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CHARACTERIZATIONS OF A CLASS OF MATRICES AND PERTURBATION OF THE DRAZIN INVERSE*

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Abstract. Given a singular square matrix A with index r, $\operatorname{ind}(A) = r$, we establish several characterizations in the Drazin inverse framework of the class of matrices B, which satisfy the conditions $\mathcal{N}(B^s) \cap \mathcal{R}(A^r) = \{0\}$ and $\mathcal{R}(B^s) \cap \mathcal{N}(A^r) = \{0\}$ with $\operatorname{ind}(B) = s$, where $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range space of a matrix A, respectively. We give explicit representations for B^{D} and BB^{D} and upper bounds for the errors $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|/\|A^{\mathrm{D}}\|$ and $\|BB^{\mathrm{D}} - AA^{\mathrm{D}}\|$. In a numerical example we show that our bounds are better than others given in the literature.

Key words. singular matrix, Drazin inverse, eigenprojectors, perturbation

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1. Introduction and preliminaries.

Let $A \in \mathbb{C}^{n\times n}$ be any complex square matrix of order n with $\operatorname{ind}(A) = r$, where $\operatorname{ind}(A)$, the *index of* A, is the smallest nonnegative integer r such that $\operatorname{rank} A^r = \operatorname{rank} A^{r+1}$. Let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range space of A and the null space of A, respectively. In our development we consider matrices $B \in \mathbb{C}^{n\times n}$, which satisfy the following condition for some positive integer s:

$$(\mathcal{C}_s) \qquad \mathcal{R}(B^s) \cap \mathcal{N}(A^r) = \{0\} \quad \text{and} \quad \mathcal{N}(B^s) \cap \mathcal{R}(A^r) = \{0\}.$$

A particular case is when the matrix B satisfies

$$\mathcal{R}(B^s) = \mathcal{R}(A^r) \quad \text{and} \quad \mathcal{N}(B^s) = \mathcal{N}(A^r). \tag{1.1}$$

The class of perturbation matrices B related to A by the condition (1.1), which is equivalent to the fact that both matrices have equal eigenprojection at zero, $B^\pi = A^\pi$ with $A^\pi = I - AA^{\mathrm{D}}$, were characterized in [4]. The Drazin inverse of B satisfying (1.1) is given by the formula $B^{\mathrm{D}} = (I + A^{\mathrm{D}}(B - A))^{-1}A^{\mathrm{D}}$. This latter formula was given in [15] for $B = A + E$, where $E = AA^{\mathrm{D}}EAA^{\mathrm{D}}$ and E sufficiently small.

The first and third authors gave in [5] characterizations of the matrices B related to A by the condition that, involving the eigenprojections at zero, $I - (B^\pi - A^\pi)^2$ is nonsingular. Therein, it was proved that $B^{\mathrm{D}} = (I + A^{\mathrm{D}}(B - A) + S)^{-1}A^{\mathrm{D}}(I - S)$ where $S = B^\pi - A^\pi$ and an upper bound for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|/\|A^{\mathrm{D}}\|$ was given in terms of $\|A^{\mathrm{D}}(B - A)\|$ and $\|B^\pi - A^\pi\|$.

The continuity of the Drazin inverse was studied in [1, 2, 3, 11]. In [2], Campbell and Meyer established that if A_j converges to A, then A_j^{D} converges to A^{D} if and only if $\operatorname{rank} A_j^{r_j} = \operatorname{rank} A^r$ for all sufficiently large j, where $r_j = \operatorname{ind}(A_j)$. Recently, the perturbation of the Drazin inverse was studied by several authors, and upper bounds for the relative error $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|/\|A^{\mathrm{D}}\|$ were given under certain conditions [4, 5, 6, 8, 9, 12, 13, 14, 15, 16].

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In this paper, in section 2 we prove that, for a matrix B with $\operatorname{ind}(B) = s$, the fact that B satisfies condition $(\mathcal{C}_s)$ is equivalent to that $I-(B^\pi - A^\pi)^2$ is nonsingular. We establish several new characterizations of the matrices which satisfy condition $(\mathcal{C}_s)$. In terms of matrix rank, this class of matrices is characterized by the condition $\operatorname{rank} A^r = \operatorname{rank} B^s = \operatorname{rank} A^r B^s A^r$ whenever $s = \operatorname{ind}(B)$.

In section 3 we study further characterizations for the class $(\mathcal{C}_1)$, giving a representation of matrices $B \in (\mathcal{C}_1)$ such that $\operatorname{ind}(B) = 1$, with respect to the core-nilpotent block form of the matrix A. We mention that the perturbation of the group inverse is a case of special interest due to its application to stability of Markov chains [3, 10].

In section 4 we extend the characterizations for the group inverse to the general case of perturbations satisfying condition $(\mathcal{C}_s)$. We give an expression for the index 1-nilpotent decomposition of the matrices $B \in (\mathcal{C}_s)$, $\operatorname{ind}(B) = s$, which will be the main tool in the development of perturbation results.

Finally, in section 5 we give an explicit representation of B^{D}, and we derive upper bounds for the errors $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|/\|A^{\mathrm{D}}\|$ and $\|BB^{\mathrm{D}} - AA^{\mathrm{D}}\|$ in terms of norms involving the powers $B^s - A^s$. In a numerical example we compare our bounds with others given recently in [13, 14].

In relation to the study of the continuity of the Drazin inverse, we can say that if A_j converges to A and $\operatorname{rank} A_j^{r_j} = \operatorname{rank} A^r A_j^{r_j} A^r = \operatorname{rank} A^r$ for all sufficiently large j, where $r_j = \operatorname{ind}(A_j)$, then an explicit representation for A_j^{D} and an explicit error bound of $\|A_j^{\mathrm{D}} - A^{\mathrm{D}}\|/\|A^{\mathrm{D}}\|$ are provided.

We recall that the *Drazin inverse of* $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^{\mathrm{D}} \in \mathbb{C}^{n\times n}$ satisfying the relations

$$A^{\mathrm{D}} A A^{\mathrm{D}} = A^{\mathrm{D}}, \quad AA^{\mathrm{D}} = A^{\mathrm{D}} A, \quad A^{l+1} A^{\mathrm{D}} = A^l \quad \text{for all } l \geq r,$$

where $r = \operatorname{ind}(A)$. If A is nonsingular, then $\operatorname{ind}(A) = 0$ and the solution to the above equations is $A^{\mathrm{D}} = A^{-1}$. The case when $\operatorname{ind}(A) = 1$, i.e., $\operatorname{rank} A = \operatorname{rank} A^2$, the Drazin inverse is called the *group inverse of* A and is denoted by $A^\sharp$.

We denote by O a null matrix. Each $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = r$ has a unique *index* 1*-nilpotent decomposition* (see [1, Theorem 11, Chapter 4]),

$$A = C_A + N_A, \quad \operatorname{ind}(C_A) = 1, \;\; C_A N_A = N_A C_A = O, \;\; N_A^r = O. \tag{1.2}$$

Moreover, we have $A^k = C_A^k + N_A^k$ for all integers $k \geq 1$, and $A^{\mathrm{D}} = C_A^\sharp$.

The following lemma gives a condition for the existence of the group inverse of a partitioned matrix and a formula for its computation (see [3, Theorems 7.7.5 and 7.7.7]).

LEMMA 1.1. *Let* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be square with* $A \in \mathbb{C}^{d\times d}$ *nonsingular and denote* $\Psi = I + A^{-1}BCA^{-1}$. *Then*

(i) $\operatorname{rank} M = \operatorname{rank} A \Longleftrightarrow D = CA^{-1}B$.

In this case, for all integers $k \geq 1$, M^k *may be partitioned as*

$$M^k = \begin{bmatrix} I \\ CA^{-1} \end{bmatrix} (A\Psi)^{k-1} A \begin{bmatrix} I & A^{-1}B \end{bmatrix}. \tag{1.3}$$

(ii) *If* $\operatorname{rank} M = \operatorname{rank} A$, *then* $\operatorname{ind}(M) = 1 \Longleftrightarrow \Psi$ *is nonsingular.*

In this case, the group inverse of M *is given by*

$$M^{\sharp} = \begin{bmatrix} I \\ CA^{-1} \end{bmatrix} (\Psi A \Psi)^{-1} \begin{bmatrix} I & A^{-1}B \end{bmatrix}. \tag{1.4}$$

Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = r$. The *eigenprojection of* A *corresponding to the eigenvalue* 0, denoted by A^{π}, is the uniquely determined *projector* such that $\mathcal{R}(A^{\pi}) = \mathcal{N}(A^r)$ and $\mathcal{N}(A^{\pi}) = \mathcal{R}(A^r)$.

