ANALYSIS OF UNIVARIATE NON-STATIONARY SUBDIVISION SCHEMES WITH APPLICATION TO GAUSSIAN-BASED INTERPOLATORY SCHEMES

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Abstract: This paper is concerned with non-stationary subdivision schemes. First, we derive new sufficient conditions for C^{ν} smoothness of such schemes. Next, a new class of interpolatory $2m$-point non-stationary subdivision schemes based on Gaussian interpolation is presented. These schemes are shown to be $C^{L+\mu}$ with $L \in \mathbb{Z}_+$ and $\mu \in (0,1)$, where L is the integer smoothness order of the known $2m$-point Deslauiers-Dubuc interpolatory schemes.

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1. Introduction

Subdivision is a powerful tool for the fast construction of smooth curves and surfaces from a set of control points by means of iterative refinements. In this paper, we consider subdivision schemes for curves. A univariate binary uniform stationary subdivision scheme defines recursively new sets of points $P^k = \{p_j^k : j \in \mathbb{Z}\}$ at level $k > 0$ from a given set of control points at level zero $P^0 = \{p_j^0 : j \in \mathbb{Z}\}$, formally by

$$P^{k+1} = SP^k, \quad k = 0, 1, \cdots .$$

A point of P^k is defined by a finite linear combination of points of P^{k-1} with two different rules,

$$p_j^{k+1} = \sum_{n\in\mathbb{Z}} a_{j-2n} p_n^k, \quad k \in \mathbb{Z}_+,\ j \in \mathbb{Z}.$$

Non-stationary subdivision schemes consist of recursive refinements of an initial sparse sequence with the use of rules that may vary from level to level but are the same everywhere on the same level. Therefore, in the binary case, starting with the control points $P^0 = \{p_n^0 : n \in \mathbb{Z}\}$, we define new sets of points $P^k = \{p_n^k : n \in \mathbb{Z}\}$ generated by the relation

$$p_j^{k+1} = \sum_{n\in\mathbb{Z}} a_{j-2n}^{[k]} p_n^k, \quad k \in \mathbb{Z}_+,\ j \in \mathbb{Z}. \tag{1.1}$$

It is common to assume that for each level k, only a finite number of coefficients $a_n^{[k]} \in \mathbb{R}$ are non-zero so that changes in a control point affect only its local neighborhood. This property clearly

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facilitates the practical implementation of (1.1). A subdivision scheme is said to be stationary when the masks $a_n^{[k]}$ are independent of the levels; then we use the notation $a_n := a_n^{[k]}$. Non-stationary subdivision schemes are useful because they can provide design flexibility and the mask can be adapted to the geometrical configuration of the given data. Non-stationary subdivision schemes are studied in [2, 8, 10, 18], while a general treatment of stationary schemes can be found in [1, 5, 6, 7, 9].

The analysis of a subdivision scheme can be reduced to the case of initial control points in $\mathbb{R}$ since each component of the curve is a scalar function generated by the same subdivision scheme. Therefore, starting with values $f^0 = \{f_n^0 \in \mathbb{R} : n \in \mathbb{Z}\}$, we consider $f^k = \{f_n^k \in \mathbb{R} : n \in \mathbb{Z}\}$ generated by the relation

$$f_j^{k+1} = \sum_{n\in\mathbb{Z}} a_{j-2n}^{[k]} f_n^k, \quad k \in \mathbb{Z}_+.$$

Definition 1.1. A binary subdivision scheme is said to be C^ν if for the initial data $\delta = \{f_n^0 = \delta_{n,0} : n \in \mathbb{Z}\}$, there exists a limit function $\phi_0 \in C^\nu(\mathbb{R})$, $\phi_0 \not\equiv 0$, satisfying

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}} |f_n^k - \phi_0(2^{-k}n)| = 0. \tag{1.2}$$

Natural questions in the analysis of subdivision schemes are the conditions for convergence and the conditions for the limit functions to be C^ν. In particular, in this study, we are interested in the class of interpolatory subdivision schemes which refine data by inserting values corresponding to intermediate points, using linear combinations of neighboring points. The general form of their refinement rules is as follows:

$$\begin{aligned} f_{2j}^{k+1} &= f_j^k, \\ f_{2j+1}^{k+1} &= \sum_{n\in\mathbb{Z}} a_{2n+1}^{[k]} f_{j-n}^k, \quad j \in \mathbb{Z},\ k \in \mathbb{Z}_+. \end{aligned}$$

Examples of a such stationary schemes are the four-point scheme by Dyn, Gregory, and Levin ([7]) and the Deslauriers-Dubuc schemes ([6]), where finer level points are determined by local polynomial interpolation of the coarse level points. When the finer level points are determined by $2m$-point interpolation from a space of exponential polynomials the resulting scheme is non-stationary, and have smoothness properties as the $2m$-point Deslauriers-Dubuc scheme [10].

An analysis of the smoothness of non-stationary subdivision schemes is discussed in [8], however, the conditions given in [8] are too strong. Thus, the first objective of this paper is to provide a new tool for the regularity analysis, improving the condition in [8]. It can be applied to a wide class of non-stationary subdivision schemes, interpolatory and non-interpolatory. Further, the results can be used directly for the smoothness analysis of non-stationary wavelet systems, which is one of the important issues in wavelet construction. Second, in this paper, we study a new class of non-stationary interpolatory subdivision schemes, where the value at the inserted point is obtained by radial basis function (RBF) interpolation to data at $2m$ points symmetric to the inserted one. Among the many possible radial basis functions, we employ the Gaussian function $G(x) = e^{-|x|^2/\lambda^2}$ with λ as a shape parameter. We show that the resulting $2m$-point subdivision scheme converges and has the smoothness $C^{L+\mu}$ with $L \in \mathbb{Z}_+$ and $\mu \in (0, 1)$, where L is the integer smoothness order of the $2m$-point Deslauiers-Dubuc scheme. The proof of these results are based on the new sufficient condition for smoothness of non-stationary schemes. Moreover, we will see that the scheme itself has its own advantages in view of approximation.

The paper is organized as follows: In section 2, we present known conditions for the convergence and smoothness of non-stationary schemes, and derive new sufficient conditions for the smoothness of such schemes. In section 3, along with the basic setting of RBF interpolation, we present a new family of interpolatory subdivision schemes based on Gaussian interpolation. Next, we show using the results of section 2 that the new $2m$-point schemes have the same integer smoothness as the $2m$-point Deslauriers-Dubuc interpolatory scheme. Finally, in section 4, we illustrate the performance of the new interpolatory schemes by some numerical examples.

