THOMASSEN'S CHOOSABILITY ARGUMENT REVISITED

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Abstract. Thomassen (1994) proved that every planar graph is 5-choosable. This result was generalised by Škrekovski (1998) and He et al. (2008), who proved that every K_5-minor-free graph is 5-choosable. Both proofs rely on the characterisation of K_5-minor-free graphs due to Wagner (1937). This paper proves the same result without using Wagner's structure theorem or even planar embeddings. Given that there is no structure theorem for graphs with no K_6-minor, we argue that this proof suggests a possible approach for attacking the Hadwiger Conjecture.

1. Introduction

In 1943, Hadwiger [2] made the following conjecture, which is widely considered to be one of the most important open problems in graph theory[1]; see [14] for a survey.

Hadwiger Conjecture. Every K_t-minor-free graph is $(t-1)$-colourable.

The Hadwiger Conjecture is true for $t \leq 6$ [9, 10] and unsolved for $t \geq 7$. In general, $ct\sqrt{\log t}$ is the best known upper bound on the chromatic number of K_t-minor-free graphs, for some constant c [7, 12]. This result is proved as follows. A graph G is *d-degenerate* if every subgraph of G has a vertex of degree at most d. Every d-degenerate graph is $(d+1)$-colourable—choose a vertex v of degree at most d, apply induction to $G - v$, and colour v with one of the colours not present in its neighbourhood. Kostochka [7] and Thomason [12] independently proved that every K_t-minor-free graph is $ct\sqrt{\log t}$-degenerate, and is thus $ct\sqrt{\log t}$-colourable. The following conjecture remains unsolved.

Weak Hadwiger Conjecture. There is a constant c such that every K_t-minor-free graph is ct-colourable.

There are (at least) two major obstacles to overcome in a proof of the Hadwiger Conjecture or the Weak Hadwiger Conjecture:

- There are K_t-minor-free graphs with minimum degree $ct\sqrt{\log t}$ for some constant c. Therefore the above degeneracy-based algorithm fails.

[1]All graphs in this paper are undirected, simple and finite. We employ standard graph-theoretic terminology and notation [1].

- For $t \geq 6$, there is no known precise structural characterisation of K_t-minor-free graphs (and even for $t = 6$ the situation seems hopeless).

This paper suggests a possible approach around these two obstacles.

A *list-assignment* of a graph G is a function L that assigns to each vertex v of G a set $L(v)$ of colours. G is *L-colourable* if there is a colouring of G such that the colour assigned to each vertex v is in $L(v)$. G is *k-choosable* if G is L-colourable for every list-assignment L with $|L(v)| \geq k$ for each vertex v of G. If G is k-choosable then G is also k-colourable—just use the same set of k colours for each vertex. Also note that every d-degenerate graph is $(d + 1)$-choosable. See [19] for a survey on list colourings. Kawarabayashi and Mohar [6] made the following conjecture:

Weak List Hadwiger Conjecture. There is a constant c such that every K_t-minor-free graph is ct-choosable.

Kawarabayashi and Mohar [6] wrote that they believe the Weak List Hadwiger Conjecture holds for $c = \frac{3}{2}$. Wood [18] conjectured it with $c = 1$.

List Hadwiger Conjecture. Every K_t-minor-free graph is t-choosable.

For $t \in \{2, 3, 4\}$, every K_t-minor-free graph is $(t - 2)$-degenerate, and thus is $(t - 1)$-choosable. Now consider the $t = 5$ case. Thomassen [13] proved that every planar graph is 5-choosable, and Voigt [15] constructed planar graphs that are not 4-choosable. Thomassen's result was generalised by Škrekovski [11] and He et al. [3] as follows:

Theorem 1 ([3, 11]). *Every K_5-minor-free graph is 5-choosable.*

One feature of Thomassen's proof is that it does not depend on the degeneracy of planar graphs. Thus list colourings provide a potential route around the first obstacle above. See [6, 18] for more concrete examples of this idea. The second obstacle remains. In particular, Thomassen's proof relies heavily on the structure of planar graphs, as do the proofs of Theorem 1, both of which employ the structural characterisation of K_5-minor graphs in terms of planar graphs due to Wagner [16]. The main contribution of this paper is to prove Theorem 1 without using Wagner's characterisation—even without planar embeddings. Given that there is no precise structure theorem for K_t-minor-free graphs for $t \geq 6$, we consider this a first step towards proving the (Weak) List Hadwiger Conjecture for $t \geq 6$.

