))$. Thus, by the proof of Lemma 8, Ψ_β preserves $g \notin \langle \bar{p} \rangle$. In other words, $\Psi_\beta(g) \notin \langle \Psi_\beta(\bar{p}) \rangle$. We get a contradiction with the following claim.

Claim 12 *Choose $\beta \in U \setminus B$. If $\Psi_\beta(C) = 0$, then $\Psi_\beta(g) \in \langle \Psi_\beta(\bar{p}) \rangle$.*

Proof. Observe that $C_{[i]} \equiv 0 \pmod{\bar{p}}$, implying $\Psi_\beta(C_{[i]}) \equiv 0 \pmod{\Psi_\beta(\bar{p})}$. We get $0 = \Psi_\beta(C) = \Psi_\beta(C_{[i]}) + \Psi_\beta(C_{[i]'})$. Going modulo \bar{p} and applying Equation (4), $\Psi_\beta(T_{i+1}) \equiv 0 \pmod{\Psi_\beta(\bar{p})}$. In terms of ideals, $\Psi_\beta(T_{i+1}) \in \langle \Psi_\beta(\bar{p}) \rangle$. Consider any form $\ell \in L(T_{i+1})$ such that $\ell \notin \text{radsp}(\bar{p})$. We will show that $\Psi_\beta(T_{i+1})/\Psi_\beta(\ell) \in \langle \Psi_\beta(\bar{p}) \rangle$.

We have $\text{rk}(\{\ell\} \cup \text{radsp}(\bar{p})) = \text{rk}(\text{radsp}(\bar{p})) + 1$, by the choice of ℓ . Since $\beta \notin B$, $\text{rk}(\{\Psi_\beta(\ell)\} \cup \text{radsp}(\Psi_\beta(\bar{p}))) = \text{rk}(\text{radsp}(\Psi_\beta(\bar{p}))) + 1$. This implies $\Psi_\beta(\ell) \notin \text{radsp}(\Psi_\beta(\bar{p}))$. Since $\Psi_\beta(T_{i+1}) \in \langle \Psi_\beta(\bar{p}) \rangle$, Lemma 10 tells us that $\Psi_\beta(T_{i+1})/\Psi_\beta(\ell) \in \langle \Psi_\beta(\bar{p}) \rangle$. We can iteratively repeat this process for all such forms ℓ . We will end up with $\Psi_\beta(g) \in \langle \Psi_\beta(\bar{p}) \rangle$. ■

4.3 The final hitting set

Let (k, d, n) be the triple of natural numbers given in the input. We design a simple hitting set \mathcal{H} . We will generate a set of vectors $\bar{\delta} \in \mathbb{F}^n$ that make \mathcal{H} .

- Let $S \subseteq \mathbb{F}$ be an arbitrary set of size $dnk^2 + 1$.
- Let $T \subseteq \mathbb{F}$ be an arbitrary set of size $d + 1$.
- For each $\beta \in S$ and each vector $(\gamma_1, \dots, \gamma_k) \in T^k$, define the vector $\bar{\delta}$ in \mathbb{F}^n as follows:

$$\delta_i := \sum_{j \in [k]} \beta^{ij} \gamma_j.$$

We will use the classical Schwartz-Zippel theorem.

Theorem 13 (*Schwartz-Zippel*) *Let $f(y_1, \dots, y_k)$ be a polynomial of degree d . Let T be a finite subset of \mathbb{F} . The probability that f is zero on a random point in T^k is at most $d/|T|$.*

Thus, for $|T| > d$, T^k is a hitting set for all k -variate polynomials of degree d .

We are now all set to finish the proof of Theorem 3.

Theorem 14 *The set \mathcal{H} is a hitting set for $\Sigma\Pi\Sigma(k, d, n)$ circuits. It can be generated in $\text{poly}(nd^k)$ time.*

Proof. The latter statement is quite clear, given the construction of \mathcal{H} . Consider a non-zero $\Sigma\Pi\Sigma(k, d, n)$ circuit C . We need to show the existence of some $\bar{\delta} \in \mathcal{H}$ such that $C(\bar{\delta}) \neq 0$. By Lemma 11, there exists a $\beta \in S$ such that $\Psi_\beta(C) \neq 0$. Since $\Psi_\beta(C)$ is a $\Sigma\Pi\Sigma(k, d, k)$ circuit, Theorem 13 tells us that there is some $\bar{\gamma} \in T^k$ such that $\Psi_\beta(C)(\bar{\gamma}) \neq 0$. Consider the $\bar{\delta}$ corresponding to this β and $\bar{\gamma}$. By construction of $\bar{\delta}$ and the definition of Ψ_β (Equation 2), $C(\bar{\delta}) = \Psi_\beta(C)(\bar{\gamma}) \neq 0$.