2. Sufficient Conditions for Smoothness of Non-stationary Schemes

In this section, we improve the results in [8] on the smoothness of non-stationary subdivision schemes. Non-stationary subdivision schemes define recursively values $f^k := \{f_n^k : n \in \mathbb{Z}\}$ by rules depending on the level k:

$$f_j^{k+1} = \sum_{n\in\mathbb{Z}} a_{j-2n}^{[k]} f_n^k, \qquad k \in \mathbb{Z}_+, \quad j \in \mathbb{Z}, \tag{2.1}$$

where the set of coefficients $a^{[k]} := \{a_n^{[k]}\}$ is termed the mask of the rule at level k. We denote this rule by $S_{a^{[k]}}$ and the corresponding non-stationary scheme by $\{S_{a^{[k]}}\}$. To simplify the presentation of a subdivision scheme and its analysis, it is convenient to assign to each rule, defined by a mask $a^{[k]} = \{a_n^{[k]}\}$, the Laurent polynomial

$$a^{[k]}(z) := \sum_{n\in\mathbb{Z}} a_n^{[k]} z^n.$$

Assume here that for each level k, $supp(a^{[k]}) \subset [-N, N]$ for some integer $N > 0$. This implies that the Laurent polynomials $a^{[k]}(z)$ have a finite degree. A stationary uniform subdivision scheme is a scheme for which $a_n^{[k]} = a_n$ for all $k \in \mathbb{Z}_+$. We denote the rule at each level by S_a and have the formal relation $f^k = S_a^k f^0$. The limit function of a C^0 stationary scheme is denoted by $S_a^\infty f^0$. In particular, for the given data $\delta = \{\delta_{0,n} : n \in \mathbb{Z}\}$ at level 0, with the Kronecker delta $\delta_{n,0}$, the basic limit function of $\{S_a\}$ is defined by

$$\phi = S_a^\infty \delta.$$

For a non-stationary subdivision scheme $\{S_{a^{[k]}}\}$, we have the formal relation

$$f^k = S_{a^{[k-1]}} \cdots S_{a^{[0]}} f^0.$$

Further, for a convergent scheme $\{S_{a^{[k]}}\}$, its *basic limit function* is the function

$$\phi_0 := \lim_{k\to\infty} S_{a^{[k]}} \cdots S_{a^{[0]}} \delta.$$

It clearly follows from the linearity of (2.1) that for any initial data $f^0 = \{f_n^0 : n \in \mathbb{Z}\} \in \ell^\infty(\mathbb{Z})$, the limit function of $\{S_{a^{[k]}}\}$ can be written as

$$f^\infty = \sum_{n\in\mathbb{Z}} f_n^0 \phi_0(\cdot - n).$$

First we cite a basic result about the smoothness of stationary subdivision schemes.

Theorem 2.1. (*Smoothing factors in stationary schemes* [5]). *Consider a stationary subdivision scheme $\{S_a\}$ with the Laurent polynomial*

$$a(z) = \frac{1}{2}(1+z)b(z),$$

where the subdivision scheme $\{S_b\}$ corresponding to $b(z)$ is C^γ. Then the scheme $\{S_a\}$ is convergent, and its basic limit function ϕ is in $C^{\gamma+1}$.

Now, for the analysis of the smoothness of non-stationary schemes, we adopt the notion of asymptotically equivalent schemes [8]: Two (non-stationary) subdivision schemes $\{S_{a_k}\}$ and $\{S_{\bar{a}^{[k]}}\}$ are *asymptotically equivalent*, $\{S_{a_k}\} \approx \{S_{\bar{a}^{[k]}}\}$, if

$$\sum_{k\in\mathbb{Z}_+} \|S_{a^{[k]}} - S_{\bar{a}^{[k]}}\|_\infty < \infty, \tag{2.2}$$

where

$$\|S_{a^{[k]}}\|_\infty = \max\left\{\sum_{n\in\mathbb{Z}} |a^{[k]}_{2n}|,\ \sum_{n\in\mathbb{Z}} |a^{[k]}_{1+2n}|\right\}.$$

Theorem 2.2. [8] *Let $\{S_a\}$ be a C^0 stationary subdivision scheme, and let $\{S_{a^{[k]}}\} \approx \{S_a\}$ with supp(a) = supp$(a^{[k]})$ for $k \in \mathbb{Z}_+$. Then $\{S_{a^{[k]}}\}$ is C^0, and if*

$$\|S_{a^{[k]}} - S_a\|_\infty \le c2^{-k}, \quad k \in \mathbb{Z}_+,$$

then the basic limit function ϕ_0 of $\{S_{a^{[k]}}\}$ is Hölder continuous with some exponent $\nu > 0$.

An analysis of the smoothness of non-stationary subdivision schemes is also discussed in [8], however, the conditions given in [8] are too strong. Thus, the purpose of this section is to provide less restrictive sufficient conditions for the smoothness of non-stationary schemes. Furthermore, we infer results on the smoothness of an interpolatory non-stationary scheme from the smoothness of a stationary scheme, which is asymptotically equivalent to it.

Let $a(z)$ be the Laurent polynomial associated with a stationary scheme $\{S_a\}$ with the property $a^{(\ell)}(-1) = 0$ for $\ell = 0, \cdots, M-1$ and $a^{(M)}(-1) \neq 0$. Accordingly, it can be written as

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^n = 2^{-M}(1+z)^M b(z) \tag{2.3}$$

with $b(-1) \neq 0$. When a scheme $\{S_{a^{[k]}}\}$ is asymptotically equivalent to $\{S_a\}$, by definition, $|a^{[k]}_n - a_n| = o(1)$ as k tends to ∞. Hence, the Laurent polynomial $a^{[k]}(z)$ associated with $S_{a^{[k]}}$ has M-roots in the neighborhood $z = -1$ in the complex plane, and it can be written in the form of

$$a^{[k]}(z) = b^{[k]}(z) \prod_{n=1}^{M} \frac{1}{2}(1 + r_{k,n}z), \quad b^{[k]}(-1) = c + o(1), \quad c \neq 0, \tag{2.4}$$

with $r_{k,n}$ complex numbers such that $r_{k,n} \to 1$ as k tends to ∞. Moreover, since $a^{(\ell)}(-1) = 0$ for $\ell = 0, \cdots, M-1$ (see (2.3)), it is easy to see that $D^\ell a^{[k]}(-1) = o(1)$ as $k \to \infty$ where D^ℓ indicates the differential operator of order ℓ. In this study, we require $D^\ell a^{[k]}(-1)$ to satisfy the following stronger condition:

Condition A. *A non-stationary subdivision scheme* $\{S_{a^{[k]}}\}$ *satisfies Condition A if the corresponding Laurent polynomials* $\{a^{[k]}(z)\}$ *are of the form (2.4) and if*

$$|D^\ell a^{[k]}(-1)| \le c2^{-(M-\ell)k}, \quad \ell = 0, \cdots, M-1, \quad k \in \mathbb{Z}_+.$$

In what follows, we show that if

$$a^{[k]}(z) = c^{[k]}(z) \prod_{n=1}^{L} \frac{1}{2}(1 + r_{k,n}z)$$

and if $\{S_{c^{[k]}}\}$ is $C^{N+\mu}$ with $N \in \mathbb{Z}_+$ and $\mu \in (0,1)$, then $\{S_{a^{[k]}}\}$ with Laurent polynomials of the form (2.4) satisfying Condition A has the smoothness $C^{N+L+\nu}$ with $\nu \in (0,1)$. First we show that a factor $(1+r_k z)$ in the Laurent polynomials of a non-stationary scheme with $|1-r_k| \le c2^{-k}$ is a smoothing factor. For this, we cite:

Lemma 2.3. [8] *Consider a non-stationary subdivision scheme* $\{S_{a^{[k]}}\}$ *with Laurent polynomials of the form*

$$a^{[k]}(z) = \frac{1}{2}(1 + r_k z) b^{[k]}(z).$$

Let ϕ_a*,* ϕ_b *and* h *are the basic limit functions of* $\{S_{a^{[k]}}\}$*,* $\{S_{b^{[k]}}\}$ *and* $\{S_{1+r_k z}\}$ *respectively. Then*

$$\phi_a = \int_{\mathbb{R}} \phi_b(\cdot - t) h(t)\, dt,$$

For the following analysis, it is necessary to remark that the basic limit function h of $\{S_{1+r_k z}\}$ is bounded and satisfies the properties:

$$\begin{aligned} &(a)\ \ \operatorname{supp} h = [0,1), \\ &(b)\ \ h\big((j+2^{-1})2^{-k}\big) = r_k h(j2^{-k}), \quad k \in \mathbb{Z}_+,\ j = 0, \cdots, 2^k - 1; \end{aligned} \tag{2.5}$$

see Example 2 in [8] for the details.

