# Errata to " Asymptotic Achievability of the Cramér -Rao Bound for Noisy Compressive Sampling "

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***Abstract*—Given $N$ noisy measurements denoted by $\mathbf{y}$ and an overcomplete Gaussian dictionary, $\mathbf{A}$, the authors in [1] establish the existence and the asymptotic statistical efficiency of an unbiased estimator unaware of the locations of the non-zero entries, collected in set $\mathcal{I}$, in the deterministic $L$-sparse signal $\mathbf{x}$. More precisely, there exists an estimator $\hat{\mathbf{x}}(\mathbf{y}, \mathbf{A})$ unaware of set $\mathcal{I}$ with a variance reaching the oracle-CRB (Cramér-Rao Bound) in the doubly asymptotic scenario, *i.e.*, for $N, L \to \infty$ and $L/N \to \alpha \in (0, 1)$. As was noted in [2] the result remains true even though the proposed closed-form expression of the variance of the estimator $\hat{\mathbf{x}}(\mathbf{y}, \mathbf{A})$ is incorrect. In this note, we correct this expression by providing an explicit formula and discuss its practical usefulness. Finally, the new expression allows to correct the misleading comprehension of the sparse signal estimation performance suggested in [1].**

## I. Main result of [1]

Let $\mathbf{y}$ be the $N \times 1$ noisy measurement vector given by

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$

where $\mathbf{A}$ is a non-stochastic $N \times M$ matrix with controlled growing dimensions according to $\lim_{N,L\to\infty} L/N = \alpha \in (0, 1)$. An entry of matrix $\mathbf{A}$ is generated as a single realization of an i.i.d. Normal distribution $\mathcal{N}(0, 1)$, $\mathbf{x}$ is a deterministic $L$-sparse vector on set $\mathcal{I}$ and $\mathbf{n}$ is a centered circular white Gaussian noise of variance $\sigma^2$. The definition of the oracle (doubly) asymptotic CRB is given hereafter.

*Definition 1.1:* The oracle-CRB in the doubly asymptotic scenario is defined according to

$$\mathcal{C}^\infty \stackrel{\text{def.}}{=} \lim_{N,L\to\infty} \mathcal{C}_\mathcal{I} \quad \text{s.t.} \quad \frac{L}{N} \to \alpha \in (0, 1)$$

where $\mathcal{C}_\mathcal{I}$ stands for the oracle-CRB for finite $N$ and $L$ on vector $\mathbf{x}_\mathcal{I}$ consisting of the $L$ non-zero values in $\mathbf{x}$. The term "oracle" means that the knowledge of support $\mathcal{I}$ is *a priori* provided thanks to a genie.

*Definition 1.2:* For an unbiased oracle-estimator, denoted by $\hat{\mathbf{x}}(\mathbf{y}, \mathbf{A}_\mathcal{I})$, and for an unbiased estimator $\hat{\mathbf{x}}(\mathbf{y}, \mathbf{A})$ *unaware* of the set $\mathcal{I}$, their MSE are respectively defined according to

$$e_G \stackrel{\text{def.}}{=} \lim_{N,L\to\infty} \mathbb{E}_{\mathbf{y}|\mathbf{x},\mathbf{A}_\mathcal{I}} \|\mathbf{x}_\mathcal{I} - \hat{\mathbf{x}}(\mathbf{y}, \mathbf{A}_\mathcal{I})\|^2$$

$$e_\delta \stackrel{\text{def.}}{=} \lim_{N,L\to\infty} \mathbb{E}_{\mathbf{y}|\mathbf{x},\mathbf{A}} \|\mathbf{x} - \hat{\mathbf{x}}(\mathbf{y}, \mathbf{A})\|^2$$

subject to $\frac{L}{N} \to \alpha \in (0, 1)$.

