SWITCHING OPERATIONS FOR HADAMARD MATRICES

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Abstract. We define several operations that switch substructures of Hadamard matrices thereby producing new, generally inequivalent, Hadamard matrices. These operations have application to the enumeration and classification of Hadamard matrices. To illustrate their power, we use them to greatly improve the lower bounds on the number of equivalence classes of Hadamard matrices in orders 32 and 36 to 3,578,006 and 4,745,357.

1. Introduction

Two matrices, A and B, with entries in the set $\{-1, 1\}$ are *Hadamard equivalent* if B can be obtained from A by some sequence of

- row negations,
- column negations,
- row permutations, and
- column permutations.

Hadamard equivalence is so named because of its connection with Hadamard matrices, defined as square matrices with elements equal to ± 1 whose rows are mutually orthogonal. The listed moves all preserve the property of being a Hadamard matrix.

In this paper, we describe some additional moves, called *switching operations,* that preserve the property of being a Hadamard matrix. These operations, when applied over and over again to a seed matrix, generally produce many inequivalent Hadamard matrices.

Furthermore, adjoining the new operations to the list above gives new notions of equivalence. These weaker notions of equivalence may be useful in the classification of Hadamard matrices since they partition the set of Hadamard matrices into a much smaller number of equivalence classes than does Hadamard equivalence, but at the same time provide an effective method for enumerating the elements of these newly defined equivalence classes.

Extensive calculation indicates that the number of Hadamard equivalence classes that can be constructed using the new operations is enormous. This is a big step forward since, although complete enumerations up to order 28 suggest that the number of equivalence classes grows rapidly in higher order, up till now there has been no general method for producing the vast numbers of equivalence classes that we expect to exist. The many

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known Hadamard matrix construction techniques typically apply only in scattered orders, or tend to produce Hadamard matrices with special features such as large automorphism groups, large Hadamard submatrices, or self-duality.

The most prolific method for constructing Hadamard matrices has been to use two Hadamard matrices of size n, A and B, to build Hadamard matrices of size $2n$

$$H = \begin{bmatrix} A & PB \\ A & -PB \end{bmatrix} \quad \text{and} \quad \widetilde{H} = \begin{bmatrix} A & A \\ BP & -BP \end{bmatrix} \tag{1.1}$$

where P is any permutation matrix. Both A and B can be taken from any equivalence class. In order 32, Lin, Wallis, and Lie [22] produced at least 66099 inequivalent matrices from the 5 equivalence classes in order 16. Since the resulting matrices contain Hadamard submatrices of order 16, however, they cannot be considered generic. In contrast, the new operations produce at least 3.57 million equivalence classes, most of which do not contain Hadamard submatrices of order 16.

Lam, Lam, and Tonchev have exercised great ingenuity in deriving lower bounds on the number of Hadamard matrices of size $2n$ of the form (1.1), and have produced spectacularly large bounds in orders 40 and higher [20, 21]. If the lessons learned from order 32 are any guide, the true numbers of Hadamard equivalence classes in these orders are far greater still.

Our results are even more striking in orders congruent to 4 mod 8 since the construction (1.1) does not apply. The previously known equivalence classes in order 36 numbered in the hundreds. By the new methods, at least 4.74 million classes can be produced.

In orders 4, 8, 12, 16, 20, 24, 28 the numbers of Hadamard equivalence classes are known to be 1, 1, 1, 5, 3, 60, 487 [10, 11, 12, 16, 17]. We define a weaker notion of equivalence, which we call *Q-equivalence*, by adjoining the new operations to the operations that define Hadamard equivalence. The numbers of Q-equivalence classes are 1, 1, 1, 1, 1, 2, 2. In order 32, we find that the 3.57 million known Hadamard equivalence classes are grouped into 11 Q-equivalence classes, and that in order 36, the 4.74 million known equivalence classes are grouped into 21 Q-equivalence classes.

2. Overview of switching

Suppose that an $n \times n$ Hadamard matrix can be put in the form

$$\begin{bmatrix}
\mathbf{1} & \cdots & \mathbf{1} & - & \cdots & - & - & \cdots & - & 1 & \cdots & 1 \\
\mathbf{1} & \cdots & \mathbf{1} & - & \cdots & - & 1 & \cdots & 1 & - & \cdots & - \\
\mathbf{1} & \cdots & \mathbf{1} & 1 & \cdots & 1 & - & \cdots & - & - & \cdots & - \\
\mathbf{1} & \cdots & \mathbf{1} & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
 & a_5 & & & b_5 & & & c_5 & & & d_5 & \\
 & \vdots & & & \vdots & & & \vdots & & & \vdots & \\
 & a_n & & & b_n & & & c_n & & & d_n &
\end{bmatrix} \tag{2.1}$$

where a_i, b_i, c_i, and d_i are $\{-1, 1\}$-vectors of length $n/4$. The columns of the matrix have been grouped into 4 sets of $n/4$ columns each. A new, generally inequivalent, Hadamard matrix can be obtained by negating the $4 \times \frac{n}{4}$ block of 1s in the upper left corner (shown in boldface). We call this operation *switching a closed quadruple.*

Suppose instead that we can put the matrix in the form

$$\begin{bmatrix}
1 & - & - & - & \mathbf{1} & \cdots & \mathbf{1} & - & \cdots & - & - & \cdots & - & 1 & \cdots & 1 \\
- & 1 & - & - & \mathbf{1} & \cdots & \mathbf{1} & - & \cdots & - & 1 & \cdots & 1 & - & \cdots & - \\
- & - & 1 & - & \mathbf{1} & \cdots & \mathbf{1} & 1 & \cdots & 1 & - & \cdots & - & - & \cdots & - \\
- & - & - & 1 & \mathbf{1} & \cdots & \mathbf{1} & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & & & & & & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & A_{11} & & & A_{12} & & & A_{13} & & & A_{14} & \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & & & & & & & & & & & & \\
1 & 1 & - & - & & & & & & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & A_{21} & & & A_{22} & & & A_{23} & & & A_{24} & \\
1 & 1 & - & - & & & & & & & & & & & & \\
1 & - & 1 & - & & & & & & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & A_{31} & & & A_{32} & & & A_{33} & & & A_{34} & \\
1 & - & 1 & - & & & & & & & & & & & & \\
- & 1 & 1 & - & & & & & & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & A_{41} & & & A_{42} & & & A_{43} & & & A_{44} & \\
- & 1 & 1 & - & & & & & & & & & & & &
\end{bmatrix} \tag{2.2}$$

where the A_{ij} are square matrices of size $(n-4)/4$. A new, often inequivalent, matrix can be obtained by negating the all 1 block of size $4 \times \frac{n-4}{4}$ contained in the first four rows, and the all 1 block of size $\frac{n-4}{4} \times 4$ contained in the first four columns (both shown in boldface). We call this operation *switching a Hall set.*

Justification for these claims and further elaboration are given in the subsequent sections.

