

Randomized Communication and Implicit Representations for Matrices and Graphs of Small Sign-Rank

Nathaniel Harms*
EPFL
nathaniel.harms@epfl.ch

Viktor Zamaraev
University of Liverpool
viktor.zamaraev@liverpool.ac.uk

July 11, 2023

Abstract

We prove a characterization of the structural conditions on matrices of sign-rank 3 and unit disk graphs (UDGs) which permit *constant-cost* public-coin randomized communication protocols. Therefore, under these conditions, these graphs also admit implicit representations.

The *sign-rank* of a matrix $M \in \{\pm 1\}^{N \times N}$ is the smallest rank of a matrix R such that $M_{i,j} = \text{sign}(R_{i,j})$ for all $i, j \in [N]$; equivalently, it is the smallest dimension d in which M can be represented as a point-halfspace incidence matrix with halfspaces through the origin, and it is essentially equivalent to the *unbounded-error communication complexity*. Matrices of sign-rank 3 can achieve the maximum possible *bounded-error* randomized communication complexity $\Theta(\log N)$, and meanwhile the existence of implicit representations for graphs of bounded sign-rank (including UDGs, which have sign-rank 4) has been open since at least 2003. We prove that matrices of sign-rank 3, and UDGs, have *constant* randomized communication complexity if and only if they do not encode arbitrarily large instances of the GREATER-THAN communication problem, or, equivalently, if they do not contain large half-graphs as semi-induced subgraphs. This also establishes the existence of implicit representations for these graphs under the same conditions.

arXiv:2307.04441v1 [cs.CC] 10 Jul 2023

*Some of this work was done while the author was at the University of Waterloo, and visiting the University of Liverpool. Partly supported by an NSERC MSFSS, an NSERC Postdoctoral Fellowship, and the Swiss State Secretariat for Education, Research and Innovation (SERI) under contract number MB22.00026.

Contents

1	Introduction	1
1.1	Constant-Cost Communication and Implicit Graph Representations	2
1.2	Results and Techniques	3
1.3	Discussion and Open Problems	6
2	Preliminaries	6
2.1	Notation	6
2.2	Sign-Rank	7
2.3	Communication Complexity and Margin	7
2.4	Constant-Cost Communication Reductions and Equality	8
2.5	From Communication Protocols to Implicit Representations	10
3	Communication Bounds for Excluded Cycles and Subdivided Stars	10
3.1	Gyárfás Decomposition: Definition, Existence, and Properties	10
3.2	Excluding Long Cycles	12
3.3	Excluding Subdivisions of Stars	13
3.4	A Communication Protocol for the Gyárfás Decomposition	14
4	Application to Sign-Rank 3 and Unit Disk Graphs	16
4.1	Sign-Rank 3	17
4.2	Unit Disk Graphs	19
5	The Sign-Rank Hierarchy	20
5.1	Sign-Rank 5: Point-Box Incidences	21
5.2	Sign-Rank 6: Point-Line Incidences	22
A	On the number of edges in weakly-sparse point-halfspace incidence graphs	25
A.1	On the line	25
A.2	On the plane	25
A.3	In \mathbb{R}^3	26

1 Introduction

Consider a sign matrix $M \in \{\pm 1\}^{N \times N}$. In communication complexity, learning theory, and graph theory, it is often useful to represent M as a point-halfspace incidence matrix of the following form. To each row $x \in [N]$, assign a point $p_x \in \mathbb{R}^d \setminus \{0\}$, and to each row $y \in [N]$ assign a unit vector $h_y \in \mathbb{R}^d$, such that $M(x, y) = \text{sign}(\langle p_x, h_y \rangle)$. In other words, $M(x, y) = 1$ if and only if the point p_x belongs to the halfspace $H_y := \{p \in \mathbb{R}^d \mid \langle p, h_y \rangle \geq 0\}$ whose boundary hyperplane goes through the origin. It is always possible to find such a representation, but, naturally, we wish to accomplish it in the simplest way. Here are two common ways to measure the complexity of this representation:

Sign-rank. We might want to minimize the dimension d of the representation. The minimum possible d where M admits such a representation is called the *sign-rank* of M and denoted $\text{rank}_{\pm}(M)$. It is equivalent to the smallest rank d of a matrix R such that $M(x, y) = \text{sign}(R(x, y))$ for all $x, y \in [N]$. Thinking of the rows of M as a fixed domain \mathcal{X} , and the columns as a hypothesis class (i.e. subsets of \mathcal{X}), a standard technique in learning theory is to transform the domain into points in \mathbb{R}^d , and the hypothesis class into halfspaces; $\text{rank}_{\pm}(M)$ is the smallest dimension such that this transformation is possible. Since halfspaces through the origin in \mathbb{R}^d have VC dimension d , sign-rank is lower bounded by the VC dimension of the hypothesis class. In communication complexity, sign-rank is essentially equivalent to the *unbounded-error* communication complexity of M [PS86], where the two players have access to *private* randomness and wish to succeed with probability strictly better than $1/2$. A set \mathcal{M} of matrices has *bounded sign-rank* if there exists a constant d such that all matrices $M \in \mathcal{M}$ have sign-rank at most d . This is equivalent to having *constant* unbounded-error communication cost. In graph theory, finding *implicit representations* (defined below) for graphs whose adjacency matrices have bounded sign-rank is an open problem since at least 2003 [Spi03].

Margin. We might want to maximize the *margin* of the representation. For a fixed representation $\{p_x\}_{x \in [N]}$ and $\{h_y\}_{y \in [N]}$, we define the *margin* as $\min_{x, y} \frac{|\langle p_x, h_y \rangle|}{\|p_x\| \cdot \|h_y\|}$. Write $\text{mar}(M)$ for the maximum m such that there is a representation with margin m ; the dimension of this representation is irrelevant. The complexity of various learning algorithms like SVM or perceptron can be bounded in terms of the margin. It is also known that $\text{mar}(M)$ is functionally equivalent to the *two-way, public-coin randomized communication complexity* (Proposition 2.3). A set \mathcal{M} of matrices has *bounded margin* if there is some constant m such that all $M \in \mathcal{M}$ have $\text{mar}(M) \geq m$, and having bounded margin is equivalent to having *constant* public-coin randomized communication cost. Therefore, graphs whose adjacency matrices have bounded margin admit implicit representations, due to the observation of [Har20, HWZ22].

One of the main goals in communication complexity is to understand the power of randomness, and both of the above measures of complexity capture a type of randomized communication. A rapidly-growing body of work on *constant-cost communication* [Har20, HHH22b, HWZ22, HHH22a, EHK22, HHL22, HHP⁺22, ACHS23, CHHS23, HHM23] studies the properties of matrices with bounded margin or bounded sign-rank, but the relationship between these two measures is not well understood. In one direction, it is believed that there exist sets of matrices \mathcal{M} with bounded margin but unbounded sign-rank, but all known lower bounds fail to prove this [HHP⁺22] (although it was proven for *partial* matrices [HHM23]). In this paper, we are interested in the other direction:

*For matrices \mathcal{M} of bounded sign-rank, under what conditions does \mathcal{M} also have bounded margin?*¹

It is known that *some* conditions are required. Write $R(M)$ for the two-way, public-coin randomized communication cost of a matrix $M \in \{\pm 1\}^{N \times N}$ (which we will refer to simply as *communication cost*) and $R(\mathcal{M})$ for the communication cost of matrices $M \in \mathcal{M}$ as a function of their size N (see Section 2.3 for formal definitions). The GREATER-THAN communication problem, defined by the matrices $\text{GT} \in \{\pm 1\}^{N \times N}$ where $\text{GT}_{i, j} = 1$ if and only if $i > j$, has sign-rank 2 but communication cost² $R(\text{GT}) = \Theta(\log \log N)$ and therefore unbounded margin. When sign-rank increases to 3, matrices

¹Note that a matrix having bounded sign-rank *and* bounded margin does not mean that sign-rank and margin are bounded *simultaneously* by the same point-halfspace representation.

²Standard notation in the literature uses n as the number of bits in the input; we use N for the domain size, so

can achieve the maximum possible communication cost $R(\mathcal{M}) = \Theta(\log N)$ [HHL22, ACHS23], far exceeding the complexity of GREATER-THAN. However, one of our main results is that, for sign-rank 3, GREATER-THAN is the only barrier to constant-cost communication:

Theorem 1.1 (Informal). *A set \mathcal{M} of matrices with sign-rank 3 has $R(\mathcal{M}) = O(1)$ (and therefore constant margin) if and only if it does not contain arbitrarily large instances of GREATER-THAN.*

We prove a similar theorem for the adjacency matrices of unit-disk graphs (UDGs), which have sign-rank 4, and these results establish the existence of implicit representations when the condition on the GREATER-THAN instances is satisfied. We also exhibit a fundamental gap between sign-rank 4 and 5 which shows that the “type” of randomness used in our communication protocols cannot succeed in sign-rank 5 and above. Theorem 1.1 is a consequence of more general results whose motivation and applications we elaborate upon below.

1.1 Constant-Cost Communication and Implicit Graph Representations

The study of constant-cost randomized communication was initiated independently in [HHH22b, HWZ22]. One motivation of [HWZ22] was that constant-cost communication is a special case of a well-studied open problem in structural graph theory and distributed computing, which asks to characterize the hereditary graph classes \mathcal{G} that admit *implicit representations* (see e.g. [KNR92, Spi03, ACLZ15, HH22, AAA+23, Cha23]).

Implicit representations. A class of graphs is a set \mathcal{G} of (labeled) graphs that is closed under isomorphism. It is *hereditary* if it is closed under taking induced subgraphs. A hereditary class \mathcal{G} admits an *implicit representation* if there exists a decoder $D : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}$ such that, for every N -vertex graph $G \in \mathcal{G}$, each vertex v of G can be assigned an encoding $\text{enc}(v)$ of $O(\log N)$ bits, where $D(\text{enc}(u), \text{enc}(v))$ outputs the adjacency of vertices u, v ; the decoder D depends on the class \mathcal{G} but *not* the specific graph G . Implicit representations were introduced in [KNR92], who observed that they are equivalent to a graph U of size $\text{poly}(N)$, called a *universal graph*, that contains every N -vertex graph $G \in \mathcal{G}$ as an induced subgraph. Since a graph of size $\text{poly}(N)$ has at most $2^{O(N \log N)}$ N -vertex induced subgraphs, a necessary condition for the existence of implicit representations is that \mathcal{G} contains at most $2^{O(N \log N)}$ N -vertex graphs, in which case \mathcal{G} is said to have *factorial speed*.

The communication problem defined by any matrix $M \in \{\pm 1\}^{N \times N}$ is equivalent to the problem of deciding adjacency in the (bipartite) graph whose adjacency matrix is M , where each player is given a vertex. Building on [Har20], [HWZ22] observed that constant-cost communication problems \mathcal{M} are *equivalent* to hereditary graph classes \mathcal{G} that admit an *adjacency sketch*, which is a randomized version of an implicit representation, where the encodings $\text{enc}(v)$ are assigned by a randomized algorithm and have *constant size* (independent of the number of vertices), in such a way that

$$\forall u, v : \quad \mathbb{P}[D(\text{enc}(u), \text{enc}(v)) \text{ correctly outputs adjacency of } u, v] \geq 2/3.$$

Adjacency sketches for trees also appeared earlier in [FK09]. As noted in [Har20, HWZ22], adjacency sketches can be derandomized (see Section 2.5) to obtain implicit representations, making constant-cost randomized communication protocols a stronger type of implicit representation.

Unit disk graphs. This again motivates our focus on sign-rank. Graphs whose adjacency matrices have bounded sign-rank are among the most important types of graphs for which implicit representations are not known to exist in general: to obtain implicit representations for geometric intersection graphs (more precisely, semi-algebraic graphs), it suffices to study graphs of bounded sign-rank (see e.g. [Fit19]). Any class of bounded sign-rank satisfies the necessary condition of *factorial speed* [Spi03], which was conjectured to be sufficient in [KNR92, Spi03]. Until this conjecture was refuted in [HH22] by a non-constructive argument, classes of bounded sign-rank were considered promising candidates for a counterexample [Spi03]. The best known implicit representations for classes of bounded sign-rank in general use $O(N^{1-\epsilon})$ bits per vertex where $\epsilon > 0$ is a constant [Fit19, Alo22].

$\Theta(\log \log N)$ corresponds to the more commonly-stated bound $\Theta(\log n)$.

A canonical example is the unit disk graphs (UDGs). UDGs admit an “implicit representation” in the sense that each vertex may be encoded with the coordinates of its disk in \mathbb{R}^2 . However, this encoding requires exponentially-many bits [MM13], and it is a central open problem whether this difficulty can be sidestepped to obtain encodings of size $O(\log N)$; our understanding is that this is not widely believed to be possible. In this paper, we resolve the *randomized* version of the question by giving a complete characterization of the UDGs which admit constant-size adjacency sketches. To state this result, we require the notion of *stability* (see e.g. [HWZ22, POS22]).

Stability. The *chain-index* $\text{ch}(G)$ of a graph G is the largest k such that there exist disjoint sets of vertices $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ where, for any $i < j$, a_i, b_j are adjacent but b_i, a_j are not. In the terminology of [GPT22], a graph class \mathcal{G} is *graph-theoretically stable* if there is a constant k such that $\text{ch}(G) \leq k$ for all $G \in \mathcal{G}$; we will say simply *stable*³. The chain-index is essentially⁴ the largest instance of the GREATER-THAN communication problem that appears in G , and therefore a class that is *not* stable must have non-constant communication cost (see [HWZ22, HHH22a] for more on the stability condition in communication). For a graph class \mathcal{G} , write $R(\mathcal{G})$ for the function $N \mapsto \max_G R(\text{ADJ}_G)$ where G ranges over the N -vertex graphs in \mathcal{G} and ADJ_G is the adjacency matrix of G (if \mathcal{G} is a class of bipartite graphs, we take the bipartite adjacency matrix). Stability is necessary for $R(\mathcal{G}) = O(1)$; for UDGs and graphs of sign-rank 3, we show it is also *sufficient*:

Theorem 1.2. *Let \mathcal{G} be either a subclass of UDGs, or a class of sign-rank at most 3. Then $R(\mathcal{G}) = O(1)$ if and only if \mathcal{G} is stable. As a consequence, stable subclasses of UDGs and graphs of sign-rank 3 admit implicit representations.*

1.2 Results and Techniques

Theorem 1.2 follows from a more general result that has other implications for implicit graph representations and which unifies and generalizes a number of previous results. We also complement it with an impossibility result that rules out using the type of randomized techniques in this paper to prove similar results in sign-rank 5 and above. Let us now explain these results in more detail and give a brief summary of the techniques.