Lemma 2.4. *Let* ϕ_b *and* h *be the basic limit functions of* $\{S_{b^{[k]}}\}$ *and* $\{S_{1+r_k z}\}$ *repectively. Suppose that*

$$|1 - r_k| \le c2^{-k}, \quad k \ge K \in \mathbb{Z}_+. \tag{2.6}$$

For each $k \in \mathbb{Z}_+$*, define the sequence of functions*

$$I_k(x) = \int_{x-1}^{x} \phi_b(t) h_k(x-t)\, dt, \tag{2.7}$$

where

$$h_k(t) = h(j2^{-k}), \quad j2^{-k} \le t < (j+1)2^{-k}, \quad j = 0, \cdots, 2^k - 1. \tag{2.8}$$

If ϕ_b *is Hölder continuous with some exponent* $\nu > 0$*, then* I_k *satisfies the following properties:*

(a) *For any* $k, \ell \ge K \in \mathbb{Z}_+$ *with* $\ell > k$*,*

$$|I'_\ell(x) - I'_k(x)| \le c2^{-\nu k} \tag{2.9}$$

(b) *There exists* $\delta_0 > 0$ *such that for any* $\delta < \delta_0$*.*

$$|I'_k(x+\delta) - I'_k(x)| \le c\delta^\mu, \quad k \ge K \in \mathbb{Z}_+,$$

for some $\mu \in (0,1)$*.*

Proof. (a) By (2.8), we have

$$I_k'(x) = \sum_{j=0}^{2^k-1} h(j2^{-k})[\phi_b(x - j2^{-k}) - \phi_b(x - (j+1)2^{-k})] \in C(\mathbb{R}), \tag{2.10}$$

and after some calculations, we get

$$\begin{aligned} I_{k+1}'(x) - I_k'(x) &= \sum_{j=0}^{2^k-1} [h((j+2^{-1})2^{-k}) - h(j2^{-k})] \\ &\quad \cdot [\phi_b(x - (j+2^{-1})2^{-k}) - \phi_b(x - (j+1)2^{-k})]. \end{aligned}$$

Here, since ϕ_b is Hölder continuous with exponent $\nu > 0$,

$$|\phi_b(x - (j+2^{-1})2^{-k}) - \phi_b(x - (j+1)2^{-k})| \le c2^{-\nu k} \tag{2.11}$$

with a constant $c > 0$ independent of j and x. Thus, in view of (2.5), (2.6) and the boundedness of h, we obtain the expression

$$|I_{k+1}'(x) - I_k'(x)| \le c2^{-\nu k} \sum_{j=0}^{2^k-1} |(r_k - 1)h(j2^{-k})| \le c'2^{-\nu k}. \tag{2.12}$$

It clearly induces the required result of (a).

(b) For the given Hölder exponent $\nu > 0$ and $\delta > 0$, choose $p = \frac{1}{\nu}$ and an integer $\tau > 0$ such that

$$(2^{-4p})^\tau \le \delta \le (2^{-2p})^\tau.$$

Note that from this inequality we can also obtain,

$$\delta^{\frac{1}{2p}} \le 2^{-\tau} \le \delta^{\frac{1}{4p}} \tag{2.13}$$

Then, by the triangle inequality,

$$I_k'(x+\delta) - I_k'(x)| \le |I_k'(x+\delta) - I_\tau'(x+\delta)| + |I_\tau'(x+\delta) - I_\tau'(x)| + |I_\tau'(x) - I_k'(x)|. \tag{2.14}$$

Assuming $k > \tau$, we apply (2.9) and (2.13) to obtain

$$|I_\tau'(x) - I_k'(x)| \le c2^{-\tau\nu} \le c\delta^{\frac{\nu}{4p}} \le c\delta^{\frac{\nu^2}{4}}, \tag{2.15}$$

which estimates the first and last terms on the right-hand side of (2.14). Next, recalling that $\operatorname{supp} h = [0, 1)$, the summation in (2.10) can be rewritten as follows (with k replaced by τ):

$$I_\tau'(x) = \sum_{j=0}^{2^\tau} [h(j2^{-\tau}) - h((j-1)2^{-\tau})]\phi_b(x - j2^{-\tau}).$$

Therefore,

$$|I_\tau'(x+\delta) - I_\tau'(x)| \le \sum_{j=0}^{2^\tau} |h(j2^{-\tau}) - h((j-1)2^{-\tau})||\phi_b(x+\delta - j2^{-\tau}) - \phi_b(x - j2^{-\tau})|.$$

Here, h is bounded and ϕ_b is Hölder continuous with exponent $\nu > 0$. Thus, due to (2.13) and the fact that $p = 1/\nu$, we get

$$|I_\tau'(x+\delta) - I_\tau'(x)| \le c2^\tau\delta^\nu \le c2^{-\tau} \le c\delta^{\frac{\nu}{4}}.$$

Finally combining this bound with (2.14) and (2.15), we conclude that

$$|I_k'(x+\delta) - I_k'(x)| \le c\delta^{\frac{\nu^2}{4}}$$

Taking $\mu := \frac{\nu^2}{4}$, we finish the proof. $\square$

Lemma 2.5. *Consider a non-stationary subdivision scheme $\{S_{a^{[k]}}\}$ with Laurent polynomials of the form*

$$a^{[k]}(z) = \frac{1}{2}(1 + r_k z) b^{[k]}(z).$$

Suppose that

$$|1 - r_k| \le c2^{-k}, \quad k \ge K \in \mathbb{Z}_+, \tag{2.16}$$

and that the scheme corresponding to $\{S_{b^{[k]}}\}$ is $C^{L+\nu}$ with $L \in \mathbb{Z}_+$ and $\nu \in (0,1)$. Then $\{S_{a^{[k]}}\}$ is $C^{L+1+\mu}$ for some $\mu \in (0,1)$.

Proof. Due to Lemma 2.3, we find that

$$\phi_a = \int_{\mathbb{R}} \phi_b(\cdot - t) h(t)\, dt,$$

where ϕ_a, ϕ_b and h are the basic limit functions of $\{S_{a^{[k]}}\}$, $\{S_{b^{[k]}}\}$ and $\{S_{1+r_k z}\}$ respectively. Note that h is bounded and $\mathrm{supp}\{h\} = [0,1)$ [10]. It is sufficient to prove the lemma for $\ell = 0$, since

$$D^\ell \phi_a = \int_{\mathbb{R}} D^\ell \phi_b(\cdot - t) h(t)\, dt.$$

To this end, invoking the definition of the function I_k in (2.7), we find that $I_k(x) \to \phi_a(x)$ uniformly as $k \to \infty$. Further, according to Lemma 2.4 (a), $\{I_k'\}$ is uniformly convergent, which means that the limit of $\{I_k'\}$ is continuous and that is ϕ_a'. Using this fact, we can conclude from Lemma 2.4 (b) that ϕ_a' is Hölder continuous with exponent $\mu > 0$. It completes the proof. $\square$

We now show that Condition A on $\{S_{a^{[k]}}\}$ implies the condition (2.16) for all the factors in the representation (2.4). To prove this we need the following two lemmas. Without loss of generality, we rearrange the set $r_{k,n}$ in (2.4) such that

$$|1 - r_{k,n}| = \max\{|1 - r_{k,\ell}| : \ell = n, \cdots, M\}, \quad n = 1, \cdots, M, \tag{2.17}$$

that is, $|1 - r_{k,n}| \ge |1 - r_{k,n+1}|$. The following lemma shows that if Condition A is satisfied, $|1 - r_{k,n}| \le c2^{-k}$. For this proof, we use the notation

$$\{x_k\} \asymp \{y_k\}$$

for two sequences of non-zero reals, if there exist some constants $c_1, c_2 > 0$ such that $c_1 \le |x_k\, y_k^{-1}| \le c_2$ for all k.