3. Closed quadruples and Hall sets

3.1. **3-normalization.** Let H be a Hadamard matrix of size n. Denote its rows by h_i and its elements by h_{ij}. Define the Hadamard product of two vectors to be

$$(a_1, \ldots, a_n) \circ (b_1, \ldots, b_n) := (a_1 b_1, \ldots, a_n b_n).$$

Let j_k be the all 1 vector of length k.

Definition. A Hadamard matrix is *3-normalized on rows* (i, j, k) if, for every column ℓ, the set $\{h_{i\ell}, h_{j\ell}, h_{k\ell}\}$ contains an even number of -1s, or equivalently if $h_i \circ h_j \circ h_k = j_n$.

3-normalization is a normalization of the columns. A 3-normalized matrix remains 3-normalized if any two of the rows i, j, k or of any single row other than i, j, k is negated. Note that 3-normalization was introduced in ref. [28]. The definition given here is slightly weaker in that it makes no stipulation that the row sums be positive, and does not impose any particular ordering on the columns. In the next paragraph we restate some needed results from [28].

The *field structure* (C_1, C_2, C_3, C_4) of a 3-normalized Hadamard matrix is the partition of the set of columns c into 4 classes, C_i, accordingly as $(h_{jc}, h_{kc}, h_{\ell c}) = (1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$, or $(1,-1,-1)$. The 4 classes are called *fields* and are all of length $n/4$. In a row $r \notin \{j,k,\ell\}$ the sum of the elements in a field is the same for each of the 4 fields in the row. This follows from orthogonality of row r with rows j, k, ℓ. Since the sum of the entries in a field is even if $n/4$ is even, and odd if $n/4$ is odd, the row sum of row $r \notin \{j,k,\ell\}$ must be congruent to n mod 8.

A quadruple of rows, (i,j,k,ℓ) of a Hadamard matrix H is said to be of *type* r, $0 \leq r \leq n/8$, if exactly $4r$ of the entries in $h_i \circ h_j \circ h_k \circ h_\ell$ equal -1 or exactly $4r$ entries equal $+1$. This notion was introduced by Kimura [17].

Definition. A quadruple of rows, (i,j,k,ℓ) of H, is *closed* if $h_i \circ h_j \circ h_k \circ h_\ell = \pm j_n$.

A closed quadruple is a quadruple of type 0. Thus if H is 3-normalized on three rows of a closed quadruple, then the fourth will consist entirely of 1s or entirely of -1s. The field structure is independent of which three rows are chosen.

Quadruples of type 1 will also play an important role in what follows. They were used extensively by Hall in the classification of Hadamard matrices of order 20 [11] and by Kimura in the classification for order 28 [18, 17]. If H is 3-normalized on three rows of a type-1 quadruple, then the fourth row will contain one odd-sign entry in each of the fields induced by the 3-normalization. Kimura and Ohmori referred to such quadruples as *Hall sets* [19].

Proposition 3.1. *If a Hadamard matrix of size n has a closed quadruple, then $n = 4$ or $n \equiv 0$ mod 8.*

Proof. Let (i,j,k,ℓ) be the closed quadruple. 3-normalize the matrix on rows i, j, k so that $h_\ell = \pm j_n$. Orthogonality implies that all rows except for h_ℓ have row sum 0. All row sums of rows other than i, j, k must be congruent to n mod 8. If $n > 4$ this can only happen when $n \equiv 0$ mod 8. □

3.2. **Obtaining new Hadamard matrices by switching closed quadruples.**

Definition. Let H be a Hadamard matrix of size n which has a closed quadruple, Q. Let (C_1, C_2, C_3, C_4) be the partition of columns induced by 3-normalization on Q. *Switching the closed quadruple* Q means negating all the elements h_{rc}, where $r \in Q$ and $c \in C_i$ for some $i \in \{1,2,3,4\}$.

Proposition 3.2. *The matrix produced by switching a closed quadruple Q in a Hadamard matrix H is a Hadamard matrix.*

Proof. Any matrix containing a closed quadruple is Hadamard equivalent to one of the form (2.1). It is evident that switching preserves orthogonality of the columns in that matrix. Since column orthogonality is preserved under the operations needed to put H in the form (2.1), the conclusion holds generally. □

It appears that when $n > 8$, switching always produces a Hadamard matrix that is inequivalent to the original Hadamard matrix.

Note that the equivalence class of the Hadamard matrix produced by switching Q is independent of which of the four fields C_i we choose to negate. To see this, note that negating the closed quadruple elements in C_2 is equivalent to first negating the closed quadruple elements in C_1, then negating all four rows of the closed quadruple, and finally performing a certain permutation of the rows of Q. The same holds for C_3 and C_4.

3.3. **More general row switching operations.** It was observed by Denniston [6] in connection with with symmetric $(25, 9, 3)$ designs that, starting from a design, a new inequivalent design can be obtained by switching a substructure known as an oval. In a more general setting, suppose that a certain type of design is defined by a set of properties $\mathcal{P}$ on the rows of the incidence matrix R—for example that the elements -1 and 1 occur with certain frequencies—and by the additional property

$$R^{\mathrm{T}}R = M$$

where M is some fixed matrix. Symmetric balanced incomplete block designs, Hadamard matrices, and certain D-optimal designs can all be defined in this manner.

Partition the incidence matrix into two submatrices, A and X,

$$R = \begin{bmatrix} A \\ X \end{bmatrix}.$$

Now suppose that B is a matrix of the same dimensions as A satisfying the same properties $\mathcal{P}$, and that $B^{\mathrm{T}}B = A^{\mathrm{T}}A$. Then the matrix obtained from R by replacing A with B is also a design of the original type.

Suppose for example, that R is an $n \times n$ Hadamard matrix and that A is an $m \times n$ submatrix whose columns are all columns in an $m \times m$ Hadamard matrix H_m or negations of such columns. Let H'_m be another $m \times m$ Hadamard matrix. Denote column j of H_m by v_j and column j of H'_m by v'_j. Suppose that column j of A is $\sigma_j v_{a(j)}$ where σ_j is a sign and a maps the integers from 1 to n onto the integers from 1 to m. Construct B by making the substitution $v \to v'$. Then B will automatically satisfy $B^{\mathrm{T}}B = A^{\mathrm{T}}A$, and so we may use it to obtain a new Hadamard matrix of order n.

