Technical Report: An MGF-based Unified Framework to Determine the Joint Statistics of Partial Sums of Ordered i.n.d. Random Variables

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Abstract

The joint statistics of partial sums of ordered random variables (RVs) are often needed for the accurate performance characterization of a wide variety of wireless communication systems. A unified analytical framework to determine the joint statistics of partial sums of ordered independent and identically distributed (i.i.d.) random variables was recently presented. However, the identical distribution assumption may not be valid in several real-world applications. With this motivation in mind, we consider

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in this paper the more general case in which the random variables are independent but not necessarily identically distributed (i.n.d.). More specifically, we extend the previous analysis and introduce a new more general unified analytical framework to determine the joint statistics of partial sums of ordered i.n.d. RVs. Our mathematical formalism is illustrated with an application on the exact performance analysis of the capture probability of generalized selection combining (GSC)-based RAKE receivers operating over frequency-selective fading channels with a non-uniform power delay profile. We also discussed a couple of other sample applications of the generic results presented in this work.

Index Terms

Order statistics, Joint statistics, Non-identical distribution, Moment generating function (MGF), Probability density function (PDF), Exponential distribution.

I. Introduction

The subject of order statistics deals with the properties and distributions of the ordered random variables (RVs) and their functions. It has found applications in many areas of statistical theory and practice [1], with examples in life-testing, quality control, radar, as well as signal and image processing [2]–[8]. Order statistics has made over the last decade an increasing number of appearances in the design and analysis of wireless communication systems, specifically for the performance analysis of advanced diversity techniques, adaptive transmission techniques, and multiuser scheduling techniques (see for example [9]–[22]). In these performance analysis exercises, the joint statistics of partial sums of ordered RVs are often necessary for the accurate characterization of system performance [12], [19], [23]. Note that even if the original unordered RVs are independently distributed, their ordered versions are dependent due to the inequality relations among them, which makes it challenging to such joint statistics. Recently, a successive conditioning approach was used to convert dependent ordered random variables into independent unordered ones [10], [11]. However, this approach requires some case-specific manipulations, which may not always be generalizable.

Motivated by these facts, we introduced in [24] a unified analytical framework to determine the joint statistics of partial sums of ordered independent and identically distributed (i.i.d.) RVs by extending the interesting results published in [4], [25], [26]. More specifically, our approach can be applied not only to the cases when all the N ordered RVs are involved but also to the cases when only the N_s $(N_s < N)$ best RVs are considered. With the proposed approach, we can systematically derive the joint statistics of any partial sums of ordered statistics, in terms of the moment generating function (MGF) and the probability density function (PDF). These statistical results can be used for the performance analysis of various wireless communication systems over generalized fading channels [9]. However, the identical fading assumption on all diversity branches is not always valid in real-life applications. The average fading power may vary from one path to the other because the branches of a diversity system are sometimes unbalanced and the communication system is sometimes operating over frequency-selective channels with a non-uniform power delay profile or channel multipath intensity profile (i.e. the average SNR of the diversity paths are not necessary the same).

We therefore introduce in this paper an unified analytical framework to determine the joint statistics of partial sums of ordered independent non-identically distributed (i.n.d.) RVs by extending our previous work for i.i.d. fading scenarios [24]. More specifically, we use an MGF based systematic analytical approach to investigate the joint statistics of any partial sums of ordered statistics for general i.n.d. fading, in terms of MGF and the PDF. We would like to emphasize that such generalization.The main challenge for generalizing the work in [24] to i.n.d. general fading cases is that joint PDF of ordered i.n.d. RVs is much more complicated than that of ordered i.i.d. RVs. We need to carry out more detailed manipulation and introduce new mathematical representation to obtain the generic results (e.g. joint MGF and related joint PDF) for i.n.d. general cases in a compact form. In addition, we present the closed-form expressions for the exponential RV special case, which is most widely used in wireless literature. For other type of RVs, our approach will lead to much simpler results than the conventional approach involving multiple-fold integration. Furthermore, the exponential distribution is frequently used in the performance evaluation analysis of networks and telecommunication systems. It is also used to model the waiting times between occurrences of rare events, lifetimes of electrical or mechanical devices [2], [3], [27], [28]. Finally, as an application of our analytical framework, we generalize the performance results of GSC-based RAKE receivers in [23] by maintaining the assumption of independence among the diversity paths but relaxing the identically distributed assumption. We also discussed a couple of other sample applications of the generic results presented in this work.