2. Proof of Theorem 1

Our proof of Theorem 1 is inspired by Thomassen's proof for planar graphs. This remarkable inductive argument allows two adjacent vertices on the outerface to be pre-coloured (that is, have a list of one colour), the remaining vertices on the outerface have a list of three colours, and the other vertices have a list of five colours. The dependence on the outerface is an obstacle to generalising Thomassen's proof, and motivates the following definition.

Definition 2. Let $\mathcal{M}$ be a minor-closed class of graphs. Let $B \subseteq V(G)$ for some graph $G \in \mathcal{M}$. Let $G^{B+\alpha}$ be the graph obtained from G by adding a new vertex α adjacent to each vertex in B. Then B is an $\mathcal{M}$-*boundary* of G if $G^{B+\alpha}$ is also in $\mathcal{M}$.

This definition generalises the outerface, since $G^{B+\alpha}$ is planar if and only if all the vertices in B are on the outerface of some planar embedding of G.

Lemma 3. *Let $\mathcal{M}$ be a minor-closed class of graphs. Let B be an $\mathcal{M}$-boundary of some graph $G \in \mathcal{M}$. Let $v \in B$. Then $C := (B \setminus \{v\}) \cup N_G(v)$ is an $\mathcal{M}$-boundary of $H := G - v$.*

Proof. Observe that $H^{C+\alpha}$ is isomorphic to the graph obtained from $G^{B+\alpha}$ by contracting the edge $v\alpha$. Since $G^{B+\alpha} \in \mathcal{M}$ and $\mathcal{M}$ is minor-closed, $H^{C+\alpha}$ is also in $\mathcal{M}$. That is, C is an $\mathcal{M}$-boundary of H. □

The following lemma is a corollary of a more general result by Mader [8]; we include the following simple proof for completeness.

Lemma 4. *Let v be a vertex in a 2-connected graph G. Then G/vw is 2-connected for some edge vw incident to v.*

Proof. Suppose on the contrary that G/vw is not 2-connected for each edge vw incident to v; thus $\{v, w\}$ is a cut set. Choose such an edge vw to minimise the order of a smallest component H of $G - \{v, w\}$. Since G is 2-connected, v has a neighbour x in H. Thus $G - \{v, x\}$ contains a component that is a proper subgraph of H, which contradicts the choice of vw. □

Let x, y, z be distinct vertices in a graph G. A K_3-*minor rooted* at x, y, z consists of three connected subgraphs X, Y, Z of G that are pairwise disjoint and pairwise adjacent, such that $x \in V(X)$, $y \in V(Y)$ and $z \in V(Z)$. See [4, 5, 17] for more on rooted minors. A vertex v of G is *good* (with respect to x, y, z) if at least two of x, y, z are in the same component of $G - v$, otherwise v is *bad*. Note that if v is a vertex in a 2-connected graph G, then $G - v$ is connected, and all the vertices in $\{x, y, z\} \setminus \{v\}$ are in the same component of $G - v$; thus at least two of x, y, z are in one component of $G - v$. That is, every vertex is good in a 2-connected graph.

Lemma 5. *Let x, y, z be distinct vertices in a graph G. Then G has a K_3-minor rooted at x, y, z if and only if every vertex in G is good.*

Proof. ($\Longrightarrow$) Let X, Y, Z be the branch sets of a K_3-minor rooted at x, y, z. Let $v \in V(G)$. Without loss of generality, $v \notin X \cup Y$. Since $G[X \cup Y]$ is connected, x and y are in the same component of $G - v$. Thus v is good.

($\Longleftarrow$) We proceed by induction. Let x, y, z be distinct vertices in a graph G in which every vertex is good. If $|V(G)| = 3$ and $G \not\cong K_3$, then without loss of generality, G is a subgraph of the path (x, y, z), implying y is bad. Thus, if $|V(G)| = 3$ then $G \cong K_3$, and we are done. Now assume that $|V(G)| \geq 4$.