■

5 Conclusion

We show that $\Sigma\Pi\Sigma(k, d, n)$ identity is only as complicated as a $\Sigma\Pi\Sigma(k, d, k)$ identity. We prove this fact by observing that there is a “low rank” homomorphism that preserves the ideal structure in depth-3 circuits. Since this low rank homomorphism is easily computable, we get a $\text{poly}(nd^k)$ time blackbox test. Can we identify properties of k -variate fanin k identities to develop faster PIT algorithms? Currently, no PIT algorithm is able to beat the exponential dependence on k .

This work also raises a question for depth-4 circuits: are there analogous low rank homomorphisms for $\Sigma\Pi\Sigma\Pi(k)$ circuits? Such results would open the door for interesting PIT algorithms for higher depth circuits.

Can this approach be used beyond PIT? In particular, there are results known about learning $\Sigma\Pi\Sigma(k)$ circuits where PIT methods have turned out to be useful [KS09a]. The variable reduction techniques might have some utility for these problems.

Acknowledgements

We are grateful to Hausdorff Center for Mathematics, Bonn for its kind support, especially in hosting the second author when part of the work was done. The first author thanks Malte Beecken and Johannes Mittmann for several interesting discussions.

References

- [AB99] M. Agrawal and S. Biswas. Primality and identity testing via Chinese remaindering. In *Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 202–209, 1999.
- [Agr05] M. Agrawal. Proving lower bounds via pseudo-random generators. In *Proceedings of the 25th Annual Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, pages 92–105, 2005.
- [Agr06] M. Agrawal. Determinant versus permanent. In *Proceedings of the 25th International Congress of Mathematicians (ICM)*, volume 3, pages 985–997, 2006.
- [AL86] L. M. Adleman and H. W. Lenstra. Finding irreducible polynomials over finite fields. In *Proceedings of the 18th Annual ACM Symposium on Theory of Computing (STOC)*, pages 350–355, 1986.
- [AM07] V. Arvind and P. Mukhopadhyay. The monomial ideal membership problem and polynomial identity testing. In *Proceedings of the 18th International Symposium on Algorithms and Computation (ISAAC)*, pages 800–811, 2007.
- [AS09] M. Agrawal and R. Saptharishi. Classifying polynomials and identity testing. Technical report, IIT Kanpur, <http://www.cse.iitk.ac.in/manindra/survey/Identity.pdf>, 2009.
- [AV08] M. Agrawal and V. Vinay. Arithmetic circuits: A chasm at depth four. In *Proceedings of the 49th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 67–75, 2008.
- [BOT88] M. Ben-Or and P. Tiwari. A deterministic algorithm for sparse multivariate polynomial interpolation. In *Proceedings of the 20th annual ACM symposium on Theory of computing (STOC)*, pages 301–309, 1988.
- [CDGK91] M. Clausen, A. W. M. Dress, J. Grabmeier, and M. Karpinski. On zero-testing and interpolation of k -sparse multivariate polynomials over finite fields. *Theoretical Computer Science*, 84(2):151–164, 1991.
- [CK97] Z. Chen and M. Kao. Reducing randomness via irrational numbers. In *Proceedings of the 29th annual ACM symposium on Theory of computing (STOC)*, pages 200–209, 1997.
- [DS05] Z. Dvir and A. Shpilka. Locally decodable codes with 2 queries and polynomial identity testing for depth 3 circuits. In *STOC*, pages 592–601, 2005.
- [GKS90] D. Grigoriev, M. Karpinski, , and M. F. Singer. Fast parallel algorithms for sparse multivariate polynomial interpolation over finite fields. *SIAM Journal on Computing*, 19(6):1059–1063, 1990.
- [GR05] A. Gabizon and R. Raz. Deterministic extractors for affine sources over large fields. In *Proceeding of the 46th FOCS*, pages 407–416, 2005.
- [HS80] J. Heintz and C. P. Schnorr. Testing polynomials which are easy to compute (extended abstract). In *Proceedings of the twelfth annual ACM Symposium on Theory of Computing*, pages 262–272, New York, NY, USA, 1980.
- [KI03] V. Kabanets and R. Impagliazzo. Derandomizing polynomial identity tests means proving circuit lower bounds. In *STOC*, pages 355–364, 2003.