Note that if $m = 1$ we may take H_1 to be $\begin{bmatrix} 1 \end{bmatrix}$. If H'_1 is taken to be $\begin{bmatrix} -1 \end{bmatrix}$ then the above operation amounts to negation of a row. Likewise, if $m = 2$ and $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ while $H'_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ then the above operation amounts to swapping two rows.

Switching closed quadruples is an instance of the $n = 4$ case. Let A be a $4 \times n$ matrix whose columns, or their negations, are columns of H_4, a 4×4 Hadamard matrix. Orthogonality of the rows of A implies that A is a closed quadruple. Now negating one column of H_4 and using the resulting matrix to construct B, has the effect of switching the closed quadruple formed by the rows of A. Thus, in some sense switching closed quadruples is a natural extension of the operations of row negation and row permutation.

3.4. **Closed quadruples and Hadamard submatrices.** There is an additional sense in which switching closed quadruples is a natural extension of the operation of row permutation. Consider the matrix H of size $2n$ defined in equation (1.1). One may negate or permute the columns of A or B without changing the equivalence class of H. One may also negate a row of PB (or of A) without changing the equivalence class of H. The reason is that negating row j of PB amounts to swapping rows j and $n+j$ of H.

On the other hand, changing the permutation P, for example by performing the additional row swap (i,j), usually does change the equivalence class of H. The additional swap will affect four rows of H, namely i, j, $i+n$, $j+n$. These four rows form a closed quadruple. One of the four fields of this quadruple is the set of columns of H in which rows i and j of PB differ. We make the switch that negates the entries in rows i, j, $i+n$, and $j+n$ that lie in this field. The result is identical to the result of swapping rows i and j of PB. Therefore, in this context, switching a closed quadruple amounts to swapping a pair of rows in one of the two matrices from which H was constructed.

3.5. **Properties of Hall sets.** Hall sets play the role for matrices of order $n \equiv 4 \bmod 8$ that closed quadruples play for matrices of order $n \equiv 0 \bmod 8$.

Hall sets can be found both in Hadamard matrices of order $n \equiv 0 \bmod 8$ and in those of order $n \equiv 4 \bmod 8$. Four columns are singled out in the definition of a Hall set, namely the columns of opposite sign in the Hadamard product. When $n \equiv 4 \bmod 8$ these form a Hall set in the columns as shown by Kimura and Ohmori [19]. For convenience of the reader, we reprove this here. We include the corresponding result for $n \equiv 0 \bmod 8$ for good measure. First define the *Hall columns* to be the four columns whose entries have odd sign in the Hadamard product. There is one Hall column in each field.

Proposition 3.3. *Let H be a Hadamard matrix of order n. If $n \equiv 0 \bmod 8$ then the Hall columns form a closed quadruple. If $n \equiv 4 \bmod 8$ then the Hall columns form a Hall set.*

Proof. We assume without loss of generality that H is 3-normalized on three rows of the Hall set. Consider a row not contained in the Hall set. Let x_i denote the element of that row in the Hall column of field i. Let a_i denote the sum of the remaining elements of field i. Then orthogonality with the Hall set rows implies

$$x_1 + a_1 = x_2 + a_2 = x_3 + a_3 = x_4 + a_4 \tag{3.1}$$

$$a_1 + a_2 + a_3 + a_4 = x_1 + x_2 + x_3 + x_4. \tag{3.2}$$

which implies that the row sum, which must be congruent to $n \bmod 8$, equals $2(x_1 + x_2 + x_3 + x_4)$. Hence the product $x_1x_2x_3x_4$ is positive for $n \equiv 0 \bmod 8$ and negative for $n \equiv 4 \bmod 8$. In each row of the Hall set, the product of the four elements in Hall columns is always positive, so the result follows. ∎

Henceforth we will consider the $n \equiv 4 \bmod 8$ case, and when we speak of a Hall set, we will mean both the four rows of the set and the four corresponding Hall columns.

By permuting the Hall rows and columns to the top- and leftmost positions and normalizing appropriately we obtain the form

$$H = \begin{bmatrix} H_4 & F_1 & F_2 & F_3 & F_4 \\ G_1 & A_{11} & A_{12} & A_{13} & A_{14} \\ G_2 & A_{21} & A_{22} & A_{23} & A_{24} \\ G_3 & A_{31} & A_{32} & A_{33} & A_{34} \\ G_4 & A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \tag{3.3}$$

where

$$H_4 = \begin{bmatrix} 1 & - & - & - \\ - & 1 & - & - \\ - & - & 1 & - \\ - & - & - & 1 \end{bmatrix} \qquad F_1 = \begin{bmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ 1 & \dots & 1 \\ 1 & \dots & 1 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 1 & \dots & 1 \\ - & \dots & - \\ - & \dots & - \\ 1 & \dots & 1 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 1 & \dots & 1 \\ - & \dots & - \\ 1 & \dots & 1 \\ - & \dots & - \end{bmatrix} \qquad F_4 = \begin{bmatrix} - & \dots & - \\ - & \dots & - \\ 1 & \dots & 1 \\ 1 & \dots & 1 \end{bmatrix}, \tag{3.4}$$

$G_1 = F_1^{\mathrm{T}}$, $G_j = -F_j^{\mathrm{T}}$ for $j \in \{2,3,4\}$, and A_{ij} are submatrices whose row and column sums equal 2 when $i = j$ and 0 when $i \neq j$.

Definition. By *switching a Hall set* in the matrix H defined in eqn. (3.3) we mean the operation of replacing F_i by its negation and G_i by its negation for one of the choices $i = 1, 2, 3, 4$.

The four possible negations of the definition produce equivalent matrices. The proof of this is similar to the proof of the analogous property of closed quadruples given in the discussion following Proposition (3.2). Switching is well defined even when the Hall rows and columns do not appear in positions 1–4 or when the normalization is different from the one in (3.3). We need only apply a signed permutation to put the matrix into the form (3.3), switch as in the definition, and then apply the inverse signed permutation.