Constant-cost reductions. We require the notion of *constant-cost reductions* and the EQUALITY oracle. The EQUALITY communication problem is the standard example of the power of (public-coin) randomized communication. Two players are given inputs $x, y \in [N]$, respectively, and they must decide if $x = y$. By random hashing, this can be done with success probability $3/4$ using only 2 bits of communication. The success probability can be improved to any arbitrary constant by increasing the number of bits by a constant factor.

One way to design a constant-cost communication protocol is to design a *deterministic* communication protocol with constant cost, which has access to an oracle that computes EQUALITY. This means that the two players can, at any time, supply the oracle with arbitrary values a, b and receive, at unit cost, the answer to the query “ $a = b$?” The power of the EQUALITY oracle has been studied in several works [BFS86, CLV19, HHH22b, HWZ22, EHK22, AY22, CHHS23, PSS23]. One may think of these protocols as the ones that can be implemented using standard practical hash functions like SHA256. Constant-cost protocols of this form are examples of *constant-cost reductions*, a type of reduction that is natural for both constant-cost communication complexity and implicit graph representations; we formally define constant-cost reductions in general in Section 2.4. Along with the algorithmic definition of reductions to EQUALITY, there is an equivalent structural definition (see e.g. [HWZ22, AY22]): if a graph class \mathcal{G} admits a constant-cost protocol for computing adjacency in graphs $G \in \mathcal{G}$, using EQUALITY oracles, then there exists a constant t such that the adjacency matrix ADJ_G of every graph

³We use *stable* in this paper but we note that the disambiguation *graph-theoretically stable* in [GPT22] is necessary to avoid confusion with stability in the literature on model theory.

⁴Not exactly: we have no restriction on the adjacency between a_i, b_i , which helps the analysis but is not qualitatively important.

$G \in \mathcal{G}$ (or bipartite adjacency matrix, if \mathcal{G} is a class of bipartite graphs) can be written as

$$\forall x, y : \quad \text{ADJ}_G(x, y) = f(Q_1(x, y), Q_2(x, y), \dots, Q_t(x, y)),$$

where $f : \{0, 1\}^t \rightarrow \{0, 1\}$ and each Q_i is the bipartite adjacency matrix of a bipartite *equivalence graph* (disjoint union of bicliques). We write $D^{\text{Eq}}(M)$ for the minimum cost of a 2-way deterministic protocol with EQUALITY oracles. For computing adjacency in *monotone* graph classes (closed under edge & vertex deletions), all constant-cost randomized protocols can be put in this form [EHK22], but in general they cannot [HHH22b, HWZ22]. [HHP⁺22] showed that $D^{\text{Eq}}(\mathcal{G}) = O(1)$ implies that \mathcal{G} has bounded sign-rank; our results explore the converse.

Forbidden cycles and subdivided stars. Our Theorem 1.2 is a consequence of a more general result, Theorem 1.4 below, which also makes some progress towards characterizing the *finitely-defined* bipartite graph classes for which constant-cost communication and implicit representations are possible. For any set \mathcal{H} of bipartite graphs, a class \mathcal{G} of bipartite graphs is \mathcal{H} -free if no graph $G \in \mathcal{G}$ contains any $H \in \mathcal{H}$ as an induced subgraph. Every hereditary class of bipartite graphs is \mathcal{H} -free for some unique but possibly infinite set \mathcal{H} . For fixed \mathcal{H} , write $\mathcal{G}_{\mathcal{H}}$ for the \mathcal{H} -free bipartite graphs. For a bipartite graph $G = (U, W, E)$ with a fixed bipartition, we write \tilde{G} for the bipartite complement of G , i.e. $\tilde{G} = (U, W, (U \times W) \setminus E)$.

The condition that \mathcal{G} is *stable* is equivalent to the condition that it is H_k -free for some constant k , where H_k denotes the *half-graph* (see [HWZ22]), so $R(\mathcal{G}_{\mathcal{H}}) = O(1)$ requires that \mathcal{H} contain some half-graph H_k . When \mathcal{H} is *finite*, it is also necessary that \mathcal{H} contain both a tree and the bipartite complement of a tree, otherwise the number of graphs in $\mathcal{G}_{\mathcal{H}}$ is too large [LZ17]. In the case $|\mathcal{H}| = 2$, it is therefore necessary for $R(\mathcal{G}_{\mathcal{H}}) = O(1)$ that $\mathcal{H} = \{H_k, T\}$ where T and its bipartite complement \tilde{T} are both trees; it was proved in [HWZ22] that this is also *sufficient*. We believe these conditions remain sufficient for larger (but still finite) \mathcal{H} , i.e. $R(\mathcal{G}_{\mathcal{H}}) = O(1)$ whenever $\mathcal{H} = \{H_k, T_1, \tilde{T}_2\}$ for some trees T_1 and T_2 . When T_1 and T_2 are *subdivided stars*, our result confirms this.

Definition 1.3 (Subdivided Star). For $s, t \in \mathbb{N}$, we write $S_{s,t}$ for the *subdivided star*, which is obtained by taking the star graph with s leaves and subdividing each edge $t - 1$ times.

As usual, we denote by C_t the cycle on t vertices. Our main technical result is:

Theorem 1.4. *Let \mathcal{G} be a stable class of bipartite graphs that satisfies either of these conditions:*

1. *There exist constants s, t such that \mathcal{G} is $(S_{s,t}, \tilde{S}_{s,t})$ -free; or*
2. *There exists a constant t such that \mathcal{G} is $\{C_{t'}, \tilde{C}_{t'} \mid t' \geq t \text{ and } t' \text{ is even}\}$ -free.*

Then $D^{\text{Eq}}(\mathcal{G}) = O(1)$.

We use Theorem 1.4 to prove Theorem 1.2 by decomposing UDGs or graphs of sign-rank 3 into bipartite graphs that are both $(S_{3,3}, \tilde{S}_{3,3})$ -free and $\{C_t, \tilde{C}_t \mid t \geq 10 \text{ and } t \text{ is even}\}$ -free (which, to clarify, is stronger than necessary to apply the theorem). We remark that the implicit representation implied by Theorem 1.4 can be efficiently computed, meaning that the labels can be constructed in time $\text{poly}(N)$ and decoded in time $\text{poly} \log N$. This efficiency is inherited by the implicit representations of UDGs and graphs of sign-rank 3, provided that the encoder is given the geometric representation of the input graph.

Theorem 1.4 is much more general, and also allows us to recover several prior results. Analogs of Theorems 1.2 and 1.4 for the classes of permutation graphs, interval graphs, and P_7 -free and $S_{1,2,3}$ -free bipartite graphs were proved in [HWZ22]. All of these results, which in [HWZ22] each required different proof strategies, follow as corollaries of Theorems 1.2 and 1.4. Likewise, [AAA⁺23] showed the existence of implicit representations for stable, chordal bipartite graphs, which is also implied by Theorem 1.4.

	Sign-rank			
	3	4	5	6
Weakly-sparse	$D^{\text{Eq}} = O(1)$	$D^{\text{Eq}} = O(1)$	$D^{\text{Eq}} = \omega(1)$	$D^{\text{Eq}} = \omega(1)$
	$R = O(1)$	$R = O(1)$	$R = ?$	$R = \text{conjectured } \omega(1)$
Stable	$D^{\text{Eq}} = O(1)$	$D^{\text{Eq}} = O(1)$ for UDGs	$D^{\text{Eq}} = \omega(1)$	$D^{\text{Eq}} = \omega(1)$
	$R = O(1)$	$R = O(1)$ for UDGs	$R = ?$	$R = \text{conjectured } \omega(1)$

Table 1: The landscape of communication complexity for small sign-ranks; see [Section 5](#). Cells in green are results of this paper. The conjectures for sign-rank 6 are due to [\[CHHS23\]](#). The upper bounds for weakly-sparse and sign-rank ≤ 4 follow from [\[CH23\]](#) (and we also give an alternate proof of a weaker statement in [Appendix A](#)).

Higher sign-ranks and weakly-sparse graphs. To advance beyond sign-rank 3, it is helpful to compare the *stability* condition with the stronger *weakly-sparse* condition. A class of graphs \mathcal{G} is *weakly-sparse* if there is a constant t such that no graph $G \in \mathcal{G}$ contains $K_{t,t}$ as a subgraph. Any weakly-sparse class is also stable. It is known and not difficult to prove that any weakly-sparse subclass of UDGs has bounded degeneracy, and therefore the analog of [Theorem 1.2](#) for weakly-sparse UDGs is trivial (because $D^{\text{Eq}}(\mathcal{G}) = O(1)$ for any \mathcal{G} of bounded degeneracy). For weakly-sparse graph classes, we present a proof in [Section 5](#) that reductions to EQUALITY are *equivalent* to bounded degeneracy:

Theorem 1.5. *Let \mathcal{G} be a hereditary class of bipartite graphs that is weakly-sparse. Then $D^{\text{Eq}}(\mathcal{G}) = O(1)$ if and only if \mathcal{G} has bounded degeneracy.*

In [\[CHHS23\]](#), it is conjectured that the point-line incidence graphs \mathcal{PL} satisfy $R(\mathcal{PL}) = \omega(1)$. [Theorem 1.5](#) shows the weaker result $D^{\text{Eq}}(\mathcal{G}) = \omega(1)$, because point-line incidences are $K_{2,2}$ -free and have unbounded degeneracy. They also have sign-rank at most 6, which means that the EQUALITY oracle does not suffice to extend [Theorem 1.4](#) to sign-rank 6 and above, even if the stability condition is replaced with the much stronger *weakly-sparse* condition. Combining known results in the literature, we also give in [Section 5](#) an example ($K_{2,2}$ -free point-box incidence graphs) with sign-rank 5 that is $K_{2,2}$ -free but has unbounded degeneracy, showing in fact that the EQUALITY oracle does not suffice to extend [Theorem 1.4](#) to sign-rank 5. It may be the case that reductions to EQUALITY are the only type of constant-cost communication possible for matrices of bounded sign-rank, see [Question 1.6](#). We summarize the known results for low sign-ranks in [Table 1](#).

Proof overview. We briefly summarize the proofs of [Theorems 1.2](#) and [1.4](#). Although UDGs and graphs of sign-rank 3 do not satisfy the conditions of [Theorem 1.4](#), we prove that two parties with access to an EQUALITY oracle can agree on a graph decomposition into pieces that avoid *edge-asteroid triple* structures (used in e.g. [\[FHH99, STU10, AZ18\]](#)), which guarantees that these pieces satisfy the conditions of [Theorem 1.4](#).

Our main tool to prove [Theorem 1.4](#) is the *Gyárfás decomposition*, which we take from [\[POS22\]](#). The Gyárfás decomposition partitions a bipartite graph into bags of vertices with a tree-like structure on the bags that controls the edges between the bags. In particular, every root-to-leaf path on the bags induces a path in the original graph. For this reason, the Gyárfás method has previously been used (as in [\[POS22\]](#)) to analyze P_t -free graphs, i.e. graphs which forbid long induced paths, where the depth of the decomposition is constant.

However, in our case, the depth of the decomposition is unbounded. Instead, we show that, under the conditions of [Theorem 1.4](#), each bag has edges to only a bounded number of its ancestors. Using this guarantee, we show that a communication protocol on input vertices x, y may use the EQUALITY oracle to either determine the adjacency, or agree on a subset of bags that contains x and y . The protocol may then recurse on these bags, sometimes switching to the bipartite complement of the graph when it does so (this is why e.g. we require both $S_{s,t}$ and $\tilde{S}_{s,t}$ to be forbidden). Due to arguments of [\[POS22\]](#), this recursion will reduce the chain-index of the graph and is therefore guaranteed to terminate after a constant number of iterations.

1.3 Discussion and Open Problems

Communication complexity. An intriguing possibility arises from this work, in conjunction with other recent work on bounded sign-rank. Adapting (or abusing) some notation of [BFS86], write $\text{UPP}[1]$ for the set of communication problems with bounded sign-rank (i.e. constant *unbounded-error communication cost* [PS86]), write $\text{BPP}[1]$ for the set of communication problems with constant public-coin randomized communication cost, and write $\text{D}^{\text{Eq}}[1]$ for the set of communication problems with a constant-cost reduction to EQUALITY. With these definitions of communication complexity classes, we can ask:

Question 1.6. Is it the case that $\text{D}^{\text{Eq}}[1] = \text{UPP}[1] \cap \text{BPP}[1]$?

A positive answer to this question would “explain” all of the known results and conjectures relating these classes. It is proved in [CHHS23] that $\text{D}^{\text{Eq}}[1] \subseteq \text{UPP}[1] \cap \text{BPP}[1]$. In the other direction, there are communication problems in $\text{BPP}[1]$ that do not belong to $\text{D}^{\text{Eq}}[1]$, which was proved independently in [HHH22b] and [HWZ22], but the example in both cases, the 1-HAMMING DISTANCE problem (adjacency in the hypercube), is believed not to belong to $\text{UPP}[1]$ [HHP⁺22], which is implied by a positive answer to Question 1.6. In Section 5, we give two explicit examples ($K_{2,2}$ -free point-box incidences, and point-line incidences) in $\text{UPP}[1]$ that do not belong to $\text{D}^{\text{Eq}}[1]$, which could possibly provide a negative answer to Question 1.6 if they belong to $\text{BPP}[1]$, but point-line incidences are conjectured not to belong to $\text{BPP}[1]$ in [CHHS23]. On the other hand, a negative answer to Question 1.6 seems to require a substantially different type of randomized protocol than the ones which have so far been discovered⁵, and would therefore be very interesting.

Implicit representations. An obvious question is whether the stability condition in our positive result for implicit representations can be dropped. This cannot be accomplished by reductions to EQUALITY, for which stability is necessary. We have shown that the GREATER-THAN problem is the only barrier to constant-cost communication, so one idea for generalizing our result is to allow the more powerful GREATER-THAN oracles in the communication protocol. Constant-cost reductions to GREATER-THAN are equally good for the purpose of finding implicit representations (we may think of some standard implicit representations, like for interval graphs [KNR92] and point-box incidences [Spi03], as protocols of this form). But this cannot succeed: a constant-cost reduction to GREATER-THAN for graphs \mathcal{G} of sign-rank 3 would imply $R(\mathcal{G}) = \Theta(\log \log N)$ which contradicts the known bound of $\Theta(\log N)$ [HHL22, ACHS23]. This answers an open question asked in independent and concurrent work [Cha23] whether (in our terminology) reductions to GREATER-THAN suffice to obtain implicit representations for geometric intersection graphs with small sign-rank realized by integer coordinates⁶. This at least demonstrates that communication complexity lower bounds can be used against certain natural types of implicit representation, although it remains open how to prove *any* explicit, non-trivial lower bounds for implicit representations.