If $\operatorname{ind}(A) = r > 0$, then there exists a nonsingular matrix P such that we can write A in the *core-nilpotent block form*

$$A = P \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} P^{-1} \quad A_1 \in \mathbb{C}^{d\times d} \text{ nonsingular}, \ d = \operatorname{rank} A^r, \ A_2^r = O. \tag{1.5}$$

By [3, Theorem 7.2.1], relative to the form (1.5), the Drazin inverse of A and the eigenprojection of A at zero are given by

$$A^{\mathrm{D}} = P \begin{pmatrix} A_1^{-1} & O \\ O & O \end{pmatrix} P^{-1}, \quad A^{\pi} = I - AA^{\mathrm{D}} = P \begin{pmatrix} O & O \\ O & I \end{pmatrix} P^{-1}.$$

The case when $\operatorname{ind}(A) = 1$ is equivalent to having $A_2 = O$ in (1.5), and so $A^{\pi}A = AA^{\pi} = O$. Moreover, we have $\mathcal{N}(A^{\pi}) = \mathcal{R}(A)$ and $\mathcal{R}(A^{\pi}) = \mathcal{N}(A)$.

LEMMA 1.2. *Let* $A, C \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = r$ *and* C *nonsingular. Then*

$$I - A^{\pi} + CA^{\pi}C^{-1}A^{\pi} \text{ is nonsingular } \iff I - A^{\pi} + C^{-1}A^{\pi}CA^{\pi} \text{ is nonsingular}.$$

Proof. Write

$$C = P \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} P^{-1} \quad \text{and} \quad C^{-1} = P \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} P^{-1},$$

where C_{11}, X_{11}, and A_1 as in (1.5) are the same size. Then

$$I - A^{\pi} + CA^{\pi}C^{-1}A^{\pi} = P \begin{pmatrix} I & C_{12}X_{22} \\ O & C_{22}X_{22} \end{pmatrix} P^{-1},$$

$$I - A^{\pi} + C^{-1}A^{\pi}CA^{\pi} = P \begin{pmatrix} I & X_{12}C_{22} \\ O & X_{22}C_{22} \end{pmatrix} P^{-1}.$$

Hence, since $C_{22}X_{22}$ is nonsingular $\iff$ $X_{22}C_{22}$ is nonsingular, the equivalence given in this lemma follows. ∎

The following lemma is concerned with the rank of a product of matrices (see [17, sec. 2.4]).

LEMMA 1.3. *Let* $A, B, C \in \mathbb{C}^{n\times n}$. *Then*

$$\operatorname{rank} AB = \operatorname{rank} B - \dim(\mathcal{R}(B) \cap \mathcal{N}(A)), \tag{1.6}$$

$$\operatorname{rank} ABC \geq \operatorname{rank} AB + \operatorname{rank} BC - \operatorname{rank} B. \tag{1.7}$$

2. Characterizations of matrices satisfying condition ($\mathcal{C}_s$). First, for a matrix B with $\operatorname{ind}(B) = s$ we establish the equivalence among condition ($\mathcal{C}_s$) and conditions involving the matrix rank, and other conditions expressed in terms of the eigenprojections at zero.

THEOREM 2.1. *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = r$. Then the following statements on $B \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(B) = s$ are equivalent:*
(a) *B satisfies condition ($\mathcal{C}_s$).*
(b) $\operatorname{rank} B^s = \operatorname{rank} A^r = \operatorname{rank} A^r B^s = \operatorname{rank} B^s A^r$.
(c) $\operatorname{rank} B^s = \operatorname{rank} A^r = \operatorname{rank} A^r B^s A^r$.
(d) $\operatorname{rank} B^s = \operatorname{rank} A^r$, *$I - A^\pi + B^\pi A^\pi$ is nonsingular.*
(e) *$I - (B^\pi - A^\pi)^2$ is nonsingular.*
(f) *$I - B^\pi - A^\pi$ is nonsingular.*

Proof. (a) $\Rightarrow$ (b). From the space decomposition $\mathbb{C}^n = \mathcal{R}(A^r) \oplus \mathcal{N}(A^r) = \mathcal{R}(B^s) \oplus \mathcal{N}(B^s)$ and the conditions $\mathcal{N}(B^s) \cap \mathcal{R}(A^r) = \{0\}$ and $\mathcal{R}(B^s) \cap \mathcal{N}(A^r) = \{0\}$, it is clear that $\operatorname{rank} B^s = \operatorname{rank} A^r$. Moreover, using Lemma 1.3, identity (1.6), we get

$$\operatorname{rank} A^r B^s = \operatorname{rank} B^s - \dim \mathcal{R}(B^s) \cap \mathcal{N}(A^r)$$

and

$$\operatorname{rank} B^s A^r = \operatorname{rank} A^r - \dim \mathcal{R}(A^r) \cap \mathcal{N}(B^s).$$

Hence, $\operatorname{rank} A^r B^s = \operatorname{rank} B^s$ and $\operatorname{rank} B^s A^r = \operatorname{rank} A^r$. So, (b) is proved.

(b) $\Rightarrow$ (c). Applying Lemma 1.3, formula (1.7), we get

$$\operatorname{rank} A^r B^s A^r \geq \operatorname{rank} A^r B^s + \operatorname{rank} B^s A^r - \operatorname{rank} B^s.$$

Hence $\operatorname{rank} A^r B^s A^r \geq \operatorname{rank} B^s$. We also have $\operatorname{rank} A^r B^s A^r \leq \operatorname{rank} A^r = \operatorname{rank} B^s$, so we conclude that $\operatorname{rank} A^r B^s A^r = \operatorname{rank} B^s$.

(c) $\Rightarrow$ (d). From condition $\operatorname{rank} A^r B^s A^r = \operatorname{rank} A^r = \operatorname{rank} B^s$, using Lemma 1.3, identity (1.6), we easily derive $\mathcal{R}(A^r) \cap \mathcal{N}(B^s) = \{0\}$ and $\mathcal{N}(A^r) \cap \mathcal{R}(B^s) = \{0\}$. Now, let $(I - A^\pi + B^\pi A^\pi)x = 0$. Then $(I - A^\pi)x = -B^\pi A^\pi x$. From this latter relation it follows that $(I - A^\pi)x \in \mathcal{R}(A^r) \cap \mathcal{N}(B^s)$, and thus $(I - A^\pi)x = 0$. Further, we also have $B^\pi A^\pi x = 0$. Hence $A^\pi x \in \mathcal{R}(B^s) \cap \mathcal{N}(A^r)$ and, consequently, $A^\pi x = 0$. Therefore $x = 0$, and $I - A^\pi + B^\pi A^\pi$ is nonsingular.

(d) $\Rightarrow$ (e). Since $I - (B^\pi - A^\pi)^2 = (I - A^\pi + B^\pi A^\pi)(I - B^\pi + A^\pi B^\pi)$, we have to prove that $I - B^\pi + A^\pi B^\pi$ is nonsingular. We write the core-nilpotent block forms, as in (1.5), $A = P\left(\begin{smallmatrix} A_1 & O \\ O & A_2 \end{smallmatrix}\right)P^{-1}$ and $B = Q\left(\begin{smallmatrix} B_1 & O \\ O & B_2 \end{smallmatrix}\right)Q^{-1}$ with A_1 and B_1 nonsingular matrices. We note that A_1 and B_1 have the same size because $\operatorname{rank} B^s = \operatorname{rank} A^r$. Moreover, $(Q^{-1}B^\pi Q = \begin{smallmatrix} O & O \\ O & I \end{smallmatrix}) = P^{-1}A^\pi P$ and, thus, $B^\pi = QP^{-1}A^\pi PQ^{-1}$. Hence $I - A^\pi + B^\pi A^\pi = I - A^\pi + QP^{-1}A^\pi PQ^{-1}A^\pi$. So $I - A^\pi + QP^{-1}A^\pi PQ^{-1}A^\pi$ is nonsingular, and by Lemma 1.2 we conclude that $PQ^{-1}(I - B^\pi + A^\pi B^\pi)QP^{-1} = I - A^\pi + PQ^{-1}A^\pi QP^{-1}A^\pi$ is also nonsingular.

(e) $\Rightarrow$ (f). Let $(I - B^\pi - A^\pi)x = 0$. Then $(I - B^\pi + A^\pi)x = 2A^\pi x$, and hence $(I + B^\pi - A^\pi)(I - B^\pi + A^\pi)x = 2B^\pi A^\pi x = 0$. So, we have $(I - (B^\pi - A^\pi)^2)x = 0$. This implies that $x = 0$, and therefore $I - B^\pi - A^\pi$ is nonsingular.

(f) $\Rightarrow$ (a). This equivalence follows from [7, Theorem 1.2], applying the equivalence of (iii) and (iv) given therein with the projectors $I - A^\pi$ and B^π. ∎

The next lemma gives properties that are needed in what follows.