Proposition 3.4. *The matrix produced by switching a Hall set in a Hadamard matrix is a Hadamard matrix.*

Proof. We will assume the form (3.3) since the conclusion is unaffected by the permutations and negations needed to convert the matrix to that form. When $i \neq j$, the rows of F_j are orthogonal to the rows of A_{ij} as the latter have row sum 0. Therefore, negating F_j does not alter the orthogonality of rows 1–4 of H with with the rows of H contained in the block $\begin{bmatrix} G_i & A_{i1} & A_{i2} & A_{i3} & A_{i4} \end{bmatrix}$. Row k ($k = 1,2,3,4$) of F_j has inner product ± 2 with any of the rows of A_{jj} while row k of H_4 has inner product ∓ 2 with any of the rows of G_j. Negating both F_j and G_j produces sign changes in these inner products that produce opposite contributions to any of the inner products of rows 1–4 of H with the rows of H contained in the block $\begin{bmatrix} G_j & A_{j1} & A_{j2} & A_{j3} & A_{j4} \end{bmatrix}$. □

Examples are known where switching a Hall set in a Hadamard matrix H produces a matrix equivalent to H. In general, however, one obtains an inequivalent matrix.

4. Invariants

First we describe an a property of Hadamard matrices that is invariant under switching of closed quadruples when the order is of the form $16k + 8$. Then we exhibit an invariant under switching of Hall sets.

4.1. A closed quadruple switching invariant for $n \equiv 8 \bmod 16$.

We will need to understand the ways that closed row quadruples may overlap within a Hadamard matrix.

Proposition 4.1. *Suppose (i, j, k, ℓ) and (i', j', k', ℓ') are distinct closed quadruples with non-null intersection. Then the number of rows common to the two quadruples is 2 if $n \equiv 8 \bmod 16$ and 1 or 2 if $n \equiv 0 \bmod 16$.*

Proof. The number of common rows cannot be 3 since the fourth row of a closed quadruple is determined, up to sign, by the other three, and the two quadruples are assumed distinct. Therefore the number of common rows must be either 1 or 2.

We will show that if the number of common rows is 1, then $n \equiv 0 \bmod 16$.

Assume the number of common rows to be 1 and let $n = 8r$. Take the two quadruples to be $(1, 2, 3, 4)$ and $(1, 5, 6, 7)$, and 3-normalize the matrix on rows 2, 3, 4. Normalize row 1 to have positive entries. By suitable column permutations, the structure of the first 5 rows can be brought to the form:

$$\begin{array}{lrrrrrrrr}
1. & 1_r & 1_r & 1_r & 1_r & 1_r & 1_r & 1_r & 1_r \\
2. & 1_r & 1_r & -1_r & -1_r & -1_r & -1_r & 1_r & 1_r \\
3. & 1_r & 1_r & -1_r & -1_r & 1_r & 1_r & -1_r & -1_r \\
4. & 1_r & 1_r & 1_r & 1_r & -1_r & -1_r & -1_r & -1_r \\
5. & 1_r & -1_r & 1_r & -1_r & 1_r & -1_r & 1_r & -1_r
\end{array}$$

The form of row 5 is a consequence of the fact that the sum of elements in each of the four fields must be zero. Since $(1, 5, 6, 7)$ is closed, the Hadamard product of rows 6 and 7 equals either row 5 or its negation. By normalizing row 7 appropriately we may assume the former. Consider the two subfields that compose the first field in the above structure. They will be further subdivided as

$$\begin{array}{lrrrrl}
5. & 1_a & 1_{r-a} & -1_b & -1_{r-b} & \dots \\
6. & 1_a & -1_{r-a} & 1_b & -1_{r-b} & \dots \\
7. & 1_a & -1_{r-a} & -1_b & 1_{r-b} & \dots
\end{array}$$

The subfields composing the remaining three fields will be subdivided similarly. Because there are r 1s per field in rows 6 and 7, just as in row 5, we have the constraints $a + b = r$

and $a + (r - b) = r$. Therefore $a = b = r - a = r - b = r/2$ and hence r is even. Consequently $n \equiv 0 \bmod 16$. □

Note that all of the degrees of overlap between closed quadruples allowed by the Proposition occur in practice.

Proposition 4.2. *Let $n \equiv 8 \bmod 16$. Let H be a Hadamard matrix of size n which has a closed row quadruple Q. Switching Q does not change the number of closed row quadruples in H.*

Proof. In the matrix obtained from H by switching Q, the rows of Q still form a closed quadruple. Also, any quadruple, whether closed or not, that doesn't involve any rows of Q is unaffected by switching. The only way the number of closed quadruples could change is if a closed quadruple were created or destroyed by switching Q. Such a closed quadruple would have to overlap Q (either before or after switching) and would therefore share exactly two of Q's rows. However, the Hadamard product of any pair of rows in Q is not altered by negation of any of the fields of Q. Hence the Hadamard product of the four rows of a putative overlapping quadruple would be unchanged by such a negation. Therefore, any closed quadruple overlapping Q in two rows remains closed after switching Q. Likewise, any quadruple overlapping Q in two rows which is not closed initially, will not be closed after switching Q. □

It is worth pointing out that switching a closed *column* quadruple does change the number of closed row quadruples in general.

4.2. **A Hall set switching invariant.** An important notion used in the classification of Hadamard matrices is that of integer equivalence.

Definition. Two integer matrices A and B are *integer equivalent* if A can be converted to B by some sequence of the following row and column operations:

- permutation of rows (columns)
- negation of rows (columns)
- addition of an integer multiple of a row (column) to another row (column).

Associated to the integer equivalence class of A is a set of integers $s_1, \ldots, s_n$ called *invariant factors* satisfying

(1) $\mathrm{diag}(s_1, \ldots, s_n) \sim A$
(2) $s_i | s_{i+1}$ for $i \leq r \leq n$ and $s_{r+1} = \ldots = s_n = 0$
(3) $s_1 s_2 \ldots s_i =$ the greatest common divisor of the $i \times i$ minors of A.

The matrix $\mathrm{diag}(s_1, \ldots, s_n)$ is called the *Smith normal form* of A. Two integer equivalent matrices have the same Smith normal form.

A number of properties of the Smith normal form of a Hadamard matrix have been proved [33, 27]:

(1) $s_1 = 1$, $s_2 = \ldots = s_{\alpha+1} = 2$, $\alpha \geq \lfloor \log_2 n \rfloor + 1$
(2) $s_i s_{n+1-i} = n$.