2 Preliminaries

Let us define some notation and formalize the notions we have discussed in the introduction. We intend this paper to be accessible to readers in graph theory or communication complexity who may not have a background in both, so we make an attempt to make the terminology explicit. We will also define a general notion of *constant-cost reductions* which has not yet appeared explicitly in the literature.

2.1 Notation

For a matrix $M \in \mathbb{R}^{X \times Y}$, row $x \in X$, and column $y \in Y$, we will write either $M_{x,y}$ or $M(x,y)$ for the entry at x and y .

⁵By this we mean that it seems unlikely to us that a negative answer to the question would be achieved by a reduction to any currently-known constant-cost problem, most of which can be found in [HWZ22].

⁶The bounds in [HHL22, ACHS23] hold for constructions with integer coordinates.

For a graph G , we write \overline{G} for the complement of G . For a bipartite graph $G = (X, Y, E)$ with a fixed bipartition, write \widetilde{G} for the bipartite complement, which has edge xy if and only if xy is not an edge of G . The adjacency matrix of a graph $G = (V, E)$ is the matrix $\text{ADJ}_G \in \{\pm 1\}^{V \times V}$ with $\text{ADJ}_G(x, y) = 1$ if and only if $xy \in E$. For a bipartite graph $G = (X, Y, E)$ with a fixed bipartition, the *bipartite* adjacency matrix is the matrix $\text{ADJ}_G \in \{\pm 1\}^{X \times Y}$ with $\text{ADJ}_G(x, y) = 1$ iff $xy \in E$, where we note that the rows are indexed by X instead of the full set of vertices $X \cup Y$ (and similar for the columns).

For a graph G and disjoint sets $X, Y \subseteq V(G)$, we will write $G[X, Y]$ for the *semi-induced* bipartite subgraph, which is the bipartite graph $G[X, Y] = (X, Y, E)$ defined by putting an edge between $x \in X$ and $y \in Y$ if and only if xy are adjacent in G . (In particular, any edges within X or Y in G are not present in $G[X, Y]$.)

2.2 Sign-Rank

For a matrix $M \in \{\pm 1\}^{N \times N}$, the *sign-rank* of M is denoted $\text{rank}_\pm(M)$ and it is the minimum $d \in \mathbb{N}$ such that there exists a matrix $R \in \mathbb{R}^{N \times N}$ of rank d with $M = \text{sign}(R)$, where $\text{sign}(R) \in \{\pm 1\}^{N \times N}$ is the matrix with entries

$$\forall i, j \in [N] : \text{sign}(R)_{i,j} = \text{sign}(R_{i,j}).$$

Equivalently, $\text{rank}_\pm(M)$ is the minimum d such that each row $i \in [N]$ may be associated with a unit vector $p_i \in \mathbb{R}^d$ (which we think of as a point) and each column $j \in [N]$ may be associated with a unit vector $h_j \in \mathbb{R}^d$ (which we think of as the normal vector for a halfspace), such that $M_{i,j} = \text{sign}(\langle p_i, h_j \rangle)$. In this way, the sign-rank of M is equivalent to the minimum dimension d such that M is the incidence matrix between a set of points X and a set of halfspaces Y , where the hyperplane boundaries of the halfspaces contain the origin.

We require a notion of sign-rank for graphs, which we will define separately for bipartite graphs with a fixed bipartition, and for general graphs. For a bipartite graph $G = (X, Y, E)$ with a fixed bipartition, its sign-rank $\text{rank}_\pm(G)$ is defined as the sign-rank $\text{rank}_\pm(\text{ADJ}_G)$ of its bipartite adjacency matrix $\text{ADJ}_G \in \{\pm 1\}^{X \times Y}$. For a general graph $G = (V, E)$, we define its *partial* adjacency matrix $A \in \{\pm 1, \star\}^{V \times V}$ to be

$$\text{ADJ}_G^*(x, y) := \begin{cases} \star & \text{if } x = y \\ 1 & \text{if } xy \in E \\ -1 & \text{otherwise.} \end{cases}$$

We then define the sign-rank $\text{rank}_\pm(G)$ as the minimum rank of a matrix R such that

$$\forall i \neq j : \text{sign}(R_{i,j}) = \text{ADJ}_G^*(i, j).$$

Specifically, we do not make any requirement on the diagonal entries.

2.3 Communication Complexity and Margin

For a matrix $M \in \{\pm 1\}^{N \times N}$, we will write $\text{R}(M)$ for the *public-coin randomized communication complexity* of M , with success probability $2/3$. In this model, Alice receives a row $x \in [N]$ and Bob receives a column $y \in [N]$ and they must output $M(x, y)$. They are given shared access to a string of random bits, and they take turns sending messages that depend on their respective inputs and the random string. They must output the correct answer with probability at least $2/3$ over the random string, and the complexity of a protocol is the total number of bits communicated between the players on the worst-case inputs x, y . $\text{R}(M)$ is the minimum complexity of any such protocol computing M . See [KN96, RY20].

The standard notion of a (total, Boolean-valued) *communication problem* is a sequence $\mathcal{P} = (P_N)_{N \in \mathbb{N}}$ of matrices, where $P_N \in \{\pm 1\}^{N \times N}$, and the complexity of the problem, denoted $\text{R}(\mathcal{P})$, is the function $N \mapsto \text{R}(P_N)$. However, we are interested in the complexity of *classes* of matrices (specifically adjacency matrices of graphs belonging to some graph class), not merely sequences of matrices, where there is a

variety of $N \times N$ matrices instead of just one. So we define communication problems more generally, as in e.g. [HWZ22, EHK22].

Definition 2.1 (Communication Problem). A *communication problem* is a set $\mathcal{P} = \bigcup_{N \in \mathbb{N}} \mathcal{P}_N$ of Boolean matrices, where \mathcal{P}_N is a finite set of matrices in $\{\pm 1\}^{N \times N}$. We then define the communication complexity $R(\mathcal{P})$ as the function

$$N \mapsto \max_{P \in \mathcal{P}_N} R(P).$$

For a class \mathcal{G} of graphs, we write $\text{ADJ}_{\mathcal{G}}$ for the communication problem that is the set of adjacency matrices of graphs in \mathcal{G} . If \mathcal{G} is a class of bipartite graphs, we take the bipartite adjacency matrices. We abuse notation and write $R(\mathcal{G}) = R(\text{ADJ}_{\mathcal{G}})$, so that $R(\mathcal{G})$ is the function

$$N \mapsto \max\{R(\text{ADJ}_G) \mid G \in \mathcal{G} \text{ has } N \text{ vertices}\}.$$

Communication complexity is always upper bounded by the number of bits n in the input, or in our notation, by $\lceil \log N \rceil$. We are interested in determining which communication problems have *constant cost*, which means that there exists a constant c such that $R(M) \leq c$ for all $M \in \mathcal{P}$. One way to rule out a constant-cost protocol for a problem \mathcal{P} is if the GREATER-THAN communication problem appears as a subproblem of \mathcal{P} . Formally, this is captured by the stability condition (see [HWZ22]):

Proposition 2.2. *Let \mathcal{G} be any graph class which is not stable. Then $R(\mathcal{G}) = \omega(1)$.*

As mentioned in the introduction, having constant communication cost is equivalent to having constant *margin*, due to the following inequality, which follows from results of [LS09]:

Proposition 2.3. *Let $M \in \{\pm 1\}^{N \times N}$. Then*

$$\Omega\left(\log \frac{1}{\text{mar}(M)}\right) \leq R(M) \leq O\left(\frac{1}{\text{mar}(M)^2}\right).$$

2.4 Constant-Cost Communication Reductions and Equality

One way to obtain constant-cost protocols is by reduction to the EQUALITY problem, for which we require the definitions of the EQUALITY problem and a notion of reduction.

Definition 2.4 (The EQUALITY Communication Problem). The EQUALITY communication problem is the set $\text{EQ} := \{I_{N \times N} : N \in \mathbb{N}\}$ where $I_{N \times N}$ denotes the $N \times N$ identity matrix.

In other words, for input size N , Alice and Bob receive elements $x, y \in [N]$ and wish to decide whether $x = y$. It is well-known that $R(\text{EQ}) = 2$.

Constant-cost communication reductions, specifically to the EQUALITY problem, have been used implicitly in several prior works. Here we choose to explicitly define *constant-cost reductions* in general⁷. For this, we require the notion of a *query set*.

Definition 2.5. A *query set* \mathcal{Q} is a set of matrices that is closed under the following operations:

1. For every $Q \in \mathcal{Q}$ and any Q' obtained by row and column permutations of Q , $Q' \in \mathcal{Q}$.
2. For every $Q \in \mathcal{Q}$, if Q' is any submatrix of Q then $Q' \in \mathcal{Q}$.
3. For every $Q \in \mathcal{Q}$, if Q' is obtained by duplicating a row or a column of Q , then $Q' \in \mathcal{Q}$.

For a set \mathcal{P} of matrices, we define $\text{QS}(\mathcal{P})$ to be the closure of \mathcal{P} under these operations.

In the communication complexity literature, $\text{QS}(\text{EQ})$ was recently named the set of *blocky matrices* [HHH22b, AY22]. In graph theory, $\text{QS}(\text{EQ})$ are the adjacency matrices of disjoint unions of bicliques, also called *bipartite equivalence graphs*. It is easily verified that for any constant c , if $R(\mathcal{P}) \leq c$ then $R(\text{QS}(\mathcal{P})) \leq c$. However, we caution that $R(\text{QS}(\mathcal{P})) \leq R(\mathcal{P})$ does not hold for *non-constant*

⁷This general definition of constant-cost reductions has arisen out discussions with several other researchers.

complexities, because $\text{QS}(\mathcal{P})$ includes all submatrices of \mathcal{P} and $\text{R}(\cdot)$ takes the *maximum* complexity over all size- N matrices (see [HWZ22] for examples).

We now give two equivalent definitions for reductions between problems; one algorithmic and one structural.

Definition 2.6 (Communication with Oracles). Let \mathcal{B} be a communication problem and let $P \in \{\pm 1\}^{N \times N}$. A *deterministic protocol computing P with \mathcal{B} oracles* is a rooted binary tree T where each leaf ℓ is assigned a value $b(\ell) \in \{\pm 1\}$ and inner node v is assigned an $N \times N$ matrix $Q_v \in \text{QS}(\mathcal{B})$, with the following conditions. On each pair of inputs $x, y \in [N]$ the protocol begins at the root node v of T . At each node v , if $Q_v(x, y) = -1$ then the protocol proceeds by advancing the current node v to its left child, and if $Q_v(x, y) = 1$ then the protocol proceeds by advancing the current node v to its right child, until v becomes a leaf, at which point the protocol outputs $b(v)$. It is required that $b(v) = P_{x,y}$ for all inputs x, y .

The *cost* of the protocol is the depth of the tree. We write $\text{D}^{\mathcal{B}}(P)$ for the minimum cost of a protocol which computes P with \mathcal{B} oracles. For a communication problem \mathcal{A} , we write $\text{D}^{\mathcal{B}}(\mathcal{A})$ for the function

$$N \mapsto \max_{P \in \mathcal{A}_N} \text{D}^{\mathcal{B}}(P).$$

In other words, a communication protocol with \mathcal{B} oracles is a *deterministic* protocol where in each round, Alice and Bob transform their inputs x, y into inputs to a problem in \mathcal{B} and receive the answer from an oracle computing \mathcal{B} at unit cost. Observe that, as long as \mathcal{B} is non-trivial (i.e. does not contain only all-1 and all-(-1) matrices), the definition of $\text{QS}(\mathcal{B})$ allows any single round of deterministic communication to be simulated by an oracle, so without loss of generality we may assume that every inner node of the protocol is an oracle call.

If there is a constant c such that $\text{D}^{\mathcal{B}}(\mathcal{A}) \leq c$, then we say that \mathcal{A} *constant-cost reduces* (or just *reduces*) to \mathcal{B} . The following proposition is easily obtained by standard error-boosting techniques:

Proposition 2.7. *Suppose $\text{R}(\mathcal{B}) = O(1)$ and \mathcal{A} reduces to \mathcal{B} . Then $\text{R}(\mathcal{A}) = O(1)$. In particular, if \mathcal{A} reduces to EQ then $\text{R}(\mathcal{A}) = O(1)$.*

The second, structural definition of reduction is as follows. We say \mathcal{A} reduces to \mathcal{B} if there exists a constant t such that, for every $A \in \mathcal{A}$, there exists:

1. a function $f : \{\pm 1\}^t \rightarrow \{\pm 1\}$; and
2. matrices $Q_1, \dots, Q_t \in \text{QS}(\mathcal{B})$,

such that $A = f(Q_1, \dots, Q_t)$, meaning that $A(i, j) = f(Q_1(i, j), Q_2(i, j), \dots, Q_t(i, j))$ for all $i, j \in [N]$. In the special case when \mathcal{B} is the set of identity matrices, this definition appeared independently in [HHH22b, HWZ22] and subsequently in [EHK22], and the minimum t such that the above conditions hold is a “functional” analog of rank, recently called the *functional blocky-rank* in [AY22]. It is not difficult to show that this structural definition of constant-cost reductions is equivalent to the algorithmic one. One may easily derive a constant-cost protocol with oracles Q_i from the structural definition, and in the other direction one may simply let the set of matrices Q_i be the inner nodes of the communication protocol and define f as the function that simulates the protocol on these queries. In the structural definition it is not hard to see an analog of Proposition 2.7 for implicit representations. A similar⁸ notion of reductions for implicit representations appeared independently and concurrently in [Cha23], which included reductions to EQUALITY and GREATER-THAN as parts of a complexity hierarchy of implicit representations.

Proposition 2.8. *Suppose \mathcal{B} is the set of adjacency matrices for a hereditary graph class that admits an implicit representation, and suppose \mathcal{A} is the set of adjacency matrices for a hereditary graph class \mathcal{G} . If \mathcal{A} reduces to \mathcal{B} then \mathcal{G} admits an implicit representation.*

⁸There are some technicalities involved in translating between the two.