First suppose that G is disconnected. If x, y and z are all in the same component H of G, then by induction, H and hence G has a K_3-minor rooted at x, y, z. Otherwise some component contains at most one of x, y, z, say x. Then y and z are both bad.

Now assume that G is connected. Suppose that G contains a cut-vertex v. Since v is good, at least two of x, y, z, say x and y, are in the same component of $G - v$. Let w be a neighbour of v in a component of $G - v$ not containing x and y. Let G' be the graph obtained from G by contracting vw into a vertex v'. We may consider x, y, z to be vertices of G'. (It is possible that $w = z$ and $v' = z$.) In G', the vertex v' is good since x and y remain in the same component of $G' - v'$. If some other vertex in G' is bad, then it would be bad in G. Thus every vertex in G' is good. By induction, G' and hence G contains a K_3-minor rooted at x, y, z.

Now assume that G is 2-connected. Choose $v \in V(G) \setminus \{x, y, z\}$. By Lemma 4, G/vw is 2-connected for some edge vw incident to v. Thus every vertex is good in G/vw. Since x, y, z are distinct vertices in G/vw, by induction, G/vw and hence G has a K_3-minor rooted at x, y, z.   □

A graph G is said to contain *every rooted K_3-minor* if G contains a K_3-minor rooted at x, y, z for all distinct $x, y, z \in V(G)$.

Proposition 6. *A graph G contains every rooted K_3-minor if and only if G is 2-connected.*

Proof. Since every vertex is good in a 2-connected graph, by Lemma 5, a 2-connected graph contains every rooted K_3-minor. For the converse, let G be a graph that contains every rooted K_3-minor. If G is disconnected, then there is no K_3-minor rooted at x, y, z, whenever x and y are in distinct components. Hence G is connected. If G has a cut-vertex x, then G contains no K_3-minor rooted at x, y, z, whenever y and z are in distinct components of $G - x$. Hence G is 2-connected.   □

Let G_1 and G_2 be subgraphs of a graph G, such that $G = G_1 \cup G_2$ and $V(G_1) \setminus V(G_2) \neq \emptyset$ and $V(G_2) \setminus V(G_1) \neq \emptyset$. In particular, there is no edge between $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$. Then $\{G_1, G_2\}$ is a *separation* of order $|V(G_1) \cap V(G_2)|$.

Theorem 1 is a consequence of the following lemma (with $A = B = \emptyset$).

Lemma 7. *Let $\mathcal{M}$ be the class of K_5-minor-free graphs. Let $G \in \mathcal{M}$. Let $A \subseteq B \subseteq V(G)$, such that A is a clique, and B is an $\mathcal{M}$-boundary of G. Let L be a list-assignment of G such that:*

- *$|L(x)| = 1$ for each vertex $x \in A$,*
- *$L(x) \neq L(y)$ for distinct $x, y \in A$,*
- *$|L(x)| \geq 3$ for each vertex $x \in B \setminus A$,*
- *$|L(x)| \geq 5$ for each vertex $x \in V(G) \setminus B$.*

Then G is L-colourable.

Proof. Let (G, A, B, L) be a counterexample with $|V(G)|$ minimum, and then with $|A|$ maximum. Clearly $|V(G)| \geq 4$.

Case 1. $B = \emptyset$: Choose $v \in V(G)$. Then $G^{\{v\}+\alpha}$, which is obtained from G by adding a new vertex α adjacent to v, is K_5-minor-free. Let $L'(v) := \{c\}$ for some colour $c \in L(v)$. Let $L'(x) := L(x)$ for every other vertex x. By the choice of (G, A, B, L), the

instance $(G, \{v\}, \{v\}, L')$ is not a counterexample, and G is L-colourable. Now assume that $B \neq \emptyset$.

Case 2. $A = \emptyset$: Choose $v \in B$. Let $L'(v) := \{c\}$ for some colour $c \in L(v)$. Let $L'(x) := L(x)$ for every other vertex x. Again $(G, \{v\}, B, L')$ is not a counterexample, and G is L-colourable. Now assume that $A \neq \emptyset$.