In order 36, for example, we have [5]

- $s_1 = 1$
- $s_i = 2$ for the next α values of i. $(2 \leq i \leq \alpha + 1)$
- $s_i = 6$ for the next $34 - 2\alpha$ values of i
- $s_i = 18$ for the next α values of i
- $s_{36} = 36$

where $6 \leq \alpha \leq 17$. The single parameter α determines the integer equivalence class of a Hadamard matrix H in order 36, and we say that H is in *Smith class* α.

That the Smith class is invariant under switching Hall sets is implied by the following:

Proposition 4.3. *If B is obtained from A by switching a Hall set, then B is integer equivalent to A.*

Proof. Switching a Hall set can be achieved by a sequence of integer row and column operations. Let the order of the matrix in (3.3) be $4k+4$. Adding each of rows 1 through 4 to each of the k rows 5 through $k+4$, and then adding each of columns 1 through 4 to each of columns 5 through $k+4$ has the effect of negating F_1 and G_1. $\square$

5. Equivalence relations

Hadamard equivalence, usually simply called equivalence, was defined in the introduction. We will call Hadamard equivalence classes *H-classes.* By adjoining additional operations to the list of operations given there, we can define new equivalence relations. We did this in the previous section when we defined integer equivalence whose equivalence classes are the Smith classes. Here we define some other notions of equivalence.

Definition. If $n \equiv 0 \bmod 8$ then two Hadamard matrices A and B are *Q-equivalent* if B can be obtained from A by some sequence of the operations

- row or column negation
- row or column permutation
- switching a closed quadruple of rows
- switching a closed quadruple of columns.

If the last operation is disallowed, then A and B are said to be QR-equivalent; if the third operation is disallowed then A and B are said to be QC-equivalent. When $n \equiv 4 \bmod 8$, Q-equivalence is defined by replacing the last two operations with

- switching a Hall set.

Associated with these equivalence relations are equivalence classes, called *Q-classes, QR-classes,* and *QC-classes.*

Codes can be associated with Hadamard matrices and have been used in classification (see [2]). When two matrices have the same code, we say they are *equivalent with respect to the associated code.* Note that in general there are several ways of associating a code with a Hadamard matrix.

Hadamard equivalence is stronger than Q-equivalence and therefore has a more refined equivalence class structure. In other words, there are at least as many H-classes as there are Q-classes, and each H-class is contained entirely within a particular Q-class. QR-equivalence (or QC equivalence) is intermediate in strength between H-equivalence and Q-equivalence, and will therefore have an intermediate number of equivalence classes.

By Proposition 4.3, Q-equivalence is stronger than integer equivalence when $n \equiv 4 \bmod 8$ which implies that there are at least as many Q-classes as there are Smith classes in those orders.

Doubly-even binary codes can be associated with Hadamard matrices. The following observations are due to Jennifer Key: closed row quadruples correspond to codes words of weight 4 in the associated code; in the case $n = 24$ the code is uniquely determined (up to equivalence) by the number of code words of weight 4. When $n \equiv 8 \bmod 16$ Proposition 4.2 implies that this number is an invariant under switching a closed row quadruple. Hence the associated code is not changed by switching. What happens for $n = 24$ is most likely a general phenomenon. If so, then when $n \equiv 8 \bmod 16$ the QR-classes will be a refinement of the equivalence classes associated to the doubly even binary code.

An equivalence class, of any type, may or may not be *self-dual.* The *dual* of a set of matrices is the set containing their transposes. A set that equals its own dual is self-dual. Many but not all Q-classes turn out to be self-dual. In other words, many matrices are Q-equivalent to their transposes. From the row-column symmetry in the definition of Q-equivalence it follows that if a Q-class contains at least one self-dual matrix, then that Q-class is self-dual.

We will see examples of these phenomena in the next section.

6. Application to the enumeration of inequivalent Hadamard matrices

We remind the reader that Hadamard matrices have been completely classified up to order 28. There are 5 H-classes in order 16 [10], 3 in order 20 [11], 60 in order 24 [12, 16], and 487 in order 28 [17]. Using the available lists of H-classes, we will be able to determine the structure of the Q-classes in these orders. The classification of H-classes in orders 32 and higher appears to be very difficult. We will content ourselves with identifying the Q-classes of all known Hadamard matrices in orders 32 and 36, and completely enumerating those Q-classes that are small enough for this to be feasible.

Our procedure requires that we maintain a database of inequivalent matrices. As new matrices are generated, they are put in a canonical form and compared with known matrices to prevent duplication in the database. To put the matrices in canonical form, we followed the suggestion of Brendan McKay [23], converting $n \times n$ matrices to graphs on $4n$ vertices and then using the graph isomorphism program *nauty* that he developed [24]. The canonical form of the graph computed by *nauty* was then converted back into a matrix. As suggested in the *nauty* User's Guide [25], we used the vertex invariant *cellquads* at level 2, which improves the efficiency in processing this type of graph.

To generate lists of inequivalent Hadamard matrices of order n we carried out the following procedure, which requires a seed Hadamard matrix of order n as input:

(1) Initialize `hadList` to null list.
(2) Compute canonical form of seed matrix using *nauty*. Append it to `hadList`.
(3) Compute canonical form of transpose of seed matrix. If it differs from canonical form of seed matrix, append it to `hadList`.
(4) Initialize `ctr` to 1.
(5) Let H be matrix number `ctr` on `hadList`. If $n \equiv 4 \bmod 8$ and this matrix is in the H-class of the transpose of the previous one, skip to Step 7.
(6) For each closed row quadruple ($n \equiv 0 \bmod 8$) or Hall set ($n \equiv 4 \bmod 8$) in H,
    (a) Switch the quadruple (Hall set) and compute the canonical form of the resulting matrix to obtain H'.
    (b) If H' differs from all matrices on `hadList`, append it to `hadList`. Then if the canonical form of the transpose of H' differs from H', append it to `hadList` as well.
(7) Increment `ctr`. If `hadList` is not exhausted, return to Step 5.

Note that this procedure generates the Q-class of the seed matrix unless the Q-class happens to be non-self-dual, in which case it generates the union of the Q-class and its dual. This is due to the use of the transposition operation in Step 6(b). Non-self-dual Q-classes always turn out to be small, and when the situation arises, we partition the union into two Q-classes by hand. (We could use column quadruple switching in the $n \equiv 0 \bmod 8$ case and dispense with transposition in both cases, thereby avoiding this issue, but we found it convenient to use transposition to keep track of duality.) We can also modify the procedure by simply eliminating the transposition step, in which case the procedure generates the QR-class of the seed matrix in the $n \equiv 0 \bmod 8$ case.