2.5 From Communication Protocols to Implicit Representations

An observation of [Har20, HWZ22] is that any hereditary graph class \mathcal{G} for which $R(\mathcal{G}) = O(1)$ must also have an implicit representation (and any constant-cost communication problem may be transformed into a hereditary graph class). Therefore, as argued in [HWZ22], constant-cost communication is essentially the probabilistic version of implicit representations.

We will present our proofs as upper bounds on communication complexity, which imply implicit representations. The general correspondence between constant-cost communication and implicit representations is non-constructive (by the probabilistic method), but for the sake of clarity and completeness, we briefly describe how to directly translate a communication protocol that uses EQUALITY oracles (as ours will do) into an implicit representation.

Recall that, for a graph $G = (V, E)$, if $D^{\text{Eq}}(G) \leq c$ then there exists a binary communication tree of depth c with each inner node v assigned to a matrix $Q_v \in \text{QS}(\text{Eq})$, which means that Q_v is the adjacency matrix of a bipartite equivalence graph. In other words, there are functions $a_v, b_v : V \rightarrow [N]$ such that

$$Q_v(x, y) = \begin{cases} 1 & \text{if } a_v(x) = b_v(y) \\ 0 & \text{otherwise.} \end{cases}$$

To obtain an implicit representation, we need to define a decoder D and encodings $\text{enc}(\cdot)$ for each graph $G \in \mathcal{G}$. We define $\text{enc}(x)$ for each $x \in V$ by writing down the values $a_v(x), b_v(y)$ for each inner node v of the tree, together with the output values at the leaves of the tree. Each value $a_v(x)$ and $b_v(x)$ requires at most $\lceil \log N \rceil$ bits, and there are at most 2^c nodes in the tree, which is constant, so the size of the encoding is $O(\log N)$. The decoder D , on inputs $\text{enc}(x)$ and $\text{enc}(y)$ for $x, y \in V$, may use the values of $a_v(x)$ and $b_v(y)$ for each node v , together with the outputs on the leaves, to simulate the communication protocol.

3 Communication Bounds for Excluded Cycles and Subdivided Stars

Our results for unit disk graphs and matrices of sign-rank 3 will follow from a more general result on bipartite graphs excluding either long cycles or subdivided stars, which we prove in this section. Recall the definition of the subdivided star $S_{s,t}$, Definition 1.3.

Theorem 1.4. *Let \mathcal{G} be a stable class of bipartite graphs that satisfies either of these conditions:*

1. *There exist constants s, t such that \mathcal{G} is $(S_{s,t}, \tilde{S}_{s,t})$ -free; or*
2. *There exists a constant t such that \mathcal{G} is $\{C_{t'}, \tilde{C}_{t'} \mid t' \geq t \text{ and } t' \text{ is even}\}$ -free.*

Then $D^{\text{Eq}}(\mathcal{G}) = O(1)$.

Our main tool will be the *Gyárfás decomposition*, which we borrow from [POS22], defined below.

3.1 Gyárfás Decomposition: Definition, Existence, and Properties

The following definition of the Gyárfás decomposition is taken from [POS22]. We will only apply the Gyárfás decomposition to bipartite graphs in this paper, so we state the special case of the decomposition for bipartite graphs. See Figure 1 for an illustration.

Definition 3.1 (Gyárfás Decomposition). A *Gyárfás decomposition* of a connected bipartite graph G is a rooted tree Y satisfying the following properties:

1. Each node of Y is a subset of $V(G)$, called a *bag*, and the nodes of Y form a partition of $V(G)$. For each vertex $v \in V(G)$, write $\text{bag}_Y(v)$ for the unique bag in Y that contains v . We will drop the subscript Y when the decomposition is clear from context.
2. The root bag of Y is a singleton containing the *root vertex*.
3. If $u, v \in V(G)$ are adjacent then $\text{bag}(u)$ is an ancestor of $\text{bag}(v)$ or vice-versa.

4. For every bag B of Y , the subgraph of G induced by B together with all of its descendants is connected.
5. For every non-root bag B of Y , there exists a vertex $h(B)$, called the *hook* of B , which belongs to the parent bag of B and has the property that $h(B)$ is adjacent to all vertices of B and non-adjacent to all vertices in the strict descendants of B .

For each bag B , we write $\text{depth}(B)$ for the length of the path from the root bag to B in Y (where the depth of the root bag is 0). For each $\ell \in \mathbb{N}$, we say that *level* ℓ of Y is the set of all bags B with $\text{depth}(B) = \ell$. A Gyárfás decomposition for a disconnected bipartite graph G is the union of Gyárfás decompositions for its connected components.

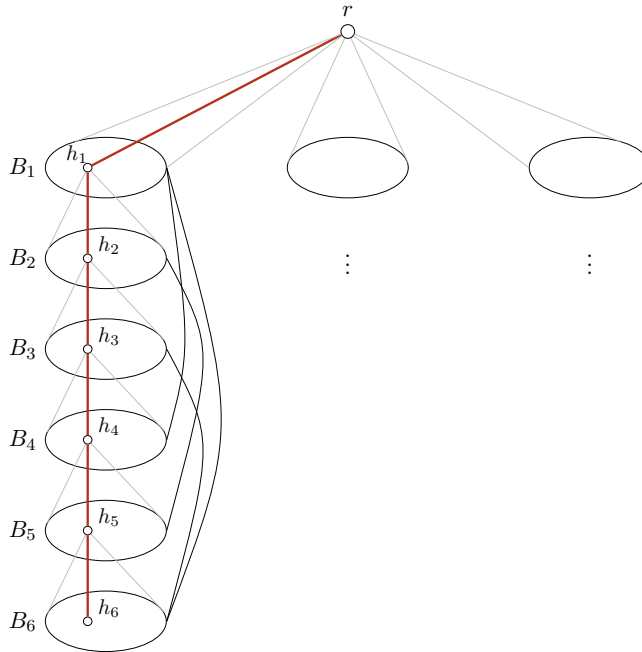


Figure 1: Gyárfás decomposition. The vertices r and h_i , $i \in [6]$, are hooks. The hook vertices induce a hook path highlighted with bold edges. Lines between two bags indicate that there are edges between some vertices in those bags.

There is a simple algorithmic proof that such decompositions always exist [POS22].

Lemma 3.2. *For every connected bipartite graph G and vertex $r \in V(G)$, there exists a Gyárfás decomposition of G with root vertex r . Given G and r , this decomposition can be computed in polynomial time.*

A path $P = (v_0, v_1, v_2, \dots, v_k)$ in G is a *hook path* (with respect to Y) if v_i is the hook of $\text{bag}(v_{i-1})$ for every $i \in [k]$. Observe that any hook path with respect to a Gyárfás decomposition is an induced path. Gyárfás decompositions are typically used in the case where some induced path P_t is forbidden, in which case the depth of the decomposition is bounded. In our case, we will not necessarily have a forbidden P_t or bounded depth of the decomposition, but we will see that the decomposition has a different structure that will permit efficient communication protocols. For this we define the notion of *back degree*.

Definition 3.3 (Back-Degree). Given a Gyárfás decomposition Y of G . We say that a bag B of Y has an edge to another bag B' in Y if there exist a vertex in B and a vertex in B' that are adjacent. The *back-degree* of a bag B in Y is the number of ancestor bags of B to which B has an edge. The *maximum back-degree* of Y is the maximum back-degree of any of its bags.

Note that Gyárfás decomposition of a P_t -free graph has depth at most t , and therefore the maximum back-degree of the decomposition is also bounded by t .

In the next two sections we show that if a graph has bounded chain-index and either

1. does not contain long induced cycles (Section 3.2), or
2. does not contain a fixed subdivision of a star (Section 3.3),

then its Gyárfás decompositions have bounded maximum back-degree. In Section 3.4, we give a general communication protocol for Gyárfás decompositions with bounded maximum back-degree.

Before proceeding Section 3.2 and Section 3.3, we introduce some notation and properties of the interactions between bags in Gyárfás decompositions that are used in both sections. Let Y be a Gyárfás decomposition of a bipartite graph $G = (X, Y, E)$, and let B be a bag of Y with $\text{depth}(B) > 0$. Write h for the hook of B . Let A_1, A_2, \dots, A_r be some ancestors of B , excluding the immediate parent of B , to which B has an edge. Then the following properties are easy to verify:

Proposition 3.4. *Let $s \in \mathbb{N}$ and suppose that $\text{depth}(A_1) < \text{depth}(A_2) < \dots < \text{depth}(A_r)$ and $\text{depth}(A_{i+1}) - \text{depth}(A_i) \geq s$ for all $i \in [r-1]$. For $i \in [r]$, we define $h_{i,1}$ to be the hook of A_i , and for $z \in [s-1]$, inductively define $h_{i,z}$ as the hook of $\text{bag}(h_{i,z-1})$. For each $i \in [r]$, let $a_i \in A_i$ be a neighbour of some $b_i \in B$. Then the following properties hold:*

- (1) *The hook h of B is adjacent to each b_i . For each $i \in [r]$ and $z \geq 1$, a_i is not adjacent to h , because they are on the same side of the bipartition of G , and $h_{i,z}$ is not adjacent to h , because $h_{i,z}$ is a hook in an ancestor bag of $\text{bag}(h)$ that is not the parent of $\text{bag}(h)$.*
- (2) *For each $i, j \in [r]$, a_i is not adjacent to a_j , because they are on the same side of the bipartition of G .*
- (3) *For each $1 \leq i < j \leq r$ and each $z \geq 1$, $h_{i,z}$ is not adjacent to a_j , because $h_{i,z}$ is a hook that is not in the parent bag of A_j .*
- (4) *For each $i \in [r]$ and each $z \geq 2$, we have $h_{i,z}$ not adjacent to a_i because $h_{i,z}$ is a hook that is not in the parent bag of $\text{bag}(a_i)$.*
- (5) *For each $i, j \in [r]$, and $z \geq 1$, $h_{i,z}$ is not adjacent to b_j because $h_{i,z}$ is a hook that is not in the parent bag of B .*

3.2 Excluding Long Cycles

Lemma 3.5. *For any $t, k \in \mathbb{N}$, there exists a constant ℓ such that the following holds. Let $G = (X, Y, E)$ be any $(C_t, C_{t+1}, C_{t+2}, \dots)$ -free bipartite graph with $\text{ch}(G) < k$. Let Y be a Gyárfás decomposition of G . Then Y has maximum back-degree at most ℓ .*

Proof. Without loss of generality we assume that $t \geq 4$. Let R be the Ramsey number that guarantees that a complete graph on R vertices with edges colored by 2^{t-4} colors has a monochromatic clique of size $r := \max\{2, k\}$.

Let $\ell = (t-3) \cdot R$ and let B be a bag of Y . If B has depth at most ℓ in Y the result holds trivially, so we will assume that B has depth greater than ℓ . Let $A_1'', A_2'', \dots, A_m''$ be the ancestors of B , excluding the immediate parent of B , to which B has an edge. For each $i \in [m]$, let $a_i'' \in A_i''$ be a neighbour of some $b_i'' \in B$.

Assume for the sake of contradiction that B has edges to more than ℓ ancestors, so $m \geq \ell$. Then there is a subsequence of ancestor bags A_1', \dots, A_R' such that $\text{depth}(A_{i+1}') - \text{depth}(A_i') \geq t-3$, for each $i \in [R-1]$, so that there are at least $t-4$ levels of the Gyárfás decomposition separating each bag in this subsequence. We will write $a_i' := a_{i^*}''$ and $b_i' := b_{i^*}''$, where i^* is the index of the bag satisfying $A_{i^*}'' = A_i'$.

For each A_i' , let $h_{i,1}'$ be the hook of A_i' , and for $1 < z \leq t-3$, inductively define $h_{i,z}'$ as the hook of $\text{bag}(h_{i,z-1}')$. For each pair $\{A_i', A_j'\}$ with $i < j$ we assign a color $\text{col}\{A_i', A_j'\} \in \{0, 1\}^{t-4}$ as follows:

the z^{th} bit is 1 if and only if a_i' is adjacent to $h_{j,z}'$, for $z \in [t-4]$.

By Ramsey's theorem, we may now choose a subsequence A_1, \dots, A_r of ancestor bags, where for each $i \in [r]$ there is a corresponding $i^* \in [R]$ such that $A_i = A_{i^*}$, and each pair $\{A_i, A_j\}$ with $1 \leq i < j \leq r$ has the same color. We will now obtain a contradiction for each possibility of this color. We will write $a_i := a_{i^*}$, $b_i := b_{i^*}$, and $h_{i,z} := h_{i^*,z}$, for each $i \in [r]$ and $z \in [t-3]$, and use the notation and the properties from [Section 3.1](#).

Case 1: There is $z \in [t-4]$ such that the z^{th} bit of the color is 1. Consider the subgraph H induced by the vertices $\{a_1, a_2, \dots, a_r\} \cup \{h_{1,z}, h_{2,z}, \dots, h_{r,z}\}$. For $1 \leq i < j \leq r$, we have a_i adjacent to $h_{j,z}$ due to the color, and $h_{i,z}$ is not adjacent to a_j due to Property (3). Thus, by definition, $\text{ch}(G) \geq \text{ch}(H) = r \geq k$, a contradiction.

Case 2: All bits of the color are 0. Consider the hook path P from a_2 to $h_{1,1}$. Let v be the first (i.e. closest to a_2) vertex on P that is adjacent to a_1 . Such a vertex exists because a_1 is adjacent to the last vertex $h_{1,1}$ of the path. Let P' be the subpath of P from a_2 to v . By Property (2) and the color assumption, P' contains the first $t-2$ vertices of P : $a_2, h_{2,1}, h_{2,2}, \dots, h_{2,t-4}, h_{2,t-3}$. Now, if b_2 is adjacent to a_1 , then b_2, P', a_1, b_2 is an induced cycle of length at least t . Similarly, if b_1 is adjacent to a_2 , then b_1, P', a_1, b_1 is such a cycle. Finally, if neither b_2 is adjacent to a_1 , nor b_1 is adjacent to a_2 , then $b_1 \neq b_2$ and, by Properties (1) and (5), h, b_2, P', a_1, b_1, h is a forbidden induced cycle. \square

3.3 Excluding Subdivisions of Stars

Lemma 3.6. *For any $s, t, k \in \mathbb{N}$, there exists a constant ℓ such that the following holds. Let $G = (X, Y, E)$ be any bipartite graph with $\text{ch}(G) < k$ that does not contain $S_{s,t}$ as an induced subgraph. Let Y be a Gyárfás decomposition of G . Then Y has maximum back-degree at most ℓ .*

Proof. Let R be the Ramsey number that guarantees that a complete graph on R vertices with edges colored by $2^{3+(t-1)}$ colors has a monochromatic clique of size $r := \max\{s, k\}$.