Using the above definitions, the following relation holds $e_\delta \geq e_G \geq \mathcal{C}^\infty$. As an analytical expression of $e_\delta$ cannot be directly derived, the authors in [1] show that an upper bound on $e_\delta$ is given by $\mathcal{C}^\infty$. As a consequence, it is proved that the MSE for an estimator unaware of set $\mathcal{I}$ is lower and upper bounded by the doubly asymptotic oracle-CRB given in definition 1.1. This proves the important result that there exists a sparse-based estimator unaware of set $\mathcal{I}$ that meats the oracle-CRB in the doubly asymptotic scenario.

In this comment note, a corrected expression for the oracle-CRB, $\mathcal{C}^\infty$, is proposed. Note that this comment correspondance is also of interest for reference [3] which suffers from the same problem.

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## II. Corrected Oracle-CRB

### A. Oracle-CRB derivation

The oracle-CRB for finite $N$ and $L$ admits the following relation [4,5]: $\mathcal{C}_\mathcal{I} \stackrel{\text{def.}}{=} \text{Tr}[\mathbf{F}_\mathcal{I}^{-1}]$ in which $\mathbf{F}_\mathcal{I}$ is the $L \times L$ Fisher Information Matrix (FIM). The FIM is defined as the variance of the score function according to

$$\begin{aligned}
[\mathbf{F}_\mathcal{I}]_{ij} &= \left[\text{Var}\left(\frac{\partial \log p(\mathbf{y}|\mathbf{x}_\mathcal{I}, \mathbf{A}_\mathcal{I})}{\partial \mathbf{x}_\mathcal{I}}\right)\right]_{ij} \\
&\stackrel{\text{def.}}{=} \mathbb{E}_{\mathbf{y}|\mathbf{x}_\mathcal{I},\mathbf{A}_\mathcal{I}}\left(\frac{\partial \log p(\mathbf{y}|\mathbf{x}_\mathcal{I}, \mathbf{A}_\mathcal{I})}{\partial x_i}\frac{\partial \log p(\mathbf{y}|\mathbf{x}_\mathcal{I}, \mathbf{A}_\mathcal{I})}{\partial x_j}\right) \\
&- \mathbb{E}_{\mathbf{y}|\mathbf{x}_\mathcal{I},\mathbf{A}_\mathcal{I}}\left(\frac{\partial \log p(\mathbf{y}|\mathbf{x}_\mathcal{I}, \mathbf{A}_\mathcal{I})}{\partial x_i}\right) \\
&\times \mathbb{E}_{\mathbf{y}|\mathbf{x}_\mathcal{I},\mathbf{A}_\mathcal{I}}\left(\frac{\partial \log p(\mathbf{y}|\mathbf{x}_\mathcal{I}, \mathbf{A}_\mathcal{I})}{\partial x_j}\right),
\end{aligned}$$

where $i, j \in \mathcal{I}$ and $p(\mathbf{y}|\mathbf{x}_\mathcal{I}, \mathbf{A}_\mathcal{I})$ is assumed to be of class $C^1$. Thanks to the model assumptions in [1], $\mathbf{y}|\mathbf{x}_\mathcal{I}, \mathbf{A}_\mathcal{I} \sim \mathcal{N}(\mathbf{A}_\mathcal{I}\mathbf{x}_\mathcal{I}, \sigma^2\mathbf{I})$ and the score function is zero-mean and of class $C^2$. Consequently, the oracle-CRB can be simplified according to

$$\begin{aligned}
\mathcal{C}_\mathcal{I} = \text{Tr}\left[\mathbf{F}_\mathcal{I}^{-1}\right] &= \text{Tr}\left[\left(\mathbb{E}_{\mathbf{y}|\mathbf{x}_\mathcal{I},\mathbf{A}_\mathcal{I}}\left(-\frac{\partial^2 \log p(\mathbf{y}|\mathbf{A}_\mathcal{I}, \mathbf{x}_\mathcal{I})}{\partial \mathbf{x}_\mathcal{I}\partial \mathbf{x}_\mathcal{I}^T}\right)\right)^{-1}\right] \\
&= \sigma^2 \text{Tr}\left[\left(\mathbf{A}_\mathcal{I}^T\mathbf{A}_\mathcal{I}\right)^{-1}\right]. \qquad (1)
\end{aligned}$$

The last expression is obtained thanks to the Slepian-Bang formula [2,5,6].