Case 3. G is not connected: Then G contains a separation $\{G_1, G_2\}$ with $V(G_1 \cap G_2) = \emptyset$. Since A is a clique, without loss of generality, $A \subseteq V(G_1)$. Let $B_i := B \cap V(G_i)$. Then B_i is an $\mathcal{M}$-boundary of G_i (since $G_i^{B_i+\alpha} \subseteq G^{B+\alpha}$). Define $L_i(x) := L(x)$ for each vertex x in G_i. Hence (G_1, A, B_1, L_1) is not a counterexample, and G_1 is L_1-colourable. Also $(G_2, \emptyset, B_2, L_2)$ is not a counterexample, and G_2 is L_2-colourable. Hence G is L-colourable. Now assume that G is connected.

Case 4. G contains a cut-vertex $v \in B$: Then G contains a separation $\{G_1, G_2\}$ with $V(G_1 \cap G_2) = \{v\}$. Since A is a clique, without loss of generality, $A \subseteq V(G_1)$. Let $B_i := B \cap V(G_i)$. Then B_i is an $\mathcal{M}$-boundary of G_i. Define $L_1(x) := L(x)$ for each vertex x in G_1. Thus (G_1, A, B_1, L_1) is not a counterexample, and G_1 is L_1-colourable. Let $L_2(v) := \{c\}$ where v is coloured c in G_1. Let $L_2(x) := L(x)$ for every other vertex x in G_2. Then $(G_2, \{v\}, B_2, L_2)$ is not a counterexample (since $v \in B_2$), and G_2 is L_2-colourable. Hence G is L-colourable (since v receives the same colour in G_1 and in G_2, and each edge of G is in G_1 or G_2). Now assume that $G - v$ is connected for every vertex $v \in B$.

Case 5. G contains a cut-vertex v separating two vertices in B: Then G contains a separation $\{G_1, G_2\}$ with $V(G_1 \cap G_2) = \{v\}$, such that $B \cap V(G_1 - v) \neq \emptyset$ and $B \cap V(G_2 - v) \neq \emptyset$. Since A is a clique, without loss of generality, $A \subseteq V(G_1)$. Let $B_1 := B \cap V(G_1)$. Then B_1 is an $\mathcal{M}$-boundary of G_1. Define $L_1(x) := L(x)$ for each vertex x in G_1. Thus (G_1, A, B_1, L_1) is not a counterexample, and G_1 is L_1-colourable.

Since G is connected, G_1 is connected. Let $B_2 := (B \cap V(G_2)) \cup \{v\}$. Then B_2 is an $\mathcal{M}$-boundary of G_2, since $G_2^{B_2+\alpha}$ is a minor of $G^{B+\alpha}$ obtained by contracting G_1 into v (since α has a neighbour in $G_1 - v$). Let $L_2(v) := \{c\}$ where v is coloured c in G_1. Let $L_2(x) := L(x)$ for every other vertex x in G_2. Then $(G_2, \{v\}, B_2, L_2)$ is not a counterexample (since $v \in B_2$), and G_2 is L_2-colourable. Hence G is L-colourable. Now assume that G contains no cut-vertex separating two vertices in B.

Case 6. G contains a cut-set $\{v, w\}$ separating two vertices in B, where $v \in B$: Thus G has a separation $\{G_1, G_2\}$ with $V(G_1) \cap V(G_2) = \{v, w\}$ and $B \cap V(G_1 - \{v, w\}) \neq \emptyset$ and $B \cap V(G_2 - \{v, w\}) \neq \emptyset$.

Suppose that $vw \notin E(G)$. We claim that adding the edge vw creates no K_5-minor in $G^{B+\alpha}$. Let G' be the graph obtained from $G^{B+\alpha}$ by adding the edge vw. Let H be a 4-connected minor in G'. Since $\{\alpha, v, w\}$ is a separator in G', there are no two branch sets of H, with one contained in $V(G_1) \setminus V(G_2)$, and the other contained in $V(G_2) \setminus V(G_1)$. Thus, without loss of generality, every branch set of H intersects G_1. By Cases 4 and 5, neither v nor w are cut-vertices in G. Thus there is vw-path P in G_2. Hence the edge vw in our H-minor can be replaced by P to obtain an H-minor in $G^{B+\alpha}$ (without vw). Since K_5 is 4-connected and $G^{B+\alpha}$ is K_5-minor-free, G' is also K_5-minor-free. That is, adding

vw does not create a K_5-minor in $G^{B+\alpha}$ (and also not in G). Since A is a clique, adding vw does not break any of the assumptions in the lemma. Now assume that $vw \in E(G)$.