Lemma 2.6. *Suppose that Condition A holds for the scheme $\{S_{a^{[k]}}\}$. Then*

$$|1 - r_{k,n}| \le c2^{-k}, \quad k \ge K \in \mathbb{Z}_+. \tag{2.18}$$

Proof. Denote $|1 - r_{k,1}| =: \omega_k$. Since $|1 - r_{k,n}| \le \omega_k$ for any $n \le M$, it is sufficient to show that $\sup_k |2^k \omega_k| \le c$ for a constant $c > 0$. Now, suppose that $\sup_k |2^k \omega_k| = \infty$, which means that there exists a sequence $\{k_\ell\}$ such that

$$|2^{k_\ell} \omega_{k_\ell}| \le |2^{k_{\ell+1}} \omega_{k_{\ell+1}}| \to \infty, \quad \text{as} \quad k_\ell \to \infty. \tag{2.19}$$

Then, recalling $|1-r_{k,n+1}| \le |1-r_{k,n}|$, we will derive a contradiction by considering the following two cases:

Case 1: $\{\omega_{k_\ell}\} \asymp \{|1-r_{k_\ell,n}|\}$, for $n = 1, \cdots, M$.

In this case, it is clear from (2.4) that $\{a_{k_\ell}(-1)\} \asymp \{\omega_{k_\ell}^M\}$. By Condition A, $|a_{k_\ell}(-1)| \le c2^{-k_\ell M}$, we get the bound $|2^{k_\ell}\omega_{k_\ell}| \le c$ for any k_ℓ, in contradiction to (2.19).

Case 2: $\{\omega_{k_\ell}\} \asymp \{|1-r_{k_\ell,n}|\}$, for $n = 1, \cdots, s < M$.

That is, there exists a subsequence $\{k_j\} \subset \{k_\ell\}$ such that for any $n > s$, $|1-r_{k_j,n}|\omega_{k_j}^{-1} \to 0$ as $k_j \to \infty$, i.e.,

$$|1-r_{k_j,n}| = o(\omega_{k_j}), \quad n > s. \tag{2.20}$$

Then we use the lemma:

Lemma 2.7. *Let*

$$F_{k_j}(z) := \prod_{n=1}^{M} \frac{1}{2}(1 + r_{k_j,n}z).$$

Under the condition of Case 2, we have

$$\{F_{k_j}^{(M-s)}(-1)\} \asymp \{\omega_{k_j}^s\} \quad \text{and} \quad |F_{k_j}^{(M-s-\ell)}(-1)| = o(\omega_{k_j}^{s+\ell}), \quad \forall \ell > 0.$$

Proof. For the given $s < M$, denote $I_s := \{1, 2, \cdots, s\}$ and let Λ_s be the collection of all subsets of $\{1, 2, \cdots, M\} = I_M$ with cardinality s, i.e.,

$$\Lambda_s := \{I \subset I_M \ : \ \#I = s\}.$$

Then,

$$F_{k_j}^{(M-s)}(-1) \;=\; \left(\prod_{n\in I_s}(1-r_{k_j,n}) + \sum_{I\in\Lambda_s\setminus I_s}\prod_{n\in I}(1-r_{k_j,n})\right)\left(\frac{1}{2^M} + o(1)\right). \tag{2.21}$$

Since $|1-r_{k_j,n}| \ge |1-r_{k_j,n+1}|$,

$$\left\{\prod_{n\in I_s}|1-r_{k_j,n}|\right\} \asymp \{\omega_{k_j}^s\} \quad \text{and} \quad \prod_{n\in I}|1-r_{k_j,n}| = o(\omega_{k_j}^s).$$

Thus,

$$\{F_{k_j}^{(M-s)}(-1)\} \asymp \{\omega_{k_j}^s\}.$$

In a similar way, we can prove the relation $|F_{k_j}^{(M-s-\ell)}(-1)| = o(\omega_{k_j}^{s+\ell})$ for all $\ell > 0$. □

Now, we turn to the proof of Lemma 2.6 in Case 2. It follows from (2.4) that for some suitable constants c_ℓ with $\ell = 0, \cdots, M-s$, we have

$$\begin{aligned} a_{k_j}^{(M-s)}(-1) &= \sum_{\ell=0}^{M-s}\binom{2m-s}{\ell} b_{k_j}^{(\ell)}(-1)F_{k_j}^{(M-s-\ell)}(-1) \\ &= b_{k_j}(-1)F_{k_j}^{(M-s)}(-1) + \sum_{\ell=1}^{M-s}\binom{2m-s}{\ell} b_{k_j}^{(\ell)}(-1)F_{k_j}^{(M-s-\ell)}(-1). \end{aligned} \tag{2.22}$$

Since $b_{k_j}(-1) = c + o(1)$ with a constant $c \neq 0$, the identity (2.22) leads to $\{a_{k_j}^{(M-s)}(-1)\} \asymp \{\omega_{k_j}^s\}$ by Lemma 2.7. Also, from Condition A, $|a_{k_j}^{(M-s)}(-1)| \le c2^{-k_j s}$, yielding $|2^{k_j}\omega_{k_j}| \le c$ for any k_j, in a contradiction to (2.19). (Here c is a generic constant). □

We are now ready to provide the main theorem of this section.

Theorem 2.8. (Smoothing factor in non-stationary subdivision schemes). *Consider a non-stationary subdivision scheme $\{S_{a^{[k]}}\}$ satisfying Condition A. If*

$$a^{[k]}(z) = \frac{1}{2}(1 + r_k z)c^{[k]}(z), \quad k > K \in \mathbb{Z}_+,$$

where $\{S_{c^{[k]}}\}$ is of compact support and $C^{L+\nu}$ with $L \in \mathbb{Z}_+$ and $\nu \in (0, 1)$, then $\{S_{a^{[k]}}\}$ is $C^{L+1+\mu}$ for some $\mu \in (0, 1)$.

Proof. From Lemma 2.6 and Lemma 2.5, the proof is immediate. □

For interpolatory schemes, we have the stronger result.

Theorem 2.9. *Let $\{S_{a^{[k]}}\}$ be a non-stationary interpolatory subdivision scheme satisfying Condition A. Assume that $\{S_{a^{[k]}}\}$ is asymptotically equivalent to a stationary subdivision scheme $\{S_a\}$. Then if $\{S_a\}$ is $C^{L+\nu}$ with $L \in \mathbb{Z}_+$ and $\nu \in (0, 1)$, then $\{S_{a^{[k]}}\}$ is $C^{L+\mu}$ for some $\mu \in (0, 1)$.*

Proof. Assume that $\{S_a\}$ is $C^{L+\nu}$. Since S_a is interpolatory $a(z) = 2^{-L}(1 + z)^L c(z)$ with $\{S_c\}$ a C^ν with $\nu \in (0, 1)$ ([5]). From the fact that $\{S_a\}$ and $\{S_{a^{[k]}}\}$ are asymptotically equivalent, we conclude that $L < M$, and we can write

$$a^{[k]}(z) = \prod_{n=1}^{L} \frac{1}{2}(1 + r_{k,n} z)c^{[k]}(z)$$

with $\{S_{c^{[k]}}\}$ symptotically equivalent to $\{S_c\}$. By [8], the scheme $\{S_{c^{[k]}}\}$ is Hölder continuous with some positive exponent. From Condition A and Lemmas 2.5, 2.6, we conclude that $\{S_{a^{[k]}}\}$ is $C^{L+\mu}$ with $\mu \in (0, 1)$. □

In what follows, we use the results of this section to analyze a new family of interpolatory schemes.

