

Capacity of the Two-Hop Relay Channel with Wireless Energy Transfer from Relay to Source and Energy Transmission Cost

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Abstract

In this paper, we investigate a communication system comprised of an energy harvesting (EH) source which harvests radio frequency (RF) energy from an out-of-band full-duplex relay node and exploits this energy to transmit data to a destination node via the relay node. We assume two scenarios for the battery of the EH source. In the first scenario, we assume that the EH source is not equipped with a battery and thereby cannot store energy. As a result, the RF energy harvested during one symbol interval can only be used in the following symbol interval. In the second scenario, we assume that the EH source is equipped with a battery having unlimited storage capacity in which it can store the harvested RF energy. As a result, the RF energy harvested during one symbol interval can be used in any of the following symbol intervals. For both system models, we derive the channel capacity subject to an average power constraint at the relay and an additional energy transmission cost at the EH source. We compare the derived capacities to the achievable rates of several benchmark schemes. Our results show that using the optimal input distributions at both the EH source and the relay is essential for high performance. Moreover, we demonstrate that neglecting the energy transmission cost at the source can result in a severe overestimation of the achievable performance.

I. INTRODUCTION

Future wireless communication devices are expected to be powered by harvesting freely available ambient energy, such as solar, thermal, and electro-magnetic, and/or radio frequency (RF) energy transmitted from dedicated wireless energy transmitters [2]. Since energy harvesting (EH) from natural resources is usually climate and location dependent, it may not be suitable for small and

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mobile wireless communication devices. For such devices, wireless energy transfer (WET) from WET transmitters is an appealing solution for providing a perpetual power supply [3]. However, although WET is a very useful technology, due to the high path loss attenuation, the RF energy emitted by the WET transmitter is severely diminished when it is received at EH device [3]. As a result, EH information sources powered by WET are constrained to communicate over short distances and with low data rates. One possible solution to overcome this limitation is to utilize the WET transmitters also as information relays which forward the information received from the EH information sources to the intended destinations. This is appealing since WET transmitters are expected to have a perpetual energy supply which can be used to facilitate both WET and information forwarding (i.e., relaying). A practical example where this architecture might be beneficial is the case where a sensor (i.e., the EH source) is embedded in a concrete wall to monitor the quality of the concrete, and a relay powers up the sensor using WET before relaying the measurement information received from the sensor to a WiFi access-point. In this paper, we investigate the simplest system model for such a scenario comprising an EH source, an out-of-band¹ full-duplex (FD) relay performing WET and information forwarding for the EH source, and a destination, cf. Fig. 1. The FD relay transmits energy to the EH source and information to the destination in one frequency band and receives information from the EH source in another frequency band. Thereby, we consider two scenarios for the battery of the EH source node. In the first scenario, we assume that the EH source is too small to be equipped with a battery and, as a result, cannot store energy. In this case, the RF energy harvested during one symbol interval can only be used in the following symbol interval. In the second scenario, we assume that the EH source is equipped with an “unlimited battery”² in which it can store the harvested RF energy. In this case, the RF energy harvested during one symbol interval can be used in any of the following symbol intervals. Moreover, for both scenarios, we assume that a part of the transmit energy is dissipated at the EH source, and only the remaining part can be used for information transmission, i.e., a non-zero energy transmission cost is incurred. For this relay channel, we investigate the channel capacity for the case when the source-relay and relay-destination channels are both non-fading additive white Gaussian noise (AWGN) channels, and the relay has an average power constraint.

Communication systems with EH and WET have recently attracted significant interest, which has led to the study of different types of channel models, including the point-to-point channel, multiple-access channel, broadcast channel, and relay channel [2]-[21]. In particular, the capacity

¹An out-of-band FD relay receives and transmits signals at the same time but in different frequency bands.

²The term “unlimited battery” is used to denote a battery with unlimited storage capacity.

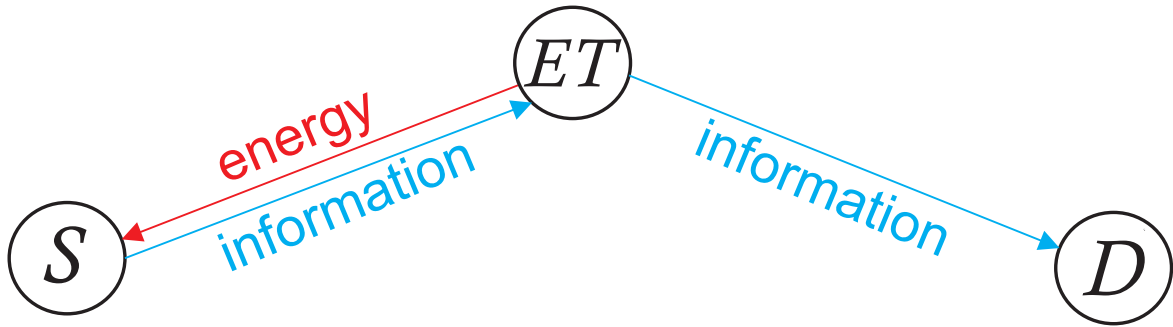


Fig. 1: System model comprised of an EH source (S), a wireless energy transmitter (ET) acting also as a relay, and a destination (D).

of the EH AWGN point-to-point channel³, comprised of an EH source equipped with an unlimited battery and a destination, was derived in [4], [5]. Whereas, the capacity of the EH AWGN point-to-point channel with a batteryless source was derived in [6], where it was shown that the capacity achieving input distribution is discrete and amplitude constrained. Receiver designs for simultaneous wireless information and energy transfer were proposed in [7]. Achievable rates for the EH AWGN multiple-access channel (MAC) were studied in [8], where it was shown that under certain asymptotic conditions the achievable rate converges to the capacity of the non-EH AWGN MAC. Achievable rates for different types of EH AWGN broadcast channels were studied in [9]-[11]. On the other hand, relay channels with EH and WET have been investigated in [12]-[21], where [12]-[14] assumed EH from natural sources, [15]-[17] assumed WET from source to relay, and [18]-[21] investigated WET from relay to source. In particular, in [12], the authors considered cooperative EH communications where, in each time slot, the EH source transmits its data to the destination either directly or with the help of an EH relay. The authors in [13] investigated the outage probability of a general wireless cooperative network where multiple pairs of sources and destinations communicate through a single EH relay. Reference [14] investigated achievable rates for a buffer-aided relay channel comprised of an EH source, an EH buffer-aided relay, and a destination. For relay channels with WET from source to relay, [15] studied the corresponding achievable rates and outage probabilities, [16] derived throughputs for instantaneous transmission, delay-constrained transmission, and delay tolerant transmission, and [17] studied throughput maximization for an FD multi-antenna relay channel where a time-switching protocol was employed by the relay.

On the other hand, [18]-[21] studied achievable rates for relay channels with WET from relay to

³We note that the point-to-point channel and the relay channel differ significantly from an information-theoretic perspective. As a result, the capacity expressions and the corresponding capacity-achieving coding schemes for the point-to-point channel are not directly applicable to the relay channel.

source, which is similar to the relay channel considered in this paper. However, different from the study conducted in this paper, [18]-[21] considered half-duplex relays, did not take into account the energy transmission cost at the EH source, and only derived achievable rates, not capacities. Moreover, the achievable rates presented in [18]-[21] are achieved with heuristic protocols which do not provide insight into the capacity of the considered relay channels.

As can be seen from the discussion above, an explicit characterization of the capacity and the capacity-achieving coding scheme of the considered relay channel with WET is not available in the literature. This motivates the information-theoretic analysis presented in this paper, which offers a more fundamental insight into the limits of communications for the considered relay channel. Our information-theoretic analysis reveals that the capacity is achieved when the transmit signal of the relay is used to simultaneously power the EH source and to convey information to the destination. In addition, we show that the EH source has to be silent in a fraction of the symbol intervals to conserve energy. Moreover, for the case when the EH source is equipped with an unlimited battery, we show that the EH source should use these silent symbol intervals for encoding additional information for the relay. As numerical examples, we compare the derived capacities to the achievable rates of several benchmark schemes. The comparison reveals that using the optimal input distributions at both the EH source and the relay is essential for achieving high performance. Moreover, we illustrate that neglecting the additional energy transmission cost at the EH source can result in a severe overestimation of the achievable performance.

The remainder of this paper is organized as follows. In Section II, we introduce the system and channel models. In Sections III and IV, we present the capacities of the considered relay channel for the cases of a batteryless EH source and an EH source with an unlimited battery, respectively. In Section V, we provide numerical results. Finally, Section VI concludes the paper.

II. SYSTEM AND CHANNEL MODELS

In the following, we formally introduce the considered system and channel models.

A. System Model

We consider a two-hop FD relay channel, comprised of an EH source S , an out-of-band FD relay R , and a destination D , where a direct source-destination link does not exist due to e.g. the large distance/heavy blockage between the source and the destination, cf. Fig. 1. For this relay channel, we assume that the source-relay and relay-destination links are not impaired by fading. Moreover, we assume that the source is an EH node that is powered wirelessly by the RF energy received from the relay, which has an average transmit power constraint denoted by P_R . Let $E_{H,i}$ denote

the harvested energy at the EH source during symbol interval⁴ i . The source uses the harvested RF energy to transmit information to the relay which then forwards the received information to the destination. We assume that the EH source can simultaneously harvest energy and transmit information since the simultaneous reception and transmission occurs in different frequency bands. Similarly, we assume that the WET relay can simultaneously receive information and transmit energy since the simultaneous reception and transmission occurs in different frequency bands. In addition, for the considered relay channel, we assume that the transmission of a symbol with non-zero energy at the EH source incurs an energy transmission cost, denoted by P_C , due to dissipation of the input energy in the transmitter's circuitry. On the other hand, we assume that the EH source can transmit a zero-energy symbol without any additional cost just by staying silent during a given symbol interval, i.e., without applying any input energy to the transmitter's circuitry⁵ [5], [22].

Furthermore, we consider two cases for the battery of the EH source, namely a batteryless EH source and an EH source equipped with an unlimited battery. In the first case, the EH source cannot store the harvested energy. In particular, for the batteryless EH source, the energy harvested during symbol interval $i - 1$, $E_{H,i-1}$, can be used for transmission *only* in symbol interval i . If the energy harvested during symbol interval $i - 1$, $E_{H,i-1}$, is not used during symbol interval i , then we assume that the energy $E_{H,i-1}$ is lost and cannot be used in future symbol intervals. In the second case, the EH source with unlimited battery can store any amounts of harvested energy for unlimited amount of time. As a result, the EH source can drain energy from its battery and use it for transmission of information in any future symbol interval.

Remark 1: By performing an information-theoretic analysis for the two extreme cases in terms of the energy storage capacity of the EH source, namely a batteryless EH source and an EH source with unlimited battery, we obtain channel capacities which constitute lower and upper bounds on the channel capacity for the case when the EH source can store a finite amount of energy.