- [KMSV10] Z. Karnin, P. Mukhopadhyay, A. Shpilka, and I. Volkovich. Deterministic identity testing of depth-4 multilinear circuits with bounded top fan-in. In *Proceedings of the 42nd ACM symposium on Theory of computing (STOC)*, pages 649–658, 2010.
- [KS96] M. Karpinski and I. Shparlinski. On some approximation problems concerning sparse polynomials over finite fields. *Theoretical Computer Science*, 157(2):259–266, 1996.
- [KS01] A. Klivans and D. A. Spielman. Randomness efficient identity testing of multivariate polynomials. In *Proceedings of the 33rd Symposium on Theory of Computing (STOC)*, pages 216–223, 2001.
- [KS06] N. Kayal and N. Saxena. Polynomial identity testing for depth 3 circuits. In *IEEE Conference on Computational Complexity*, pages 9–17, 2006.
- [KS08] Z. Karnin and A. Shpilka. Deterministic black box polynomial identity testing of depth-3 arithmetic circuits with bounded top fan-in. In *Proceedings of the 23rd Annual Conference on Computational Complexity (CCC)*, pages 280–291, 2008.
- [KS09a] Z. S. Karnin and A. Shpilka. Reconstruction of generalized depth-3 arithmetic circuits with bounded top fan-in. In *Proceedings of the 24th Annual Conference on Computational Complexity (CCC)*, pages 274–285, 2009.
- [KS09b] N. Kayal and S. Saraf. Blackbox polynomial identity testing for depth 3 circuits. In *Proceedings of the 50th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 198–207, 2009.
- [LV98] D. Lewin and S. Vadhan. Checking polynomial identities over any field: Towards a derandomization? In *Proceedings of the 30th Annual Symposium on the Theory of Computing (STOC)*, pages 428–437, 1998.
- [Raz10] R. Raz. Tensor-rank and lower bounds for arithmetic formulas. In *Proceedings of the 42nd Annual ACM Symposium on Theory of Computing (STOC)*, 2010.
- [RS05] R. Raz and A. Shpilka. Deterministic polynomial identity testing in non-commutative models. *Computational Complexity*, 14(1):1–19, 2005.
- [Sax08] N. Saxena. Diagonal circuit identity testing and lower bounds. In *Proceedings of the 35th Annual International Colloquium on Automata, Languages and Programming (ICALP)*, pages 60–71, 2008.
- [Sax09] N. Saxena. Progress on polynomial identity testing. *Bulletin of the European Association for Theoretical Computer Science (EATCS)- Computational Complexity Column*, (99):49–79, 2009.
- [Sch80] J. T. Schwartz. Fast probabilistic algorithms for verification of polynomial identities. *Journal of the ACM*, 27(4):701–717, 1980.
- [SS96] R. E. Schapire and L. M. Sellie. Learning sparse multivariate polynomials over a field with queries and counterexamples. *Journal of Computer and System Sciences*, 52(2):201–213, 1996.
- [SS09] N. Saxena and C. Seshadhri. An almost optimal rank bound for depth-3 identities. In *Proceedings of the 24th Annual Conference on Computational Complexity (CCC)*, pages 137–148, 2009.
- [SS10a] N. Saxena and C. Seshadhri. From Sylvester-Gallai configurations to rank bounds: Improved black-box identity test for depth-3 circuits. In *Proceedings of the 51st Annual Symposium on Foundations of Computer Science (FOCS)*, 2010.

- [SS10b] N. Saxena and C. Seshadhri. From Sylvester-Gallai configurations to rank bounds: Improved black-box identity test for depth-3 circuits. Technical Report TR10-013 (Revision #1), ECCC, <http://eccc.hpi-web.de/report/2010/013/>, 2010.
- [SV09] A. Shpilka and I. Volkovich. Improved polynomial identity testing for read-once formulas. In *Proceedings of the 13th International Workshop on Randomization and Computation (RANDOM)*, pages 700–713, 2009.
- [Wer94] K. Werther. The complexity of sparse polynomial interpolation over finite fields. *Applicable Algebra in Engineering, Communication and Computing*, 5:91–103, 1994.
- [Zip79] R. Zippel. Probabilistic algorithms for sparse polynomials. In *Proceedings of the International Symposium on Symbolic and Algebraic Manipulation (EUROSAM)*, pages 216–226, 1979.

A A Vandermonde-inspired Linear Transformation

Lemma 7. Let $\Psi_\beta : \mathcal{R} \rightarrow \mathcal{R}'$ be the linear homomorphism defined by Equation (2). Let $S \subseteq L(\mathcal{R})$ be a subset of linear forms with $\text{rk}(S) \leq k$. For all but nk^2 values of β , $\text{rk}(\Psi_\beta(S)) = \text{rk}(S)$.

The homomorphism Ψ_β depends solely on the triple of natural numbers (k, d, n) and is computable in $\text{poly}(kdn)$ time.