LEMMA 2.2. *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = r$. If $B \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(B) = s$ satisfies condition ($\mathcal{C}_s$), then*

(i) *for any integer* $l \geq s$, $I + (A^{\mathrm{D}})^l(B^l - A^l)$ *is nonsingular.*

(ii) $I - (I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi - A^\pi(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}$ *is nonsingular.*

Proof. (i) Let $l \geq s$ and $(I+(A^{\mathrm{D}})^l(B^l-A^l))x = 0$. Then, $A^\pi x = -(A^{\mathrm{D}})^l B^l x = 0$. Hence, $x \in \mathcal{N}(A^\pi) = \mathcal{R}(A^r)$ and $B^l x \in \mathcal{N}\left((A^{\mathrm{D}})^l\right) = \mathcal{N}(A^r)$. Since $\mathcal{R}(B^l) = \mathcal{R}(B^s)$, then $B^l x \in \mathcal{R}(B^s) \cap \mathcal{N}(A^r)$. So, $B^l x = 0$. Therefore, $x \in \mathcal{N}\left(B^l\right) \cap \mathcal{R}(A^r)$, and thus $x = 0$. So, $I + (A^{\mathrm{D}})^l(B^l - A^l)$ is nonsingular.

(ii) Let $x - (I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi x - A^\pi(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}x = 0$. Then $(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}(A^{\mathrm{D}})^s B^s x = A^\pi(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}x$. From this identity and the fact that $(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}(A^{\mathrm{D}})^s = (A^{\mathrm{D}})^s(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}$, we conclude that $(I+(A^{\mathrm{D}})^s(B^s-A^s))^{-1}(A^{\mathrm{D}})^s B^s x = 0$ and $A^\pi(I+(B^s-A^s)(A^{\mathrm{D}})^s)^{-1}x = 0$. Therefore, $(A^{\mathrm{D}})^s B^s x = 0$ and so $B^s x \in \mathcal{R}(B^s) \cap \mathcal{N}(A^r)$. Thus, $B^s x = 0$. Moreover, since $(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}x \in \mathcal{R}(A^r)$, $(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}x = A^r y$ for some y. This implies that $x = B^s(A^{\mathrm{D}})^s A^r y$, and so $x \in \mathcal{R}(B^s) \cap \mathcal{N}(B^s)$. Hence, $x = 0$ because $\mathrm{ind}(B) = s$. So, (ii) is proved. ∎

In the following theorem, we will derive a formula for the eigenprojection of B at zero, B^π.

THEOREM 2.3. *Let* $A \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(A) = r$. *If* $B \in \mathbb{C}^{n\times n}$ *with* $\mathrm{ind}(B) = s$ *satisfies condition* $(\mathcal{C}_s)$, *then*

$$B^\pi = -(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi X^{-1} = -X^{-1}A^\pi(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1},$$

where

$$X = I - (I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi - A^\pi(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}.$$

Proof. From Lemma 2.2 we know that $I+(A^{\mathrm{D}})^s(B^s-A^s)$ and X are nonsingular. Using that $A^\pi(I+(A^{\mathrm{D}})^s(B^s-A^s))^{-1} = A^\pi = (I+(B^s-A^s)(A^{\mathrm{D}})^s)^{-1}A^\pi$, it is easily checked that

$$\begin{aligned}(2.1)\qquad & X(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi \\ &= -A^\pi(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi \\ &= A^\pi(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}X.\end{aligned}$$

Hence

$$(2.2)\qquad (I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi X^{-1} = X^{-1}A^\pi(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}.$$

Let $Q = -(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi X^{-1}$. We observe that

$$\mathcal{R}(Q) = \mathcal{R}((I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi)$$

because X is nonsingular. Let us show that Q is the projector with $\mathcal{N}(Q) = \mathcal{R}(B^s)$ and $\mathcal{R}(Q) = \mathcal{N}(B^s)$. First, using (2.2) and (2.1) we see that

$$Q^2 = X^{-1}A^\pi(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi X^{-1} = Q.$$

Now, let us assume that $x \in \mathcal{N}(B^s)$. Then $A^\pi x + (A^{\mathrm{D}})^s B^s x = A^\pi x$. From this relation it follows that $x = (A^\pi + (A^{\mathrm{D}})^s B^s)^{-1}A^\pi x$ and, thus, $x \in \mathcal{R}(Q)$. Conversely, assuming $x \in \mathcal{R}(Q)$ we get $(A^\pi + (A^{\mathrm{D}})^s B^s)x = A^\pi y$ for some $y \in \mathbb{C}^n$. Hence $(A^{\mathrm{D}})^s B^s x = A^\pi(y - x)$. Then $(A^{\mathrm{D}})^s B^s x = 0$. Therefore, $B^s x \in \mathcal{R}(B^s) \cap \mathcal{N}(A^r)$. So $B^s x = 0$. Consequently, $\mathcal{R}(Q) = \mathcal{N}(B^s)$.

By (2.2) we have that $\mathcal{N}(Q) = \mathcal{N}\left(X^{-1}A^{\pi}(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}\right)$. Hence it follows that $\mathcal{N}(Q) = \mathcal{N}\left(A^{\pi}(I + (B^s - A^s)(A^{\mathrm{D}})^s)^{-1}\right)$ because X is nonsingular. Let us assume that $x \in \mathcal{N}(Q)$. Then

$$A^{\pi}(A^{\pi} + B^s(A^{\mathrm{D}})^s)^{-1}x = (I - B^s(A^{\mathrm{D}})^s(A^{\pi} + B^s(A^{\mathrm{D}})^s)^{-1})x = 0.$$

Hence, $x = B^sA^{\mathrm{D}}(A^{\pi} + B^s(A^{\mathrm{D}})^s)^{-1}x$, and thus $x \in \mathcal{R}(B^s)$. Since $\mathcal{N}(Q) \subseteq \mathcal{R}(B^s)$, and $\mathbb{C}^n = \mathcal{R}(Q) \oplus \mathcal{N}(Q) = \mathcal{R}(B^s) \oplus \mathcal{N}(B^s)$ because $\operatorname{ind}(B) = s$, we conclude that $\mathcal{N}(Q) = \mathcal{R}(B^s)$. So we have $B^{\pi} = Q$, which is the desired result. ∎

3. The class ($\mathcal{C}_1$). We shall first give further characterizations of matrices B satisfying condition ($\mathcal{C}_1$) and $\operatorname{ind}(B) = 1$. We obtain a representation of B with respect to the core-nilpotent block form of the matrix A.

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = r$. Then the following conditions on $B \in \mathbb{C}^{n\times n}$ are equivalent:*

(a) *B satisfies condition ($\mathcal{C}_1$) and $\operatorname{ind}(B) = 1$.*

(b) *$B(I + A^{\mathrm{D}}(B - A))^{-1}A^{\pi} = O$, $I + A^{\mathrm{D}}(B - A)$ and $I + (A^{\mathrm{D}})^2(B^2 - A^2)$ are nonsingular.*

(c) *Relative to the core-nilpotent block form of A in (1.5), B has the following representation:*

$$B = P\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{21}B_{11}^{-1}B_{12}\end{pmatrix}P^{-1}, \tag{3.1}$$

where B_{11} and $I + B_{11}^{-1}B_{12}B_{21}B_{11}^{-1}$ are nonsingular.

(d) *$\operatorname{rank} B = \operatorname{rank} A^r$, $I + A^{\mathrm{D}}(B - A)$ and $I + (A^{\mathrm{D}})^2(B^2 - A^2)$ are nonsingular.*

Proof. (a) ⇒ (b). Since $\operatorname{ind}(B) = 1$, from Lemma 2.2(i) we get that $I + A^{\mathrm{D}}(B-A)$ and $I + (A^{\mathrm{D}})^2(B^2 - A^2)$ are nonsingular. Finally, using that $BB^{\pi} = O$ and applying Theorem 2.3, we conclude that $B(I + A^{\mathrm{D}}(B - A))^{-1}A^{\pi} = O$.