B. Channel Model

The time-discrete memoryless two-hop FD relay channel is defined by \mathcal{X}_S , \mathcal{X}_R , $\bar{\mathcal{Y}}_R$, $\bar{\mathcal{Y}}_D$, and $p(\bar{y}_R, \bar{y}_D | x_S, x_R)$, where \mathcal{X}_S and \mathcal{X}_R are the input alphabets at the source and the relay, respec-

⁴We use the terms ‘‘symbol interval’’ and ‘‘channel use’’ interchangeably, i.e., during one symbol interval the channel can be used only once for sending one symbol. Hence, one symbol spans one symbol interval. On the other hand, a codeword is comprised of many symbols and thereby spans many symbol intervals.

⁵The adopted constant energy cost model for non-zero symbols was introduced in [22] for conventional (i.e., non-EH) and in [5] for EH communication systems. In general, this energy cost model can be seen as a first order approximation of the actual energy cost incurred in communication systems.

tively, $\bar{\mathcal{Y}}_R$ and $\bar{\mathcal{Y}}_D$ are the output alphabets at the relay and the destination, respectively, and $p(\bar{y}_R, \bar{y}_D | x_S, x_R)$ is the probability distribution on $\bar{\mathcal{Y}}_R \times \bar{\mathcal{Y}}_D$ for given $x_S \in \mathcal{X}_S$ and $x_R \in \mathcal{X}_R$. In symbol interval i , let $X_{S,i}$ and $X_{R,i}$ denote the random variables (RVs) modeling the transmit symbols of the source and the relay, respectively, and let $\bar{Y}_{R,i}$ and $\bar{Y}_{D,i}$ denote the RVs modeling the received symbols at the relay and destination, respectively. Then, $x_{S,i} \in \mathcal{X}_S$, $x_{R,i} \in \mathcal{X}_R$, $\bar{y}_{R,i} \in \bar{\mathcal{Y}}_R$, and $\bar{y}_{D,i} \in \bar{\mathcal{Y}}_D$ are the realizations of $X_{S,i}$, $X_{R,i}$, $\bar{Y}_{R,i}$, and $\bar{Y}_{D,i}$, respectively.

The considered channel is memoryless in the sense that given the input symbols for the i -th channel use, the i -th output symbols are independent from all previous input symbols. As a result, $p(\bar{y}_R^n, \bar{y}_D^n | x_S^n, x_R^n)$, where the notation a^n is used to denote the ordered sequence $a^n = (a_1, a_2, \dots, a_n)$, can be factorized as $p(\bar{y}_R^n, \bar{y}_D^n | x_S^n, x_R^n) = \prod_{i=1}^n p(\bar{y}_{R,i}, \bar{y}_{D,i} | x_{S,i}, x_{R,i})$. Moreover, since the considered two-hop FD relay channel does not have a direct source-destination link, this relay channel belongs to the class of degraded relay channels defined in [23]. As a result, $p(\bar{y}_R, \bar{y}_D | x_S, x_R)$ can also be written as $p(\bar{y}_R, \bar{y}_D | x_S, x_R) = p(\bar{y}_R | x_S, x_R) p(\bar{y}_D | x_R)$, where we have used $p(\bar{y}_D | x_S, x_R, \bar{y}_R) = p(\bar{y}_D | x_R)$.

Let $w \in \{1, 2, \dots, W\}$ be the message that the EH source wants to transmit to the destination. Let us define the encoding function for w at the source for channel use i as

$$x_{S,i} = \begin{cases} g_{S,i}(w, E_{H,i-1}, P_C) & \text{for the batteryless EH source} \\ g_{S,i}(w, E_H^{i-1}, P_C) & \text{for the unlimited battery EH source.} \end{cases} \quad (1)$$

In (1), $x_{S,i}$ is the output of the source's encoder, whereas message w , harvested energy $E_{H,i-1}$ or the sequence of harvested energies E_H^{i-1} , and energy transmission cost P_C are the inputs at the source's encoder. On the other hand, the encoding function at the relay for channel use i is defined as $x_{R,i} = g_{R,i}(\bar{Y}_R^{i-1})$, where $x_{R,i}$ is the output of the relay's encoder for channel use i and the sequence \bar{Y}_R^{i-1} is the input of the relay's encoder. Finally, the decoding function at the destination is defined as $\hat{w} = g_D(\bar{Y}_D^n)$, where n denotes the total number of channel uses, \hat{w} is an estimate of transmitted message w , and \bar{Y}_D^n is the input sequence of the decoder at the destination.

Here, we assume that the source-relay and relay-destination links are real-valued AWGN channels with constant channel gains h_{SR} and h_{RD} , respectively, and noise variances $\bar{\sigma}_R^2$ and $\bar{\sigma}_D^2$, respectively⁶. Since the channel gains h_{SR} and h_{RD} are assumed to be constant, i.e., they do not vary with time,

⁶Similar to [23], as a first step for investigating the capacity of the considered relay channel, we do not consider fading and assume real-valued channel inputs and outputs. Deriving the channel capacity when fading is present is a much more difficult task since the source-relay and relay-destination channel gains vary from one channel use to the next. As a result, the fading case is left for future investigation. Similarly, deriving the capacity for the multi-antenna case, though of high interest, is beyond the scope of this paper.

given sufficient time, the channel gains h_{SR} and h_{RD} can be estimated almost perfectly using pilot symbols, see [24]. Hence, we assume that h_{SR} and h_{RD} are perfectly known at all three nodes and are fixed during the entire transmission. In symbol interval i , let $\bar{Z}_{R,i}$ and $\bar{Z}_{D,i}$ denote the RVs modeling the AWGN at relay and destination, respectively. Consequently, the RVs modeling the received symbols at relay and destination in channel use i , $\bar{Y}_{R,i}$ and $\bar{Y}_{D,i}$, are given by

$$\bar{Y}_{R,i} = h_{SR}X_{S,i} + \bar{Z}_{R,i} \quad \text{and} \quad \bar{Y}_{D,i} = h_{RD}X_{R,i} + \bar{Z}_{D,i}. \quad (2)$$

For notational simplicity and without loss of generality, instead of studying the capacity for the input-output model in (2), we can study instead the capacity using the following input-output model

$$Y_{R,i} = X_{S,i} + Z_{R,i} \quad \text{and} \quad Y_{D,i} = X_{R,i} + Z_{D,i}, \quad (3)$$

where

$$Y_{R,i} = \frac{\bar{Y}_{R,i}}{h_{SR}}, \quad Y_{D,i} = \frac{\bar{Y}_{D,i}}{h_{RD}}, \quad Z_{R,i} = \frac{\bar{Z}_{R,i}}{h_{SR}}, \quad \text{and} \quad Z_{D,i} = \frac{\bar{Z}_{D,i}}{h_{RD}}. \quad (4)$$

Using (4), we can obtain the variances of the equivalent noises at relay and destination, $Z_{R,i}$ and $Z_{D,i}$, as $\sigma_R^2 = \bar{\sigma}_R^2/h_{SR}^2$ and $\sigma_D^2 = \bar{\sigma}_D^2/h_{RD}^2$, respectively, where $\bar{\sigma}_R^2$ and $\bar{\sigma}_D^2$ are the variances of the AWGNs $\bar{Z}_{R,i}$ and $\bar{Z}_{D,i}$, respectively.

C. Energy Harvesting Model

We assume that the harvested energy at the EH source in symbol interval i , $E_{H,i}$, is a deterministic function of the transmit symbol at the relay in symbol interval i , $X_{R,i}$, which we write explicitly as $E_{H,i} = E_H(X_{R,i})$, where $E_H(X_{R,i})$ is a deterministic function of $X_{R,i}$. Since $E_H(X_{R,i})$ represents the harvested energy, we assume that $E_H(X_{R,i})$ is a non-decreasing function of $X_{R,i}^2$. For the presented analysis, we do not need to impose any additional constraints on $E_H(X_{R,i})$. Hence, the model adopted for the energy harvested at the EH source during symbol interval i is very general. One example for $E_H(X_{R,i})$, which was considered in numerous previous works [4]-[21] and is included in our general model as a special case, is

$$E_H(X_{R,i}) = \eta h_{RS}^2 X_{R,i}^2, \quad (5)$$

where h_{RS} is the relay-source channel gain and $0 < \eta < 1$ denotes the energy harvesting efficiency.

Having defined the considered relay channel and the harvested energy, in the following, we derive the channel capacities for the two different battery scenarios at the EH source.

III. CHANNEL CAPACITY WHEN THE SOURCE IS BATTERYLESS

In the following, we study the capacity for the case when the EH source is batteryless. For clarity of presentation, we first provide a summary of the procedure adopted for deriving the capacity.

A. Summary of the Capacity Derivation

The approach employed for derivation of the capacity of the considered two-hop FD relay channel with WET and batteryless EH source is summarized in the following five steps.

1) The two-hop FD relay channel with WET and batteryless EH source is written equivalently as a conventional (i.e., non-EH) two-hop FD relay channel with fading source-relay channel, where the fading states of the source-relay channel are known completely at both the source and the relay, and the source has a constraint on the amplitude of its input symbols.

2) Since the equivalent conventional two-hop FD relay channel does not have a direct source-destination link, this relay channel belongs to the class of degraded relay channels as defined in [23]. Consequently, the converse of the capacity for the degraded relay channel derived in [23] is also a converse of the capacity for the equivalent conventional two-hop FD relay channel. As a result, the general capacity expression for the degraded relay channel in [23] is also the capacity expression for the equivalent conventional two-hop FD relay channel.

3) Next, we simplify the general capacity expression for the degraded relay channel in [23] exploiting the following characteristics of the equivalent conventional two-hop FD relay channel: (a) AWGN source-relay and relay-destination channels, (b) fading source-relay channel with fading states that are known at source and relay, and (c) amplitude constrained inputs at the source.

4) Subsequently, we derive the optimal input distributions at the source and the relay for the equivalent conventional two-hop FD relay channel and insert them into the general capacity expression for the degraded relay channel in [23]. As a result, we obtain the final capacity expression in Theorem 1.

5) For the achievability of the derived capacity, we resort to the general capacity-achieving coding scheme for the degraded relay channel presented in [23]. This coding scheme requires $N \rightarrow \infty$ time slots, where each time slot is comprised of $k \rightarrow \infty$ symbol intervals with $Nk = n$, and a decode-and-forward (DF) relay. Furthermore, in each time slot, the source and the relay transmit with the minimum of the capacities of the corresponding source-relay and relay-destination channels. Since the equivalent conventional two-hop FD relay channel has a fast-fading⁷ AWGN source-relay channel and full CSI is available at source and relay, we use the coding scheme⁸ in [25] in order to achieve the capacity of the source-relay channel in each time slot.

⁷A fast-fading channel is a channel where the fading gain changes from one symbol interval to the next.

⁸The coding scheme in [25] is a capacity-achieving coding scheme for the fast-fading AWGN channel with full CSI at both the transmitter and the receiver.