3. Subdivision Schemes based on Gaussian-interpolation and their Analysis

3.1 Construction

Radial basis function (RBF) interpolation is a very strong and convenient tool for interpolation in the multivariate setting ([4, 12, 13, 15]). In this section, we apply RBF interpolation in the univariate setting to construct interpolatory subdivision schemes. Given data $(x_j, f(x_j))$, $j = 1, \cdots, n$, where $X := \{x_1, \cdots, x_N\}$ is a subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, we consider interpolants to the data of the form

$$\mathcal{R}_{f,X}(x) := \sum_{n=1}^{N} \alpha_n G(x - x_n), \tag{3.1}$$

where G is the Gaussian function

$$G(x) = e^{-|x|^2/\lambda^2},$$

with λ a parameter (λ can serve as a shape parameter in the resulting subdivision scheme). The coefficients $\alpha_1, \cdots, \alpha_N$ are determined by the interpolation condition

$$\mathcal{R}_{f,X}(x_j) = f(x_j), \quad j = 1, \cdots, N. \tag{3.2}$$

It is well-known ([13]) that the linear system (3.2) is non-singular for any choice of X consisting of distint points. The interpolant $\mathcal{R}_{f,X}$ in (3.1) has a Lagrange-type representation:

$$\mathcal{R}_{f,X}(x) := \sum_{n=1}^{N} u_n(x) f(x_n), \quad u_n(x_\ell) = \delta_{n,\ell}, \tag{3.3}$$

where u_n are the Lagrange functions from the space $G_X := \mathrm{span}\{G(\cdot - x_1), \cdots, G(\cdot - x_N)\}$. The coefficients $u_n(x)$, $n = 1, \cdots, N$, can be obtained as the solution of the linear system

$$\sum_{n=1}^{N} u_n(x) G(x_n - x_\ell) = G(x - x_\ell), \quad \ell = 1, \cdots, N. \tag{3.4}$$

We study interpolatory subdivision schemes based on interpolation at symmetric $2m$-points to the inserted point. By (3.4) and since $G(x) = G(-x)$, the subdivision schemes considered are non-stationary and uniform in the sense that their refinement rules depend on the level of refinement but are the same everywhere on the same level. Let

$$X_{k,j} := \{(j+n)2^{-k} : n = -m+1, \cdots, m\},$$

which is the local set of symmetric $2m$-points around $(j + 2^{-1})2^{-k}$. Then, the value f^{k+1}_{2j+1} is defined by the Gaussian-based interpolation to the data $\{(j+n)2^{-k}, f^k_{j+n}) : n = -m+1, \cdots, m\}$, denoted by $\mathcal{R}_{k,j}$. Thus,

$$\begin{aligned} f^{k+1}_{2j+1} &= \mathcal{R}_{k,j}(2^{-k}(j + 2^{-1})) \\ &= \sum_{n=-m+1}^{m} u_n^{[k,j]}(2^{-k}(j + 2^{-1})) f^k((j+n)2^{-k}) \end{aligned}$$

with the Lagrange function $u_n^{[k,j]}$ as in (3.4). Here and in the sequel, we use the notation

$$X_0 := X_{0,0} := \{-m+1, \cdots, m\}. \tag{3.5}$$

It is easy to verify from (3.4) that the $u_n^{[k,j]}(2^{-k}(j + 2^{-1}))$ with $n \in X_0$ are independent of the location j. Thus we can define

$$a^{[k]}_{1-2n} := u_n^{[k,j]}(2^{-k}(j + 2^{-1})), \quad j \in \mathbb{Z}, \tag{3.6}$$

and the mask at level k of the $2m$-point Gaussian-based interpolatory subdivision scheme by

$$a^{[k]}_{1-2n} := u_n^{[k,0]}(2^{-k-1}), \quad a^{[k]}_{2n} = \delta_{n,0}, \quad n \in X_0, \ k \in \mathbb{Z}_+. \tag{3.7}$$

Note that by construction,

$$\sum_{n \in X_0} a^{[k]}_{1-2n} G((n - \ell)2^{-k}) = G((2^{-1} - \ell)2^{-k}), \quad \ell \in X_0. \tag{3.8}$$

We denote the non-stationary scheme with mask defined in (3.7) by $\{S^G_{a^{[k]}}\}$. To study the convergence and smoothness of $\{S^G_{a^{[k]}}\}$, we use the results of section 2 and compare $\{S^G_{a^{[k]}}\}$ with the $2m$-point Deslauriers-Dubuc interpolatory subdivision scheme, which we denote by $\{S_a\}$.

The $2m$-point Deslauriers-Dubuc interpolatory subdivision scheme defines the values at the inserted point by using polynomial interpolation of degree $2m-1$ through the symmetric $2m$-points. Define the Lagrange polynomials on the set X_0 in (3.5) by

$$L_n(x) = \prod_{\substack{\ell \neq n \\ \ell \in X_0}} \frac{x-\ell}{n-\ell}, \quad n \in X_0. \tag{3.9}$$

It is obvious that $L_n(\ell) = \delta_{n,\ell}$ with $\ell \in X_0$. Then, the mask of the $2m$-point Deslauriers-Dubuc interpolatory subdivision scheme is given by

$$a_{2n} = \delta_{0,n}, \quad a_{1-2n} \ := \ L_n(2^{-1}), \quad n \in X_0. \tag{3.10}$$

One should keep in mind that S_a reproduces polynomials of degree $\leq 2m-1$. In particular, for any $k \in \mathbb{Z}_+$,

$$p(2^{-k-1}) = \sum_{n \in X_0} a_{1-2n} p(n2^{-k}), \quad p \in \Pi_{<2m}. \tag{3.11}$$

where $\Pi_{<n}$ stands for the space consisting of all univariate algebraic polynomials of degree less than n.

3.2 Analysis of Convergence

The goal of this section is to prove that the $2m$-point Gaussian-based interpolatory subdivision scheme $\{S^G_{a^{[k]}}\}$ is asymptotically equivalent to the $2m$-point Deslauriers-Dubuc interpolatory scheme $\{S_a\}$, which implies that $\{S^G_{a^{[k]}}\}$ is convergent ([8]).