(b) ⇒ (c). Write

$$B = P\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}P^{-1}.$$

We compute

$$I + A^{\mathrm{D}}(B - A) = P\begin{pmatrix} A_1^{-1}B_{11} & A_1^{-1}B_{12} \\ O & I\end{pmatrix}P^{-1}.$$

Hence B_{11} is nonsingular because $I + A^{\mathrm{D}}(B - A)$ is nonsingular. We have

$$I + (A^{\mathrm{D}})^2(B^2 - A^2) = P\begin{pmatrix} A_1^{-2}(B_{11}^2 + B_{12}B_{21}) & A_1^{-2}(B_{11}B_{12} + B_{12}B_{22}) \\ O & I\end{pmatrix}P^{-1}.$$

Thus, $B_{11}^2 + B_{12}B_{21}$ is nonsingular because $I + (A^{\mathrm{D}})^2(B^2 - A^2)$ is nonsingular. On the other hand,

$$\begin{aligned} B(I + A^{\mathrm{D}}(B - A))^{-1}A^{\pi} &= P\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix}\begin{pmatrix} B_{11}^{-1}A_1 & -B_{11}^{-1}B_{12} \\ O & I\end{pmatrix}\begin{pmatrix} O & O \\ O & I\end{pmatrix}P^{-1} \\ &= P\begin{pmatrix} O & O \\ O & -B_{21}B_{11}^{-1}B_{12} + B_{22}\end{pmatrix}P^{-1}. \end{aligned}$$

From the assumption $B(I+A^{\mathrm{D}}(B-A))^{-1}A^{\pi}=O$ it follows that $B_{22}=B_{21}B_{11}^{-1}B_{12}$.

(c) ⇔ (d). From the representation (3.1), applying Lemma 1.1, it follows that $\operatorname{rank} B = \operatorname{rank} B_{11} = \operatorname{rank} A^r$. The rest is easily seen.

Conversely, write

$$B = P\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} P^{-1}.$$

Since $I+A^{\mathrm{D}}(B-A)$ and $I+(A^{\mathrm{D}})^2(B^2-A^2)$ are nonsingular, arguing as in the proof of (b) ⇒ (c), we get that B_{11} and $I+B_{11}^{-1}B_{12}B_{21}B_{11}^{-1}$ are nonsingular. Finally, from $\operatorname{rank} B = \operatorname{rank} A^r$ we obtain that $\operatorname{rank} B = \operatorname{rank} B_{11}$, and hence by Lemma 1.1(i) we conclude that $B_{22}=B_{21}B_{11}^{-1}B_{12}$.

(c) ⇒ (a). Assume that B has the block representation (3.1). By Lemma 1.1(i), (ii), we conclude that $\operatorname{rank} B = \operatorname{rank} B_{11} = \operatorname{rank} A^r$ and $\operatorname{ind}(B)=1$. On the other hand,

$$\operatorname{rank} A^r B A^r = \operatorname{rank} P\begin{pmatrix} A_1^r B_{11} A_1^r & O \\ O & O \end{pmatrix} P^{-1} = \operatorname{rank} A_1^r B_{11} A_1^r = \operatorname{rank} A^r.$$

Hence, in view of Theorem 2.1 (a)⇔(c), we conclude that B satisfy condition $(\mathcal{C}_1)$. □

Remark 3.2. Conditions (b) and (d) in the above theorem can be replaced by the following symmetrical conditions:

(b′) $A^{\pi}(I+(B-A)A^{\mathrm{D}})^{-1}B=O$, $I+(B-A)A^{\mathrm{D}}$ and $I+(B^2-A^2)(A^{\mathrm{D}})^2$ are nonsingular.

(d′) $\operatorname{rank} B = \operatorname{rank} A^r$, $I+(B-A)A^{\mathrm{D}}$ and $I+(B^2-A^2)(A^{\mathrm{D}})^2$ are nonsingular.

Next, we state the following compact representation for B and $B^{\sharp}$.

LEMMA 3.3. *Let* $A\in\mathbb{C}^{n\times n}$, $\operatorname{ind}(A)=r$ *and let* $B\in\mathbb{C}^{n\times n}$, $\operatorname{ind}(B)=1$, *satisfying condition* $(\mathcal{C}_1)$. *Then we have the representation*

$$B = P\begin{bmatrix} I \\ S \end{bmatrix} B_1 \begin{bmatrix} I & T \end{bmatrix} P^{-1}, \tag{3.2}$$

where B_1 *and* $I+TS$ *are nonsingular. According to this expression, the group inverse of* B *can be represented in the form*

$$B^{\sharp} = P\begin{bmatrix} I \\ S \end{bmatrix} [(I+TS)B_1(I+TS)]^{-1} \begin{bmatrix} I & T \end{bmatrix} P^{-1}. \tag{3.3}$$

Proof. By Theorem 3.1 (c),

$$B = P\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{21}B_{11}^{-1}B_{12} \end{pmatrix} P^{-1},$$

where B_{11} and $I+B_{11}^{-1}B_{12}B_{21}B_{11}^{-1}$ are nonsingular. By denoting $B_1=B_{11}$, $T=B_{11}^{-1}B_{12}$, and $S=B_{21}B_{11}^{-1}$ we get the representation (3.2). Now, applying formula (1.4) given in Lemma 1.1, we obtain the representation for $B^{\sharp}$. □

4. The class $(\mathcal{C}_s)$. Next, based on Theorem 3.1, we establish the following new characterizations of B satisfying condition $(\mathcal{C}_s)$.

THEOREM 4.1. *Let* $A\in\mathbb{C}^{n\times n}$, $\operatorname{ind}(A)=r$. *Then the following conditions on* $B\in\mathbb{C}^{n\times n}$ *are equivalent:*

(a) *B satisfies condition* $(\mathcal{C}_s)$ *and* $\operatorname{ind}(B)=s$.

(b) *For the smallest positive integer* s *such that* $B^s(I+(A^{\mathrm{D}})^s(B^s-A^s))^{-1}A^\pi = O$, $I+(A^{\mathrm{D}})^s(B^s-A^s)$ *and* $I+(A^{\mathrm{D}})^{s+1}(B^{s+1}-A^{s+1})$ *are nonsingular.*

(c) *The index* 1*-nilpotent decomposition of* B *has the following representation relative to the core-nilpotent block form of* A *in* (1.5),

$$B = C_B + N_B = P\begin{bmatrix} I \\ S \end{bmatrix} B_1 \begin{bmatrix} I & T \end{bmatrix} P^{-1} + P\begin{bmatrix} T \\ -I \end{bmatrix} B_2 \begin{bmatrix} S & -I \end{bmatrix} P^{-1}, \tag{4.1}$$

where B_1 *and* $I+TS$ *are nonsingular and* $B_2(I+ST)$ *is nilpotent of index* s.

(d) *For the smallest positive integer* s *such that* $\operatorname{rank} B^s = \operatorname{rank} A^r$, $I+(A^{\mathrm{D}})^s(B^s-A^s)$ *and* $I+(A^{\mathrm{D}})^{s+1}(B^{s+1}-A^{s+1})$ *are nonsingular.*

Proof. If $\operatorname{ind}(B)=s$ and B satisfies condition $(\mathcal{C}_s)$, then s is the smallest positive integer such that B^s satisfies condition $(\mathcal{C}_1)$ and $\operatorname{ind}(B^s)=1$. Moreover, we observe that for any $k \ge s$, $I+(A^{\mathrm{D}})^k(B^k-A^k)$ is nonsingular if and only if $I+A^{\mathrm{D}}(B^k-A)$ is nonsingular. So, applying Theorem 3.1 with B^s, it follows the equivalence between condition (a) and the following:

(b′) For the smallest positive integer s such that $B^s(I+(A^{\mathrm{D}})^s(B^s-A^s))^{-1}A^\pi = O$, $I+(A^{\mathrm{D}})^s(B^s-A^s)$ and $I+(A^{\mathrm{D}})^{2s}(B^{2s}-A^{2s})$ are nonsingular.

We now note that conditions (b′) and (b) are equivalent.

A similar device proves the equivalence between conditions (a) and (d) in this theorem. Applying Theorem 3.1 with B^s we get the equivalence of (a) and the following:

(d′) For the smallest positive integer s such that $\operatorname{rank} B^s = \operatorname{rank} A^r$, we have that $I+(A^{\mathrm{D}})^s(B^s-A^s)$ and $I+(A^{\mathrm{D}})^{2s}(B^{2s}-A^{2s})$ are nonsingular.

Finally, we note that conditions (d′) and (d) are equivalent.

Now, we will prove the equivalence between (a) and (c). Suppose $B = C_B + N_B$ is the index 1-nilpotent decomposition (1.2) of B. We know that if s is the index of B, then $\mathcal{N}(C_B) = \mathcal{N}(B^s)$ and $\mathcal{R}(C_B) = \mathcal{R}(B^s)$. Hence if B satisfies condition $(\mathcal{C}_s)$, then C_B satisfies condition $(\mathcal{C}_1)$ and $\operatorname{ind}(C_B)=1$. By Lemma 3.3 it follows that

$$C_B = P\begin{bmatrix} I \\ S \end{bmatrix} B_1 \begin{bmatrix} I & T \end{bmatrix} P^{-1}, \tag{4.2}$$

where B_1 and $I+TS$ are nonsingular. We observe that $I+ST$ is also nonsingular. Now, write

$$N_B = P\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} P^{-1}.$$

Since $C_B N_B = N_B C_B = O$, by direct computations it follows that $N_{11} = TN_{22}S$, $N_{12} = -TN_{22}$, and $N_{21} = -N_{22}S$. So,

$$N_B = P\begin{bmatrix} T \\ -I \end{bmatrix} B_2 \begin{bmatrix} S & -I \end{bmatrix} P^{-1}, \tag{4.3}$$

where we have renamed $B_2 = N_{22}$. Thus, for every positive integer k,

$$N_B^k = P\begin{bmatrix} T \\ -I \end{bmatrix} (B_2(I+ST))^{k-1} B_2 \begin{bmatrix} S & -I \end{bmatrix} P^{-1}. \tag{4.4}$$

Condition $N_B^s = O$ implies that $(B_2(I+ST))^s = O$. Therefore, $B_2(I+ST)$ is nilpotent of index s. Hence, from (4.2) and (4.3) we get the representation (4.1).