### B. Closed-form expression in the doubly asymptotic scenario

In this section, we leverage on random matrix results to propose the corrected expression of the oracle-CRB.

*Result 2.1:* The oracle-CRB in the doubly asymptotic scenario takes the following closed-form expression:

$$\mathcal{C}^\infty = \sigma^2 \frac{\alpha}{1 - \alpha} \qquad (2)$$

in almost sure convergence.

**Proof** Define the matrix $\mathbf{Z}_\mathcal{I} = \frac{1}{\sqrt{N}}\mathbf{A}_\mathcal{I}$ where the i.i.d. entries of matrix $\mathbf{Z}_\mathcal{I}$ follow the distribution $\mathcal{N}(0, 1/N)$. According to [7], the empirical distribution of the eigenvalues of $\mathbf{Z}_\mathcal{I}^T\mathbf{Z}_\mathcal{I}$ converges almost surely to the Marcenko-Pastur distribution on the interval $\left[(1 - \sqrt{\alpha})^2, (1 + \sqrt{\alpha})^2\right]$. This implies that the $\mathbf{Z}_\mathcal{I}^T\mathbf{Z}_\mathcal{I}$ in the doubly asymptotic scenario is almost surely full-rank, and thus its matrix inverse exists. Using eq. (1), observe that

$$\mathcal{C}^{\infty} = \lim_{N,L\to\infty} \frac{\sigma^2}{N} \mathrm{Tr}\left[ (\mathbf{Z}_{\mathcal{I}}^T \mathbf{Z}_{\mathcal{I}})^{-1} \right].$$

Finally, according to [8], the asymptotic inverse moment is given by $\mathrm{Tr}\left[ (\mathbf{Z}_{\mathcal{I}}^T \mathbf{Z}_{\mathcal{I}})^{-1} \right]/L \to 1/(1-\alpha)$ in almost sure convergence and the claim is thus proved. □

### C. Validity of eq. (20) in [1]

The validity of eq. (20) in [1] in the derivation of the upper bound on the MSE is crucial. So, there is a need to make the assumptions more precise on $\mathcal{C}_{\mathcal{I}}$ for the validity of eq. (20) in [1]. Note that $\mathcal{C}_{\mathcal{I}}$ converges almost surely to $\mathcal{C}^{\infty}$. This implies the convergence in probability [9]. In addition, if we assume that $\mathcal{C}_{\mathcal{I}}$ is uniformly integrable, then thanks to the generalized dominated convergence theorem [10], the convergence in mean holds, *i.e.*,

$$\lim_{N,L\to\infty} \mathbb{E}_{\mathbf{A}_{\mathcal{I}}} \mathcal{C}_{\mathcal{I}} = \mathbb{E}_{\mathbf{A}_{\mathcal{I}}} \left( \lim_{N,L\to\infty} \mathcal{C}_{\mathcal{I}} \right) = \mathbb{E}_{\mathbf{A}_{\mathcal{I}}} \mathcal{C}^{\infty} = \sigma^2 \frac{\alpha}{1-\alpha}$$

since $\mathcal{C}^{\infty}$ is not a function of $\mathbf{A}_{\mathcal{I}}$ but only of the asymptotic ratio $\alpha$.

### D. Qualitative analysis and numerical illustrations

In [1], it is claimed that the oracle-CRB is given by $\sigma^2\alpha$. This expression would mean that the variance of the estimator $\hat{\mathbf{x}}(\mathbf{y},\mathbf{A})$ is always lower or equal to the noise variance for any $\alpha$. In particular, for $\alpha \to 1$ and $\sigma^2 \to 0$, the variance of the estimator $\hat{\mathbf{x}}(\mathbf{y},\mathbf{A})$ would converge to zero while the degree of freedom (DoF) per measurement[1] tends to zero. So, the incorrect oracle-CRB expression suggests the too optimistic and misleading idea that the considered estimator always exhibits a low and finite estimation accuracy even for low DoF where the number of unknown parameters and measurements are almost identical. Conversely, the corrected oracle-CRB given in Result 2.1 allows a correct qualitative analysis of the underlying sparse estimation problem as shown in Fig. 1.