II. Problem Statement and Main Idea

Order statistics deals with the distributions and statistical properties of the new random variables obtained after ordering the realizations of some random variables. Let $\{\gamma_{i_l}\}$, $i_l = 1, 2, \cdots, N$ denote N i.n.d. nonnegative random variables with PDF $p_{i_l}(\cdot)$ and CDF $P_{i_l}(\cdot)$. Let u_i denote the random variable corresponding to the i-th largest observation of the N original random variables (also called i-th order statistics), such that $u_1 \geq u_2 \geq \cdots \geq u_N$. The N-dimensional joint PDF of the ordered RVs $\{u_i\}_{i=1}^{N}$ is given by [1]

$$g(u_1, u_2, \ldots, u_N) = \sum_{\substack{i_1, i_2, \ldots, i_N \\ i_1 \neq i_2 \neq \cdots \neq i_N}}^{1,2,\ldots,N} p_{i_1}(u_1)\, p_{i_2}(u_2) \cdots p_{i_N}(u_N). \tag{1}$$

Similarly, the N_s-dimensional joint PDF of $\{u_i\}_{i=1}^{N_s}$ is given by [1]

$$\begin{aligned} g\left(u_1, u_2, \cdots, u_{N_s}\right) &= \sum_{\substack{i_1, i_2, \cdots, i_N \\ i_1 \neq i_2 \neq \cdots \neq i_N}}^{1,2,\cdots,N} p_{i_1}\left(u_1\right) p_{i_2}\left(u_2\right) \cdots p_{i_{N_s}}\left(u_{N_s}\right) \prod_{j=N_s+1}^{N} P_{i_j}\left(u_{N_s}\right) \\ &\text{or} \\ &= \sum_{\substack{i_1, i_2, \cdots, i_N \\ i_1 \neq i_2 \neq \cdots \neq i_{N_s}}}^{1,2,\cdots,N} p_{i_1}\left(u_1\right) p_{i_2}\left(u_2\right) \cdots p_{i_{N_s}}\left(u_{N_s}\right) \sum_{\substack{i_{N_s+1}, \cdots, i_N \\ i_{N_s+1} \neq \cdots \neq i_N \\ i_{N_s+1} \neq i_1, i_2, \cdots, i_{N_s} \\ \vdots \\ i_N \neq i_1, i_2, \cdots, i_{N_s}}}^{1,2,\cdots,N} \prod_{\substack{l=N_s+1 \\ \{i_{N_s+1}, \cdots, i_N\}}}^{N} P_{i_l}\left(u_{N_s}\right). \end{aligned} \quad (2)$$

The objective is to derive the joint PDF of partial sums involving either all N or the first N_s ($N_s < N$) ordered RVs for the more general case in which the diversity paths are independent but not necessarily identically distributed. Similar to [24], we adopt a general two-step approach:

- Step I: Obtain the analytical expressions of the joint MGF of partial sums (not necessarily the partial sums of interest as will be seen later).
- Step II: Apply inverse Laplace transform to derive the joint PDF of partial sums (additional integration may be required to obtain the desired joint PDF).

In step I, by interchanging the order of integration, while ensuring each pair of limits is chosen to be as tight as possible, the multiple integral can be rewritten into compact equivalent representations. After obtaining the joint MGF in a compact form, we can derive joint PDF of selected partial sum through inverse Laplace transform. For most cases of our interest, the joint MGF involves basic functions, for which the inverse Laplace transform can be calculated analytically. In the worst case, we may rely on the Bromwich contour integral. In most of the case, the result involves a single one-dimensional contour integration, which can be easily and accurately evaluated numerically with the help of integral tables [29], [30] or using standard mathematical packages such as Mathematica and Matlab.