Conversely, assume that we have the splitting $B = C_B + N_B$, where C_B and N_B have the representation given by (4.1). Clearly $C_B N_B = N_B C_B = O$. Moreover, by Theorem 3.1, equivalence between (a) and (c), it follows that C_B satisfies condition $(\mathcal{C}_1)$ and $\operatorname{ind}(C_B) = 1$. Using (4.4), we see that $N_B^s = O$. So $B = C_B + N_B$ is the core-nilpotent decomposition of B and $\operatorname{ind}(B) = s$. Since $\mathcal{R}(B^s) = \mathcal{R}(C_B)$ and $\mathcal{N}(B^s) = \mathcal{N}(C_B)$, we conclude that $\mathcal{R}(B^s) \cap \mathcal{N}(A^r) = \{0\}$ and $\mathcal{N}(B^s) \cap \mathcal{R}(A^r) = \{0\}$. Thus $B \in (\mathcal{C}_s)$ and $\operatorname{ind}(B) = s$. ∎

Remark 4.2. Conditions (b) and (d) in Theorem 4.1 can be replaced by the corresponding symmetrical conditions, as expressed in Remark 3.2.

COROLLARY 4.3. *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = r$. Then the following statements about $B \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(B) = s$ are equivalent:*

(a) *B satisfies condition $(\mathcal{C}_s)$.*
(b) *$I + (A^{\mathrm{D}})^s(B^s - A^s)$ is nonsingular and $B^s(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi = O$.*
(c) *$\operatorname{rank} B^s = \operatorname{rank} A^r$ and $I + (A^{\mathrm{D}})^s(B^s - A^s)$ is nonsingular.*

Proof. (a)⇔(b). This equivalence follows from the equivalence (a)⇔(b) established in Theorem 4.1 if we show that, under assumption $\operatorname{ind}(B) = s$, the condition (b) in this theorem implies that $I + (A^{\mathrm{D}})^{s+1}(B^{s+1} - A^{s+1})$ is nonsingular. First, we observe that $\mathcal{N}(B^s) = \mathcal{N}\left(B^{s+1}\right)$ because $\operatorname{ind}(B) = s$. Now, since $A^\pi + (A^{\mathrm{D}})^s B^s$ is nonsingular, then $\mathcal{N}(A^\pi) \cap \mathcal{N}(B^s) = \{0\}$. From $B^s(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}A^\pi = O$ it follows that $B^s = B^s(I + (A^{\mathrm{D}})^s(B^s - A^s))^{-1}(A^{\mathrm{D}})^s B^s$. So, we see that $\mathcal{N}\left((A^{\mathrm{D}})^{s+1}B^{s+1}\right) = \mathcal{N}\left((A^{\mathrm{D}})^s B^{s+1}\right) \subseteq \mathcal{N}\left(B^{s+1}\right)$. Thus $A^\pi + (A^{\mathrm{D}})^{s+1}B^{s+1}$ is nonsingular because $\mathcal{N}(A^\pi) \cap \mathcal{N}\left(B^{s+1}\right) = \{0\}$.

(a)⇔(c). This equivalence follows from the equivalence (a)⇔(d) established in Theorem 4.1. The details are omitted. ∎

Next, we give a representation for the powers of B.

LEMMA 4.4. *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = r$ and let $B \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(B) = s$, satisfying condition $(\mathcal{C}_s)$. Then, for all integer $k \geq 1$, we have the representation*

$$B^k = P\left\{\begin{bmatrix} I \\ S \end{bmatrix}(B_1(I+TS))^{k-1}B_1\begin{bmatrix} I & T \end{bmatrix} + \begin{bmatrix} T \\ -I \end{bmatrix}(B_2(I+ST))^{k-1}B_2\begin{bmatrix} S & -I \end{bmatrix}\right\}P^{-1},$$

where B_1 and $I + TS$ are nonsingular and $B_2(I + ST)$ is nilpotent of index s.

Proof. The formula for the powers B^k can be derived from the representation (4.1), using the formula (1.3) of Lemma 1.1 and the formula (4.4). ∎

5. Perturbation results. In this section we give an explicit representation of B^{D} and we derive perturbation bounds of the Drazin inverse and the eigenprojection at zero.

THEOREM 5.1. *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = r > 0$ and let $B \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(B) = s$, satisfying condition $(\mathcal{C}_s)$. Denote $E_1 = E = B - A$ and $E_s = B^s - A^s$. Assume that $I + A^{\mathrm{D}}E$ is nonsingular. Then*

(5.1)
$$\begin{aligned} B^{\mathrm{D}} =& \Phi_1^{-1}\Big(A^{\mathrm{D}} + A^{\mathrm{D}}\Psi_{ss}^{-1}\Phi_s^{-1}(A^{\mathrm{D}})^s E_s A^\pi(I - A^\pi E_s(A^{\mathrm{D}})^s\widetilde{\Phi}_s^{-1}) + A^\pi E_s(A^{\mathrm{D}})^s\widetilde{\Phi}_s^{-1}\Psi_{1s}^{-1} \\ & \times\left(A^{\mathrm{D}} - \Phi_1^{-1}A^{\mathrm{D}}EA^{\mathrm{D}} - \Phi_1^{-1}A^{\mathrm{D}}(\Psi_{ss} - I)\Psi_{ss}^{-1}\right)(I + \Phi_s^{-1}(A^{\mathrm{D}})^s E_s A^\pi)\Big), \end{aligned}$$

where $\Phi_i = I+(A^{\mathrm{D}})^i E_i$, $\widetilde{\Phi}_i = I+E_i(A^{\mathrm{D}})^i$, *and* $\Psi_{is} = I+\Phi_i^{-1}(A^{\mathrm{D}})^i E_i A^{\pi} E_s (A^{\mathrm{D}})^s \widetilde{\Phi}_s^{-1}$ *for* $i=1$ *and* $i=s$. *If* $\max\{\|A^{\mathrm{D}}E\|, \|(A^{\mathrm{D}})^s E_s\|, \|E_s(A^{\mathrm{D}})^s\|\} < 1$, *then*

(5.2)

$$\begin{aligned}
\frac{\|B^{\mathrm{D}} - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|}
&\le \frac{\|A^{\mathrm{D}}E\|}{1-\|A^{\mathrm{D}}E\|} + \frac{\|(A^{\mathrm{D}})^s E_s A^{\pi}\|\|\Psi_{ss}^{-1}\|}{(1-\|A^{\mathrm{D}}E\|)(1-\|(A^{\mathrm{D}})^s E_s\|)}\left(1+\frac{\|A^{\pi}E_s(A^{\mathrm{D}})^s\|}{1-\|E_s(A^{\mathrm{D}})^s\|}\right)\\
&\quad + \frac{\|A^{\pi}E_s(A^{\mathrm{D}})^s\|\|\Psi_{1s}^{-1}\|}{(1-\|A^{\mathrm{D}}E\|)(1-\|E_s(A^{\mathrm{D}})^s\|)}\left(1+\frac{\|(A^{\mathrm{D}})^s E_s A^{\pi}\|}{1-\|(A^{\mathrm{D}})^s E_s\|}\right)\\
&\quad \times\left(1+\frac{\|A^{\mathrm{D}}E\|}{1-\|A^{\mathrm{D}}E\|} + \frac{\|(A^{\mathrm{D}})^s E_s A^{\pi}\|\|A^{\pi}E_s(A^{\mathrm{D}})^s\|\|\Psi_{ss}^{-1}\|}{(1-\|A^{\mathrm{D}}E\|)(1-\|E_s(A^{\mathrm{D}})^s\|)(1-\|(A^{\mathrm{D}})^s E_s\|)}\right).
\end{aligned}$$