Since A is a clique, without loss of generality, $A \subseteq V(G_1)$. Let $B_1 := B \cap V(G_1)$. Then B_1 is an $\mathcal{M}$-boundary of G_1. Define $L_1(x) := L(x)$ for each vertex x in G_1. Thus (G_1, A, B_1, L_1) is not a counterexample, and G_1 is L_1-colourable.

Since G is connected and v is not a cut-vertex, $G_1 - v$ is connected. Let $B_2 := (B \cap V(G_2)) \cup \{w\}$. Then B_2 is an $\mathcal{M}$-boundary of G_2, since $G_2^{B_2+\alpha}$ is a minor of $G^{B+\alpha}$ obtained by contracting $G_1 - v$ into w (since α has a neighbour in $G_1 - v$). Let $L_2(v) := \{c\}$ where v is coloured c in G_1. Let $L_2(w) := \{d\}$ where w is coloured d in G_1. Let $L_2(x) := L(x)$ for every other vertex x in G_2. Then $(G_2, \{v, w\}, B_2, L_2)$ is not a counterexample (since $\{v, w\} \subseteq B_2$), and G_2 is L_2-colourable. Hence G is L-colourable. Now assume that G contains no such cut-set $\{v, w\}$.

Case 7. Some vertex $v \in B$ has degree at least 3 in $G[B]$: Let x, y, z be three neighbours of v in B. If $G - v$ contains a K_3-minor rooted at x, y, z, then adding v and α gives a K_5-minor in $G^{B+\alpha}$, which contradicts the assumption that B is an $\mathcal{M}$-boundary of G. Thus $G - v$ contains no K_3-minor rooted at x, y, z. By Lemma 5, for some vertex w in $G - v$, all the vertices in $\{x, y, z\} \setminus \{w\}$ are in distinct components of $G - \{v, w\}$. Thus $\{v, w\}$ is a cut-set satisfying Case 6 or 7. Now assume that $G[B]$ has maximum degree at most 2.

Case 8. $G[A]$ is a component of $G[B]$: Choose $v \in A$. Let $G' := G - v$ and $A' := A \setminus \{v\}$ and $B' := (B \setminus \{v\}) \cup N_G(v)$. By Lemma 3, B' is an $\mathcal{M}$-boundary of G'. Let $L'(u) := L(u) \setminus L(v)$ for each vertex $u \in N_G(v) \setminus B$. Since $|L(v)| = 1$ and v has no neighbour in $B \setminus A$, we have $|L'(x)| \geq 3$ for each $x \in B'$. Let $L'(x) := L(x)$ for every other vertex x. Then (G', A', B', L') is not a counterexample, and G' is L'-colourable. Assign v the colour in $L(v)$. This colour is not in $L'(u)$ for each $u \in N_G(v)$. Thus G is L-colourable.

Case 9. $G[A]$ is not a component of $G[B]$: Choose $v \in B \setminus A$ adjacent to some vertex $p \in A$. Since $G[B]$ has maximum degree at most 2, v has at most one other neighbour in B; let w be this neighbour (if it exists). Let $G' := G - v$ and $B' := (B \setminus \{v\}) \cup N_G(v)$. By Lemma 3, B' is an $\mathcal{M}$-boundary of G'. Let c, d be distinct colours in $L(v) \setminus L(p)$. Let $L'(u) := L(u) \setminus \{c, d\}$ for each vertex $u \in N_G(v) \setminus B$; thus $|L'(u)| \geq 5 - 2 = 3$. Let $L'(x) := L(x)$ for every other vertex x. Then (G', A, B', L') is not a counterexample, and G' is L'-colourable. Assign v colour c or d different from the colour assigned to w (if w exists). Hence G is L-colourable. $\square$

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