Theorem 3.1. *Let $\{a^{[k]}_n\}$ be the mask at level k of $\{S^G_{a^{[k]}}\}$, and let $\{a_n\}$ be the mask of $\{S_a\}$. Then, there exists a constant $c_{2m} > 0$ such that*

$$\max_{n \in \mathbb{Z}} |a^{[k]}_n - a_n| \leq c_{2m} 2^{-2k} \quad k \geq K \in \mathbb{Z}_+.$$

Proof. Since $a_{2n} = a^{[k]}_{2n} = \delta_{n,0}$, we need to estimate only the difference $a_{1-2n} - a^{[k]}_{1-2n}$. Recall from (3.7) that $a^{[k]}_{1-2n} = u^{[k,0]}_n(2^{-k-1})$, $n \in X_0$, with $u^{[k,0]}_n$ the Lagrange function of the Gaussian-based interpolation on $X_{k,0} = \{\ell 2^{-k} : \ell \in X_0\}$, satisfying

$$u^{[k,0]}_n(2^{-k}\ell) = \delta_{n,\ell}, \quad \ell \in X_0. \tag{3.12}$$

Further, since $u^{[k,0]}_n(x) \in \operatorname{span}\{G(\cdot - \ell 2^{-k}) : \ell \in X_0\}$, there exist constants $\alpha^{[k]}_\ell$, $\ell \in X_0$, such that

$$u_{k,n}(x) := u^{[k,0]}_n(2^{-k}x) = \sum_{\ell \in X_0} \alpha^{[k]}_\ell G(2^{-k}(x-\ell)), \tag{3.13}$$

yielding $u_{k,n}(\ell) = \delta_{n,\ell}$ for any $\ell \in X_0$. Thus, $u_{k,n}(x)$ can be considered as the RBF interpolant to the data $\{\delta_{n,\ell} : \ell \in X_0\}$ on X_0 by $G(2^{-k}\cdot)$. On the other hand, the mask of Deslauriers-Dubuc scheme is given by $a_{1-2n} = L_n(2^{-1})$, where the function $L_n(x)$ is also a polynomial interpolant to the data $\{\delta_{n,\ell} : \ell \in X_0\}$ on X_0, which means

$$u_{k,n}(\ell) = L_n(\ell) = \delta_{n,\ell}, \quad \ell \in X_0.$$

Very recently, it has been proved in [16] that the Gaussian interpolant of the form $u_{k,n}(x)$ converges uniformly to the polynomial interpolant $L_n(x)$ as $k \to \infty$, with the convergence rate $O(2^{-2k})$. In particular,

$$|u_{k,n}(2^{-1}) - L_n(2^{-1})| = O(2^{-2k}).$$

Thus, since $a^{[k]}_{1-2n} = u_{k,n}(2^{-1})$ and $a_{1-2n} = L_n(2^{-1})$, we arrive at the required conclusion. $\square$

Corollary 3.2. *The scheme $\{S^G_{a^{[k]}}\}$ is convergent and is C^0.*

3.3 Analysis of Smoothness

First, we give a simple proof of C^1 smoothness based on a result from [8]:

Result A: The scheme $\{S_{a^{[k]}}\}$ is in C^γ if a scheme $\{S_a\}$ is in C^γ and

$$\sum_{k\in\mathbb{Z}_+} 2^{\gamma k}\|S_a - S_{a^{[k]}}\|_\infty < \infty. \tag{3.14}$$

Theorem 3.3. *Let $\{S^G_{a^{[k]}}\}$ be the $2m$-point Gaussian-based interpolatory subdivision scheme. Then, if $m \geq 2$, $\{S^G_{a^{[k]}}\}$ is at least C^1.*

Proof. Let S_a be the $2m$-point Deslauriers-Dubuc interpolatory scheme, it is clear from Theorem 3.1 that

$$\sum_{k\in\mathbb{Z}_+} 2^{k}\|S_a - S^G_{a^{[k]}}\|_\infty < \infty.$$

Since S_a is at least C^1 for $m \geq 2$, the C^1-smoothenss of $\{S^G_{a^{[k]}}\}$ is an immediate consequence of Result A. $\square$

Next, we show that if the $2m$-point Deslauriers-Dubuc interpolatory scheme $\{S_a\}$ is $C^{L+\nu}$ with $L \in \mathbb{Z}_+$ and $\nu \in (0, 1)$, then the non-stationary $2m$-point Gaussian-based interpolatory subdivision scheme $\{S^G_{a^{[k]}}\}$ is $C^{L+\mu}$ for some $\mu \in (0, 1)$. It implies that both $2m$-point schemes have the same integer smoothness as the $2m$-point Deslauriers-Dubuc interpolatory scheme. This is proved by Theorem 2.9 and by verifying first that $\{S^G_{a^{[k]}}\}$ satisfies Condition A of section 2. Our proof relies on the following property of the Gaussian function [11]

$$\det\left(G^{(\ell+n)}(0)\right) \neq 0, \quad \ell, n = 0, \cdots, 2m-1 \tag{3.15}$$

and on the three auxiliary lemmas.

Lemma 3.4. *Let $T_{G^{(\ell)}}$ be the Taylor polynomial of $G^{(\ell)}$ of degree $2m-1$ around zero, i.e.,*

$$T_{G^{(\ell)}}(x) := \sum_{n=0}^{2m-1} \frac{x^n}{n!} G^{(\ell+n)}(0). \tag{3.16}$$

Then, $T_{G^{(\ell)}}$, $\ell = 0, \cdots, 2m-1$, are linearly independent.

Proof. It is sufficient to prove that for any distinct points $t_0, \cdots, t_{2m-1}$, the $2m \times 2m$ matrix $\mathbf{T}$ with entries

$$\mathbf{T}(\ell, n) = T_{G^{(\ell)}}(t_n), \quad \ell, n = 0, \cdots, 2m-1$$

is non-singular. We see that the matrix $\mathbf{T}$ can be decomposed as

$$\mathbf{T} = \mathbf{B} \cdot \mathbf{V},$$

where

$$\mathbf{B}(\ell, n) = G^{(\ell+n)}(0) \quad \text{and} \quad \mathbf{V}(\ell, n) = t_n^\ell/\ell!.$$

Since both matrices $\mathbf{B}$ and $\mathbf{V}$ are invertible, the non-singularity of $\mathbf{T}$ is immediate. $\square$

To prove the next lemma, we recall that the $(n-1)$-th order *divided difference* of a function $f \in C^{n-1}$ at the points $(-m+1), \cdots, (-m+n)$ is given by

$$(n-1)! f[-m+1, \cdots, -m+n] = \sum_{\alpha=1}^{n-1} c_{n,\alpha} f(-m+\alpha) = f^{(n-1)}(\xi) \tag{3.17}$$

with $\xi \in [-m+1, -m+n]$ and where

$$c_{n,\alpha} := (n-1)! \prod_{\substack{j \neq \alpha \\ j=1}}^{n} \frac{1}{\alpha - j}, \quad \alpha = 1, \cdots, n \tag{3.18}$$

Lemma 3.5. *Let $\mathbf{P}_k$ be the $2m \times 2m$ matrix with entries*

$$\mathbf{P}_k(\ell, n) := \mathbf{P}_{k,x}(\ell, n) := G^{(\ell)}(x - (n-m+1)2^{-k}), \tag{3.19}$$

where $\ell, n = 0, \cdots, 2m-1$. Then there exist $\eta > 0$ and $K \in \mathbb{Z}_+$ such that for any $x \in [-\eta, \eta]$,

$$\det \mathbf{P}_k = O(2^{-km(2m-1)}) \quad \text{and} \quad \|\mathbf{P}_k^{-1}\|_\infty = O(2^{(2m-1)k}), \quad k \geq K.$$

Here $\|\mathbf{A}\|_\infty$ indicates the ∞-norm of the matrix $\mathbf{A}$.