The above general steps can be directly applied when all N ordered RVs are considered and the RVs in the partial sums are continuous. When either of these conditions do not hold, we need to apply some extra steps in the analysis in order to obtain a valid joint MGF [24]. For example, when the RVs involved in one partial sum is not continuous, i.e., separated by the other RVs, we need to divide these RVs into smaller sums. For example in Fig. 1, we consider 3-dimensional joint PDF of $\{\gamma_{1:K}, \gamma_{2:K}, \gamma_{5:K}, \gamma_{6:K}\}$, $\{\gamma_{3:K}, \gamma_{4:K}\}$, and $\{\gamma_{7:K}, \gamma_{8:K}\}$ for $K > 8$. Note that the first group is not continuous. As a result, we will derive 5-dimensional joint MGF in step I, $\{\gamma_{1:K}, \gamma_{2:K}\}$, $\{\gamma_{3:K}, \gamma_{4:K}\}$, $\{\gamma_{5:K}, \gamma_{6:K}\}$, $\{\gamma_{7:K}\}$, $\{\gamma_{8:K}\}$. After the joint PDF of the new substituted partial sums are derived with inverse Laplace transform in step II, we can transform it to a lower dimensional desired joint PDF with finite integration.

III. Common Functions and Useful Relations

In the following sections, we present several examples to illustrate the proposed analytical framework. Our focus is on how to obtain compact expressions of the joint MGFs for i.n.d. general fading conditions, which can be greatly simplified with the application of the following function and relations.

A. Common Functions

i) A mixture of a CDF and an MGF $c_{i_l}(\gamma, \lambda)$:

$$c_{i_l}(\gamma, \lambda) = \int_0^{\gamma} p_{i_1}(x) \exp(\lambda x)\, dx, \tag{3}$$

where $p_{i_1}(x)$ denotes the PDF of the RV of interest. Note that $c_{i_l}(\gamma, 0) = c_{i_l}(\gamma)$ is the CDF and $c_{i_l}(\infty, \lambda)$ leads to the MGF. Here, the variable γ is real, while λ can be complex.

ii) A mixture of an exceedance distribution function (EDF) and an MGF, $e_{i_l}(\gamma, \lambda)$:

$$e_{i_l}(\gamma, \lambda) = \int_{\gamma}^{\infty} p_{i_1}(x) \exp(\lambda x)\, dx. \tag{4}$$

Note that $e_{i_l}(\gamma, 0) = e_{i_l}(\gamma)$ is the EDF while $e_{i_l}(0, \lambda)$ gives the MGF.

iii) An interval MGF $\mu_{i_l}(\gamma_a, \gamma_b, \lambda)$:

$$\mu_{i_l}(z_a, z_b, \lambda) = \int_{z_a}^{z_b} p_{i_1}(x) \exp(\lambda x)\, dx. \tag{5}$$

Note that $\mu_{i_l}(0, \infty, \lambda)$ gives the MGF.

Note that the functions defined in (3), (4) and (5) are related as follows:

$$\mu_{i_l}(z_a, z_b, \lambda) = c_{i_l}(z_b, \lambda) - c_{i_l}(z_a, \lambda) \tag{6}$$

$$= e_{i_l}(z_b, \lambda) - e_{i_l}(z_a, \lambda). \tag{7}$$

B. *Simplifying Relationship*

i) Integral J_m:

Based on the derivation given in Appendix I, the integral J_m defined as:

$$J_m = \sum_{\substack{i_m, i_{m+1}, \ldots, i_N \\ i_m \neq i_{m+1} \neq \cdots \neq i_N \\ i_m \neq i_1, i_2, \ldots, i_{m-1} \\ i_{m+1} \neq i_1, i_2, \ldots, i_{m-1} \\ \vdots \\ i_N \neq i_1, i_2, \ldots, i_{m-1}}}^{1,2,\ldots,N} \int_0^{u_{m-1}} du_m p_{i_m}(u_m) \exp(\lambda u_m) \int_0^{u_m} du_{m+1} p_{i_{m+1}}(u_{m+1}) \exp(\lambda u_{m+1}) \cdots \int_0^{u_{N-1}} du_N p_{i_N}(u_N) \exp(\lambda u_N), \tag{8}$$

can be simply expressed in terms of the function $c_{i_l}(\gamma, \lambda)$ as

$$J_m = \sum_{\{i_m, i_{m+1}, \ldots, i_N\} \in \mathrm{P}_{N-m+1}(I_N - \{i_1, i_2, \ldots, i_{m-1}\})} \prod_{\substack{l=m \\ \{i_m, i_{m+1}, \ldots, i_N\}}}^{N} c_{i_l}(u_{m-1}, \lambda). \tag{9}$$

In here, the complicated summation notation used in eq. (8) is simplified based on the following power set definition. We define index set I_N as $I_N = \{1, 2, \cdots, N\}$. The subset of I_N with n $(n \leq N)$ elements is denoted by $\mathcal{P}_n(I_N)$. The remaining index can be grouped in the set $I_N - \mathcal{P}_n(I_N)$. Based on these definitions, a summation in (8) includes all possible subsets of the index set I_N $(I_N = \{i_1, i_2, \cdots, i_N\})$ excluding the subset $\{i_1, i_2, \ldots, i_{m-1}\}$

with $N-(m-1)$ elements and these subsets with $N-(m-1)$ elements can be denoted by $\mathcal{P}_{N-m+1}\left(I_N-\{i_1,i_2,\ldots,i_{m-1}\}\right)$.

ii) Integral J'_m:

Following the similar derivation as given in Appendix II, the integral J'_m, defined as

$$J'_m = \sum_{\substack{i_1,i_2,\ldots,i_m \\ i_1\neq i_2\neq\cdots\neq i_m \\ i_1\neq i_{m+1},i_{m+2},\ldots,i_N \\ i_2\neq i_{m+1},i_{m+2},\ldots,i_N \\ \vdots \\ i_m\neq i_{m+1},i_{m+2},\ldots,i_N}}^{1,2,\ldots,N} \int_{u_{m+1}}^{\infty} du_m p_{i_m}(u_m)\exp(\lambda u_m) \int_{u_m}^{\infty} du_{m-1} p_{i_{m-1}}(u_{m-1})\exp(\lambda u_{m-1}) \cdots \int_{u_2}^{\infty} du_1 p_{i_1}(u_1)\exp(\lambda u_1), \tag{10}$$

can be simply re-written in terms of the function $e_{i_l}(\gamma,\lambda)$ with the help of the definition of power set used in III-B-i) as

$$J'_m = \sum_{\{i_1,i_2,\ldots,i_m\}\in \mathrm{P}_m\left(I_N-\{i_{m+1},i_{m+2},\ldots,i_N\}\right)} \prod_{\substack{l=1 \\ \{i_1,i_2,\ldots,i_m\}}}^{m} e_{i_l}(u_{m+1},\lambda). \tag{11}$$

iii) Integral $J''_{a,b}$:

Based on the derivation given in Appendix III, the integral $J''_{a,b}$, defined as

$$J'_{a,b} = \sum_{\substack{i_{a+1},\ldots,i_{b-1} \\ i_{a+1}\neq i_{a+2}\neq\cdots\neq i_{b-1} \\ i_{a+1}\neq i_1,\cdots,i_a,i_b,\ldots,i_N \\ i_{a+2}\neq i_1,\cdots,i_a,i_b,\ldots,i_N \\ \vdots \\ i_{b-1}\neq i_1,\cdots,i_a,i_b,\ldots,i_N}}^{1,2,\ldots,N} \int_{u_b}^{u_a} du_{b-1} p_{i_{b-1}}(u_{b-1})\exp(\lambda u_{b-1}) \int_{u_{b-1}}^{u_a} du_{b-2} p_{i_{b-2}}(u_{b-2})\exp(\lambda u_{b-2}) \cdots \int_{u_{a+2}}^{u_a} du_{a+1} p_{i_{a+1}}(u_{a+1})\exp(\lambda u_{a+1}), \tag{12}$$