Here are the results on the Q-classes and QR-classes for orders 16 and 24:

- $n = 16$: The 5 H-classes are all Q-equivalent. Even more strongly, they are all QR-equivalent.
- $n = 24$: Of the 60 H-classes, 59 are Q-equivalent. The H-class missing from the main Q-class is that of the Paley matrix which has no closed quadruples and is self-dual. It forms a Q-class all by itself.

  Assmus and Key classified the 60 H-classes according to their doubly-even binary codes. (See Table 1 in [3] or Table 7.1 in [2], but beware that 42^{32}_{D}, listed with the code D, should be listed with the code C, and that 32^{42}_{D} in line 3 of the table should be changed to 32^{42}_{C}.) We use QR-equivalence to refine this classification.

  Assmus and Key found that 6 codes, labeled A, C, D, E, F, and G, occur. They are distinguished by the number of code words of weight 4 and therefore by the number of closed row quadruples in the associated matrices. (See the discussion in the previous section.) By Proposition 4.2 the operations allowed by QR-equivalence do not change the number of closed row quadruples, and therefore do not change the code. For example, the matrices associated with the code D

code	# weight-4 code words	size of code class	sizes of QR-classes
A	30	8	8
C	18	17	17
D	12	15	5, 10
E	66	8	8
F	6	10	5, 5
G	0	2	1, 1

TABLE 1. The 6 codes associated with the 60 Hadamard matrices of order 24.

all have 12 closed row quadruples. Switching any of these quadruples produces another matrix with code the D. Depending on these matrices one starts with, switching row quadruples produces a QR-class of size 5 or of size 10. These two QR-classes together account for all 15 H-classes associated with the code D. Results for all the codes appear in Table 1.

Note that the matrices associated with the $[24, 12]$ extended Golay code G do not contain closed row quadruples. One class of such matrices must be that of the Paley Hadamard matrix as we have already stated that it has no closed quadruples. There is a second class of matrices with no closed row quadruples. The matrices in this class, however, do each have 66 closed column quadruples, since their duals turn out to be in the class of the code E.

The results on Q-classes in orders 20 and 28 are:

- $n = 20$: The 3 H-classes are Q-equivalent.
- $n = 28$: Of the 487 H-classes, 486 of them (the ones containing Hall sets [18]) are Q-equivalent. The Paley matrix (generated from quadratic residues in $\mathrm{GF}(3^3)$) contains no Hall set and therefore its H-class forms a Q-class by itself.

Before presenting our results in orders 32 and 36, we ask what might the results so far lead us to expect in higher order? It is striking that except for a small number of exceptions (the H-classes of the Paley matrices in orders 24 and 28), all Hadamard matrices of given order are Q-equivalent. Could this be a general phenomenon?

In order 36, a difficulty arises. By Proposition 4.3 the Smith class is invariant under the defining operations of Q-equivalence. We will see that at least 6 different Smith classes occur, and so there must be at least 6 Q-classes, each possibly containing many H-classes. The reason the multiplicity of Smith classes was not an issue in order 28 is that $7 = 28/4$ is an odd square free number. By a result in [34] this implies that all Hadamard matrices in order 28 lie in a single Smith class. From the foregoing discussion, the best we can hope for for general $n \equiv 4 \bmod 8$ is that within each Smith class there will be a single dominant Q-class, and that the total number of Q-classes will still be small.

The results we have obtained so far in orders 32 and 36 appear to support these speculations, but given the lack of a complete classification of Hadamard matrices in these orders, it is not possible to be definitive. Our method was to collect as many Hadamard matrices as possible from the literature or using known construction techniques, and then to apply our algorithm to each of these matrices in order to obtain its Q-class. In fortunate cases our program terminated in a reasonable time, giving us a complete enumeration of the elements of the Q-class of the given seed matrix. In less fortunate cases—and if our speculations are correct, this is expected to be the usual situation—the Q-class was too big to enumerate completely. Instead, we compared partially constructed Q-classes with each other, and looked for overlaps. By so doing, we managed to identify unambiguously the Q-class of every Hadamard matrix in orders 32 and 36 known to us, to enumerate the smaller of these Q-classes, and to obtain lower bounds on the sizes of the larger Q-classes. It is a near certainty, however, that there are Q-classes we have missed.

6.1. **Order 32.**

Proposition 6.1. *All Hadamard matrices of either of the forms*

$$H = \begin{bmatrix} A & B \\ A & -B \end{bmatrix}, \qquad \widetilde{H} = \begin{bmatrix} A & A \\ B & -B \end{bmatrix},$$

where A and B are any Hadamard matrices of order 16, are Q-equivalent.

Proof. From the discussion in Section 3.4 it follows that, from the matrix $\begin{bmatrix} A & B \\ A & -B \end{bmatrix}$, with A and B fixed, we may obtain any matrix of the form $\begin{bmatrix} A & PB \\ A & -PB \end{bmatrix}$, by switching closed row quadruples.

To show that all matrices of the form H are Q-equivalent we need only to show that we can change the H-class of A or of B to any of the 5 classes in order 16 by switching closed quadruples. Since all Hadamard matrices of order 16 are QR- and QC-equivalent, we can achieve this by switching closed column quadruples in the A columns of H only or in the B columns of H only. (Closed column quadruples of A or of B extend to closed column quadruples of H and switching transforms the top and bottom halves of a column the same way.)

Analogous arguments, with rows and columns interchanged, show that all matrices of the form $\widetilde{H}$ are Q-equivalent. To show that matrices of the form H and of the form $\widetilde{H}$ are Q-equivalent to each other, simply note that both sets contain the Sylvester Hadamard matrix. □

Thus the 66099 H-classes identified in [22] are Q-equivalent. We call the Q-class of these matrices the *Sylvester Q-class*. We now turn to other known Hadamard matrices in order 32:

- the Paley matrix,
- 13 matrices from generalized Legendre (GL) pairs [8],
- 4 matrices listed in [1] and their transposes,
- the maximal excess matrix in [7],

- 4 matrices from Construction II in [22],
- 2 Williamson matrices,
- 8 Goethals-Seidel matrices constructed from circulant blocks,
- 18 Goethals-Seidel matrices constructed from negacyclic blocks,
- 10 matrices constructed from two circulants,
- 17 matrices constructed from two negacyclic matrices,
- a matrix from the appendix of [21] and its transpose.