Furthermore, if

$$\max\{\|A^{\mathrm{D}}E\|, \|(A^{\mathrm{D}})^s E_s\|, \|E_s(A^{\mathrm{D}})^s\|\} < \frac{1}{1+\sqrt{\|A^{\pi}\|}},$$

then we have the following upper bounds for $i=1$ *and* $i=s$*:*

$$\|\Psi_{is}^{-1}\| \le \frac{(1-\|(A^{\mathrm{D}})^i E_i\|)(1-\|E_s(A^{\mathrm{D}})^s\|)}{(1-\|(A^{\mathrm{D}})^i E_i\|)(1-\|E_s(A^{\mathrm{D}})^s\|) - \|(A^{\mathrm{D}})^i E_i\|\|A^{\pi}E_s(A^{\mathrm{D}})^s\|}. \tag{5.3}$$

Proof. From Theorem 4.1(c), we have that the index 1-nilpotent decomposition of B is given by $B = C_B + N_B$, with $C_B = P\begin{bmatrix} I \\ S\end{bmatrix} B_1 \begin{bmatrix} I & T\end{bmatrix} P^{-1}$ and $N_B = P\begin{bmatrix} T \\ -I\end{bmatrix} B_2 \begin{bmatrix} S & -I\end{bmatrix} P^{-1}$, where B_1 and $I+TS$ are nonsingular and $B_2(I+ST)$ is nilpotent of index s. Hence, applying Lemma 3.3, formulae (3.3), we obtain

$$B^{\mathrm{D}} = C_B^{\sharp} = P\begin{bmatrix} I \\ S\end{bmatrix}[(I+TS)B_1(I+TS)]^{-1}\begin{bmatrix} I & T\end{bmatrix} P^{-1}. \tag{5.4}$$

Furthermore, we can write $E = B - A$ as

$$E = P\begin{pmatrix} B_1 + TB_2S - A_1 & B_1T - TB_2 \\ SB_1 - B_2S & SB_1T + B_2 - A_2\end{pmatrix} P^{-1}. \tag{5.5}$$

In view of this latter representation we get

$$I + A^{\mathrm{D}}E = P\begin{pmatrix} A_1^{-1}(B_1 + TB_2S) & A_1^{-1}(B_1T - TB_2) \\ O & I\end{pmatrix} P^{-1}. \tag{5.6}$$

From the assumption that $I + A^{\mathrm{D}}E$ is nonsingular, it follows that $B_1 + TB_2S$ is nonsingular. Using (5.6) and (5.4) we obtain

(5.7)

$$(I+A^{\mathrm{D}}E)B^{\mathrm{D}} = P\begin{pmatrix} A_1^{-1}(I+TS)^{-1} & A_1^{-1}(I+TS)^{-1}T \\ S((I+TS)B_1(I+TS))^{-1} & S((I+TS)B_1(I+TS))^{-1}T\end{pmatrix} P^{-1}.$$

By denoting $\Phi_1 = I + A^{\mathrm{D}}E$, in view of (5.6) we obtain

$$\Phi_1^{-1} = P\begin{pmatrix}(B_1+TB_2S)^{-1}A_1 & -(B_1+TB_2S)^{-1}(B_1T-TB_2)\\ O & I\end{pmatrix}P^{-1}. \tag{5.8}$$

Utilizing the representations of the powers of B given in Lemma 4.4, we write $E_s = B^s - A^s$ as

$$E_s \;=\; P\begin{pmatrix}(B_1(I+TS))^{s-1}B_1 - A_1^s & (B_1(I+TS))^{s-1}B_1T\\ S(B_1(I+TS))^{s-1}B_1 & S(B_1(I+TS))^{s-1}B_1T - A_2^s\end{pmatrix}P^{-1}\ .$$

By denoting $\Phi_s = I + (A^{\mathrm{D}})^sE_s$ and $\widetilde{\Phi}_s = I + E_s(A^{\mathrm{D}})^s$ we get

$$\begin{aligned}\Phi_s^{-1} &= P\begin{pmatrix}B_1^{-1}\big((B_1(I+TS))^{(s-1)}\big)^{-1}A_1^s & -T\\ O & I\end{pmatrix}P^{-1},\\ \widetilde{\Phi}_s^{-1} &= P\begin{pmatrix}A_1^sB_1^{-1}\big((B_1(I+TS))^{(s-1)}\big)^{-1} & O\\ -S & I\end{pmatrix}P^{-1},\end{aligned} \tag{5.9}$$

and, hence,

$$\Phi_s^{-1}(A^{\mathrm{D}})^s = (A^{\mathrm{D}})^s\widetilde{\Phi}_s^{-1} = P\begin{pmatrix}B_1^{-1}\big((B_1(I+TS))^{(s-1)}\big)^{-1} & O\\ O & O\end{pmatrix}P^{-1}.$$

Furthermore,

$$\Phi_s^{-1}(A^{\mathrm{D}})^sE_sA^{\pi} = P\begin{pmatrix}O & T\\ O & O\end{pmatrix}P^{-1},\;\; A^{\pi}E_s(A^{\mathrm{D}})^s\widetilde{\Phi}_s^{-1} = P\begin{pmatrix}O & O\\ S & O\end{pmatrix}P^{-1}. \tag{5.10}$$

Let $\Psi_{is} = I + \Phi_i^{-1}(A^{\mathrm{D}})^iE_iA^{\pi}E_s(A^{\mathrm{D}})^s\widetilde{\Phi}_s^{-1}$ for $i=1$ and $i=s$. Using (5.10) we see that

$$\Psi_{ss}^{-1} = P\begin{pmatrix}(I+TS)^{-1} & O\\ O & I\end{pmatrix}P^{-1}, \tag{5.11}$$

and, using (5.5), (5.6), and (5.10) we obtain

$$\begin{aligned}\Psi_{1s} &= P\left[\begin{pmatrix}I & O\\ O & I\end{pmatrix}\right.\\ &\qquad\left.+\begin{pmatrix}I-(B_1+TB_2S)^{-1}A_1 & (B_1+TB_2S)^{-1}(B_1T-TB_2)\\ O & O\end{pmatrix}\begin{pmatrix}O & O\\ S & O\end{pmatrix}\right]P^{-1}\\ &= P\begin{pmatrix}(B_1+TB_2S)^{-1}B_1(I+TS) & O\\ O & I\end{pmatrix}P^{-1},\end{aligned}$$

and, thus,

$$\Psi_{1s}^{-1} = P\begin{pmatrix}(I+TS)^{-1}B_1^{-1}(B_1+TB_2S) & O\\ O & I\end{pmatrix}P^{-1}. \tag{5.12}$$

Now, let us introduce

$$\begin{aligned}
&\Sigma_1 = A^{\mathrm{D}} + A^{\mathrm{D}}\Psi_{ss}^{-1}\Phi_s^{-1}(A^{\mathrm{D}})^s E_s A^\pi (I - A^\pi E_s (A^{\mathrm{D}})^s \widetilde{\Phi}_s^{-1}),\\
&\Omega = A^{\mathrm{D}} - \Phi_1^{-1} A^{\mathrm{D}} E A^{\mathrm{D}} - \Phi_1^{-1} A^{\mathrm{D}} (\Psi_{ss} - I)\Psi_{ss}^{-1},\\
&\Sigma_2 = A^\pi E_s (A^{\mathrm{D}})^s \widetilde{\Phi}_s^{-1} \Psi_{1s}^{-1}\Omega(I + \Phi_s^{-1}(A^{\mathrm{D}})^s E_s A^\pi).
\end{aligned} \tag{5.13}$$

In order to verify identity (5.1) we will see that the matrix representation of $\Sigma_1 + \Sigma_2$ is equal to the right-hand side of (5.7). We compute

$$\begin{aligned}
\Sigma_1 &= P\left[\begin{pmatrix} A_1^{-1} & O \\ O & O\end{pmatrix} + \begin{pmatrix} O & A_1^{-1}(I+TS)^{-1}T \\ O & O\end{pmatrix}\begin{pmatrix} I & O \\ -S & I\end{pmatrix}\right]P^{-1}\\
&= P\begin{pmatrix} A_1^{-1}(I+TS)^{-1} & A_1^{-1}(I+TS)^{-1}T \\ O & O\end{pmatrix}P^{-1}.
\end{aligned}$$

On the other hand, utilizing (5.5), (5.8), and (5.11) we see that

$$\Omega = P\begin{pmatrix} (B_1 + TB_2S)^{-1}(I+TS)^{-1} & O \\ O & O\end{pmatrix}P^{-1},$$

and, hence, using (5.12), we get

$$\Psi_{1s}^{-1}\Omega = P\begin{pmatrix} (I+TS)^{-1}B_1^{-1}(I+TS)^{-1} & O \\ O & O\end{pmatrix}P^{-1}.$$