can be simply re-written in terms of the function $\mu(\cdot,\cdot)$ as

$$J''_{a,b} = \sum_{\{i_{a+1},\ldots,i_{b-1}\}\in \mathrm{P}_{b-a+1}\left(I_N-\{i_1,\cdots,i_a,i_b,\ldots,i_N\}\right)} \prod_{\substack{l=a+1 \\ \{i_{a+1},\ldots,i_{b-1}\}}}^{b-1} \mu_{i_l}(u_b,u_a,\lambda) \qquad \text{for } b>a. \tag{13}$$

IV. Sample Cases when All N Ordered RVs are Considered

Theorem 4.1: (PDF of $\sum_{n=1}^{N} u_n$ among N ordered RVs)

Let $Z_1 = \sum_{n=1}^{N} u_n$ for convenience. We can derive the PDF of $Z = [Z_1]$ as

$$\begin{aligned} p_Z(z_1) &= L_{S_1}^{-1}\{\mu_Z(-S_1)\} \\ &= \sum_{\{i_1,i_2,\ldots,i_N\}\in \mathrm{P}_N(I_N)} L_{S_1}^{-1}\left\{ \prod_{\substack{l=1 \\ \{i_1,i_2,\ldots,i_N\}}}^{N} c_{i_l}(\infty,-S_1) \right\}, \end{aligned} \tag{14}$$

where $\mathcal{L}_{S_1}^{-1}\{\cdot\}$ denotes the inverse Laplace transform with respect to S_1.

Proof: The MGF of $Z = [Z_1]$ is given by the expectation

$$\begin{aligned} MGF_Z(\lambda_1) &= E\{\exp(\lambda_1 z_1)\} \\ &= \sum_{\substack{i_1,i_2,\cdots,i_N \\ i_1\neq i_2\neq\cdots\neq i_N}}^{1,2,\cdots,N} \int_0^{\infty} du_1 p_{i_1}(u_1)\exp(\lambda_1 u_1) \int_0^{u_1} du_2 p_{i_2}(u_2)\exp(\lambda_1 u_2) \\ &\quad \times\cdots\times \int_0^{u_{N-1}} du_N p_{i_N}(u_N)\exp(\lambda_1 u_N), \end{aligned} \tag{15}$$

where $\mathrm{E}\{\cdot\}$ denotes the expectation operator. By applying (9), we can obtain the MGF of $Z_1 = \sum_{n=1}^{N} u_n$ as

$$MGF_Z(\lambda_1) = \sum_{\{i_1,i_2,\ldots,i_N\}\in \mathrm{P}_N(I_N)} \prod_{\substack{l=1 \\ \{i_1,i_2,\ldots,i_N\}}}^{N} c_{i_l}(\infty,\lambda_1). \tag{16}$$

Therefore, we can derive the PDF of $Z_1 = \sum_{m=1}^{N} u_n$ by applying the inverse Laplace transform as

$$\begin{aligned} p_Z(z_1) &= L_{S_1}^{-1}\{\mu_Z(-S_1)\} \\ &= \sum_{\{i_1,i_2,\ldots,i_N\}\in \mathrm{P}_N(I_N)} L_{S_1}^{-1}\left\{ \prod_{\substack{l=1 \\ \{i_1,i_2,\ldots,i_N\}}}^{N} c_{i_l}(\infty,-S_1) \right\}. \end{aligned} \tag{17}$$

■

Theorem 4.2: (Joint PDF of $\sum_{n=1}^{m} u_n$ and $\sum_{n=m+1}^{N} u_n$)

Let $Z_1 = \sum_{n=1}^{m} u_n$ and $Z_2 = \sum_{n=m+1}^{N} u_n$ for convenience, then we can derive the 2-dimensional joint