Let $\ell = t \cdot R + 1$ and let B be a bag of Y . If B has depth at most ℓ in Y the result holds trivially, so we will assume that B has depth greater than ℓ . Let $A_1'', A_2'', \dots, A_m''$ be the ancestors of B , excluding the immediate parent of B , to which B has an edge, meaning that for each $i \in [m]$, there exists a vertex $b_i'' \in B$ with an edge to a vertex $a_i'' \in A_i''$.

Assume for the sake of contradiction that B has edges to more than ℓ ancestors, so $m \geq \ell$. Then there is a subsequence of ancestor bags A_1', \dots, A_R' such that $\text{depth}(A_1') \geq t$ and $\text{depth}(A_{i+1}') - \text{depth}(A_i') \geq t$, for each $i \in [R-1]$; in particular there are at least $t-1$ levels of the Gyárfás decomposition separating each bag in this subsequence. We will write $a_i' := a_{i^*}''$ and $b_i' := b_{i^*}''$, where i^* is the index of the bag satisfying $A_{i^*}'' = A_i'$.

For each A_i' , let $h_{i,1}'$ be the hook of A_i' , and for $1 < z \leq t-1$, inductively define $h_{i,z}'$ as the hook of $\text{bag}(h_{i,z-1}')$. For each pair $\{A_i', A_j'\}$ with $i < j$ we assign a color $\text{col}\{A_i', A_j'\} \in \{0, 1\}^{3+(t-1)}$ as follows:

1. The first bit indicates whether $b_i' = b_j'$ (i.e. set the bit to 1 if $b_i' = b_j'$ and 0 otherwise).
2. The second bit indicates whether b_i' is adjacent to a_j' .
3. The third bit indicates whether b_j' is adjacent to a_i' .
4. The remaining t bits indicates whether a_i' is adjacent to $h_{j,z}'$, for $z \in [t-1]$.

By Ramsey's theorem, we may now choose a subsequence A_1, \dots, A_r of ancestor bags, where for each $i \in [r]$ there is a corresponding $i^* \in [R]$ such that $A_i = A_{i^*}'$, and each pair $\{A_i, A_j\}$ with $i < j$ has the same color. We will now obtain a contradiction for each possibility of this color. We will write $a_i := a_{i^*}'$, $b_i := b_{i^*}'$, and $h_{i,z} := h_{i^*,z}'$, for each $i \in [r]$ and $z \in [t-1]$, and use the notation and the properties from [Section 3.1](#).

Case 1: There is $z \in [t-1]$ such that the $(3+z)^{\text{th}}$ bit of the color is 1. The argument is exactly as in Case 1 of [Lemma 3.5](#). Consider the subgraph H induced by the vertices $\{a_1, a_2, \dots, a_r\} \cup$

$\{h_{1,z}, h_{2,z}, \dots, h_{r,z}\}$. For $1 \leq i < j \leq r$, we have a_i adjacent to $h_{j,z}$ due to the color, and $h_{i,z}$ is not adjacent to a_j due to Property (3). Thus, by definition, $\text{ch}(G) \geq \text{ch}(H) = r \geq k$, a contradiction.

Case 2: The first bit or second bit of the color is 1, and the $(3+z)^{\text{th}}$ color is 0 for all $z \in [t-1]$. Consider the subgraph induced by the vertices

$$\{b_1\} \cup \bigcup_{i=1}^s \{a_i, h_{i,1}, h_{i,2}, \dots, h_{i,t-1}\}.$$

Since each A_i is separated by at least $t-1$ levels of Y , each of the above named vertices are distinct. If the first bit of the color is 1, then we have b_1 adjacent to each a_i by definition, since $b_1 = b_2 = \dots = b_s$. If the first bit of the color is 0 but the second bit of the color is 1, then we have b_1 adjacent to each a_i because of the color.

For each $1 \leq i < j \leq s$ and $z \in [t-1]$, we have a_i not adjacent to $h_{j,z}$ because the associated bit of the color is set to 0, and we have a_j not adjacent to $h_{i,z}$ by Property (3). For $z \geq 2$, we have a_i not adjacent to $h_{i,z}$ by Property (4). We have a_i not adjacent to a_j by Property (2). And we have b_1 not adjacent to $h_{i,z}$ by Property (5). But we have b_1 adjacent to each a_i , as well as edges $a_i h_{i,1}$ and $h_{i,1} h_{i,2}, h_{i,2} h_{i,3}, \dots, h_{i,z-1} h_{i,z}$ by definition. So the subgraph induced by the considered vertices is $S_{s,t}$, which is a contradiction.

Case 3: The first two bits of the color are 0, the third bit is 1, and the $(3+z)^{\text{th}}$ bit is 0 for each $z \in [t-1]$. Consider the subgraph H induced by the vertices $\{b_1, \dots, b_k\} \cup \{a_1, \dots, a_k\}$. For each $i < j$, a_i is adjacent to b_j due to the third bit of the color, but b_i not adjacent to a_j due to the second bit of the color. Then we have $\text{ch}(G) \geq \text{ch}(H) = r \geq k$, a contradiction.

Case 4: All bits of the color are 0. Consider the subgraph induced by the vertices

$$\{h\} \cup \bigcup_{i=1}^s \{b_i, a_i, h_{i,1}, \dots, h_{i,t-2}\}.$$

Since each bag A_i in the sequence is separated by at least $t-1$ levels of Y and the first bit of the color is 0, each of the named vertices above is distinct. By Property (1), h is adjacent to none of the vertices $a_i, h_{i,1}, \dots, h_{i,t-2}$ for every $i \in [s]$. For each $i < j$, we have b_i not adjacent to a_j and a_i not adjacent to b_j due to the color. For each $z \in [t-1]$, we have $h_{i,z}$ not adjacent to b_i or b_j due to Property (5); and we have $h_{i,z}$ not adjacent to a_j due to Property (3) and a_i not adjacent to $h_{j,z}$ due to the color. For $z \geq 2$ we have $h_{i,z}$ not adjacent to a_i due to Property (4).

On the other hand, we have edges hb_i for each $i \in [s]$ by definition, along with edges $b_i a_i, a_i h_{i,1}$, and $h_{i,1} h_{i,2}, \dots, h_{i,t-3} h_{i,t-2}$. Therefore the induced subgraph is $S_{s,t}$, which is a contradiction. \square

3.4 A Communication Protocol for the Gyárfás Decomposition

Let G be a connected bipartite graph and let Y be a Gyárfás decomposition of G . For any bag B of Y with depth $d = \text{depth}(B)$, let G_B denote the subgraph of G induced by B together with all of the descendent bags of B in Y with depth $d' \not\equiv d \pmod{2}$. We require the next two lemmas of [POS22].

Lemma 3.7. *Let B be a bag of Y with $\text{depth}(B) = d$ for any $d \geq 2$. Then $\text{ch}(G_B) < \text{ch}(G)$.*

Lemma 3.8. *Let B be a bag of Y with $\text{depth}(B) = 1$. Let C be a connected component of \widetilde{G}_B and Y_C be a Gyárfás decomposition of C rooted at a vertex $r_C \in V(C) \cap B$. Let B' be a bag of Y_C with $\text{depth}(B') = d \geq 1$. Then $\text{ch}(C_{B'}) < \text{ch}(G)$.*

We will also require the following easy fact.

Proposition 3.9. *Let \mathcal{G} be the class of bipartite graphs G with $\text{ch}(G) = 1$. Then $D^{\text{Eq}}(\mathcal{G}) \leq 2$.*

Proof. Note that the chain-index of P_4 , the 4-vertex path, is 2. Thus each $G \in \mathcal{G}$ is P_4 -free and therefore is a disjoint union of bicliques, i.e. an equivalence graph. Therefore Alice and Bob may

compute adjacency in G by using 1 bit of communication to ensure that their input vertices x and y are on opposite sides of the bipartition, and using 1 call to the EQ oracle to check if x, y are in the same biclique. \square

Our first main result, [Theorem 1.4](#), follows from the next lemma, applied together with [Lemmas 3.5](#) and [3.6](#).

Lemma 3.10. *Let \mathcal{G} be a hereditary class of bipartite graphs that is closed under bipartite complementation, and which satisfies the following conditions:*

1. *There exists a constant k such that $\text{ch}(\mathcal{G}) \leq k$.*
2. *There exists a constant ℓ such that for any $G = (X, Y, E) \in \mathcal{G}$, any Gyárfás decomposition of G has back-degree bounded by ℓ .*

Then there exists a constant c such that $\text{D}^{\text{EQ}}(\mathcal{G}) \leq c$.

Proof. We prove the theorem by induction on k . The base case $k = 1$ is established in [Proposition 3.9](#). Let $x, y \in V(G)$ be Alice's and Bob's inputs, respectively. We may assume without loss of generality that G is connected and that x and y are in opposite parts of the bipartition of G , since Alice and Bob may use one EQ oracle call to check whether their inputs x and y are in the same connected component, and use 1 bit of communication to determine whether x and y are in opposite parts.

Let Y be a Gyárfás decomposition of G . The communication protocol proceeds as follows. We will assume that the root vertex of Y is on the left side of the bipartition of G , and that Alice's input x is on the left side and Bob's input y is on the right side of the bipartition.

1. Using 1 bit of communication, Alice tells Bob whether x is the root vertex of Y . If so, Bob outputs 1 if y has depth 1 in Y and the protocol terminates. The protocol is correct in this case, since by [Definition 3.1](#), all vertices at depth 1 are adjacent to the root vertex.
2. Using 1 bit of communication, Bob tells Alice whether y has depth 1 in Y . If so, they perform the following:
 - (a) Using 1 call to the EQ oracle, Alice and Bob decide if $\text{bag}(x)$ is a descendent of $\text{bag}(y)$. This is possible because Alice and Bob each know the set of level 1 bags of Y . If $\text{bag}(x)$ is not a descendent of $\text{bag}(y)$, they output 0 and the protocol terminates. The protocol is correct in this case, since by [Definition 3.1](#), if x and y are adjacent then $\text{bag}(x)$ must be the descendent of $\text{bag}(y)$ or vice versa.
 - (b) Alice and Bob now agree on $B = \text{bag}(y)$, so they each compute the connected components C_1, \dots, C_m of \widetilde{G}_B and agree on Gyárfás decompositions Y_1, \dots, Y_m of these components, respectively, where the root vertex of each decomposition is on the right side of the bipartition. Using 1 call to the EQ oracle, they decide if x, y belong to the same connected component of \widetilde{G}_B . If not, they output 1 and terminate the protocol. The protocol is correct in this case by definition.
 - (c) Let i be the index of the component C_i of \widetilde{G}_B containing both x and y . Using 1 bit of communication, Bob tells Alice whether y is the root vertex of Y_i . If so, Alice outputs 0 if x has depth 1 in Y_i and the protocol terminates. The protocol is correct in this case, since by [Definition 3.1](#) all vertices of depth 1 in Y_i are adjacent to the root vertex y in \widetilde{G}_B (and therefore non-adjacent in G).
 - (d) By the assumption of bounded back-degree, $\text{bag}_{Y_i}(x)$ has edges to at most ℓ of its ancestors in Y_i . Call these ancestors $A_1, \dots, A_{\ell'}$ where $\ell' \leq \ell$. Using ℓ calls to the EQ oracle, Alice and Bob determine whether $A_j = B'$ for some $j \leq \ell'$, where $B' := \text{bag}_{Y_i}(y)$.
 - i. If $A_j = B'$ then Alice and Bob inductively compute adjacency in the graph $(C_i)_{B'}$, which is the bipartite complement of a graph in \mathcal{G} and therefore is contained in \mathcal{G} , and

which by [Lemma 3.8](#) satisfies $\text{ch}((C_i)_{B'}) < \text{ch}(G)$. They then output the opposite value and terminate the protocol. The protocol is correct in this case since, by induction, they will compute adjacency of x, y in $(C_i)_{B'}$, which is an induced subgraph of \tilde{G} , so x, y have the opposite adjacency as in G .

ii. If $A_j \neq B'$ for all j , the protocol proceeds as below.

(e) Similar to step [2d](#), $\text{bag}_{Y_i}(y)$ has edges to at most ℓ of its ancestors in Y_i . Call these ancestors $A_1, \dots, A_{\ell'}$ where $\ell' \leq \ell$. Using ℓ calls to the EQ oracle, Alice and Bob determine whether $A_j = B'$ for some $j \leq \ell'$, where $B' := \text{bag}_{Y_i}(x)$.

i. If $A_j = B'$ then Alice and Bob inductively compute adjacency in the graph $(C_i)_{B'}$, which again is contained in \mathcal{G} and by [Lemma 3.8](#) satisfies $\text{ch}((C_i)_{B'}) < \text{ch}(G)$. They then output the opposite value and terminate the protocol. The protocol is correct in this case since, by induction, they will compute adjacency of x, y in $(C_i)_{B'}$, which is an induced subgraph of \tilde{G} , so x, y have the opposite adjacency as in G .

ii. If $A_j \neq B'$ for all j , then Alice and Bob output 1 and the protocol terminates. The protocol is correct in this case because x, y are adjacent in G if and only if they are non-adjacent in C_i . By [Definition 3.1](#), if they are adjacent in C_i then either $\text{bag}_{Y_i}(x)$ is an ancestor of $\text{bag}_{Y_i}(y)$ or vice versa. From step [2d](#), we know that if $\text{bag}_{Y_i}(y)$ is an ancestor of $\text{bag}_{Y_i}(x)$, then $\text{bag}_{Y_i}(x)$ has no edges to $\text{bag}_{Y_i}(y)$, so x, y are non-adjacent in C_i and therefore adjacent in G . From the current step, we know similarly that if $\text{bag}_{Y_i}(x)$ is an ancestor of $\text{bag}_{Y_i}(y)$ then x, y are again non-adjacent in C_i and therefore adjacent in G .