Some of these matrices were provided by Hadi Kharaghani. Discarding duplicates (which occur due to accidental equivalences) and matrices that happen to have one of the forms in Proposition 6.1, we are left with a list of 59 matrices. Of these, 49 are in the Sylvester Q-class. Using these matrices, and some matrices from Proposition 6.1 as seeds, we have managed to generate 3,577,996 H-classes in the Sylvester Q-class by using our program and then piecing together the results. This is certainly a gross underestimate of the actual number.

The 10 exceptional matrices among the 59 all lack closed quadruples either in rows or in columns, and therefore form Q-classes by themselves. Of the 10 exceptional matrices, 6 are constructed from GL pairs, and 4 are constructed from two negacyclic blocks. The matrices from GL pairs are listed on the web page [29] as P12–P19 (with transposes of non-self-dual matrices omitted). The exceptional GL pair matrices are P13, P15 and its transpose, P17, and P19 and its transpose. Matrix P17 is Hadamard equivalent to the Paley matrix. Of the matrices constructed from two negacyclic blocks, the exceptional ones come in 2 dual pairs.

The Sylvester Q-class and the 10 singleton Q-classes give total of 11 known Q-classes in order 32, containing at least 3,578,006 Hadamard equivalence classes.

6.2. **Order 36.** As noted above, in order 36 we must consider each Smith class separately. Although Smith classes $\alpha = 6, 7, \dots, 17$ are allowed, the only Smith classes known to be nonempty are $\alpha = 11, 12, 13, 14, 15, 17$.

A complete summary of the seed matrices we compiled in order 36 follows:

- Ted Spence's 180 matrices related to regular 2-graphs (S1–S180) [30, 26, 31],
- the 24 matrices of Goethals-Seidel type classified by Spence and Turyn (GS1–GS24) [31],
- the 11 matrices with automorphism of size 17 classified by Tonchev (T1–T11) [32],
- the Bush-type Hadamard matrix found by Janko (B1) [13],
- a regular Hadamard matrix found by Jennifer Seberry and listed on her web page (R1) [29], (She actually lists four, but two are duplicates, and two are of Goethals-Seidel type.)
- 4 Williamson Hadamard matrices (W1–W4), (There is a fifth, but it is equivalent to one of Tonchev's.)
- the $(35, 17, 8)$-difference set construction (D1),
- 7 matrices of the type defined by Whiteman (a Goethals-Seidel array bordered by a Hall set) (Wh1–Wh7) [35],

- 2 block negacyclic Bush-type Hadamard matrices, the first given in the paper of Janko and Kharaghani (NB1, NB2) [14],
- a matrix in Smith class 11, found in the course of a (fruitless) search for block circulant Bush-type matrices (O1),
- a skew Bush-type Hadamard matrix found by Leif Jørgensen and its transpose (J1, J2) [15],
- a matrix listed in the appendix of [21] (LLT1).

Reference [5] was helpful in assembling the above list, but the reader should note that the 80 matrices from Steiner triple systems, which are a subset of Spence's 180 matrices, are in Smith class 13, not 12 as stated there. We have not made a serious effort to credit the original author of every matrix on our list, as we were more concerned with compiling as complete a list as possible from readily obtainable sources. We should note, however, that many of these matrices derive from the important work of Goethals and Seidel [9], including the 80 matrices from Steiner triple systems mentioned above, and 11 matrices derived from Latin squares of order 6, which are also a subset of Spence's list.

The structure we have uncovered in Smith class 13 is interesting, so we describe it in detail. Hadamard matrices in this class include 179 of Ted Spence's 180 matrices. (His matrix 137 is in Smith class 11.) Two other matrices in Smith class 13 were previously known: the regular Hadamard matrix constructed by Seberry, and the block negacyclic Bush-type Hadamard matrix constucted by Janko and Kharaghani. Seberry's matrix and 172 of Spence's fall into the same Q-class which we found has size 3425. Two of Spence's matrices (179 and 180) and the Bush-type matrix form singleton Q-classes. They have no Hall sets. The remaining 5 of Spence's matrices lie in a Q-class of size 6.

Spence's matrix 137, which is in Smith class 11 and is one of the matrices derived from a Latin square of order 6, is intriguing. It has 9 Hall sets, but switching any of these produces a matrix H-equivalent to the original.

Only two matrices on our list are in Smith class 14, B1 and LLT1. They are Q-equivalent. A major success of our program has been the complete enumeration of their Q-class, which has 954,254 elements. In each of Smith classes 15 and 17 there is one known Q-class of size above 1 million while all other known Q-classes are of size no greater than 5. The two large Q-classes have not yet been completely enumerated. At present, there is no evidence for more than one large Q-class in any Smith class.

The 235 matrices we compiled represent 6 different Smith classes, and lie in 20 different Q-classes. Some details are given in Table 2. The union of the known Q-classes contains at least 3,734,467 Hadamard equivalence classes of order 36.

Note: After this work was completed, Bouyukliev, Fack, and Winne announced the complete classification of 2-$(35, 17, 8)$ designs with an automorphism of order 3 fixing 2 points and blocks. From these designs, they find 7238 H-equivalence classes [4]. Most of these matrices have not yet been analyzed. Such an analysis should provide a good test of the ideas of this paper regarding using Q-equivalence in classifying Hadamard matrices. Among the matrices of Bouyukliev, Fack, and Winne are the first examples

α	Q-classes
11	1 (S137), 1 (O1)
12	1 (D1)
13	1 (S179), 1 (S180), 1 (NB1), 6 (S172), 3425 (S1)
14	$954,254$ (B1)
15	5 (W3), 5 (W4), $\geq 1,291,413$ (GS1)
17	1 (GS11), 1 (GS12), 1 (T1), 1 (T2), 1 (T5), 1 (T6), 1 (T7), $\geq 1,485,346$ (GS4)

TABLE 2. Sizes of known Q-classes in order 36 for the 6 known Smith classes, α. A representative matrix is listed for each Q-class.

known (to this author) in Smith class 16. One of these was used to generate a Q-class of size at least 1,010,890. This brings the number of known Smith classes to 7, the number of known Q-classes to 21, and the number of known H-equivalence classes to 4,745,357. Preliminary analysis of a sampling of the the new matrices of Bouyukliev, Fack, and Winne indicates the presence of a number of new small Q-classes. We intend to make a complete enumeration of these, and a full analysis of all the new matrices. The results will be presented in a follow-up to the present paper.

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References

[1] Makoto Araya, Masaaki Harada, and Hadi Kharaghani. Some Hadamard matrices of order 32 and their binary codes. *J. Combin. Des.*, 12(2):142–146, 2004.