B. Equivalent Relay Channel

In this subsection, we model the considered two-hop FD relay channel with WET and batteryless EH source as an equivalent conventional two-hop FD relay channel with fading source-relay channel, where the fading states are known at source and relay, and an amplitude constrained source.

Since the source is batteryless, the energy harvested during symbol interval $i - 1$, $E_H(X_{R,i-1})$, cannot be stored at the source and can only be used for transmission of the source symbol during symbol interval i , $X_{S,i}$. On the other hand, since the energy of symbol $X_{S,i}$ is $X_{S,i}^2$, we require that $X_{S,i}^2 \leq E_H(X_{R,i-1})$ has to hold, i.e., the source cannot transmit more energy in symbol interval i than what it has harvested during symbol interval $i - 1$. However, because of the additional energy transmission cost that incurs during transmission, P_C , the energy of symbol $X_{S,i}$, $X_{S,i}^2$, has to satisfy the following more stringent constraint

$$X_{S,i}^2 \leq \max\{0, E_H(X_{R,i-1}) - P_C\} \triangleq f(X_{R,i-1}), \quad (6)$$

where function $f(X_{R,i-1})$ is introduced for notational simplicity. Condition (6) means that if the energy harvested during symbol interval $i - 1$, $E_H(X_{R,i-1})$, exceeds the energy transmission cost, P_C , then the energy of symbol $X_{S,i}$, $X_{S,i}^2$, can take any value between zero and $E_H(X_{R,i-1}) - P_C$, i.e., $0 \leq X_{S,i}^2 \leq f(X_{R,i-1})$. Otherwise, if the energy harvested during symbol interval $i - 1$, $E_H(X_{R,i-1})$, is smaller than the energy transmission cost, P_C , then the energy of symbol $X_{S,i}$, $X_{S,i}^2$, can only be zero, i.e., $X_{S,i}^2 = 0$. Hence, $f(X_{R,i-1}) = \max\{0, E_H(X_{R,i-1}) - P_C\}$ represents an upper bound on the available transmission energy for symbol $X_{S,i}$.

Since the absolute value of symbol $X_{S,i}$ is uniquely determined by its energy, $X_{S,i}^2$, from (6), we can obtain the limits for the value of source symbol $X_{S,i}$ as

$$-\sqrt{f(X_{R,i-1})} \leq X_{S,i} \leq \sqrt{f(X_{R,i-1})}. \quad (7)$$

From (7), we can conclude that if $P_C \geq E_H(X_{R,i-1})$, i.e., if $f(X_{R,i-1}) = 0$ holds, then the source can *only* transmit the symbol zero (i.e., it can only be silent) in symbol interval i since the source does not have enough energy for the transmission of any other symbol. On the other hand, if $P_C < E_H(X_{R,i-1})$, i.e., if $f(X_{R,i-1}) > 0$ holds, then the source can transmit any symbol in the range between $-\sqrt{f(X_{R,i-1})}$ and $\sqrt{f(X_{R,i-1})}$. Hence, the effect of WET for a batteryless EH source can be modeled by an amplitude constrained input at the EH source, cf. (7). This is also in line with the model and results for the batteryless EH source presented in [6]. Hence, the capacity of the considered relay channel with batteryless EH source can be found using the input-output relations in (3), where the input at the source, $X_{S,i}$, has to meet the amplitude constraint in (7). In

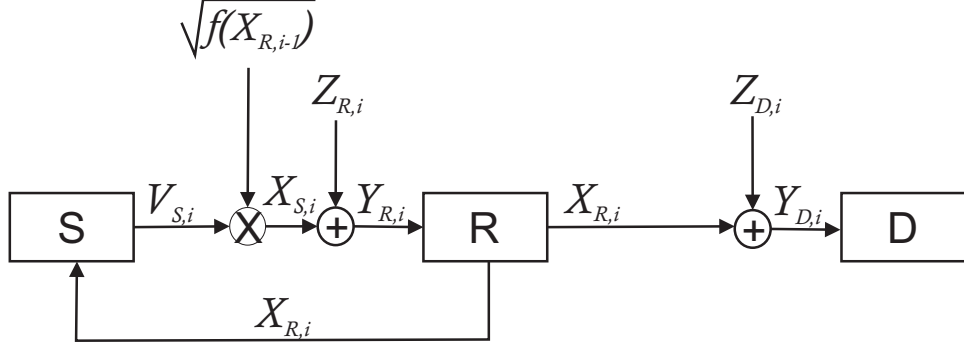


Fig. 2: An equivalent model of the considered relay channel, where $V_{S,i}$ is constrained as in (9).

the following, we provide an equivalent expression for the amplitude constraint in (7), and thereby obtain an equivalent channel model.

In particular, (7) can be represented equivalently as

$$X_{S,i} = V_{S,i} \sqrt{f(X_{R,i-1})}, \quad (8)$$

where $V_{S,i}$ is an RV, which satisfies

$$-1 \leq V_{S,i} \leq 1. \quad (9)$$

Eqns. (8) and (9) provide a different perspective on how source symbol $X_{S,i}$ can be generated for channel use i . In particular, the source can first generate the symbol $V_{S,i}$, which can take any value between -1 and 1 , and then multiply $V_{S,i}$ with the coefficient $\sqrt{f(X_{R,i-1})}$ in order to produce $X_{S,i}$. Inserting (8) into (3), we obtain

$$Y_{R,i} = V_{S,i} \sqrt{f(X_{R,i-1})} + Z_{R,i}, \text{ where } -1 \leq V_{S,i} \leq 1, \text{ and } Y_{D,i} = X_{R,i} + Z_{D,i}. \quad (10)$$

Hence, instead of using (3) and constraint (7) for deriving the capacity, we can equivalently use (10). On the other hand, the input-output relation for the source-relay channel in (10) resembles a fast-fading AWGN channel, where the source imposes an amplitude constraint on its transmit symbols, $V_{S,i}$, as in (9), and where the “fading channel gain” in channel use i is $\sqrt{f(X_{R,i-1})}$, which is known at source and relay, i.e., the source and the relay have full CSI⁹.

Hence, the considered two-hop FD relay channel with WET and batteryless EH source can equivalently be represented as a conventional two-hop FD relay channel with fast-fading on the source-relay channel, where the fading states are known at source and relay, and amplitude constrained inputs at the source, cf. (9). This equivalent relay channel is illustrated in Fig. 2. Since the two relay channels are equivalent, the capacities of the two relay channels are identical. Hence,

⁹Since the function $f(X_{R,i-1})$ is a deterministic function of the relay’s transmit symbol $X_{R,i-1}$, it is clear that the relay knows the value of $f(X_{R,i-1})$ even before the start of symbol interval i . On the other hand, since the source knows how much energy it harvested during symbol interval $i-1$, it can calculate $f(X_{R,i-1})$ in symbol interval i using (6).

by deriving the capacity of the relay channel shown in Fig. 2, we also obtain the capacity of the original two-hop FD relay channel with WET and batteryless EH source.

A general expression for the capacity of the considered relay channel with WET and batteryless EH source is given in the following lemma.

Lemma 1: The capacity of the two-hop FD relay channel with batteryless EH source is given by the following general expression

$$C = \max_{p(x_R) \in \mathcal{P}} \min \left\{ \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) p(x_R) ; I(X_R; Y_D) \right\}$$

Subject to C1 : $-1 \leq V_S \leq 1$,

$$\text{C2 : } \sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) \leq P_R, \quad (11)$$

where \mathcal{P} is the set of all possible probability distributions, P_R is the average transmit power constraint at the relay, $p(v_S|x_R)$ is the conditional probability distribution of V_S given X_R , and $p(x_R)$ is the probability distribution of X_R .

Proof: Please refer to Appendix A. ■

From Lemma 1, we see that in order to obtain a more explicit expression for the capacity of the considered relay channel with a batteryless EH source, we need to obtain the optimal input distributions $p(v_S|x_R)$ and $p(x_R)$, denoted by $p^*(v_S|x_R)$ and $p^*(x_R)$, respectively, as the solutions of (11). These distributions are provided in the following two subsections.

C. Optimal $p^*(v_S|x_R)$

The optimal input distribution at the EH source, $p^*(v_S|x_R)$, and the corresponding expression for the mutual information $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R)$ are provided in the following lemma. To this end, let us first define function $\delta(\cdot)$ as follows: $\delta(x) = 0$ if $x \neq 0$ and $\delta(x) = 1$ if $x = 0$.

Lemma 2: The optimal input distribution at the EH source, $p^*(v_S|x_R)$ is discrete with a finite number of probability mass points, and can be written in a general form as

$$p^*(v_S|x_R) = \sum_{k=1}^{K^*(x_R)} p_{S,k}^*(x_R) \delta(v_S - v_k^*(x_R)), \quad -1 \leq v_k^*(x_R) \leq 1, \quad \forall k, \quad (12)$$

where $v_k^*(x_R)$ are the optimal values that v_S can assume, $p_{S,k}^*(x_R) = \Pr\{V_S = v_k(x_R) | X_R = x_R\}$, where $\Pr\{\cdot|\cdot\}$ denotes conditional probability, is the conditional probability of the outcome $V_S = v_k^*(x_R)$, and $K^*(x_R)$ is the optimal number of possible values that v_S can assume. The corresponding

expression for the mutual information $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R)$ is given by

$$\begin{aligned} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R) &= - \int_{-\infty}^{\infty} \sum_{k=1}^{K^*(x_R)} \frac{p_{S,k}^*(x_R)}{\sqrt{2\pi\sigma_R^2}} \exp\left(-\frac{(y_R - v_k^*(x_R)\sqrt{f(x_R)})^2}{2\sigma_R^2}\right) \\ &\times \log_2 \left(\sum_{k=1}^{K^*(x_R)} \frac{p_{S,k}^*(x_R)}{\sqrt{2\pi\sigma_R^2}} \exp\left(-\frac{(y_R - v_k^*(x_R)\sqrt{f(x_R)})^2}{2\sigma_R^2}\right) \right) dy_R - \frac{1}{2} \log_2(2\pi e\sigma_R^2). \end{aligned} \quad (13)$$

Proof: Please refer to Appendix B. ■

There are special cases for which the optimal input distribution $p^*(v_S|x_R)$ and the corresponding maximal mutual information $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R)$ are known in closed form. In particular, $p^*(v_S|x_R)$ and $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R)$ are known in closed form if $f(x_R)/\sigma_R^2$ is sufficiently small and large¹⁰, respectively. These closed-form expressions are provided in the following corollary.