3. Now guaranteed that x and y are each in bags at depth 2 or higher, Alice and Bob proceed similarly as in steps [2d](#) and [2e](#), with the following differences. Here, Y is used instead of Y_i . In step [2\(e\)ii](#), the protocol outputs 0 instead of 1, because they are operating on the graph G itself instead of an induced subgraph of the bipartite complement \tilde{G} . When applying the inductive hypothesis, we use [Lemma 3.7](#) instead of [Lemma 3.8](#), and the players do not flip the output of the protocol applied to $G_{B'}$. Correctness again follows by induction.

This concludes the proof. □

4 Application to Sign-Rank 3 and Unit Disk Graphs

We now prove our results [Theorems 4.10](#) and [4.13](#) for graphs of sign-rank 3 and unit disk graphs. This will require the notion of *edge-asteroid triples* (see e.g. [\[FHH99, STU10, AZ18\]](#)).

Definition 4.1 (Edge-Asteroid Triples). A set of three edges in a graph is called an *edge-asteroid triple* if for each pair of the edges, there is a path containing both of the edges that avoids the neighbourhoods of the end-vertices of the third edge (see [Figure 2](#) for an illustration). We say that a graph class \mathcal{G} is *edge-asteroid-triple-free* if no $G \in \mathcal{G}$ contains an edge-asteroid triple.



Figure 2: Both $S_{3,3}$ and C_{10} contain an edge-asteroid triple (bold edges).

Since $S_{3,3}$ and C_t for $t \geq 10$ contain edge-asteroid triples, we make the following simple observation:

Observation 4.2. *Let G be any bipartite graph that is edge-asteroid-triple-free. Then G is both $S_{3,3}$ -free, and C_t -free for all $t \geq 10$.*

This observation will allow us to apply our [Theorem 1.4](#), but it requires that our graphs G and their complements both be edge-asteroid-triple-free. Unit disk graphs and graphs of sign-rank 3 are not necessarily edge-asteroid-triple-free. But we show that we can decompose these graphs into pieces which satisfy the necessary conditions.

4.1 Sign-Rank 3

To apply [Theorem 1.4](#) to graphs of sign-rank 3, we will decompose these graphs into pieces which are edge-asteroid-triple-free. We achieve this by interpreting graphs of sign-rank 3 as point-halfspace incidences and projecting down into dimension 2.

Definition 4.3 (Point-Halfspace and -Halfplane Incidence Graphs). Let P be a set of points in \mathbb{R}^d and H be a set of halfspaces in \mathbb{R}^d . The *incidence graph* of P and H is the bipartite graph

$$G(P, H) = (P, H, \{ph \mid p \in P, h \in H, \text{ and } p \in h\}).$$

A bipartite graph G is a *point-halfspace incidence graph* in \mathbb{R}^d if it can be represented as an incidence graph in \mathbb{R}^d ; more specifically, if there exist a set P of points and a set H of halfspaces both in \mathbb{R}^d such that G is isomorphic to $G(P, H)$. If $d = 2$ we call the graph a *point-halfplane incidence graph*.

If there exists such a representation of G , where in addition all pairwise dot products of the norm vectors of the hyperplanes defining the halfspaces in H are non-negative, then G is called a *positive point-halfspace incidence graph* in \mathbb{R}^d .

Proposition 4.4. *Any bipartite graph $G = (U, W, E)$ of sign-rank d admits a partition $U = U_1 \cup U_2$ such that $G[U_1, W]$ and $G[U_2, W]$ are point-halfspace incidence graphs in \mathbb{R}^{d-1} .*

Proof. Let $a : U \rightarrow \mathbb{R}^d$, $b : W \rightarrow \mathbb{R}^d$ be such that $u \in U$, $w \in W$ are adjacent if and only if $\langle a(u), b(w) \rangle \geq 0$. We assume, without loss of generality, that $a(u)_d \neq 0$ for every $u \in U$, and partition U into $U_1 = \{u \in U \mid a(u)_d > 0\}$ and $U_2 = \{u \in U \mid a(u)_d < 0\}$

We define $a' : U \rightarrow \mathbb{R}^{d-1}$ and $b' : W \rightarrow \mathbb{R}^{d-1}$ as

$$a'(u) = \frac{1}{|a(u)_d|} (a(u)_1, a(u)_2, \dots, a(u)_{d-1}), \quad b'(w) = (b(w)_1, b(w)_2, \dots, b(w)_{d-1}).$$

Further, we define the following sets of points and halfspaces in \mathbb{R}^{d-1} :

$$\begin{aligned} P_i &= \{p_u = a'(u) \mid u \in U_i\}, i = 1, 2, \\ H_1 &= \left\{ h_w \mid w \in W, h_w = \{x \in \mathbb{R}^{d-1} \mid \langle x, b'(w) \rangle \geq -b(w)_d \} \right\}, \\ H_2 &= \left\{ h'_w \mid w \in W, h'_w = \{x \in \mathbb{R}^{d-1} \mid \langle x, b'(w) \rangle \geq b(w)_d \} \right\}. \end{aligned}$$

Finally, we define two point-halfspace incidence graphs $G_1 = (U_1, W, E_1)$ and $G_2 = (U_2, W, E_2)$, where $E_1 = \{uw \mid u \in U_1, w \in W, p_u \in h_w\}$ and $E_2 = \{uw \mid u \in U_2, w \in W, p_u \in h'_w\}$.

We claim that $G = G_1 \cup G_2 = (U_1 \cup U_2, W, E_1 \cup E_2)$. Indeed, for any $u \in U$ and $w \in W$,

$$\begin{aligned} uw \in E &\iff \langle a(u), b(w) \rangle \geq 0 \iff \langle a'(u), b'(w) \rangle + \text{sign}(a(u)_d) \cdot b(w)_d \geq 0 \\ &\iff \langle a'(u), b'(w) \rangle \geq -\text{sign}(a(u)_d) \cdot b(w)_d. \end{aligned}$$

Hence, if $u \in U_1$, then $uw \in E \iff \langle a'(u), b'(w) \rangle \geq -b(w)_d \iff p_u \in h_w \iff uw \in E_1$; and if $u \in U_2$, then $uw \in E \iff \langle a'(u), b'(w) \rangle \geq b(w)_d \iff p_u \in h'_w \iff uw \in E_2$. \square

Proposition 4.5. *Any point-halfspace incidence graph $G = G(P, H)$ in \mathbb{R}^d admits a partition $H = \bigcup_{i=1}^{2^d} H_i$ such that each $G(P, H_i)$ is a positive point-halfspace incidence graph.*

Proof. For $h \in H$, let $w_h \in \mathbb{R}^d$ and $t_h \in \mathbb{R}$ be such that $h = \{x \in \mathbb{R}^d \mid \langle w_h, x \rangle \leq t_h\}$. We partition H into 2^d subsets H_α , $\alpha \in \{-1, +1\}^d$, with respect to the sign patterns of the norm vectors. More specifically, $h \in H_\alpha$ if and only if $(w_h)_i \geq 0 \iff \alpha_i = +1$ for every $i \in [d]$. Clearly, for any $\alpha \in \{-1, +1\}^d$ and any $h, h' \in H_\alpha$, we have that $\langle w_h, w_{h'} \rangle \geq 0$, i.e. $G(P, H_\alpha)$ is a positive point-halfplane incidence graph. \square

From [Proposition 4.4](#) and [Proposition 4.5](#) we obtain the following immediate

Corollary 4.6. *Any bipartite graph $G = (U, W, E)$ of sign-rank d admits a partition $U = \bigcup_{i=1}^{2^d} U_i$ such that each $G[U_i, W]$ is a positive point-halfspace incidence graph in \mathbb{R}^{d-1} .*

We now prove that positive point-halfplane incidence graphs are edge-asteroid-triple free.

Lemma 4.7. *Every positive point-halfplane incidence graph G is edge-asteroid-triple-free.*

Proof. Let G be a positive point-halfplane incidence graph and let P and H be sets of points and halfplanes respectively whose incidence graph is isomorphic to G . For a point $p \in P$ we denote by x_p and y_p its coordinates respectively; for a halfplane $h \in H$ we denote by a_h, b_h, t_h the coefficients of the halfplane inequality, i.e. $h = \{(x, y) \in \mathbb{R}^2 \mid a_h x + b_h y \leq t_h\}$. Without loss of generality, we can assume that no point in P lies on the boundary of any $h \in H$. Since G is positive, using translation and rotation, we can further assume that for every $h \in H$ both a_h and b_h are non-negative. This latter assumption implies the following useful claim which is straightforward to verify.

Claim 1. *For every $h \in H$ it holds that if $(x, y) \in h$, then $(x', y') \in h$ for every $x' \leq x$ and $y' \leq y$.*

Suppose now, towards a contradiction, that G contains an edge-asteroid triple, and let $p_1 h_1, p_2 h_2, p_3 h_3$ be its edges, where $p_i \in P, h_i \in H$. Note that the points p_1, p_2, p_3 are pairwise incomparable with respect to the coordinatewise order. Indeed, if for example $x_{p_1} \leq x_{p_2}$ and $y_{p_1} \leq y_{p_2}$, then by Claim 1 we would have $p_1 \in h_2$, i.e. p_1 and h_2 would be adjacent in G , which would contradict the assumption that the three edges form an edge-asteroid triple.

Thus, without loss of generality, we assume that $x_{p_1} \leq x_{p_2} \leq x_{p_3}$ and $y_{p_1} \geq y_{p_2} \geq y_{p_3}$.

Let $Q = (q_1, f_1, q_2, f_2, q_3, \dots, f_{k-1}, q_k)$, $q_i \in P, f_i \in H$, $q_1 = p_1$ and $q_k = p_3$, be a path containing p_1 and p_3 that avoids the neighbourhoods of both p_2 and h_2 . Since $x_{q_1} \leq x_{p_2} \leq x_{q_k}$, there exists $s \in [k-1]$ such that $x_{q_s} \leq x_{p_2} \leq x_{q_{s+1}}$. As above, using Claim 1, we can conclude q_s is incomparable with p_2 , as otherwise f_s would be adjacent to p_2 or h_2 would be adjacent to q_s , contradicting the choice of Q . Similarly, q_{s+1} is incomparable with p_2 . Hence, we have that $y_{q_s} \geq y_{p_2} \geq y_{q_{s+1}}$.

Let now $\overline{h_2}$ be the closure of the complement of h_2 , i.e. $\overline{h_2} = \{(x, y) \in \mathbb{R}^2 \mid a_{h_2} x + b_{h_2} y \geq t_{h_2}\}$, and let

$$A_{-+} = \{(x, y) \in \overline{h_2} \mid x \leq x_{p_2}, y \geq y_{p_2}\},$$

$$A_{+-} = \{(x, y) \in \overline{h_2} \mid x \geq x_{p_2}, y \leq y_{p_2}\}.$$

Note that $q_s \in A_{-+}$ and $q_{s+1} \in A_{+-}$. Hence the segment connecting q_s and q_{s+1} intersects the line $x = x_{p_2}$ that separates the two sets. Let $(x_{p_2}, y^*) \in \mathbb{R}^2$ be the point of intersection. Since both q_s and q_{s+1} are in $\overline{h_2}$, so is (x_{p_2}, y^*) , which together with Claim 1 implies that $y_{p_2} \leq y^*$. Similarly, (x_{p_2}, y^*) is in f_s because both q_s and q_{s+1} are in f_s . Consequently, by Claim 1, p_2 is also contained in f_s . This contradiction completes the proof. \square

Proposition 4.8. *Let $G(P, H)$ be a positive point-halfplane incidence graph. Then the bipartite complement of $G(P, H)$ is also a positive point-halfplane incidence graph.*

Proof. Without loss of generality, we can assume that no point in P lies on the boundary of any $h \in H$. For every point $p = (x_p, y_p) \in P$ we define $p' := (-x_p, -y_p)$, and for every $h = \{(x, y) \in \mathbb{R}^2 \mid a_h x + b_h y <$

$t_h\} \in H$ we define $h' = \{(x, y) \in \mathbb{R}^2 \mid a_h x + b_h y < -t_h\}$. Let $P' = \{p' \mid p \in P\}$ and $H' = \{h' \mid h \in H\}$. We claim that $G(P', H')$ is the bipartite complement of $G(P, H)$. Indeed, for any $p \in P$ and $h \in H$ we have that $p \in h \iff a_h x_p + b_h y_p < t_h \iff a_h(-x_p) + b_h(-y_p) > -t_h \iff p' \notin h'$. Finally, notice that the norm vector of the hyperplane defining a halfspace $h \in H$ is the same as the norm vector of the hyperplane defining $h' \in H'$. Hence, $G(P', H')$ is a positive point-halfplane incidence graph. \square

Lemma 4.9. *Let $G = (X, Y, E)$ be a bipartite graph of sign-rank 3. Then there exists a partition $Y = \bigcup_{i=1}^{2^3} Y_i$ such that each $G[X, Y_i]$ is both $(S_{3,3}, \tilde{S}_{3,3})$ -free, and (C_t, \tilde{C}_t) -free for all $t \geq 10$.*

Proof. We claim that the partition $Y = \bigcup_{i=1}^{2^3} Y_i$ given by [Corollary 4.6](#) is a desired one. Indeed, by [Corollary 4.6](#) and [Proposition 4.8](#), we conclude that for each $i \in [2^3]$, the graph $G_i := G[X, Y_i]$ and its bipartite complement are both positive point-halfplane incidence graphs. Hence, the lemma follows from [Lemma 4.7](#) and [Observation 4.2](#). \square

Theorem 4.10. *Let \mathcal{G} be a graph class with sign-rank at most 3. Then $D^{\text{Eq}}(\mathcal{G}) = O(1)$ if and only if \mathcal{G} is stable.*

Proof. It suffices to prove that $D^{\text{Eq}}(\mathcal{G}) = O(1)$ when \mathcal{G} is stable, due to [Proposition 2.2](#). On graph $G = (X, Y, E)$ and inputs $x \in X, y \in Y$, the players compute the decomposition $Y = \bigcup_{i=1}^8 Y_i$ given by [Lemma 4.9](#) and use 3 bits of communication to agree on the value i such that $y \in Y_i$. Then they compute adjacency in $G[X, Y_i]$ by applying the protocol in [Theorem 1.4](#). \square

4.2 Unit Disk Graphs

In this section we prove our result for unit disk graphs. A graph G is *unit disk* if there exists a mapping $\phi : V(G) \rightarrow \mathbb{R}^2$ such that $xy \in E(G)$ if and only if $\|\phi(x) - \phi(y)\|_2 < 2$. The mapping ϕ is called a *realisation* of G . Note that the constant 2 may be replaced with any other constant. We start by observing that unit disk graphs have sign-rank at most 4.