PDF of $Z=[Z_1,Z_2]$ as

$$\begin{aligned}p_Z(z_1,z_2) &= L^{-1}_{S_1,S_2}\left\{\mu_Z(-S_1,-S_2)\right\}\\ &= \sum_{i_m=1}^{N}\int_0^{\infty} du_m p_{i_m}(u_m)\sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})} L^{-1}_{S_1}\left\{\prod_{\substack{k=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_k}(u_m,-S_1)\exp(-S_1u_m)\right\}\\ &\quad\times\sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})} L^{-1}_{S_2}\left\{\prod_{\substack{l=m+1\\ \{i_{m+1},\ldots,i_N\}}}^{N} c_{i_l}(u_m,-S_2)\right\}\\ &\quad\text{for } z_1\geq \frac{m}{N-m}z_2. \end{aligned} \tag{18}$$

Proof: The second order MGF of $Z=[Z_1,Z_2]$ is given by the expectation

$$\begin{aligned}MGF_Z(\lambda_1,\lambda_2) &= \sum_{\substack{i_1,i_2,\cdots,i_N\\ i_1\neq i_2\neq\cdots\neq i_N}}^{1,2,\cdots,N}\int_0^{\infty} du_1 p_{i_1}(u_1)\exp(\lambda_1u_1)\cdots\int_0^{u_{m-1}} du_m p_{i_m}(u_m)\exp(\lambda_1u_m)\\ &\quad\times\int_0^{u_m} du_{m+1}p_{i_{m+1}}(u_{m+1})\exp(\lambda_2u_{m+!})\cdots\int_0^{u_{N-1}} du_N p_{i_N}(u_N)\exp(\lambda_2u_N). \end{aligned}\tag{19}$$

We show in Appendix IV that by applying (9) and [24, Eq. (2)] and then (11), we can obtain the second order MGF of Z as

$$\begin{aligned}MGF_Z(\lambda_1,\lambda_2) &= \sum_{i_m=1}^{N}\int_0^{\infty} du_m p_{i_m}(u_m)\exp(\lambda_1u_m)\\ &\quad\times\sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})}\prod_{\substack{k=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_k}(u_m,\lambda_1)\\ &\quad\times\sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})}\prod_{\substack{l=m+1\\ \{i_{m+1},\ldots,i_N\}}}^{N} c_{i_l}(u_m,\lambda_2). \end{aligned}\tag{20}$$

Again, letting $\lambda_1=-S_1$ and $\lambda_2=-S_2$, we can obtain the desired 2-dimensional joint PDF of $Z_1=\sum_{n=1}^{m}u_n$ and $Z_2=\sum_{n=m+1}^{N}u_n$ by applying the inverse Laplace transform as

$$\begin{aligned}p_Z(z_1,z_2) &= L^{-1}_{S_1,S_2}\left\{\mu_Z(-S_1,-S_2)\right\}\\ &= \sum_{i_m=1}^{N}\int_0^{\infty} du_m p_{i_m}(u_m)\sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})} L^{-1}_{S_1}\left\{\prod_{\substack{k=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_k}(u_m,-S_1)\exp(-S_1u_m)\right\}\\ &\quad\times\sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})} L^{-1}_{S_2}\left\{\prod_{\substack{l=m+1\\ \{i_{m+1},\ldots,i_N\}}}^{N} c_{i_l}(u_m,-S_2)\right\}. \end{aligned}\tag{21}$$

■

Theorem 4.3: (Joint PDF of u_m and $\sum_{\substack{n=1\\n\neq m}}^{N} u_n$)