Corollary 1: In the special case when $f(x_R)/\sigma_R^2 \ll 1$, the optimal input distribution at the source $p^*(v_S|x_R)$ is the binary distribution

$$p^*(v_S|x_R) = \frac{1}{2}\delta(v_S - 1) + \frac{1}{2}\delta(v_S + 1). \quad (14)$$

Consequently, the corresponding $\max_{p(v_S|x_R)} I(V_S; Y_R|X_R = x_R)$ is given by

$$\max_{p(v_S|x_R)} I(V_S; Y_R|X_R = x_R) = \frac{f(x_R)}{\ln(2)\sigma_R^2} - \int_{-\infty}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \log_2 \left(\cosh \left(\frac{f(x_R)}{\sigma_R^2} + \sqrt{\frac{f(x_R)}{\sigma_R^2}} t \right) \right) dt. \quad (15)$$

On the other hand, when $f(x_R)/\sigma_R^2 \gg 1$, the optimal input distribution at the source $p^*(v_S|x_R)$ is uniformly distributed with a very large number of probability mass points in the interval $[-1, 1]$. As a result, the optimal input distribution can be accurately approximated by the continuous uniform distribution given by

$$p^*(v_S|x_R) = \begin{cases} 1/2 & \text{if } -1 \leq v_S \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

Consequently, the corresponding $\max_{p(v_S|x_R)} I(V_S; Y_R|X_R = x_R)$ is given by

$$\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R) = \frac{1}{2} \log_2 \left(1 + \frac{2f(x_R)}{\pi e\sigma_R^2} \right). \quad (17)$$

¹⁰Parameter $f(x_R)/\sigma_R^2$ can be interpreted as the normalized transmit power at the EH source for $X_R = x_R$, i.e., the normalized transmit power at the EH source for a given symbol interval for which $X_R = x_R$ holds.

Proof: Proofs that the optimal input distribution for low and high SNRs for an amplitude constrained AWGN channel, i.e., when $f(x_R)/\sigma_R^2 \ll 1$ and $f(x_R)/\sigma_R^2 \gg 1$ hold, are given by (14) and (16), respectively, are given in [26] and [27]. Consequently, $\max_{p(v_S|x_R)} I(V_S; Y_R|X_R = x_R)$ for input distributions given by (14) and (16) are derived in [28, pp. 145] and [26] as (15) and (17), respectively. ■

Remark 2: As can be seen from (16), when $f(x_R)/\sigma_R^2 \gg 1$ holds, the optimal input distribution at the source $p^*(v_S|x_R)$ becomes independent of x_R , i.e., independent of the “fading gain” $f(x_R)$. This property of $p^*(v_S|x_R)$ can be exploited for designing a simple achievability coding scheme, cf. Remark 6.

The optimal input distribution at the relay, $p^*(x_R)$, is provided in the following.

D. Optimal $p^*(x_R)$ and Capacity Expressions

In order to obtain the capacity for the considered relay channel with a batteryless EH source, we first need an expression for $I(X_R; Y_D)$ for a general discrete probability distribution $p(x_R) = \sum_{m=1}^M p_{R,m} \delta(x_R - x_{R,m})$, which is given by [29]

$$I(X_R; Y_D) = - \sum_{m=1}^M \int_{-\infty}^{\infty} \frac{p_{R,m}}{\sqrt{2\pi\sigma_D^2}} \exp\left(-\frac{(y_R - x_{R,m})^2}{2\sigma_D^2}\right) \times \log_2 \left(\sum_{m=1}^M \frac{p_{R,m}}{\sqrt{2\pi\sigma_D^2}} \exp\left(-\frac{(y_R - x_{R,m})^2}{2\sigma_D^2}\right) \right) dy_R - \frac{1}{2} \log_2(2\pi e \sigma_D^2). \quad (18)$$

Using (18), the optimal $p^*(x_R)$ and the capacity are provided in the following theorem.

Theorem 1: There are three cases for the optimal input distribution of the relay, $p^*(x_R)$, and the capacity of the considered relay channel with a batteryless EH source.

Case 1: Let $p^+(x_R)$ be the optimal solution of

$$C = \max_{p(x_R) \in \mathcal{P}} \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R)} I(V_S; Y_R|X_R = x_R) p(x_R)$$

Subject to C1 : $\sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) \leq P_R,$ (19)

where $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R)$ is given in (13). Then, if condition

$$C \leq I(X_R; Y_D) \Big|_{p(x_R)=p^+(x_R)} \quad (20)$$

holds, where $I(X_R; Y_D)$ is given in (18), the capacity is given by C in (19) and the optimal $p^*(x_R)$ is $p^+(x_R)$ found as the solution of (19). In this case, the source-relay channel is the performance bottleneck. In particular, even if the relay transmits using the distribution which maximizes the average mutual information of the source-relay channel, the average mutual information of the

source-relay channel is still smaller then or equal to the average mutual information of the relay-destination channel.

Case 2: If condition

$$C = \frac{1}{2} \log_2 \left(1 + \frac{P_R}{\sigma_D^2} \right) < \int_{-\infty}^{\infty} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) \frac{1}{\sqrt{2\pi P_R}} \exp \left(-\frac{x_R^2}{2P_R} \right) dx_R \quad (21)$$

holds, where $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R)$ is given in (13), then the capacity is given by C in (21) and the optimal $p^*(x_R)$ is the zero-mean Gaussian distribution with variance P_R . In this case, the relay-destination channel is the bottleneck. In particular, even if the relay transmits Gaussian distributed symbols with which it achieves the capacity of the relay-destination channel, the rate of the relay-destination channel is still smaller than the rate of the source-relay channel. Hence, in this case, the capacity is equal to the capacity of the relay-destination link, and therefore, even equipping a battery at the source would not increase the capacity.

Case 3: If both conditions (20) and (21) do not hold, the capacity is given by

$$\begin{aligned} C &= \max_{p(x_R) \in \mathcal{P}} I(X_R; Y_D) \\ \text{Subject to C1 : } &\sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) \leq P_R \\ \text{C2 : } &\sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) p(x_R) = I(X_R; Y_D), \end{aligned} \quad (22)$$

where $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R)$ and $I(X_R; Y_D)$ are given in (13) and (18), respectively. The optimal $p^*(x_R)$ is discrete and found as the solution of (22).

Proof: Please refer to Appendix C. ■

Remark 3: The capacity and the optimal distribution $p^*(x_R)$ for Cases 1 and 3 in Theorem 1 can be obtained numerically using a numerical optimization software such as Mathematica. In particular, the optimization problem in (19) is a linear optimization problem and can be easily solved using numerical optimization software. On the other hand, using its epigraph form, the optimization problem in (22) can be written equivalently as a concave optimization problem as shown in Appendix C in (45). The equivalent concave optimization problem can then be solved using numerical optimization software.

In the following, we provide useful corollaries for the case when $f(x_R)/\sigma_R^2 \gg 1$ holds $\forall x_R \in \mathcal{X}_R$ for which $f(x_R) \neq 0$.

Corollary 2: If $f(x_R)/\sigma_R^2 \gg 1$ holds $\forall x_R \in \mathcal{X}_R$ for which $f(x_R) \neq 0$, then in the capacity expressions in Theorem 1, $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R)$ can be replaced by the expression in (17).

Proof: When $f(x_R)/\sigma_R^2 \gg 1$ holds for a given $x_R \in \mathcal{X}_R$, then $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R)$ in (13) converges to the expression in (17). Otherwise, when $f(x_R) = 0$, then $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R) = 0$. ■

Corollary 2 significantly simplifies the expressions in Theorem 1.

Since $E_H(x_R)$ given by (5) is a frequently used energy harvesting model, in the following, we derive the capacity for $E_H(x_R)$ given by (5).

Corollary 3: If $f(x_R)/\sigma_R^2 \gg 1$ holds $\forall x_R \in \mathcal{X}_R$ for which $f(x_R) \neq 0$ and if $E_H(x_R)$ is given by (5), then we have the following three cases.

Case 1: If conditions $\eta h_{RS}^2 P_R > P_C$ and

$$C = \frac{1}{2} \log_2 \left(1 + \frac{2(\eta h_{RS}^2 P_R - P_C)}{\pi e \sigma_R^2} \right) < \frac{P_R}{\ln(2)\sigma_D^2} - \int_{-\infty}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \log_2 \left(\cosh \left(\frac{P_R}{\sigma_D^2} + \sqrt{\frac{P_R}{\sigma_D^2}} t \right) \right) dt \quad (23)$$

hold, then the capacity is given by C in (23) and the optimal input distribution at the relay is

$$p^*(x_R) = \frac{1}{2} \delta(x_R - \sqrt{P_R}) + \frac{1}{2} \delta(x_R + \sqrt{P_R}). \quad (24)$$

Case 2: If conditions $\eta h_{RS}^2 P_R \leq P_C$ and

$$\begin{aligned} C &= \frac{1}{2} \log_2 \left(1 + \frac{2(\eta h_{RS}^2 x_0^2 - P_C)}{\pi e \sigma_R^2} \right) \frac{P_R}{x_0^2} \\ &< - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_D^2}} \left\{ \frac{P_R}{2x_0^2} \exp \left(-\frac{(y_R - x_0)^2}{2\sigma_D^2} \right) + \frac{P_R}{2x_0^2} \exp \left(-\frac{(y_R + x_0)^2}{2\sigma_D^2} \right) \right. \\ &\quad \left. + \left(1 - \frac{P_R}{x_0^2} \right) \exp \left(-\frac{y_R^2}{2\sigma_D^2} \right) \right\} \times \log_2 \left[\frac{1}{\sqrt{2\pi\sigma_D^2}} \left\{ \frac{P_R}{2x_0^2} \exp \left(-\frac{(y_R - x_0)^2}{2\sigma_D^2} \right) + \right. \right. \\ &\quad \left. \left. \frac{P_R}{2x_0^2} \exp \left(-\frac{(y_R + x_0)^2}{2\sigma_D^2} \right) + \left(1 - \frac{P_R}{x_0^2} \right) \exp \left(-\frac{y_R^2}{2\sigma_D^2} \right) \right\} \right] dy_R - \frac{1}{2} \log_2 (2\pi e \sigma_D^2) \end{aligned} \quad (25)$$

hold, then the capacity is given by C in (25) and the optimal input distribution at the relay is

$$p^*(x_R) = \frac{P_R}{2x_0^2} \delta(x_R - x_0) + \left(1 - \frac{P_R}{x_0^2} \right) \delta(x_R) + \frac{P_R}{2x_0^2} \delta(x_R + x_0), \quad (26)$$

where

$$x_0 = \sqrt{\frac{2P_C - e\pi\sigma_R^2}{2\eta h_{RS}^2} - \frac{1}{2\lambda \ln(2)} W \left(-\frac{e\pi\lambda\sigma_R^2 \ln(2)}{\eta h_{RS}^2} 2^{\frac{\lambda}{\eta h_{RS}^2} (2P_C - e\pi\sigma_R^2)} \right)}. \quad (27)$$

In (27), $W(\cdot)$ is the Lambert W function and λ is given by

$$\lambda = \frac{\eta h_{RS}^2}{(2P_C - e\pi\sigma_R^2) \ln(2)} W \left(\frac{2P_C - e\pi\sigma_R^2}{e^2\pi\sigma_R^2} \right). \quad (28)$$

Case 3: If (23) and (25) do not hold, the capacity is given by Case 2 or Case 3 in Theorem 1 with

$\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R)$ and $E_H(x_R)$ given by (17) and (5), respectively.