Fact 4.11. *Any unit disk graph G has sign-rank at most 4.*

Proof. Let $v \mapsto (x_v, y_v) \in \mathbb{R}^2$ for $v \in V(G)$, be a realisation of G , such that for any two distinct vertices $a, b \in V$, $ab \in E(G)$ if and only if

$$(x_a - x_b)^2 + (y_a - y_b)^2 < \sqrt{2} \iff x_a^2 - 2x_a x_b + x_b^2 + y_a^2 - 2y_a y_b + y_b^2 - \sqrt{2} < 0$$

Then, by defining $\sigma : v \mapsto (-1, 2x_v, 2y_v, -x_v^2 - y_v^2) \in \mathbb{R}^4$, $\psi : v \mapsto (x_v^2 + y_v^2 - \sqrt{2}, x_v, y_v, 1) \in \mathbb{R}^4$ for $v \in V(G)$, we see that for any *distinct* $a, b \in V(G)$

$$\langle \sigma(a), \psi(b) \rangle > 0 \iff (x_a - x_b)^2 + (y_a - y_b)^2 < \sqrt{2},$$

so $\text{sign}(\langle \sigma(a), \psi(b) \rangle) = 1$ if and only if $ab \in E$, as desired. \square

The main tool for our application to unit disk graphs is the following lemma of [\[AZ18\]](#). A graph G is *co-bipartite* if its complement \overline{G} is bipartite.

Lemma 4.12 ([\[AZ18\]](#)). *Let G be a co-bipartite unit disk graph. Then the bipartite graph \overline{G} and its bipartite complement do not contain any edge-asteroid triples. In particular, due to [Observation 4.2](#), \overline{G} is both $(S_{3,3}, \tilde{S}_{3,3})$ -free, and (C_t, \tilde{C}_t) -free for all $t \geq 10$.*

Our upper bound on the communication complexity of stable unit disk graphs will follow from a fairly straightforward decomposition of a unit disk graph into unit-length grid cells, such that between any two grid cells the graph is co-bipartite.

Theorem 4.13. *Let \mathcal{G} be a subclass of unit disk graphs. Then $D^{\text{Eq}}(\mathcal{G}) = O(1)$ if and only if \mathcal{G} is stable.*

Proof. It suffices to show that if \mathcal{G} is stable, then $D^{\text{Eq}}(\mathcal{G}) = O(1)$, due to [Proposition 2.2](#). Since \mathcal{G} is stable, there exists a constant k such that for all $G \in \mathcal{G}$, $\text{ch}(G) < k$. Fix any $G \in \mathcal{G}$ together with its realisation $\phi : V(G) \rightarrow \mathbb{R}^2$. For convenience, we will identify the vertices $x \in V(G)$ with the corresponding points $\phi(x) \in \mathbb{R}^2$. On inputs $x, y \in V(G)$, Alice and Bob will perform the following protocol.

1. Alice and Bob each partition \mathbb{R}^2 into a grid with cells $C_{i,j}$ for $i, j \in \mathbb{Z}$, where $C_{i,j} := \{(z_1, z_2) \in \mathbb{R}^2 : i \leq z_1 < i + 1, j \leq z_2 < j + 1\}$. Observe that if x, y are adjacent, then if $x \in C_{i,j}$ we must have $y \in C_{i+a, j+b}$ for some $a, b \in \{-2, -1, 0, 1, 2\}$; and if $x, y \in C_{i,j}$ then $\|x - y\|_2 < \sqrt{2}$ so x, y are adjacent. Let $i_x, j_x \in \mathbb{Z}$ be such that $x \in C_{i_x, j_x}$ and let $i_y, j_y \in \mathbb{Z}$ be such that $y \in C_{i_y, j_y}$.
2. Using 1 call to the EQ oracle, Alice and Bob check if $(i_x, j_x) = (i_y, j_y)$. If so, they output 1 and the protocol terminates. In this case, the protocol is correct due to the observation above.
3. For each $(a, b), (a', b') \in \{-2, -1, 0, 1, 2\}^2$ such that $(a, b) \neq (0, 0)$ and $(a', b') \neq (0, 0)$, Alice and Bob use 2 calls to the EQ oracle to check if both $(i_x + a, j_x + b) = (i_y, j_y)$. If so, then Alice and Bob compute adjacency in the semi-induced bipartite graph $G[X, Y]$ where $X := C_{i_x, j_x} \cap V(G)$ and $Y := C_{i_y, j_y} \cap V(G)$. This is possible because:
 - Alice and Bob each know X and Y : Alice knows $(i_x + a, j_x + b) = (i_y, j_y)$ and Bob knows $(i_y + a', j_y + b') = (i_x, j_x)$; and
 - The graph $G[X, Y]$ has $\text{ch}(G[X, Y]) \leq \text{ch}(G) \leq k$, and it is $(S_{3,3}, \tilde{S}_{3,3})$ -free, and (C_t, \tilde{C}_t) -free for all $t \geq 10$, by [Lemma 4.12](#), so we may apply [Theorem 1.4](#).
4. If $(i_x + a, j_x + b) \neq (i_y, j_y)$ for all $(a, b) \in \{-2, -1, 0, 1, 2\}^2$, then Alice and Bob output 0. In this case the protocol is correct by the observation in step 1.

This concludes the proof. □

5 The Sign-Rank Hierarchy

We have now determined exactly the conditions required for graphs of sign-rank 3, and some graphs of sign-rank 4, to have constant randomized communication cost (equivalently, constant margin). Let us now consider sign-ranks 5 and above. We will see that our techniques for the lower sign-ranks, specifically the reduction to EQUALITY, will surely fail. This witnesses a certain threshold between sign-ranks 3 and 5.

It is common to study the bipartite graphs which are $K_{t,t}$ -free, for some constant t . If a class of graphs is $K_{t,t}$ -free it is called *weakly-sparse*. This is a much stronger condition than stability: if G is $K_{t,t}$ -free then it must satisfy $\text{ch}(G) \leq 2t$.

A hereditary graph class \mathcal{G} has *bounded degeneracy* (equivalently, *bounded arboricity*) if there exists a constant d such that every $G \in \mathcal{G}$ has a vertex of degree at most d (equivalently, if there exists a constant a such that every $G \in \mathcal{G}$ on N vertices has at most $a \cdot N$ edges). The following is well-known and easy to prove (see e.g. [\[Har20, HWZ22\]](#)).

Proposition 5.1. *If a hereditary graph class \mathcal{G} has bounded degeneracy then $D^{\text{Eq}}(\mathcal{G}) = O(1)$.*

Recent results of [\[CH23\]](#) show that, for any constant t , the $K_{t,t}$ -free point-halfspace incidence graphs in dimension 3 have bounded degeneracy. Since any graph of sign-rank 4 can be written as a union of two point-halfspace incidence graphs in dimension 3 (see [Proposition 4.4](#)), we obtain the following:

Theorem 5.2. *Let \mathcal{G} be a hereditary graph class that is weakly-sparse and has sign-rank at most 4. Then $D^{\text{Eq}}(\mathcal{G}) = O(1)$.*

Our [Theorem 4.10](#) and [Theorem 4.13](#) are strengthenings of this theorem for the special cases of sign-rank 3 and unit disk graphs, where we replace the *weakly-sparse* condition with the much less restrictive stability condition. The proof of [\[CH23\]](#) uses a technique based on “shallow cuttings”. In [Appendix A](#),

we give a simpler proof, using elementary geometry, of the weaker statement that $K_{2,t}$ -free point-halfspace incidence graphs in dimension 3 have bounded degeneracy.

However, it is not possible to extend these results even to sign-rank 5. To see this, we first require a theorem which shows that, for weakly-sparse classes, bounded degeneracy is equivalent to the existence of a reduction to EQUALITY. The proof generalizes a theorem of [HWZ22] and is due to Bonamy, Esperet, & Girão, which we include here with permission and gratitude.

Theorem 1.5. *Let \mathcal{G} be a hereditary class of bipartite graphs that is weakly-sparse. Then $D^{\text{Eq}}(\mathcal{G}) = O(1)$ if and only if \mathcal{G} has bounded degeneracy.*

This theorem follows from the next two lemmas. The first lemma is implicit in [HWZ22].

Lemma 5.3. *Let \mathcal{G} be a class of bipartite graphs satisfying the following Ramsey property: for any $k, \ell \in \mathbb{N}$ there exists a graph $G \in \mathcal{G}$ such that, for any coloring of the edges of G with at most k colors, there exists a monochromatic induced path on ℓ vertices. Then $D^{\text{Eq}}(\mathcal{G}) = \omega(1)$.*

This lemma was used in [HWZ22] in conjunction with a result of [ARSV06] that established the required Ramsey property for induced subgraphs of hypercubes, which are $K_{2,3}$ -free, to show that hypercubes do not have constant-cost reductions to EQUALITY. The next lemma generalizes this result.

Lemma 5.4 (Bonamy, Esperet, & Girão). *For any $k, t, \ell \in \mathbb{N}$, there is an integer d such that, if a $K_{t,t}$ -free bipartite graph G has average degree at least d , and its edges are colored with at most k colors, then G contains a monochromatic induced path of length at least ℓ .*

Proof. We first reduce to the case $t = 2$. Suppose $t > 2$ and let $d \geq t$. A result of [McC21] shows that there is a constant d' such that any $K_{t,t}$ -free bipartite graph of average degree at least d' contains a $K_{2,2}$ -free induced subgraph of average degree at least d . Therefore it suffices to consider the case $t = 2$.

Choose $b > \ell$ and set $d = 2kb$. Consider a $K_{2,2}$ -free graph G , whose edges are colored with at most k colors. Then, if the average degree of G is at least d , there exists a color $c \in [k]$ such that the graph G_c induced by the edges with color c has average degree at least $2b$. Then G_c has an induced subgraph G'_c with *minimum* degree at least b .

We now construct a monochromatic induced path in G by induction as follows. The base case, a monochromatic path on 2 vertices, is trivial. Suppose we have obtained an induced path $P_{s-1} = \{v_1, \dots, v_{s-1}\}$, for $s - 1 < \ell$ where each (v_i, v_{i+1}) is an edge of G'_c . Let $N'_c(v_{s-1})$ be the neighbors of v_{s-1} in G'_c and suppose for contradiction that all vertices $u \in N'_c(v_{s-1}) \setminus P_{s-1}$ are adjacent in G to some v_i with $i < s - 1$. Since v_{s-1} has at least $b > \ell > s - 1$ neighbors, there are two vertices $u, w \in N'_c(v_{s-1}) \setminus P_{s-1}$ that are adjacent in G to both v_{s-1} and v_i for some $i < s - 1$. But then $\{v_i, v_{s-1}, u, w\}$ form an induced $K_{2,2}$, which is a contradiction. Therefore there exists a vertex $v_s \in N'_c(v_{s-1}) \setminus P_{s-1}$ which produces a monochromatic induced path $P_s = \{v_1, \dots, v_s\}$. This concludes the proof. \square

5.1 Sign-Rank 5: Point-Box Incidences

To show that our techniques cannot extend to sign-rank 5, even if we ask for the much stronger $K_{2,2}$ -free condition instead of stability, it now suffices to show that there exists a weakly-sparse class of bipartite graphs with sign-rank 5 and unbounded degeneracy. For this we use the *point-box incidence* graphs.

Definition 5.5 (Point-Box Incidences). Let P be a set of points in \mathbb{R}^2 and H a set of axis-aligned rectangles in \mathbb{R}^2 . The *incidence graph* of P and H is the bipartite graph

$$G(P, H) := (P, H, \{ph \mid p \in P, h \in H, p \in h\}).$$

The fact that these graphs have sign-rank 5 follows from a transformation of point-box incidences in dimension 2 to point-halfspace incidences in dimension 4, which appears in [PT13, CMK19]. The sign-rank of point-halfspace incidences in \mathbb{R}^4 is at most 5.

Lemma 5.6 ([CMK19]). *The class of point-box incidence graphs has sign-rank at most 5.*

What remains is the claim that *weakly-sparse* point-box incidence graphs on N vertices can have $\omega(N)$ edges. This is true even under the strongest condition of being $K_{2,2}$ -free. The lower bound of the next lemma was proved recently in [BCS⁺21], and the upper bound in [CH23]. We remark that the lemma remains true even if the boxes are restricted to be *dyadic*, i.e. the product of intervals of the form $[s2^t, (s+1)2^t)$ with integers s, t .

Lemma 5.7 ([BCS⁺21, CH23]). *The maximum number of edges in a $K_{2,2}$ -free point-box incidence graph is $\Theta\left(n \cdot \frac{\log n}{\log \log n}\right)$. As a consequence, $K_{2,2}$ -free point-box incidence graphs have unbounded degeneracy.*

Combining [Theorem 1.5](#) and [Lemmas 5.6](#) and [5.7](#), we get:

Corollary 5.8. *There is a hereditary class \mathcal{G} of $K_{2,2}$ -free bipartite graphs with sign-rank 5 and $D^{\text{Eq}}(\mathcal{G}) = \omega(1)$.*

5.2 Sign-Rank 6: Point-Line Incidences

The above result shows that reductions to EQUALITY cannot be used to prove $R(\mathcal{G}) = O(1)$ in general, even for weakly-sparse classes, let alone stable ones. This leaves open the possibility that there is another method for obtaining constant-cost randomized communication protocols for weakly-sparse or even stable graph classes with sign-rank 5, 6, or any constant. However, we discuss here a recent conjecture of [CHHS23] regarding point-line incidences suggesting that weakly-sparse graphs of sign-rank 6 have non-constant communication complexity.

Definition 5.9 (Point-Line Incidences). Let P be a set of points in \mathbb{R}^2 and L be a set of lines in \mathbb{R}^2 . The *incidence graph* of P and L is the bipartite graph

$$G(P, L) := (P, L, \{ph | p \in P, \ell \in L, p \in \ell\}).$$

Point-line incidence graphs are $K_{2,2}$ -free by definition, and it is well-known that the incidence graph between N points and N lines can have $\Theta(N^{4/3})$ edges; therefore, [Theorem 1.5](#) guarantees that they do not reduce to EQUALITY. Furthermore, it is known that point-line incidence graphs are point-halfspace incidence graphs in \mathbb{R}^5 (see e.g. [CH23]), and hence they have sign-rank at most 6:

Proposition 5.10. *Point-line incidence graphs have sign-rank at most 6.*

The communication complexity of point-line incidence graphs was recently studied in [CHHS23], but it remains unknown whether they have constant-cost. It was conjectured that they do not:

Conjecture 5.11 ([CHHS23]). *The class \mathcal{G} of point-line incidence graphs has $R(\mathcal{G}) = \omega(1)$.*

Acknowledgments

We are grateful to Marthe Bonamy, Louis Esperet, and Antonio Girão for their proof of [Lemma 5.4](#), and to Louis Esperet for communicating this proof to us and allowing us to include it here. We thank Lianna Hambardzumyan, Pooya Hatami, and Sebastian Wild for several conversations on the topic of this paper. The general definition of constant-cost reductions given in this paper has arisen partly out of collaboration with Yuting Fang, Lianna Hambardzumyan, and Pooya Hatami. We thank Mónika Csikós for telling us about [Lemma 5.6](#).