Proof. We can assume wlog that S is of rank k , and has k linearly independent forms $\ell_1, \dots, \ell_k \in L(\mathcal{R})$. Say Ψ_β maps $\ell_i = \sum_{j \in [n]} a_{i,j} x_j$ to $\sum_{j \in [k]} b_{i,j} y_j$, for all $i \in [k]$. Define matrices $B := ((b_{i,j}))_{i \in [k], j \in [k]}$ and $A := ((a_{i,j}))_{i \in [k], j \in [n]}$. By Equation (1) we deduce $B = A \cdot V_{n,k}$. We will show that B is invertible.

Since the rows of A are linearly independent over \mathbb{F} , we can apply *partial* Gaussian elimination (i.e. row operations) on A . This has the effect of *left*-multiplying A by an invertible matrix $E \in \mathbb{F}^{k \times k}$ ensuring: there are column indices $j_1 > \dots > j_k \in [n]$ such that j_i is the *maximal* index with $(EA)_{i,j_i} \neq 0$, for all $i \in [k]$. Now we consider the matrix $B' := (EA) \cdot V_{n,k}$,

$$\det(B') = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \cdot P_\sigma(\beta), \quad \text{where } P_\sigma(\beta) := \prod_{i \in [k]} B'_{i,\sigma(i)}.$$

Note that we view $B'_{i,\sigma(i)}$ as a polynomial in $\mathbb{F}[\beta]$ which, by the assumption on EA , is of degree $j_i \cdot \sigma(i)$. Thus,

$$\deg(P_\sigma(\beta)) = \sum_{i \in [k]} j_i \cdot \sigma(i).$$

Since $j_1 > \dots > j_k$, it can be easily shown that the expression above achieves its maxima (over $\sigma \in S_k$) only if $\sigma(1) > \dots > \sigma(k)$. But this uniquely specifies σ , hence there is a unique $P_\sigma(\beta)$ of the largest degree ($\leq nk^2$). Thus, $\det(B')$ is a nonzero polynomial in $\mathbb{F}[\beta]$ of degree at most nk^2 . This means that B' is invertible, hence $B = E^{-1}B'$ is invertible, for all but at most nk^2 values of β . ■

B A Cancellation Lemma

An $f \in \mathcal{R}$ is called a *zerodivisor* of an ideal I (or mod I) if $f \notin I$ and there exists a $g \in \mathcal{R} \setminus I$ such that $fg \in I$.

Let $u, v \in \mathcal{R}$. It is easy to see that if u is nonzero mod I and is a *non-zerodivisor* mod I then: $uv \in I$ iff $v \in I$. This can be seen as some sort of a “cancellation rule” for non-zerodivisors. We show such a cancellation rule in the case of ideals arising in $\Sigma\Pi\Sigma$ circuits.

Lemma 10. Let f_1, \dots, f_m be multiplication terms generating an ideal I , let $\ell \in L(\mathcal{R})$ and $g \in \mathcal{R}$. If $\ell \notin \text{radsp}(I)$ then: $\ell g \in I$ iff $g \in I$.

Proof. Assume $\ell \notin \text{radsp}(I)$. If $I = \{0\}$ then the lemma is of course true. So let us assume that $I \neq \{0\}$ and $\text{rk}(\text{radsp}(I)) =: r \in [n - 1]$. As $\ell \notin \text{radsp}(I)$ there exists an invertible linear transformation $\tau : L(\mathcal{R}) \rightarrow L(\mathcal{R})$ that maps each form of $\text{radsp}(I)$ to $\text{sp}(x_1, \dots, x_r)$ and maps ℓ to x_n . Now suppose that $\ell g \in I$. This means that there are $q_1, \dots, q_m \in \mathcal{R}$ such that $\ell g = \sum_{i=1}^m q_i f_i$. Apply τ on this to get:

$$x_n g' = \sum_{i=1}^m q'_i \tau(f_i). \quad (5)$$

We know that $\tau(f_i)$'s are free of x_n . Express g', q'_i -s as polynomials wrt x_n , say

$$g' = \sum_{j \geq 0} a_j x_n^j, \text{ where } a_j \in \mathbb{F}[x_1, \dots, x_{n-1}] \quad (6)$$

$$q'_i = \sum_{j \geq 0} b_{i,j} x_n^j, \text{ where } b_{i,j} \in \mathbb{F}[x_1, \dots, x_{n-1}] \quad (7)$$

Now for some $d \geq 1$ compare the coefficients of x_n^d on both sides of Equation (5). We get $a_{d-1} = \sum_{i=1}^m b_{i,d} \tau(f_i)$, thus a_{d-1} and $a_{d-1} x_n^{d-1}$ are in $\langle \tau(f_1), \dots, \tau(f_m) \rangle$. Doing this for all $d \geq 1$, we get $g' \in \langle \tau(f_1), \dots, \tau(f_m) \rangle$, hence $g = \tau^{-1}(g') \in \langle f_1, \dots, f_m \rangle = I$. This finishes the proof. ■