Proof: Please refer to Appendix D. ■

Remark 4: For Cases 1 and 2 in Corollary 3, we have closed-form expressions for the capacity. Moreover, the input distribution at the relay is either binary shift phase keying (BPSK) (for Case 1) or a simple three-point constellation (for Case 2), which can be considered as a BPSK with an additional zero symbol in the constellation.

Remark 5: We note that the capacity of the considered two-hop FD relay channel with batteryless EH source given in Theorem 1, and the capacity of a conventional (non-EH) AWGN two-hop FD relay channel given in [23] are very different. The differences are due to the amplitude constraint and the energy transmission cost at the EH source, which are not present for the conventional AWGN two-hop FD relay channel in [23].

In the following, we discuss the achievability of the capacity for the considered relay channel with batteryless EH source given in Theorem 1.

E. Achievability of the Capacity

Since the considered two-hop FD relay channel belongs to the class of degraded relay channels defined in [23], its capacity can be achieved by the capacity-achieving coding scheme for the degraded relay channel in [23]. The capacity-achieving coding scheme in [23] requires the transmission to be carried out over $N + 1$ time slots, where during each time slot the channel is used k times, where $N \rightarrow \infty$ and $k \rightarrow \infty$. Moreover, during each time slot, the source and the relay transmit with rates which are smaller but arbitrary close to the capacity C given in Theorem 1. On the other hand, since the source-relay channel can be modeled as a fast-fading AWGN channel with full CSI at the source and the relay, the rate C on the source-relay channel can be achieved using the capacity-achieving coding scheme for a fast-fading AWGN channel with full CSI proposed in [25].

The coding scheme in [25] requires the source to use F codebooks, where F is the number of non-zero “fading states” of the source-relay channel. For the considered source-relay channel, the number of non-zero “fading states” is equal to the number of non-zero values that $f(x_R)$ can assume. As a result, $F = |f(\mathcal{X}_R)| - 1$, where set \mathcal{X}_R contains all possible values that x_R can assume and $|\cdot|$ denotes the cardinality of a set. Each of the F codebooks is mapped to a specific non-zero “fading state” $f(x_R) \neq 0$. The codebook corresponding to “fading state” $f(x_R) \neq 0$ contains $2^{kp^*(x_R)R_S(x_R)}$ codewords comprised of $kp^*(x_R)$ symbols where each symbol is generated independently using the distribution $p^*(v_S|x_R)$, and where $R_S(x_R)$ is given by

$$R_S(x_R) = Q \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) - \epsilon. \quad (29)$$

In (29), ϵ is an arbitrarily small positive number and Q is a scaling factor given by

$$Q = \frac{C}{\sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) p^*(x_R)}, \quad (30)$$

where C is the capacity given in Theorem 1. The scaling factor Q scales the rate $R_S(x_R)$ such that the average of $R_S(x_R)$ with respect to $p^*(x_R)$ does not exceed the capacity C , i.e., the following holds $\sum_{x_R \in \mathcal{X}_R} R_S(x_R) p^*(x_R) < C$. On the other hand, the relay uses only one codebook which contains 2^{kR_R} codewords comprised of k symbols where each symbol is generated independently using the distribution $p^*(x_R)$. The rate of the relay R_R is set to $R_R = C - \epsilon$, where C is the capacity given in Theorem 1. Note that $R_S(x_R)$ and R_R are related as $R_R = \sum_{x_R \in \mathcal{X}_R} R_S(x_R) p^*(x_R) = C - \epsilon$. Having defined the above codebooks and rates, in the following, we discuss the transmission of a single message from the source via the relay to the destination in $N + 1$ time slots.

The source wants to transmit message w selected uniformly from the set $\{1, 2, \dots, 2^{nR_R}\}$, which carries nR_R bits of information. This message is split into N smaller messages, denoted by $w(1), w(2), \dots, w(N)$, where each smaller message carries kR_R bits and $n = kN$. Each of these smaller messages is transmitted in a different time slot. Thereby, in the first time slot, the source transmits to the relay message $w(1)$ whereas the relay transmits a known ‘‘dummy’’ codeword¹¹, which does not carry any information and is used for powering up the source in the first time slot. In time slot b , where $2 \leq b \leq N$, the source transmits message $w(b)$ to the relay, whereas the relay receives and retransmits to the destination message $w(b - 1)$, which it received from the source in the previous, i.e., $(b - 1)$ -th time slot. Finally, in the last, i.e., the $(N + 1)$ -th time slot, the source is silent since it has transmitted all of its messages and the relay retransmits to the destination message $w(N)$, which it received in the previous, i.e., the N -th time slot. In the following, we explain how the source and the relay transmit the corresponding messages in a specific time slot b .

The relay transmits message $w(b - 1)$ to the destination in time slot b by mapping message $w(b - 1)$ to the corresponding codeword from its codebook and transmitting this codeword to the destination during k symbol intervals. On the other hand, by employing the coding scheme in [25], in order for the source to transmit message $w(b)$ to the relay in time slot b , the source splits message $w(b)$ into F smaller messages, where each smaller message is mapped to a different ‘‘fading state’’ $f(x_R) \neq 0$ and where the message corresponding to ‘‘fading state’’ $f(x_R)$ carries $k p^*(x_R) R_S(x_R)$ bits of information. Now, when ‘‘fading state’’ $f(x_R) \neq 0$ occurs in symbol interval i of time slot b , the source selects its next untransmitted symbol from the codeword mapped to the message that

¹¹This can be any codeword from relay’s codebook and this codeword should be revealed to the source before the start of transmission.

corresponds to “fading state” $f(x_R)$, multiplies this symbol with $\sqrt{f(x_R)}$, and transmits it to the destination. If “fading state” $f(x_R) = 0$ occurs, the source is silent and the relay disregards the received symbol. Since for $k \rightarrow \infty$, “fading state” $f(x_R)$ occurs $kp^*(x_R)$ times, the source is able to transmit all of its F messages in a single time slot, see [25] for more details.

Remark 6: The coding performed at the EH source can be simplified significantly when $f(x_R)/\sigma_R^2 \gg 1$ holds $\forall x_R \in \mathcal{X}_R$ for which $f(x_R) \neq 0$, using the coding scheme for the fast-fading channel in [30], [31]. This is because, in this case, the input distribution at the relay is uniform between -1 and 1 and independent of the “fading gain”, see (14) and Remark 2. As a result, in this case, the source does not need to split $w(b)$ into F messages in each time slot b , and thereby, does not need to use F different codebooks. Instead, the source can use only one codebook which contains $2^{kp_f R_S}$ codewords comprised of kp_f symbols where each symbol is generated independently using the uniform distribution $p^*(v_S|x_R)$ given in (16), and where $p_f = \Pr\{f(x_R) \neq 0\}$ and $p_f R_S = R_R$ hold. Next, when the source transmits the codeword from its codebook mapped to message $w(b)$, then each symbol of this codeword is multiplied by the “fading state” $f(x_R) \neq 0$ corresponding to the symbol interval in which the symbol is transmitted, see [30], [31]. If $f(x_R) = 0$ occurs in a given symbol interval, the source is silent and the relay disregards the received symbol.

IV. CHANNEL CAPACITY WHEN THE SOURCE HAS AN UNLIMITED BATTERY

In the following, we determine the channel capacity of the considered relay channel for the case when the EH source is equipped with an unlimited battery.

A. Channel Capacity

To obtain the channel capacity, we exploit the results for the capacity of the degraded relay channel in [23] and the capacity of the EH AWGN channel with energy transmission cost in [5]. In this way, the capacity can be obtained in a much more straightforward manner than the capacity for the case with a batteryless EH source, and can be written as

$$\begin{aligned}
 C &= \max_{p(x_S, x_R) \in \mathcal{P}} \min \{I(X_S; Y_R | X_R) ; I(X_R; Y_D)\} \\
 \text{Subject to C1 : } &\sum_{x_S \in \mathcal{X}_S} P_S(x_S) p(x_S) \leq \sum_{x_R \in \mathcal{X}_R} E_H(x_R) p(x_R) \\
 \text{C2 : } &\sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) \leq P_R,
 \end{aligned} \tag{31}$$

where $\sum_{x_R \in \mathcal{X}_R} E_H(x_R)p(x_R)$ is the average energy harvested by the EH source and $P_S(x_S)$ is the energy spent by the EH source for transmitting symbol x_S , which is given by [5]

$$P_S(x_S) = \begin{cases} 0 & \text{if } x_S = 0 \\ x_S^2 + P_C & \text{if } x_S \neq 0. \end{cases} \quad (32)$$

Hence, as can be seen from (32), for every non-zero (non-silent) symbol transmitted by the source, an additional energy P_C is needed. Constraint C1 in (31) is due to the average energy causality constraint. Now, since we assumed out-of-band FD relaying, which does not cause self-interference, RVs $X_{S,i}$ and $Y_{R,i}$ are both independent of RV $X_{R,i}$, cf. (3). As a result, the capacity expression in (31) can be simplified as

$$\begin{aligned} C &= \min \left\{ \max_{p(x_S) \in \mathcal{P}} I(X_S; Y_R) ; \max_{p(x_R) \in \mathcal{P}} I(X_R; Y_D) \right\} \\ \text{Subject to C1 : } &\sum_{x_S \in \mathcal{X}_S} P_S(x_S)p(x_S) \leq \sum_{x_R \in \mathcal{X}_R} E_H(x_R)p(x_R) \\ \text{C2 : } &\sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) \leq P_R. \end{aligned} \quad (33)$$

The optimal input distributions $p^*(x_S)$ and $p^*(x_R)$ found as the solution of (33) depend on the function $E_H(x_R)$. Therefore, in the following, we pursue the special case when $E_H(x_R)$ is given by (5).

Lemma 3: In the special case when $E_H(x_R)$ is given by (5), the capacity is given by

$$C = \min \left\{ I(X_S; Y_R) \Big|_{p(x_S)=p^*(x_S)} ; \frac{1}{2} \log_2 \left(1 + \frac{P_R}{\sigma_D^2} \right) \right\}, \quad (34)$$

where $p^*(x_S)$ is discrete and found as the solution of

$$\begin{aligned} &\max_{p(x_S)} I(X_S; Y_R) \\ \text{Subject to C1 : } &\sum_{x_S \in \mathcal{X}_S} P_S(x_S)p(x_S) = \eta h_{RS}^2 P_R, \end{aligned} \quad (35)$$

where $P_S(x_S)$ is given in (32).

Proof: Please refer to Appendix E. ■

As shown in [5], for $P_C > 0$, the optimal input distribution at the source, $p^*(x_S)$, found as the solution of (35), always includes the zero (silent) symbol with non-zero probability. This means that in order to achieve the capacity, the source is silent in a fraction of the symbol intervals. Moreover, the source also uses these silent symbols for encoding additional information for the relay.