[2] E. F. Assmus, Jr. and J. D. Key. *Designs and their codes*, volume 103 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.

[3] E. F. Assmus, Jr. and J. D. Key. Hadamard matrices and their designs: a coding-theoretic approach. *Trans. Amer. Math. Soc.*, 330(1):269–293, 1992.

[4] Iliya Bouyukllev, Veere Fack, and Joost Winne. Hadamard matrices of order 36 and double-even self-dual $[72, 36, 12]$ codes. In Stefan Felsner, editor, *European Conference on Combinatorics, Graph*

Theory and Applications 2005, volume AE of *DMTCS Proceedings*, pages 93–98. Discrete Mathematics and Theoretical Computer Science, 2005.

[5] Joan Cooper, James Milas, and W. D. Wallis. Hadamard equivalence. In *Combinatorial mathematics (Proc. Internat. Conf. Combinatorial Theory, Australian Nat. Univ., Canberra, 1977)*, volume 686 of *Lecture Notes in Math.*, pages 126–135. Springer, Berlin, 1978.

[6] R. H. F. Denniston. Enumeration of symmetric designs $(25, 9, 3)$. In *Algebraic and geometric combinatorics*, volume 65 of *North-Holland Math. Stud.*, pages 111–127. North-Holland, Amsterdam, 1982.

[7] Hikoe Enomoto and Masahiko Miyamoto. On maximal weights of Hadamard matrices. *J. Combin. Theory Ser. A*, 29(1):94–100, 1980.

[8] Roderick J. Fletcher, Marc Gysin, and Jennifer Seberry. Application of the discrete Fourier transform to the search for generalised Legendre pairs and Hadamard matrices. *Australas. J. Combin.*, 23:75–86, 2001.

[9] J.-M. Goethals and J. J. Seidel. Strongly regular graphs derived from combinatorial designs. *Canad. J. Math.*, 22:597–614, 1970.

[10] Marshall Hall, Jr. *Research Summary No. 36-10, Volume I*, chapter Hadamard matrices of order 16, pages 21–26. Jet Propulsion Laboratory, Pasadena, 1961.

[11] Marshall Hall, Jr. Hadamard matrices of order 20. Technical Report 32-761, Jet Propulsion Laboratory, Pasadena, 1965.

[12] Noboru Ito, Jeffrey S. Leon, and Judith Q. Longyear. Classification of $3-(24, 12, 5)$ designs and 24-dimensional Hadamard matrices. *J. Combin. Theory Ser. A*, 31(1):66–93, 1981.

[13] Zvonimir Janko. The existence of a Bush-type Hadamard matrix of order 36 and two new infinite classes of symmetric designs. *J. Combin. Theory Ser. A*, 95(2):360–364, 2001.

[14] Zvonimir Janko and Hadi Kharaghani. A block negacyclic Bush-type Hadamard matrix and two strongly regular graphs. *J. Combin. Theory Ser. A*, 98(1):118–126, 2002.

[15] Leif K. Jørgensen. Non-symmetric 3-class association schemes. Technical Report R-2005-13, Aalborg University, Department of Mathematical Sciences, http://www.math.aau.dk/research/reports/R-2005-13.pdf, March 2005.

[16] Hiroshi Kimura. New Hadamard matrix of order 24. *Graphs Combin.*, 5(3):235–242, 1989.

[17] Hiroshi Kimura. Classification of Hadamard matrices of order 28. *Discrete Math.*, 133(1-3):171–180, 1994.

[18] Hiroshi Kimura. Classification of Hadamard matrices of order 28 with Hall sets. *Discrete Math.*, 128(1-3):257–268, 1994.

[19] Hiroshi Kimura and Hiroyuki Ohmori. Construction of Hadamard matrices of order 28. *Graphs Combin.*, 2(3):247–257, 1986.

[20] Clement Lam, Sigmund Lam, and Vladimir D. Tonchev. Bounds on the number of affine, symmetric, and Hadamard designs and matrices. *J. Combin. Theory Ser. A*, 92(2):186–196, 2000.

[21] Clement Lam, Sigmund Lam, and Vladimir D. Tonchev. Bounds on the number of Hadamard designs of even order. *J. Combin. Des.*, 9(5):363–378, 2001.

[22] Cantian Lin, W. D. Wallis, and Zhu Lie. Equivalence classes of Hadamard matrices of order 32. In *Proceedings of the Twenty-fourth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1993)*, volume 95, pages 179–182, 1993.

[23] Brendan D. McKay. Hadamard equivalence via graph isomorphism. *Discrete Math.*, 27(2):213–214, 1979.

[24] Brendan D. McKay. Practical graph isomorphism. In *Proceedings of the Tenth Manitoba Conference on Numerical Mathematics and Computing, Vol. I (Winnipeg, Man., 1980)*, volume 30, pages 45–87, 1981.

[25] Brendan D. McKay. *nauty User's Guide (Version 2.2)*. Computer Science Department, Australian National University, 2004.

[26] Brendan D. McKay and Edward Spence. Classification of regular two-graphs on 36 and 38 vertices. *Australas. J. Combin.*, 24:293–300, 2001.

[27] Morris Newman. Invariant factors of combinatorial matrices. *Israel J. Math.*, 10:126–130, 1971.

[28] William P. Orrick and Bruce Solomon. Large-determinant sign matrices of order $4k+1$. *preprint*, pages 1–24, 2003. math.CO/0311292.

[29] Jennifer Seberry. Library of hadamard matrices. `http://www.uow.edu.au/~jennie/hadamard.html`, 2005.

[30] Edward Spence. Regular two-graphs on 36 vertices. *Linear Algebra Appl.*, 226/228:459–497, 1995.

[31] Edward Spence. Ted spence's home page. `http://www.maths.gla.ac.uk/~es/`, 2005.

[32] Vladimir D. Tonchev. Hadamard matrices of order 36 with automorphisms of order 17. *Nagoya Math. J.*, 104:163–174, 1986.

[33] W. D. Wallis. Integral equivalence of Hadamard matrices. *Israel J. Math.*, 10:349–358, 1971.

[34] W. D. Wallis and Jennifer Wallis. Equivalence of Hadamard matrices. *Israel J. Math.*, 7:122–128, 1969.

[35] Albert Leon Whiteman. Hadamard matrices of order $4(2p+1)$. *J. Number Theory*, 8(1):1–11, 1976.

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