Therefore,

$$\begin{aligned}
\Sigma_2 =& P\begin{pmatrix} O & O \\ S & O\end{pmatrix}\begin{pmatrix} (I+TS)^{-1}B_1^{-1}(I+TS)^{-1} & O \\ O & I\end{pmatrix}\begin{pmatrix} I & T \\ O & I\end{pmatrix}P^{-1}\\
=& P\begin{pmatrix} O & O \\ S(I+TS)^{-1}B_1^{-1}(I+TS)^{-1} & S(I+TS)^{-1}B_1^{-1}(I+TS)^{-1}T\end{pmatrix}P^{-1}.
\end{aligned}$$

In view of these expressions of Σ_1 and Σ_2 we conclude the proof of the first part. From the identity $B^{\mathrm{D}} - A^{\mathrm{D}} + A^{\mathrm{D}}E(B^{\mathrm{D}} - A^{\mathrm{D}} + A^{\mathrm{D}}) = \Sigma_1 - A^{\mathrm{D}} + \Sigma_2$, taking norms we obtain

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le \|A^{\mathrm{D}}E\|\|B^{\mathrm{D}} - A^{\mathrm{D}}\| + \|A^{\mathrm{D}}E\|\|A^{\mathrm{D}}\| + \|\Sigma_1 - A^{\mathrm{D}}\| + \|\Sigma_2\|.$$

Since $\max\{\|A^{\mathrm{D}}E\|, \|(A^{\mathrm{D}})^s E_s\|, \|E_s(A^{\mathrm{D}})^s\|\} < 1$, we have

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| \le \frac{\|A^{\mathrm{D}}\|\|A^{\mathrm{D}}E\| + \|\Sigma_1 - A^{\mathrm{D}}\| + \|\Sigma_2\|}{1 - \|A^{\mathrm{D}}E\|} \tag{5.14}$$

and

$$\|\Phi_s^{-1}\| \le \frac{1}{1 - \|(A^{\mathrm{D}})^s E_s\|} \quad \text{and} \quad \|\widetilde{\Phi}_s^{-1}\| \le \frac{1}{1 - \|E_s(A^{\mathrm{D}})^s\|}. \tag{5.15}$$

Taking norms in (5.13), and using these upper bounds, we get

$$\|\Sigma_1 - A^{\mathrm{D}}\| \le \frac{\|A^{\mathrm{D}}\|\|(A^{\mathrm{D}})^s E_s A^\pi\|\|\Psi_{ss}^{-1}\|}{1 - \|(A^{\mathrm{D}})^s E_s\|}\left(1 + \frac{\|A^\pi E_s (A^{\mathrm{D}})^s\|}{1 - \|E_s(A^{\mathrm{D}})^s\|}\right)$$

and

$$\|\Sigma_2\| \leq \frac{\|A^{\mathrm{D}}\|\|A^\pi E_s(A^{\mathrm{D}})^s\|\|\Psi_{1s}^{-1}\|}{1-\|E_s(A^{\mathrm{D}})^s\|}\left(1+\frac{\|(A^{\mathrm{D}})^s E_s A^\pi\|}{1-\|(A^{\mathrm{D}})^s E_s\|}\right)$$
$$\times\left(1+\frac{\|A^{\mathrm{D}}E\|}{1-\|A^{\mathrm{D}}E\|}+\frac{\|(A^{\mathrm{D}})^s E_s A^\pi\|\|A^\pi E_s(A^{\mathrm{D}})^s\|\|\Psi_{ss}^{-1}\|}{(1-\|A^{\mathrm{D}}E\|)(1-\|E_s(A^{\mathrm{D}})^s\|)(1-\|(A^{\mathrm{D}})^s E_s\|)}\right).$$

Substituting these upper bounds of $\|\Sigma_1 - A^{\mathrm{D}}\|$ and $\|\Sigma_2\|$ in (5.14) we conclude the proof of (5.2). Finally, if $\max\{\|A^{\mathrm{D}}E\|, \|(A^{\mathrm{D}})^s E_s\|, \|E_s(A^{\mathrm{D}})^s\|\} < \frac{1}{1+\sqrt{\|A^\pi\|}}$, then

$$\|\Psi_{is} - I\| \leq \frac{\|(A^{\mathrm{D}})^i E_i\|\|A^\pi E_s(A^{\mathrm{D}})^s\|}{(1-\|(A^{\mathrm{D}})^i E_i\|)(1-\|E_s(A^{\mathrm{D}})^s\|)} < 1, \quad i=1,s.$$

Hence, it follows that

$$\|\Psi_{is}^{-1}\| \leq \frac{(1-\|(A^{\mathrm{D}})^i E_i\|)(1-\|E_s(A^{\mathrm{D}})^s\|)}{(1-\|(A^{\mathrm{D}})^i E_i\|)(1-\|E_s(A^{\mathrm{D}})^s\|)-\|(A^{\mathrm{D}})^i E_i\|\|A^\pi E_s(A^{\mathrm{D}})^s\|}, \quad i=1,s.$$

This completes the proof. ∎

Remark 5.2. If we denote $\delta_{is} = (1-\|(A^{\mathrm{D}})^i E_i\|)(1-\|E_s(A^{\mathrm{D}})^s\|) - \|(A^{\mathrm{D}})^i E_i\|$ $\|A^\pi E_s(A^{\mathrm{D}})^s\|$, then the upper bounds (5.3), for $i=1$ and $i=s$, can be expressed as

$$\|\Psi_{is}^{-1}\| \leq 1+\frac{\|(A^{\mathrm{D}})^i E_i\|\|A^\pi E_s(A^{\mathrm{D}})^s\|}{\delta_{is}} = 1+O(\|E\|^2),$$

where in the last identity we have taken into account that $\|E_s\| = O(\|E\|)$ (see [11]).

Substituting this in (5.2) we get that the upper bound of $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|$ up to the first order of $\|E\|$, has the following expression

$$\frac{\|B^{\mathrm{D}}-A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \leq \frac{\|A^{\mathrm{D}}E\|}{1-\|A^{\mathrm{D}}E\|}+\frac{\|(A^{\mathrm{D}})^s E_s A^\pi\|}{(1-\|A^{\mathrm{D}}E\|)(1-\|(A^{\mathrm{D}})^s E_s\|)}$$
$$+\frac{\|A^\pi E_s(A^{\mathrm{D}})^s\|}{(1-\|A^{\mathrm{D}}E\|)(1-\|E_s(A^{\mathrm{D}})^s\|)}+O(\|E\|^2). \tag{5.16}$$

In the following corollary we show that the matrices satisfying condition (1.1), or equivalently $B^\pi = A^\pi$, are a particular case of the matrices satisfying condition $(\mathcal{C}_s)$.

COROLLARY 5.3. *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = r > 0$, and let $B \in \mathbb{C}^{n\times n}$ $\operatorname{ind}(B) = s$ satisfying condition $(\mathcal{C}_s)$. Denote $E = B - A$. If $A^\pi E A^{\mathrm{D}} = A^{\mathrm{D}} E A^\pi$, then we have $B^{\mathrm{D}} = (I + A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}}$. Further, if $\|A^{\mathrm{D}}E\| < 1$, then*

$$\frac{\|B^{\mathrm{D}}-A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \leq \frac{\|A^{\mathrm{D}}E\|}{1-\|A^{\mathrm{D}}E\|}. \tag{5.17}$$

Proof. We have that E has the representation (5.5) given in the proof of Theorem 5.1. From condition $A^\pi E A^{\mathrm{D}} = A^{\mathrm{D}} E A^\pi$ it follows that

$$B_1 T = T B_2 \quad\text{and}\quad S B_1 = B_2 S.$$

Using these relations we get that

$$S(B_1(I+TS))^s = B_2(I+ST)S(B_1(I+TS))^{s-1} = \cdots = (B_2(I+ST))^s S.$$

Applying that $B_2(I+ST)$ is nilpotent of index s and $B_1(I+TS)$ is nonsingular we obtain that $S=O$. Analogously, we can see that $T=O$. Thus, expression (5.6) takes the form

$$I+A^{\mathrm{D}}E=P\begin{pmatrix}A_1^{-1}B_1 & O\\ O & I\end{pmatrix}P^{-1}.$$

Clearly $I+A^{\mathrm{D}}E$ is nonsingular. In view of (5.4) we get

$$B^{\mathrm{D}}=P\begin{pmatrix}B_1^{-1} & O\\ O & O\end{pmatrix}P^{-1}=(I+A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}}.$$

Hence, we get that $B^{\pi}=A^{\pi}$ and the upper bound (5.17). ◻

THEOREM 5.4. *Let $A\in\mathbb{C}^{n\times n}$, $\operatorname{ind}(A)=r>0$, and let $B\in\mathbb{C}^{n\times n}$, $\operatorname{ind}(B)=s$, satisfying condition $(\mathcal{C}_s)$. Denote $E_s=B^s-A^s$. If $\max\{\|(A^{\mathrm{D}})^sE_s\|,\|E_s(A^{\mathrm{D}})^s\|\}<1$, then*

$$\begin{aligned}\|B^{\pi}-A^{\pi}\|\le{}&\frac{\|(A^{\mathrm{D}})^sE_sA^{\pi}\|}{1-\|(A^{\mathrm{D}})^sE_s\|}\\ &+\frac{\|A^{\pi}E_s(A^{\mathrm{D}})^s\|\|\Psi_{ss}^{-1}\|}{(1-\|(A^{\mathrm{D}})^sE_s\|)(1-\|E_s(A^{\mathrm{D}})^s\|)}\left(1+\frac{\|(A^{\mathrm{D}})^sE_sA^{\pi}\|}{1-\|(A^{\mathrm{D}})^sE_s\|}\right),\end{aligned}\tag{5.18}$$

where $\Psi_{ss}=I+(I+(A^{\mathrm{D}})^sE_s)^{-1}(A^{\mathrm{D}})^sE_sA^{\pi}E_s(A^{\mathrm{D}})^s(I+E_s(A^{\mathrm{D}})^s)^{-1}$.