B. Achievability of the Channel Capacity

The capacity achieving coding scheme for this channel can be obtained by combining the coding schemes for the degraded relay channel in [23] and the coding schemes for the EH AWGN channel in [4] and [5]. For completeness, we outline the combination of the coding schemes presented in [4], [5], and [23].

The transmission is carried out in $K + N + 1$ time slots, where during each time slot the channel is used k times. The numbers N and K are chosen such that $N \rightarrow \infty$, $K \rightarrow \infty$, and $K/N \rightarrow 0$ hold. In particular, similar to the *save-and-transmit* scheme in [4], in the first K time slots, the source fills its battery without transmitting any information. To this end, the relay sends a “dummy” codeword to the source in each time slot during the first $K + 1$ time slots. The destination discards these “dummy” codewords received in the first $K + 1$ time slots, whereas the source harvests the energy from these codewords. Next, in each time slot from the $(K + 1)$ -th time slot to the $(K + N)$ -th time slot, the source transmit a message to the relay while harvesting the energy from the codeword transmitted by the relay. At the same time, the relay receives, decodes the received codewords, and then retransmits the received message in the next time slot to the destination. In the last, i.e., the $(K + N + 1)$ -th time slot, the source is silent and the relay retransmits to the destination the message received from the source in the previous, i.e., the $(K + N)$ -th time slot. The transmission rates of source and relay in each time slot from the $(K + 1)$ -th to the $(N + 1)$ -th time slot is C , where C is the channel capacity given in Section IV-A. The input distributions at source and relay are $p^*(x_S)$ and $p^*(x_R)$, respectively, and are also provided in Section IV-A. Since $K/N \rightarrow 0$ holds, the time spent for powering up the EH source has a negligible impact on the overall achieved data rate and on the average power consumed by the relay. Moreover, using the *save-and-transmit* scheme in [4], the energy causality is satisfied for each transmitted symbol, see [4] for a proof.

V. NUMERICAL EXAMPLES

In the following, we provide numerical examples to compare the derived capacities with several benchmark schemes. To this end, we first introduce the system parameters, define the benchmark schemes, and finally provide the numerical examples.

A. System Parameters

We compute the channel gains of the source-relay (SR) and relay-destination (RD) links using the standard path loss model

$$h_L^2 = \left(\frac{c}{f_c 4\pi} \right)^2 d_L^{-\alpha}, \quad \text{for } L \in \{SR, RD\}, \quad (36)$$

where c is the speed of light, f_c is the carrier frequency, d_L is the distance between the transmitter and the receiver of link L , and α is the path loss exponent. For the numerical examples in this section, we assume $\alpha = 3$, $d_{SR} = 10$ or $d_{SR} = 20$ meters, and $d_{RD} = 200$ meters. Moreover, we assume that the carrier frequencies of the transmit and receive signals of the relay are $f_c = 2.3999$ GHz and $f_c = 2.4001$ GHz¹². The transmit bandwidth is assumed to be $B = 100$ kHz. Thereby, assuming ideal Nyquist sampling, we have $2B$ independent symbols per second. Moreover, we assume that the noise power per Hz is -160 dBm, which leads to a total noise power of $10^{-19}B$ Watt. Moreover, for the harvested energy in a single symbol interval, we assume that $E_H(x_R)$ is given in (5) with $h_{RS} = h_{SR}$ and $\eta = 0.8$. Furthermore, we assume that $P_C = 1$ mWatt. Hence, in order for the source to emit P Watt, it has to spend an additional 1 mWatt. Finally, since the capacities derived throughout this paper are in bits/symbol, to plot the capacities in bits/sec, we need to multiply the corresponding capacity expressions by $2B$.

For the above set of parameters, for the case of a batteryless EH source, even for very small transmit powers at the relay, P_R , we obtain that when $f(x_R)$ is not zero, $f(x_R)/\sigma_R^2 \gg 1$ holds. As a result, we can use Corollary 3 to obtain the capacity. In particular, we obtain from Corollary 3 that the capacity is given by Case 2 in Corollary 3, i.e., as a closed-form expression. Hence, for the adopted set of parameters, the source-relay channel is the bottleneck, which is expected in general since the source is powered by WET, whereas the relay has its own power supply.

B. Benchmark Schemes

To fairly evaluate the capacity of the batteryless EH source and the capacity of the source with an unlimited battery, we compare the derived capacities with the rates achieved by several benchmark schemes as references. For the first benchmark scheme, referred to as Benchmark Scheme 1, we assume that the source is batteryless and transmits using the optimal input distribution, given in Section III-C, however, the relay does not use the optimal input distribution, given in Section III-D, and instead transmits Gaussian distributed symbols, which is a commonly used input distribution for the EH relay channel in the literature [12]-[21]. As a result, the rate achieved by Benchmark Scheme 1 is the minimum of the rates given by the expressions in the left hand side and the right hand side of the inequality in (21). For the second benchmark scheme, referred to as Benchmark Scheme 2, we assume that the source is equipped with an unlimited battery. Moreover, we assume

¹²The values for the carrier frequencies are chosen such that they are close to 2.4 GHz, which is a frequently used carrier frequency in practice. Because of the frequency separation, there is no interference between the transmit and receive signals at the relay, which both occupy a 100 kHz bandwidth. On the other hand, $d_{RD} = 200$ m is chosen to illustrate that a relatively large distance between the EH source and the destination can be bridged by using the WET transmitter as a relay.

that the transmission time is divided into slots of equal length and that one codeword spans one time slot. In addition, we assume that the source is silent for $t \geq 0$ time slots during which it harvests energy from the relay and conserves it for future transmissions. Thereby, since $E_H(x_R)$ is given by (5), in t time slots the source can harvest $t\eta h_{RS}^2 P_R$ Watt. Once the source has harvested enough energy, it transmits a Gaussian distributed codeword to the relay with power $t\eta h_{SR}^2 P_R - P_C$ spanning one time slot. Having in mind that when the source transmits information it can also harvest energy, the maximum achievable rate on the source-relay channel using Benchmark Scheme 2 is

$$\max_t \frac{1}{1+t} \frac{1}{2} \log_2 \left(1 + \frac{(1+t)\eta h_{SR}^2 P_R - P_C}{\sigma_R^2} \right). \quad (37)$$

On the other hand, for Benchmark Scheme 2, we assume that the relay is never silent and it transmits in each time slot a codeword with Gaussian distributed symbols. Thereby, the maximum achievable rate on the relay-destination channel is $(1/2) \log_2(1 + P_R/\sigma_D^2)$. Combining this rate with (37), we obtain the maximum achievable rate for Benchmark Scheme 2 as

$$R = \min \left\{ \max_t \frac{1}{1+t} \frac{1}{2} \log_2 \left(1 + \frac{(1+t)\eta h_{SR}^2 P_R - P_C}{\sigma_R^2} \right), \frac{1}{2} \log_2 \left(1 + \frac{P_R}{\sigma_D^2} \right) \right\}. \quad (38)$$

Finally, we also use the rate achieved with the protocol in [18] as a benchmark. However, for fair comparison, we modify the scheme proposed in [18] and instead of a half-duplex relay we assume an out-of-band FD relay. According to the protocol in [18], both source and relay transmit Gaussian distributed symbols. Thereby, the achievable rate of this benchmark scheme, denoted by Benchmark Scheme 3, is identical to (38) with the parameter t set to zero. Hence, Benchmark Scheme 3 is identical to Benchmark Scheme 2 if the source is not allowed to conserve energy but is forced to transmit Gaussian signals in each time slot using.

C. Numerical Results

For the above set of parameters with a source-relay distance of $d_{SR} = 10$ meter, the channel capacities of the considered relay channel with a batteryless EH source and an EH source with an unlimited battery, respectively, are shown in Fig. 3. In addition, in Fig. 3, we also show the rates achieved with the three benchmark schemes. As can be seen from Fig. 3, for P_R in the range from zero to five Watts, the channel capacity for the case when the source is equipped with an unlimited battery is more than four times higher than the channel capacity for the case when the source is batteryless. This is expected since, in the former case, the source can store energy in its battery and then use it later to transmit information to the relay. In fact, for the capacity of the EH source with unlimited battery, in a large portion of the transmission time, the EH source is silent and conserves energy. Moreover, the EH source uses the silent symbols for encoding additional information for the

relay. Although the batteryless EH source is also silent in a large portion of the symbol intervals, it cannot use the silent symbols for encoding additional information for the relay since the relay knows when these silent symbol intervals occur. On the other hand, in Benchmark Scheme 2, the source is also silent during t time slots and conserves energy. However, since for Benchmark Scheme 2 the source does not use the silent symbols for encoding additional information for the relay, the rate of Benchmark Scheme 2, given by (38), is much lower than the derived capacity when the source has an unlimited battery and is only slightly larger than the derived capacity when the source is batteryless. This shows that the encoding of information in the silent symbols at the EH source with a battery has a large impact on the achievable data rate. Moreover, Fig. 3 also shows that using the optimal coding scheme with the optimal input distributions at the source and the relay is essential for high performance for both battery scenarios. For example, if non-optimal Gaussian signaling is used at the relay for the case of the batteryless EH source, as in Benchmark Scheme 1, or if non-optimal Gaussian signaling is used at the source for the case when the source is equipped with an unlimited battery, as in Benchmark Scheme 3, the data rate is zero for the adopted range of P_R , i.e., no information can be transmitted by the source to the relay for Benchmark Schemes 1 and 3 for $P_R \leq 5$ Watt. The poor performance is a consequence of the fact that Benchmark Schemes 1 and 3 do not take into account the energy transmission cost. More precisely, for Benchmark Scheme 1, due to the non-optimal signaling used by the relay, the source is not able to harvest more energy than what is consumed by the energy transmission cost P_C . Hence, since every attempt to emit a non-zero symbol incurs an energy transmission cost, the energy left for information transmission after subtracting the energy transmission cost is zero. On the other hand, for Benchmark Scheme 3, due to the non-optimal signaling used by the source, the source is forced to transmit a non-zero symbol in every symbol interval without having the chance to be silent and to conserve energy. Due to the energy transmission cost, the energy left for information transmission is again zero for $P_R \leq 5$ Watt.

For Fig. 4, we use the same parameters as in Fig. 3 but the distance between the source and relay is increased to $d_{SR} = 20$ meter. Comparing Fig. 3 and Fig. 4, we can see that the doubling of the source-relay distance results in a tenfold reduction of the channel capacity for both the batteryless EH source and the source with an unlimited battery.