Proof. Denote by $\mathbf{p}_n$, $n = 1, \cdots, 2m$, be the column vectors of the matrix $\mathbf{P}_k$. Since the determinant of a matrix is invariant under elementary column operations, we perform the following column operations:

$$\mathbf{p}'_n := \mathbf{p}_n + \sum_{\alpha=1}^{n-1} c_{n,\alpha} \mathbf{p}_\alpha, \quad n = 2m, 2m-1, \cdots, 1,$$

with $c_{n,\alpha}$ defined as in (3.17). Defining $\mathbf{P}'_k$ to be the matrix with columns $(\mathbf{p}'_1, \cdots, \mathbf{p}'_{2m})$, we observe that $\det(\mathbf{P}_k) = \det(\mathbf{P}'_k)$. Further, applying (3.17), we get

$$\mathbf{P}'_k(\ell, n) = 2^{-nk} G^{(\ell+n)}(x - \xi_{\ell,n} 2^{-k}), \tag{3.20}$$

with $0 \leq \ell, n \leq 2m-1$ and $\xi_{\ell,n} \in [-m+1, -m+1+n]$. Thus, from (3.15), we can deduce that there exist $\eta > 0$ and $K \in \mathbb{Z}_+$ such that $\det \mathbf{P}_k = O(2^{-km(2m-1)}) \neq 0$ for any $x \in [-\eta, \eta]$ and $k \geq K$. Further, a direct calculation from (3.20) easily leads to the estimate $\|\mathbf{P}_k^{-1}\|_\infty = O(2^{(2m-1)k})$ as $k \to \infty$. $\square$

For any $\beta = 0, \cdots, 2m-1$, define the function

$$\Phi_{k,\beta}(x) := \sum_{n \in X_0} g_\beta^{[k]}(n) G(x - 2^{-k} n) \tag{3.21}$$

so that the coefficient vector $\mathbf{g}_\beta^{[k]} := (g_\beta^{[k]}(n) : n \in X_0)$ is obtained by solving the linear system

$$\Phi_{k,\beta}^{(\ell)}(2^{-k-1}) = \delta_{\beta,\ell}(-1)^\ell \ell! \tag{3.22}$$

which can be written in the matrix form

$$\mathbf{P}_k \cdot \mathbf{g}_\beta^{[k]} = \mathbf{c}.$$

where $\mathbf{P}_k = \mathbf{P}_{k,2^{-k-1}}$ with $\mathbf{P}_{k,x}$ defined in (3.19) and $\mathbf{c}(\ell) := \delta_{\beta,\ell}(-1)^\ell \ell!$ with $\ell = 0, \cdots, 2m-1$. The following estimates are central to the proof that $\{S_{a^{[k]}}^G\}$ satisfies Conditon A.

Lemma 3.6. *For all $\beta = 0, \cdots, 2m-1$,*

$$(i)\ \|\mathbf{g}_\beta\|_\infty = O(2^{k(2m-1)}) \quad \text{as} k \to \infty, \tag{3.23}$$
$$(ii)\ \|\Phi_{k,\beta}^{(\ell)}(2^{-k-1})| \leq c, \quad \ell = 2m, \cdots, 4m-2.$$

Proof. Since $\mathbf{g}_\beta^{[k]} = \mathbf{P}_k^{-1} \cdot \mathbf{c}$, the estimate $\|\mathbf{g}_\beta^{[k]}\|_\infty = O(2^{k(2m-1)})$ follows immediately from Lemma 3.5. Next, recall that $T_{G^{(\ell)}}$ indicates the Taylor polynomial of degree $2m-1$ of $G^{(\ell)}$ around zero with $\ell = 2m, \cdots, 4m-2$. By Lemma 3.4, there exist some suitable constants $\gamma_{\ell,0}, \cdots, c\gamma_{\ell,2m-1}$ such that $T_{G^{(\ell)}} =: \sum_{\alpha=0}^{2m-1} \gamma_{\ell,\alpha} T_{G^{(\alpha)}}$. Thus, we get from (3.21),

$$\begin{aligned}
\Phi_{k,\beta}^{(\ell)}(2^{-k-1}) &= \sum_{n \in X_0} g_\beta^{[k]}(n) G^{(\ell)}(2^{-k-1} - n2^{-k}) \\
&= \sum_{n \in X_0} g_\beta^{[k]}(n) \big(T_{G^{(\ell)}}(2^{-k-1} - n2^{-k}) + O(2^{-2mk})\big) \\
&= \sum_{\alpha=0}^{2m-1} \gamma_{\ell,\alpha} \sum_{n \in X_0} g_\beta^{[k]}(n) \big[T_{G^{(\alpha)}}(2^{-k-1} - n2^{-k}) + O(2^{-2mk})\big].
\end{aligned}$$

Note that

$$T_{G^{(\alpha)}}(2^{-k-1} - n2^{-k}) = G^{(\alpha)}(2^{-k-1} - n2^{-k}) + O(2^{-2mk}), \quad \alpha = 0, \cdots, 2m-1.$$

Applying this identity to (3.21), we get in view of (3.23)

$$\begin{aligned}
\Big|\sum_{n \in X_0} g_\beta^{[k]}(n) T_{G^{(\ell)}}(2^{-k-1} - n2^{-k})\Big| &\leq \big|\Phi_{k,\beta}^{(\ell)}(2^{-k-1})\big| + O(2^{-k}) \\
&\leq \ell! + O(2^{-k}),
\end{aligned}$$

as a consequence of (3.22). Also, by (i), the property $|\Phi_{k,\beta}^{(\ell)}(2^{-k-1})| \leq c$ for $\ell = 2m, \cdots, 4m-2$ is obvious. □

Now, we are ready to prove that $\{S_{a^{[k]}}^G\}$ satisfies Condition A with $M = 2m$.

Theorem 3.7. *Let $a^{[k]}(z) = \sum_{n \in \mathbb{Z}} a_n^{[k]} z^n$ be the Laurent polynomial at level k associated with $\{S_{a^{[k]}}^G\}$. Then, for any $\beta = 0, \cdots, 2m-1$, we have*

$$|D^\beta a^{[k]}(-1)| \leq c2^{-k(2m-\beta)}, \quad k \geq K$$

for some $K \in \mathbb{Z}_+$.

Proof. Since

$$D^\beta a^{[k]}(-1) = \sum_{\ell=0}^{\beta} \gamma_{\beta,\ell} \sum_{n \in \mathbb{Z}} a_n^{[k]} n^\ell (-1)^n$$

for some constants $\gamma_{\beta,\ell}$, $\ell = 0, \cdots, \beta$, it is sufficient to prove that for any $\beta = 0, \cdots, 2m-1$,

$$s_\beta := \sum_{n \in \mathbb{Z}} (-1)^n n^\beta a_n^{[k]} = O(2^{-k(2m-\beta)}), \quad k \to \infty,$$

in order to conclude Condition A. Recalling that $a_{2n}^{[k]} = \delta_{n,0}$, observe that

$$2^{-\beta(k+1)} s_\beta = \delta_{\beta,0} - \sum_{n \in X_0} a_{1-2n}^{[k]} \big((-n + 2^{-1})2^{-k}\big)^\beta. \tag{3.24}$$

Invoking (3.8) and (3.22), we get

$$\delta_{\beta,0} = \Phi_{k,\beta}(2^{-k-1}) = \sum_{n\in X_0} a^{[k]}_{1-2n}\Phi_{k,\beta}(n2^{-k}). \tag{3.25}$$

This together with (3.24) lead to

$$2^{-\beta(k+1)}s_\beta = \sum_{n\in X_0} a^{[k]}_{1-2n}\Big(\Phi_{k,\beta}(n2^{-k}) - ((-n+2^{-1})2^{-k})^\beta\Big). \tag{3.26}$$