In Fig. 2, the oracle-CRB expression given in eq. (1) is compared to the expression in eq. (2) with respect to $1/\sigma^2$ in dB and for relatively small numerical values of $L$ and $N$. Note that the two bounds are in good agreement and thus the corrected oracle-CRB remains an accurate analytical expression even if the doubly asymptotic scenario is not rigorously respected. This numerical illustration, in our view, is important from an operational point of view.

## III. Conclusion

In this note, a corrected explicit formula for the doubly asymptotic oracle-CRB involved in [1] is derived. Regarding the original article, the main result on the existence and the efficiency of an unbiased estimator unaware of the locations of the nonzero elements remains correct but the derivation of its variance, given by the oracle-CRB, is incorrect. As illustrated in this comment correspondence, this erroneous variance was too optimistic and always lower bounded by the noise variance even for a low DoF. This produces a fundamentally misleading comprehension of the estimation performance limit of sparse signals. The corrected expression solves this issue and its practical usefulness is illustrated.

[1]Each unknown parameter can be seen as an additional DoF while each measurement can be viewed as a constraint that restricts the DoF. The asymptotic DoF per measurement of a linear system is given by $1-\frac{L}{N} \to 1-\alpha$. An unfavorable DoF is close to zero meaning that the number of unknown/desired parameters and measurements are almost equal. In contrast, a DoF close to one means that we dispose of a large number of measurements to estimate few unknown parameters.

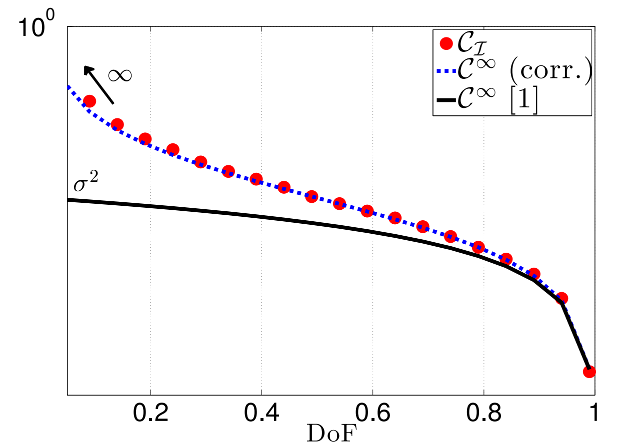


Fig. 1. Comparaison of the corrected and the erroneous oracle-CRB *vs.* the asymptotic DoF per measurement with $\sigma^2 = 1e-2$ with $M = 300$ and $N = 100$.

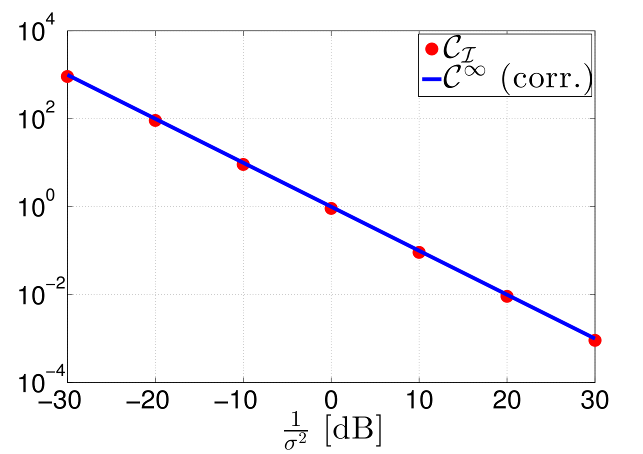


Fig. 2. Lower bounds *vs.* $1/\sigma^2$ in dB with $M = 50$, $N = 20$ and $L = 10$.

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