To illustrate the effect that the energy transmission cost has on the channel capacities, in Fig. 5,

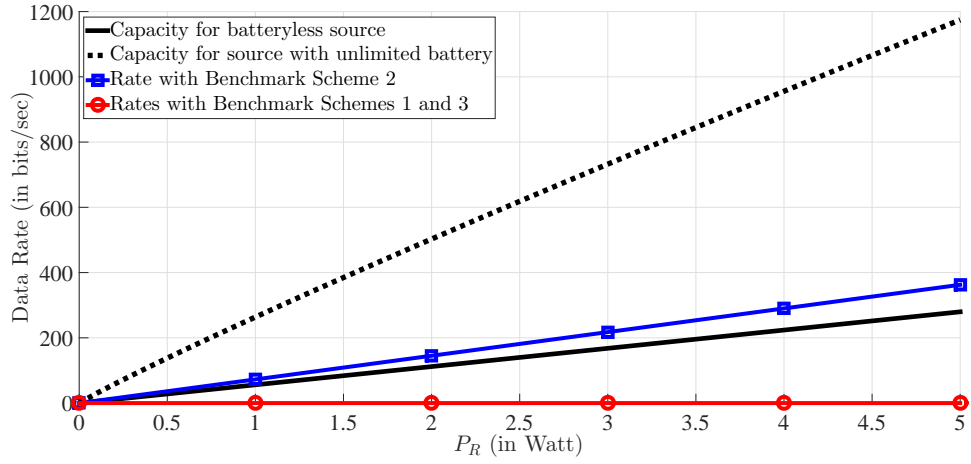


Fig. 3: Comparison of capacities and achievable rates of the benchmark schemes as a function of the relay's power P_R in Watt for $d_{SR} = 10$ meter.

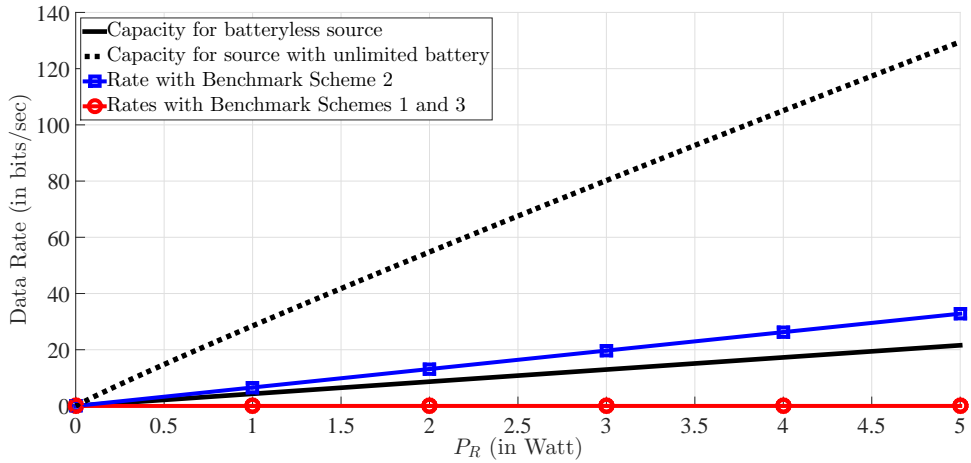


Fig. 4: Comparison of capacities and achievable rates of the benchmark schemes as a function of the relay's power P_R in Watt for $d_{SR} = 20$ meter.

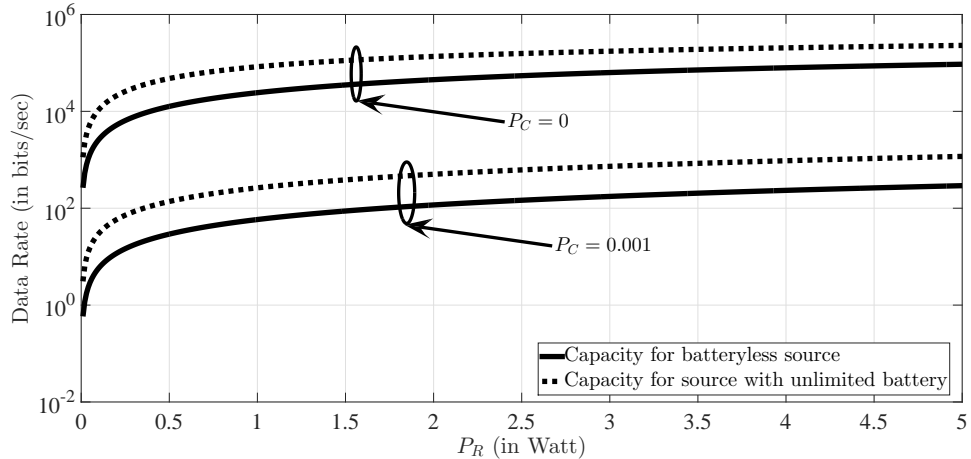


Fig. 5: Comparison of capacities for different energy transmission costs P_C as a function of the relay's power P_R in Watt.

we show the capacities¹³ for $P_C = 1$ mWatt and $P_C = 0$ Watt (zero energy transmission cost¹⁴), for the case when the distance between the source and the relay is $d_{SR} = 10$ meter. The figure shows that a non-zero energy cost has a severe impact on the channel capacity. In particular, for the considered parameters, the capacities for $P_C = 0$ Watt are approximately 10^3 times higher than the capacities for $P_C = 1$ mWatt. Hence, for the considered relay channel, any approximation of the achievable data rates made by neglecting the energy transmission cost can result in a severe overestimation of the achievable performance.

VI. CONCLUSION

We have derived the capacity of a two-hop relay channel impaired by AWGN, where an EH source is powered wirelessly by an out-of-band FD relay. We assumed that the relay has an average transmit power constraint whereas the source has an energy transmission cost constraint. Moreover, we considered two extreme cases for the battery at the EH source, a batteryless source and a source equipped with an unlimited battery. For both cases, we showed that in order to achieve the capacity of the considered relay channel, the source has to harvest the RF energy that reaches the source when the relay transmits information to the destination. Moreover, for both considered cases, we demonstrated that the capacity-achieving distribution at the source is discrete, whereas the capacity-achieving distribution at the relay can either be discrete or zero-mean Gaussian. Furthermore, our results revealed that the use of suboptimal input distributions may incur a severe degradation in performance, and neglecting the energy transmission cost at the source can result in a severe overestimation of the achievable performance.

APPENDIX

A. Proof of Lemma 1

Since the considered relay channel belongs to the class of degraded relay channels, its capacity is given by the capacity expression for the degraded relay channel in [23]. Taking the amplitude constraint at the source, given by (9), and the average power constraint at the relay, P_R , into account,

¹³We note that the rates in Figs. 3 and 4 appear to be a linear function of P_R since the source-relay channel operates in the low SNR regime due to the high-path loss attenuation associated with WET. The rates in Fig. 5 are also a linear function of P_R , however, since the y-axis in Fig. 5 is given in the logarithmic scale, this linearity is not obvious.

¹⁴For the adopted set of parameters, in the case when $P_C = 0$, the expression for the channel capacity for the batteryless case is given by Case 1 in Corollary 3 with $P_C = 0$, whereas, the capacity for the case of a source with unlimited battery is given by (34) with $I(X_S; Y_R)|_{p(x_S)=p^*(x_S)} = \frac{1}{2} \log_2 \left(1 + \frac{\eta h_{SR}^2 P_R}{\sigma_D^2} \right)$.

the capacity can be expressed in the following general form

$$C = \max_{p(v_S, x_R) \in \mathcal{P}} \min\{I(V_S; Y_R | X_R, f(X_R)); I(X_R; Y_D)\}$$

Subject to C1 : $-1 \leq V_S \leq 1$,

$$C2 : \sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) \leq P_R. \quad (39)$$

Now, since $f(X_R)$ is a deterministic function of X_R , conditioning the mutual information on X_R and $f(X_R)$ is equivalent to conditioning only on X_R . As a result, the conditioning on $f(X_R)$ in (39) can be removed. To further simplify the capacity expression in (39), note that $p(v_S, x_R)$ can be written as $p(v_S, x_R) = p(v_S | x_R)p(x_R)$. As a result, the maximization over $p(v_S, x_R)$ can be replaced by two nested maximizations, one with respect to $p(v_S | x_R)$ for a fixed $p(x_R)$, and the other one with respect to $p(x_R)$. Thereby, (39), with the conditioning on $f(X_R)$ removed, can be written equivalently as

$$C = \max_{p(x_R) \in \mathcal{P}} \max_{p(v_S | x_R) \in \mathcal{P}} \min\{I(V_S; Y_R | X_R); I(X_R; Y_D)\}$$

Subject to C1 : $-1 \leq V_S \leq 1$,

$$C2 : \sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) \leq P_R. \quad (40)$$

Now, in the capacity expression in (40), note that only $I(V_S; Y_R | X_R)$ depends on $p(v_S | x_R)$ whereas $I(X_R; Y_D)$ is not dependent on $p(v_S | x_R)$. As a result, the capacity expression in (40) can be further simplified as in (11), where we have exploited the identity $I(V_S; Y_R | X_R) = \sum_{x_R \in \mathcal{X}_R} I(V_S; Y_R | X_R = x_R)p(x_R)$.

B. Proof of Lemma 2

The optimization problem in (11) with respect to $p(v_S | x_R)$ can be resolved into the following, much simpler, optimization problem

$$\max_{p(v_S | x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R)$$

Subject to C1 : $-1 \leq V_S \leq 1$. (41)

On the other hand, from [26] it is known that the optimal input distribution that maximizes the mutual information of a point-to-point AWGN channel with a fixed channel gain and an amplitude constraint imposed at the transmitter is discrete with a finite number of probability mass points. Since $I(V_S; Y_R | X_R = x_R)$ in (41) is the mutual information of the source-relay AWGN channel for a fixed channel gain $\sqrt{f(X_R = x_R)}$ and an amplitude constraint given by C1 in (41), the optimal

input distribution $p^*(v_S|x_R)$ obtained as the solution of (41) is discrete with a finite number of probability mass points. As a result, $p^*(v_S|x_R)$ can be written in a general form as (12).