In the following, we replace $\Phi_{k,\beta}(n2^{-k})$ by its Taylor polynomial of degree $4m-2$ plus the remainder term. The Taylor expansion of $\Phi_{k,\beta}$ around 2^{-k-1} of degree $4m-2$ is

$$\Phi_{k,\beta}(n2^{-k}) = \sum_{\ell=0}^{4m-2}\big((n-2^{-1})2^{-k}\big)^\ell \frac{\Phi^{(\ell)}_{k,\beta}(2^{-k-1})}{\ell!} + R_{k,\beta,m} \tag{3.27}$$

where the remainder $R_{k,\beta,m}$ is given by

$$R_{\Phi_\beta,4m-1} = \big((n-2^{-1})2^{-k}\big)^{4m-1}\frac{\Phi^{(4m-1)}_{k,\beta}(\xi 2^{-k})}{(4m-1)!}$$

with ξ a point between 2^{-1} and n. Noting that by (3.22)

$$\sum_{\ell=0}^{2m-1}\big((n-2^{-1})2^{-k}\big)^\ell\frac{\Phi^{(\ell)}_{k,\beta}(2^{-k-1})}{\ell!} = \Big((-n+2^{-1})2^{-k}\Big)^\beta,$$

we get from (3.25) and (3.26)

$$2^{-\beta(k+1)}s_\beta = \sum_{n\in X_0} a^{[k]}_{1-2n}\Big(\sum_{\ell=2m}^{4m-2}\big((n-2^{-1})2^{-k}\big)^\ell\frac{\Phi^{(\ell)}_{k,\beta}(2^{-k-1})}{\ell!} + R_{k,\beta,m}\Big).$$

By Lemma 3.6, $|\Phi^{(\ell)}_{k,\beta}(2^{-k-1})| \le c$ for any $\ell = 2m,\cdots,4m-2$. Consequently,

$$\Big|\sum_{n\in\mathbb{Z}}(-1)^n n^\beta a^{[k]}_n\Big| \le c2^{-k(2m-\beta)}\Big(1 + 2^{-k(2m-1)}\|\Phi^{(4m-1)}_{k,\beta}\|_{L_\infty[-\eta,\eta]}\Big). \tag{3.28}$$

Since $\|\mathbf{g}_\beta\|_\infty = O(2^{k(2m-1)})$ and $G^{(4m-1)}$ is bounded,

$$2^{-k(2m-1)}\|\Phi^{(4m-1)}_\beta\|_{L_\infty[-\eta,\eta]} \le c,$$

which completes the proof of theorem. □

We are now ready to state and prove the main theorem of this section.

Theorem 3.8. *If the $2m$-point Deslauriers-Dubuc interpolatory scheme $\{S_a\}$ is $C^{L+\nu}$ with $L \in \mathbb{Z}_+$ and $\nu \in (0,1)$, then the non-stationary $2m$-point Gaussian-based interpolatory subdivision scheme $\{S^G_{a^{[k]}}\}$ is $C^{L+\mu}$ for some $\mu \in (0,1)$.*

Proof. Due to Theorem 3.1, both $2m$-point schemes $\{S^G_{a^{[k]}}\}$ and $\{S_a\}$ are asymptotically equivalent. As a consequence of Theorem 3.7 and Theorem 2.9, this theorem is immediate. □

Corollary 3.9. *The non-stationary $2m$-point Gaussian-based interpolatory subdivision scheme has the same integer smoothness as the $2m$-point Deslauriers-Dubuc interpolatory scheme.*

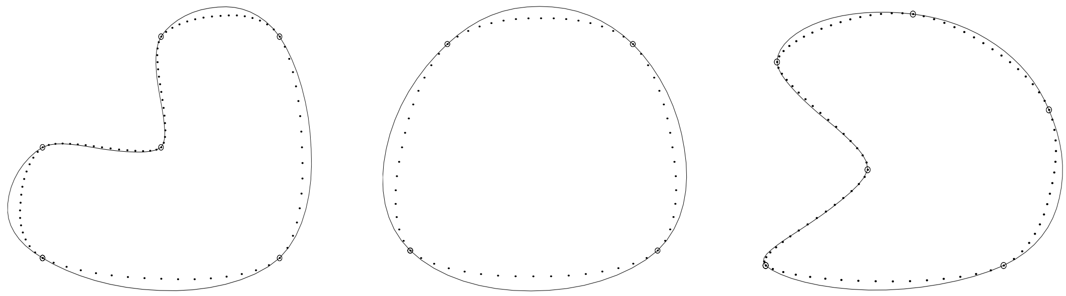

FIGURE 4.1: Interpolating curves generated by the 4-point Gaussian-based interpolation with $\lambda^{-1} = .5$, (solid) and the 4-point DD scheme (dotted).

Remark: In this section, the Gaussian function has been used. We believe that the same results can be obtained by using other smooth radial basis functions such as multiquadrics.

4. Examples

In this section, we illustrate the performance of the 4-point Gaussian-based interpolatory subdivision schemes with some numerical examples. Recall that the Gaussian function is of the form

$$G(x) = e^{-|x|^2/\lambda^2}, \quad \lambda > 0.$$

Here λ serves as a shape parameter. Having tried several alternatives for the parameter λ, we found out that good choices of λ are in the range $0 < \lambda^{-1} \leq 1.0$. The solid curves in Figure 4.1 are generated by a 4-point Gaussian-based interpolatory subdivision scheme using the parameter $\lambda^{-1} = .5$. The dotted curves in Figure 4.1 are generated by the 4-point Deslauriers-Dubuc scheme. It is known that the 4-point Deslauriers-Dubuc scheme has the smoothness C^1. Hence, due to Theorem 3.8, the 4-point Gaussian-based interpolatory subdivision scheme is also C^1. On the other hand, the 4-point Gaussian-based scheme is a first step towards the design and analysis of bivariate Gaussian-based schemes extending the interpolatory butterfly scheme. However, this requires much heavier analysis, especially at extraordinary points.

In addition, the Gaussian-based schemes have an advantage over polynomial based schemes, especially in signal processing. The following example shows an application of the 4-point Gaussian based scheme as compared to the 4-point Deslauriers-Dubuc scheme. We approximate oscillatory signals of the form [3]

$$f(t) = \cos(2\pi F t + \beta \sin(2\pi F_s t)).$$

We choose $F = 0.5$, $\beta = 5.75$, $F_s = 0.0062$ and use 311 data samples in the domain $[0, 30]$ as initial data for the subdivision. The solid curve in Figure 4.2 indicates the approximation errors by the 4-point Gaussian-based schemes as a function of λ. The dotted line is the error by the Deslauriers-Dubuc 4-point scheme. We find that by choosing a suitable parameter λ in the Gaussian $g(x) := e^{-x^2/\lambda^2}$ (around $\lambda = 2\pi$ in this example), the 4-point Gaussian-based scheme provides much better accuracy than the polynomial based 4-point scheme. For this reason, a future project would be to find an algorithm for choosing the appropriate λ for a given signal.

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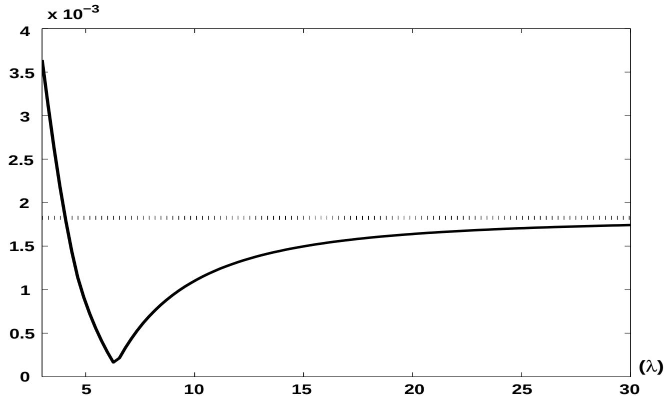

FIGURE 4.2: Approximation errors by the 4-point Gaussian-Based Scheme as a function of λ. The dotted line indicates the error by the 4-point Deslauriers-Dubuc scheme.

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