References

- [AAA⁺23] Bogdan Alecu, Vladimir E Alekseev, Aistis Atminas, Vadim Lozin, and Viktor Zamaraev. Graph parameters, implicit representations and factorial properties. *Discrete Mathematics*, 346(10):113573, 2023.
- [ACHS23] Manasseh Ahmed, Tsun-Ming Cheung, Hamed Hatami, and Kusha Sareen. Communication complexity of half-plane membership. Technical Report TR23-50, Electronic Colloquium on Computational Complexity (ECCC), 2023.
- [ACLZ15] Aistis Atminas, Andrew Collins, Vadim Lozin, and Victor Zamaraev. Implicit representations and factorial properties of graphs. *Discrete Mathematics*, 338(2):164–179, 2015.
- [Alo22] Noga Alon. Implicit representation of sparse hereditary families. *arXiv preprint arXiv:2201.00328*, 2022.
- [ARSV06] Noga Alon, Radoš Radoičić, Benny Sudakov, and Jan Vondrák. A Ramsey-type result for the hypercube. *Journal of Graph Theory*, 53(3):196–208, 2006.
- [AY22] Daniel Avraham and Amir Yehudayoff. On block ranks of matrices. Technical Report TR22-137, Electronic Colloquium on Computational Complexity (ECCC), 2022.
- [AZ18] Aistis Atminas and Viktor Zamaraev. On forbidden induced subgraphs for unit disk graphs. *Discrete & Computational Geometry*, 60(1):57–97, 2018.
- [BCS⁺21] Abdul Basit, Artem Chernikov, Sergei Starchenko, Terence Tao, and Chieu-Minh Tran. Zarankiewicz’s problem for semilinear hypergraphs. In *Forum of Mathematics, Sigma*, volume 9, page e59. Cambridge University Press, 2021.
- [BFS86] László Babai, Peter Frankl, and Janos Simon. Complexity classes in communication complexity theory. In *27th Annual Symposium on Foundations of Computer Science (SFCS 1986)*, pages 337–347. IEEE, 1986.
- [CH23] Timothy H Chan and Sariel Har-Peled. On the number of incidences when avoiding an induced biclique in geometric settings. In *Proceedings of the Symposium on Discrete Algorithms (SODA 2023)*, 2023.
- [Cha23] Maurice Chandoo. Logical labeling schemes. *Discrete Mathematics*, 346(10):113565, 2023.
- [CHHS23] Tsun-Ming Cheung, Hamed Hatami, Kaave Hosseini, and Morgan Shirley. Separation of the factorization norm and randomized communication complexity. In *Proceedings of the Computational Complexity Conference (CCC 2023)*, 2023. Available at ECCC, TR22-165.
- [CLV19] Arkadev Chattopadhyay, Shachar Lovett, and Marc Vinyals. Equality alone does not simulate randomness. In *Proceedings of the Computational Complexity Conference (CCC 2019)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2019.
- [CMK19] Mónika Csikós, Nabil H Mustafa, and Andrey Kupavskii. Tight lower bounds on the vc-dimension of geometric set systems. *The Journal of Machine Learning Research*, 2019.
- [EHK22] Louis Esperet, Nathaniel Harms, and Andrey Kupavskii. Sketching distances in monotone graph classes. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2022)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2022.
- [FHH99] Tomás Feder, Pavol Hell, and Jing Huang. List homomorphisms and circular arc graphs. *Combinatorica*, 19(4):487–505, 1999.
- [Fit19] Matthew Fitch. Implicit representation conjecture for semi-algebraic graphs. *Discrete Applied Mathematics*, 259:53–62, 2019.

- [FK09] Pierre Fraigniaud and Amos Korman. On randomized representations of graphs using short labels. In *Proceedings of the twenty-first annual symposium on Parallelism in algorithms and architectures*, pages 131–137, 2009.
- [GPT22] Jakub Gajarský, Michał Pilipczuk, and Szymon Toruńczyk. Stable graphs of bounded twin-width. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 1–12, 2022.
- [Har20] Nathaniel Harms. Universal communication, universal graphs, and graph labeling. In *11th Innovations in Theoretical Computer Science Conference (ITCS 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.
- [HH22] Hamed Hatami and Pooya Hatami. The implicit graph conjecture is false. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1134–1137. IEEE, 2022.
- [HHH22a] Lianna Hambardzumyan, Hamed Hatami, and Pooya Hatami. A counter-example to the probabilistic universal graph conjecture via randomized communication complexity. *Discrete Applied Mathematics*, 322:117–122, 2022.
- [HHH22b] Lianna Hambardzumyan, Hamed Hatami, and Pooya Hatami. Dimension-free bounds and structural results in communication complexity. *Israel Journal of Mathematics*, pages 1–62, 2022.
- [HHL22] Kaave Hosseini, Hamed Hatami, and Shachar Lovett. Sign-rank vs. discrepancy. *Theory of Computing*, 18(1):1–22, 2022.
- [HHM23] Hamed Hatami, Kaave Hosseini, and Xiang Meng. A Borsuk-Ulam lower bound for sign-rank and its applications. In *Proceedings of the 55th Annual ACM SIGACT Symposium on Theory of Computing (STOC 2023)*, 2023.
- [HHP⁺22] Hamed Hatami, Pooya Hatami, William Pires, Ran Tao, and Rosie Zhao. Lower bound methods for sign-rank and their limitations. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2022)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2022.
- [HWZ22] Nathaniel Harms, Sebastian Wild, and Viktor Zamaraev. Randomized communication and implicit graph representations. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (STOC 2022)*, pages 1220–1233, 2022.
- [KN96] Eyal Kushilevitz and Noam Nisan. *Communication Complexity*. Cambridge University Press, 1996.
- [KNR92] Sampath Kannan, Moni Naor, and Steven Rudich. Implicit representation of graphs. *SIAM Journal on Discrete Mathematics*, 5(4):596–603, 1992.
- [LS09] Nati Linial and Adi Shraibman. Learning complexity vs communication complexity. *Combinatorics, Probability and Computing*, 18(1-2):227–245, 2009.
- [LZ17] Vadim Lozin and Viktor Zamaraev. The structure and the number of P_7 -free bipartite graphs. *European Journal of Combinatorics*, 65:143–153, 2017.
- [McC21] Rose McCarty. Dense induced subgraphs of dense bipartite graphs. *SIAM Journal on Discrete Mathematics*, 35(2):661–667, 2021.
- [MM13] Colin McDiarmid and Tobias Müller. Integer realizations of disk and segment graphs. *Journal of Combinatorial Theory, Series B*, 103(1):114–143, 2013.
- [POS22] Michał Pilipczuk, Patrice Ossona de Mendez, and Sebastian Siebertz. Transducing paths in graph classes with unbounded shrubdepth. *European Journal of Combinatorics*, page 103660, 2022.

- [PS86] Ramamohan Paturi and Janos Simon. Probabilistic communication complexity. *Journal of Computer and System Sciences*, 33(1):106–123, 1986.
- [PSS23] Toniann Pitassi, Morgan Shirley, and Adi Shraibman. The strength of equality oracles in communication. In *14th Innovations in Theoretical Computer Science Conference (ITCS 2023)*, 2023.
- [PT13] János Pach and Gábor Tardos. Tight lower bounds for the size of epsilon-nets. *Journal of the American Mathematical Society*, 26(3):645–658, 2013.
- [RY20] Anup Rao and Amir Yehudayoff. *Communication Complexity: and Applications*. Cambridge University Press, 2020.
- [Spi03] Jeremy P Spinrad. *Efficient Graph Representations*, volume 19. American Mathematical Society, 2003.
- [STU10] Anish Man Singh Shrestha, Satoshi Tayu, and Shuichi Ueno. On orthogonal ray graphs. *Discrete Applied Mathematics*, 158(15):1650–1659, 2010.
- [Zie95] Günter M. Ziegler. *Lectures on Polytopes*. Vol. 152. Springer, New York, graduate texts in mathematics edition, 1995.

A On the number of edges in weakly-sparse point-halfspace incidence graphs

In this section we show that $K_{2,s}$ -free point-halfspace incidence graphs in dimensions 1,2, and 3 have linearly many edges. The same result was recently obtained by Chan and Har-Peled in [CH23] for more general classes of $K_{s,s}$ -free graphs. We present our results for two reasons. First, the proof technique is completely different and might be of independent interest. Second, our bounds are more specific for the considered cases.

To prove our upper bounds, we will show that every graph in a class has a vertex of bounded degree. Since the classes are hereditary, this will imply linear bounds on the number of edges.

A.1 On the line

In this section we will show that the $K_{s,s}$ -free point-halfline incidence graphs on \mathbb{R} have linear number of edges. In fact we will show a linear bound on the number of edges in the more general class of the $K_{s,s}$ -free point-interval incidence graphs. For this latter class, [CH23] shows that an n -vertex $K_{s,s}$ -free point-interval incidence graphs with n_p points and n_i intervals contains at most $s(n_p + 3n_i)$ edges. Our bound of $(s - 1)n = (s - 1)(n_p + n_i)$ is a slight improvement over the bound from [CH23].

Lemma A.1. *Let G be a $K_{s,s}$ -free n -vertex point-interval incidence graph. Then G has at most $(s - 1)n$ edges.*

Proof. Let P be a set of points and I be a set of intervals on the real line such that $G \simeq G(P, I)$. To prove the statement we will show that G has a vertex of degree at most $s - 1$. Suppose that all vertices of G have degree at least s and let p be the leftmost point in P . The degree assumption implies that p belongs to at least s intervals, which we denote i_1, i_2, \dots, i_s . For the same reason, each of these intervals should contain the $s - 1$ points in P closest to p , which we denote p_1, p_2, \dots, p_{s-1} . But then the vertices corresponding to i_1, i_2, \dots, i_s and $p, p_1, p_2, \dots, p_{s-1}$ induce the forbidden $K_{s,s}$. \square

A.2 On the plane

In dimensions 2 and 3, the bounds for $K_{s,s}$ -free graphs from [CH23] are $O(sn)$, and the constants in the big- O are not specified.

Our bounds in dimension 2 and 3 are respectively $3(s-1)n$ and $5(s-1)n$. To obtain them we will use the following lemma that reduces the analysis to the case where the points are in convex position.

Lemma A.2. *Let $G \simeq G(P, H)$ be the incidence graph of a set P of points and a set H of halfspaces in \mathbb{R}^d . If G is $K_{2,s}$ -free and P is not in convex position, then G has a vertex of degree at most $(d+1)(s-1)$.*

Proof. Suppose that P is not in convex position, and let $p \in P$ be a non-extremal point of the convex hull $\text{conv}(P)$. By Carathéodory's theorem, p belongs to the convex hull of at most $d+1$ extremal points of $\text{conv}(P)$. Let p_1, p_2, \dots, p_k , $k \leq d+1$ be a minimal set of such extremal points. Since p belongs to the interior of $\text{conv}(\{p_1, \dots, p_k\})$, any halfspace containing p contains one of the points p_1, \dots, p_k . Thus, if p belongs to at least $k(s-1) + 1$ halfspaces, one of the points p_1, \dots, p_k belongs to at least s of them resulting in the forbidden $K_{2,s}$. Hence, the degree of p is at most $k(s-1) \leq (d+1)(s-1)$. \square

The *polytope graph* of a polytope is the incidence graph of the extremal points and 1-dimensional faces of the polytope. We will need the following well-known fact.

Fact A.3. *Let P and H be respectively a polytope and a halfspace in \mathbb{R}^d . The subgraph of the polytope graph of P induced by the extremal point of P that belong to H is connected.*

Lemma A.4. *Let G be a $K_{2,s}$ -free n -vertex point-halfplane incidence graph. Then G has at most $3(s-1)n$ edges.*

Proof. Let P be a set of points on the plane and H be a set of halfplanes such that $G \simeq G(P, H)$. We assume without loss of generality that $|P| \geq 3$. To prove the lemma we will show that G has a vertex of degree at most $3(s-1)$. If P is not in convex position, such a vertex exists by [Lemma A.2](#), so we can assume that all points in P are extremal points of $\text{conv}(P)$. Suppose that all vertices of G have degree at least $3(s-1) + 1$ and let p be an arbitrary point in P . The polytope graph of P is a cycle, and hence p has exactly 2 neighbours in this graph. [Fact A.3](#) implies that each of the halfplanes that contain p and some other vertices in P should also contain at least one of these 2 neighbours. Thus, since p belongs to $3(s-1) + 1 \geq 2(s-1) + 1$ halfplanes in H , at least s of them contain one other fixed point in P , which witnesses a forbidden $K_{2,s}$. \square

A.3 In \mathbb{R}^3

Lemma A.5. *Let G be a $K_{2,s}$ -free n -vertex point-halfspace incidence graph in \mathbb{R}^3 . Then G has at most $5(s-1)n$ edges.*

Proof. Let P and H be respectively a set of points and a set of halfspaces in \mathbb{R}^3 such that $G \simeq G(P, H)$. As before, to prove the lemma we will show that G has a vertex of degree at most $5(s-1)$. Towards a contradiction, suppose that all vertices in G have at least $5(s-1) + 1$ neighbours. This assumption and [Lemma A.2](#) imply that P is in convex position, and hence all points in P are extremal points of $\text{conv}(P)$.

Let F be the polytope graph of P . By Steinitz's theorem (see e.g. [\[Zie95\]](#)), F is planar, and therefore has a vertex of degree at most 5. Let $p \in P$ be such a vertex. It follows from [Fact A.3](#) that any halfspace in H that contains p also contains at least one of the neighbours of p in F . Thus, by the pigeonhole principle, at least s halfspaces among those in H that contain p contain also one fixed neighbour of p , which witnesses the forbidden $K_{2,s}$. This contradiction completes the proof. \square