Now, to obtain $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R)$, which is $I(V_S; Y_R|X_R = x_R)$ for $p^*(v_S|x_R)$ given in (12), we use the following identity

$$I(V_S; Y_R|X_R = x_R) = h(Y_R|X_R = x_R) - h(Y_R|V_S, X_R = x_R), \quad (42)$$

where $h(\cdot|\cdot)$ denotes the conditional differential entropy [29]. In (42), $h(Y_R|V_S, X_R = x_R)$ is the differential entropy of the AWGN at the relay, cf. Fig. 2, which is given by

$$h(Y_R|V_S, X_R = x_R) = \frac{1}{2} \log_2 (2\pi e \sigma_R^2). \quad (43)$$

On the other hand, we can obtain $h(Y_R|X_R = x_R)$ by definition as [29]

$$h(Y_R|X_R = x_R) = - \int_{-\infty}^{\infty} \sum_{v_S \in V_S} p(y_R|v_S, x_R) p(v_S|x_R) \log_2 \left(\sum_{v_S \in V_S} p(y_R|v_S, x_R) p(v_S|x_R) \right) dy_R, \quad (44)$$

where the optimal $p(v_S|x_R)$ is given by (12) and $p(y_R|v_S, x_R)$ is a Gaussian distribution with variance σ_R^2 and mean $v_S \sqrt{f(x_R)}$, cf. Fig. 2. Inserting (43) and (44) into (42), we finally obtain

$$\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R) \text{ as in (13).}$$

C. Proof of Theorem 1

The maximization problem in (11) can be written equivalently using the epigraph form as

$$\begin{aligned} & \underset{u, p(x_R)}{\text{Maximize}} && u \\ & \text{Subject to C1:} && u - \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R) p(x_R) \leq 0 \\ & && \text{C2: } u - I(X_R; Y_D) \leq 0 \\ & && \text{C3: } \sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) - P_R \leq 0 \\ & && \text{C4: } \sum_{x_R \in \mathcal{X}_R} p(x_R) - 1 = 0. \end{aligned} \quad (45)$$

The optimization problem in (45) is a concave optimization problem since constraints C1, C3, and C4 are all affine with respect to $p(x_R)$ and constraint C2 is convex with respect to $p(x_R)$. Hence, (45) can be solved using the Lagrangian method [32]. The Lagrange function for optimization problem (45) is given by

$$\begin{aligned} \mathcal{L} = & u - \alpha_1 \left(u - \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R|X_R = x_R) p(x_R) \right) - \alpha_2 (u - I(X_R; Y_D)) \\ & - \lambda \left(\sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) - P_R \right) - \xi \left(\sum_{x_R \in \mathcal{X}_R} p(x_R) - 1 \right), \end{aligned} \quad (46)$$

where α_1 , α_2 , λ , and ξ are Lagrange multipliers, which have to satisfy the following Karush-Kuhn-Tucker (KKT) conditions

$$\alpha_1 \left(u - \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) p(x_R) \right) = 0 \text{ and } \alpha_1 \geq 0, \quad (47a)$$

$$\alpha_2 (u - I(X_R; Y_D)) = 0 \text{ and } \alpha_2 \geq 0, \quad (47b)$$

$$\lambda \left(\sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) - P_R \right) = 0 \text{ and } \lambda \geq 0, \quad \xi \left(\sum_{x_R \in \mathcal{X}_R} p(x_R) - 1 \right) = 0 \text{ and } \xi \geq 0. \quad (47c)$$

Differentiating \mathcal{L} with respect to u and equating the result to zero, we obtain that $\alpha_1 = 1 - \alpha_2 = \alpha$ has to hold in order to have a bounded solution for the dual problem (46), where $0 \leq \alpha \leq 1$. To obtain the maximum of \mathcal{L} with respect to $p(x_R)$, we need to obtain the derivative of \mathcal{L} with respect to $p(x_R)$, denoted by $\partial \mathcal{L} / \partial p(x_R)$, and equate it to zero. Thereby, we obtain

$$\alpha \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) - (1 - \alpha) I'(X_R; Y_D) - \lambda x_R^2 - \xi = 0, \quad (48)$$

where $I'(X_R; Y_D) = \frac{\partial}{\partial p(x_R)} I(X_R; Y_D)$ is given by

$$I'(X_R; Y_D) = \int_{y_D} p(y_D | x_R) p(x_R) \log_2 \left(\frac{p(y_D | X_R)}{p(y_D)} \right) - \frac{1}{\ln(2)}. \quad (49)$$

We note that there are three possible solutions for (48) depending on whether $\alpha = 0$, $\alpha = 1$, or $0 < \alpha < 1$, respectively. In the following, we analyze these solutions.

If $\alpha = 0$, we obtain from (47a) that for the optimal $p^*(x_R)$, the following has to hold

$$u < \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) p^*(x_R) \quad \text{and} \quad u = I(X_R; Y_D) \Big|_{p(x_R)=p^*(x_R)}, \quad (50)$$

which is equivalent to

$$u = I(X_R; Y_D) \Big|_{p(x_R)=p^*(x_R)} < \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) p^*(x_R). \quad (51)$$

Hence, when $\alpha = 0$, in order to maximize u , we need to maximize $I(X_R; Y_D)$. As a result, the optimal $p^*(x_R)$ in this case is found as the distribution which maximizes $I(X_R; Y_D)$ with the average power constraint P_R imposed. For the AWGN channel, this distribution is known and is the zero-mean Gaussian distribution with variance P_R , for which $I(X_R; Y_D)$ is given by [29]

$$I(X_R; Y_D) = \frac{1}{2} \log_2 \left(1 + \frac{P_R}{\sigma_D^2} \right). \quad (52)$$

Averaging $I(V_S; Y_R | X_R = x_R)$, given in (13), with respect to the zero-mean Gaussian distribution with variance P_R , and inserting the result along with (52) into (51), we obtain Case 2 in Theorem 1.

On the other hand, if $\alpha = 1$, we obtain from (47a) that for the optimal $p(x_R)$, the following has to hold

$$u = \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) p^*(x_R) \quad \text{and} \quad u < I(X_R; Y_D) \Big|_{p(x_R)=p^*(x_R)}, \quad (53)$$

which is equivalent to

$$u = \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) p^*(x_R) < I(X_R; Y_D) \Big|_{p(x_R)=p^*(x_R)}. \quad (54)$$

Hence, when $\alpha = 1$, in order to maximize u , we need to maximize $\sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R)} I(V_S; Y_R | X_R = x_R) p(x_R)$ and for the resulting $p^*(x_R)$, (54) should hold, which leads to Case 1 in Theorem 1.

Finally, if $0 < \alpha < 1$, then for the optimal $p^*(x_R)$ the following holds

$$u = I(X_R; Y_D) \Big|_{p(x_R)=p^*(x_R)} = \sum_{x_R \in \mathcal{X}_R} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) p^*(x_R). \quad (55)$$

Considering that α in (48) satisfies $0 < \alpha < 1$, we can write (48) equivalently as

$$I'(X_R; Y_D) = -\frac{\alpha}{1-\alpha} \max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R) + \frac{\lambda}{1-\alpha} x_R^2 + \frac{\xi}{1-\alpha}. \quad (56)$$

Now, using the approach in [33] it can be shown that the $p^*(x_R)$ that satisfies (56) cannot be a continuous distribution and can only be discrete. Combining this with (55), we obtain Case 3 in Theorem 1.

Remark 7: Although we derived (48) assuming that $p(x_R)$ is discrete, we would have arrived at the same result if we had assumed that $p(x_R)$ was a continuous distribution. To this end, we first would have to replace the sums in the optimization problem in (45) with integrals with respect to x_R . Next, in order to obtain the stationary points of the corresponding Lagrangian function, instead of the ordinary derivative, we would have to take the functional derivative and equate it to zero. This again would lead to the identity in (48). Hence, the conclusions drawn from the Lagrangian and (48) are also valid when $p(x_R)$ is a continuous distribution.

D. Proof of Corollary 3

Corollary 3 follows by solving (19) for $\max_{p(v_S|x_R) \in \mathcal{P}} I(V_S; Y_R | X_R = x_R)$ given in (17) and $E_H(x_R)$ given in (5). The corresponding Lagrangian of this optimization problem is

$$\begin{aligned} \mathcal{L} = & \sum_{x_R \in \mathcal{X}_R} \frac{1}{2} \log_2 \left(1 + \frac{2 \max(0, \eta h_{RS}^2 x_R^2 - P_C)}{\pi e \sigma_R^2} \right) p(x_R) - \lambda \left(\sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) - P_R \right) \\ & - \xi \left(\sum_{x_R \in \mathcal{X}_R} p(x_R) - 1 \right). \end{aligned} \quad (57)$$

Differentiating \mathcal{L} with respect to $p(x_R)$ and equating the result to zero, we obtain that for $0 < p(x_R) < 1$, the following has to hold

$$\frac{1}{2} \log_2 \left(1 + \frac{2 \max(0, \eta h_{RS}^2 x_R^2 - P_C)}{\pi e \sigma_R^2} \right) = \lambda x_R^2 + \xi. \quad (58)$$

Now, from (58) we can see that if $\xi \neq 0$, then (58) does not hold for $x_R = 0$. Furthermore, for $\xi \neq 0$, (58) has only two solutions which are in the form $\pm x_R^*$. Since there are only two solutions for x_R , in order for $p(x_R)$ to be a valid distribution, $p(x_R^*) = 1 - p(-x_R^*)$ has to hold. Moreover, in order for C1 in (19) to hold $x_R^* = \sqrt{P_R}$ has to hold. Hence, one possible solution for $p(x_R)$ is given in (24) and this solution is possible if

$$\frac{1}{2} \log_2 \left(1 + \frac{2 \max(0, \eta h_{RS}^2 P_R - P_C)}{\pi e \sigma_R^2} \right) = \lambda P_R + \xi \quad (59)$$

holds or equivalently if $\eta h_{RS}^2 P_R > P_C$ holds. On the other hand, if $\eta h_{RS}^2 P_R > P_C$ does not hold, then $\xi = 0$ and the following has to hold

$$\frac{1}{2} \log_2 \left(1 + \frac{2 \max(0, \eta h_{RS}^2 x_R^2 - P_C)}{\pi e \sigma_R^2} \right) = \lambda x_R^2. \quad (60)$$

Now, (60) can be solved in closed form. In particular, we obtain three solutions for x_R , $x_R = 0$, $x_R = -x_0$, and $x_R = x_0$, where x_0 is given in (27). For these three values of x_R , we can find $p(x_R)$ from constraint C1 in (19) and from the constraint that $p(x_R)$ has to be a valid probability distribution. This concludes the proof.

E. Proof of Lemma 3

If $E_H(x_R)$ is given by (5), then constraint C2 in (33) has to hold with equality since $\max_{p(x_R)} I(X_R; Y_D)$ is a non-decreasing function of P_R . As a result, we obtain the right hand side of constraint C1 in (33) as $\sum_{x_R \in \mathcal{X}_R} E_H(x_R) p(x_R) = \eta h_{RS}^2 P_R$. Since $\max_{p(x_S)} I(X_S; Y_R)$ is also a non-decreasing function of P_R , constraint C1 in (33) also has to hold with equality. Consequently, the optimization problem in (33) can be decomposed into two optimization problems. The first optimization problem is $\max_{p(x_R)} I(X_R; Y_D)$ subject to $\sum_{x_R \in \mathcal{X}_R} x_R^2 p(x_R) = P_R$, whose solution for $p^*(x_R)$ is the zero-mean Gaussian distribution with variance P_R and consequently $\max_{p(x_R) \in \mathcal{P}} I(X_R; Y_D) = 1/2 \log_2(1 + P_R/\sigma_D^2)$. On the other hand, the second optimization problem is given by (35). It is proven in [5] that the optimal distribution $p^*(x_S)$, obtained as the solution of (35), is discrete. As a result, this distribution can be found by solving the concave optimization problem in (35) numerically using numerical optimization software such as Mathematica. Combining the above results, for $E_H(x_R)$ given by (5), we obtain the capacity as (34).

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