If $\max\{\|(A^{\mathrm{D}})^sE_s\|,\|E_s(A^{\mathrm{D}})^s\|\}<\frac{1}{1+\sqrt{\|A^{\pi}\|}}$, then an upper bound of $\|\Psi_{ss}^{-1}\|$ is given by (5.3).

Proof. From Theorem 2.3 we have

$$B^{\pi}+(A^{\mathrm{D}})^sE_sB^{\pi}=-A^{\pi}X^{-1},\tag{5.19}$$

where $X=I-(I+(A^{\mathrm{D}})^sE_s)^{-1}A^{\pi}-A^{\pi}(I+E_s(A^{\mathrm{D}})^s)^{-1}$. Utilizing the expressions of Φ_s^{-1} and $\widetilde{\Phi}_s^{-1}$ given in the proof of Theorem 5.1 by (5.9), we can represent

$$X=P\begin{pmatrix}I & T\\ S & -I\end{pmatrix}P^{-1}\quad\text{and}\quad X^{-1}=P\begin{pmatrix}(I+TS)^{-1} & (I+TS)^{-1}T\\ S(I+TS)^{-1} & -I+S(I+TS)^{-1}T\end{pmatrix}P^{-1}.$$

Thus,

$$-A^{\pi}X^{-1}=A^{\pi}+P\begin{pmatrix}O & O\\ -S(I+TS)^{-1} & -S(I+TS)^{-1}T\end{pmatrix}P^{-1}.$$

Hence, in view of the representations (5.10) and (5.11) we may write

$$-A^{\pi}X^{-1}=A^{\pi}-A^{\pi}E_s(A^{\mathrm{D}})^s\widetilde{\Phi}_s^{-1}\Psi_{ss}^{-1}(I+\Phi_s^{-1}(A^{\mathrm{D}})^sE_sA^{\pi}).$$

Substituting the latter identity in (5.19) we obtain

$$B^{\pi}-A^{\pi}=-(A^{\mathrm{D}})^sE_s(B^{\pi}-A^{\pi}+A^{\pi})-A^{\pi}E_s(A^{\mathrm{D}})^s\widetilde{\Phi}_s^{-1}\Psi_{ss}^{-1}(I+\Phi_s^{-1}(A^{\mathrm{D}})^sE_sA^{\pi}).$$

Taking norms

$$\begin{aligned}\|B^{\pi}-A^{\pi}\|\le{}&\|(A^{\mathrm{D}})^sE_s\|\|B^{\pi}-A^{\pi}\|+\|(A^{\mathrm{D}})^sE_sA^{\pi}\|\\ &+\|A^{\pi}E_s(A^{\mathrm{D}})^s\|\|\widetilde{\Phi}_s^{-1}\|\|\Psi_{ss}^{-1}\|(1+\|\Phi_s^{-1}\|\|(A^{\mathrm{D}})^sE_sA^{\pi}\|).\end{aligned}$$

TABLE 5.1
Comparison of upper bounds of $\|BB^{\mathrm{D}} - AA^{\mathrm{D}}\|_2$.

	Exact value	[13, Thm. 5], (15)	(5.18)
$B = A + E_1$	9.99×10^{-10}	1.00×10^{-5}	1.00×10^{-9}
$B = A + E_2$	1.85×10^{-9}	2.74×10^{-5}	2.74×10^{-9}

TABLE 5.2
Comparison of upper bounds of $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|_2/\|A^{\mathrm{D}}\|_2$.

	$B = A + E_1$	$B = A + E_2$
Exact Value	1.12×10^{-10}	3.44×10^{-11}
[13, Thm. 1], (1)	0.7649	0.9008
[13, Thm. 4], (6)	$1.00 \times 10^{-5} + O(\|E\|^2)$	$2.73 \times 10^{-5} + O(\|E\|^2)$
(5.20)+(5.18)	3.41×10^{-9}	6.88×10^{-9}
(5.2)	2.41×10^{-9}	4.15×10^{-9}
(5.16)	$2.41 \times 10^{-9} + O(\|E\|^2)$	$4.15 \times 10^{-9} + O(\|E\|^2)$

TABLE 5.3
Comparison of upper bounds of $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|_F/\|A^{\mathrm{D}}\|_F$.

	Exact value	[14, Thm. 4.1], (4.1)	(5.2)
$B = A + E_1$	1.14×10^{-10}	8.39×10^{-5}	2.42×10^{-9}
$B = A + E_2$	3.47×10^{-11}	8.39×10^{-5}	4.15×10^{-9}

Since $\max\{\|(A^{\mathrm{D}})^s E_s\|, \|E_s(A^{\mathrm{D}})^s\|\} < 1$, regrouping in $\|B^\pi - A^\pi\|$ and substituting $\|\Phi_s^{-1}\|$ and $\|\widetilde{\Phi}_s^{-1}\|$ by the upper bounds (5.15), we get (5.18). □

Remark 5.5. If $\max\{\|A^{\mathrm{D}}E\|, \|(A^{\mathrm{D}})^s E_s\|, \|E_s(A^{\mathrm{D}})^s\|\} < \frac{1}{1+\sqrt{\|A^\pi\|}}$, as we have seen in Remark 5.2, the upper bound of $\|B^\pi - A^\pi\|$ up to the first order of $\|E\|$ has the following expression:

$$\|B^\pi - A^\pi\| \leq \frac{\|(A^{\mathrm{D}})^s E_s A^\pi\|}{1 - \|(A^{\mathrm{D}})^s E_s\|} + \frac{\|A^\pi E_s (A^{\mathrm{D}})^s\|}{(1 - \|(A^{\mathrm{D}})^s E_s\|)(1 - \|E_s(A^{\mathrm{D}})^s\|)} + O(\|E\|^2).$$

Remark 5.6. In [5, Theorem 3.1 and Remark 3.3], under assumption $\Delta + \|A^{\mathrm{D}}E\| < 1$, where Δ is un upper bound of $\|B^\pi - A^\pi\|$, the following estimation of the Drazin inverse was given:

$$\frac{\|B^{\mathrm{D}} - A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \leq \frac{\|A^{\mathrm{D}}E\| + 2\Delta}{1 - \|A^{\mathrm{D}}E\| - \Delta}. \tag{5.20}$$

Example 5.7. In Table 5.1 we compare the upper bound for $\|B^\pi - A^\pi\|_2$ derived in Theorem 5.4 with the upper bound given in [13, Theorem 5]. The upper bounds for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|_2/\|A^{\mathrm{D}}\|_2$ given in Theorem 5.1, Remark 5.2, and Remark 5.6, replacing Δ in (5.20) by the upper bound given in (5.18), are compared in Table 5.2 with the upper bounds given in [13]. Let

$$A = \begin{pmatrix} \frac{1}{100} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\epsilon = 10^{-9}$. We have $\operatorname{ind}(A) = \operatorname{ind}(A + E_i) = 2$ and $\operatorname{rank} A^2 = \operatorname{rank}(A + E_i)^2 = \operatorname{rank} A^2(A + E_i)^2A^2 = 3$, $i = 1, 2$. By Theorem 2.1 we have that $B = A + E_i$ satisfies condition $(\mathcal{C}_2)$.

In Table 5.3 we compare the upper bound (5.2) using the Frobenius norm with the upper bound given in [14], formula (4.1). That formula is based on the separation of matrices $\operatorname{sep}_F(C, N)$, with C and N being the matrices in the following Schur decomposition,

$$Q^H AQ = \begin{bmatrix} C & G \\ O & N \end{bmatrix},$$

where Q is an unitary matrix, C is nonsingular, and N is nilpotent of index $\operatorname{ind}(A)$. In this example $\operatorname{sep}_F(C, N) = 1.42 \times 10^{-4}$.

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