Let $Z_1 = u_m$ and $Z_2 = \sum_{\substack{n=1\\n\neq m}}^{N} u_n$ for convenience. We can obtain the 2-dimensional joint PDF of $Z = [Z_1, Z_2]$ as

$$p_Z(z_1, z_2) = L^{-1}_{S_1,S_2}\{\mu_Z(-S_1, -S_2)\}$$
$$= \sum_{i_m=1}^{N} \int_0^{\infty} du_m p_{i_m}(u_m) L^{-1}_{S_1}\{\exp(-S_1 u_m)\} \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})}$$
$$\times \sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})} L^{-1}_{S_2}\left\{ \prod_{\substack{k=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_k}(u_m, -S_2) \prod_{\substack{l=m+1\\ \{i_{m+1},\ldots,i_N\}}}^{N} c_{i_l}(u_m, -S_2) \right\}. \quad (22)$$

Proof: Similarly to *Theorem 4.1* and *4.2*, by applying (9), [24, Eq. (2)] and (11), we can obtain the second order MGF of $Z_1 = u_m$ and $Z_2 = \sum_{\substack{n=1\\n\neq m}}^{N} u_n$. Detailed derivation is omitted. ■

V. SAMPLE CASES WHEN ONLY N_s ORDERED RVS ARE CONSIDERED

Let us now consider the cases where only the best $N_s(\leq N)$ ordered RVs are involved.

Theorem 5.1: (PDF of $\sum_{n=1}^{N_s} u_n$, $N_s \geq 2$)

Let $Z' = \sum_{n=1}^{N_s} u_n$ for convenience, then we can derive the PDF of Z' as

$$p_{Z'}(x) = p_{\sum_{n=1}^{N_s} u_n}(x) = \int_0^{\frac{x}{N_s}} p_Z(x - z_2, z_2)\, dz_2 \quad \text{for } N_s \geq 2, \quad (23)$$

where

$$\begin{aligned} p_Z(z_1, z_2) &= L^{-1}_{S_1,S_2}\{\mu_Z(-S_1, -S_2)\} \\ &= \sum_{i_{N_s}=1}^{N} \int_0^{\infty} du_{N_s} p_{i_{N_s}}(u_{N_s}) L^{-1}_{S_2}\{\exp(-S_2 u_{N_s})\} \sum_{\substack{i_{N_s+1},\ldots,i_N \\ i_{N_s+1}\neq\cdots\neq i_N \\ i_{N_s+1}\neq i_{N_s} \\ \vdots \\ i_N\neq i_{N_s}}}^{1,2,\ldots,N} \prod_{\substack{k=N_s+1\\ \{i_{N_s+1},\ldots,i_N\}}}^{N} P_{i_k}(u_{N_s}) \\ &\quad \times \sum_{\{i_1,\ldots,i_{N_s-1}\}\in \mathrm{P}_{N_s-1}(I_N-\{i_{N_s}\}-\{i_{N_s+1},\ldots,i_N\})} L^{-1}_{S_1}\left\{ \prod_{\substack{l=1\\ \{i_1,\ldots,i_{N_s-1}\}}}^{N_s-1} e_{i_l}(u_{N_s}, -S_1) \right\}. \end{aligned} \quad (24)$$

Proof: We only need to consider u_{N_s} separately in this case. Let $Z_1 = \sum_{n=1}^{N_s-1} u_n$ and $Z_2 = u_{N_s}$. The target second order MGF of $Z = [Z_1, Z_2]$ is given by the expectation in

$$\begin{aligned} MGF_Z(\lambda_1,\lambda_2) &= \mathrm{E}\left\{\exp\left(\lambda_1 z_1 + \lambda_2 z_2\right)\right\} \\ &= \sum_{\substack{i_1,i_2,\cdots,i_N \\ i_1\neq i_2\neq\cdots\neq i_N}}^{1,2,\cdots,N} \int_0^\infty du_1 p_{i_1}(u_1)\exp(\lambda_1 u_1)\cdots\int_0^{u_{N_s-2}} du_{N_s-1} p_{i_{N_s-1}}(u_{N_s-1})\exp(\lambda_1 u_{N_s-1}) \\ &\quad\times \int_0^{u_{N_s-1}} du_{N_s} p_{i_{N_s}}(u_{N_s})\exp(\lambda_2 u_{N_s}) \prod_{j=N_s+1}^{N} P_{i_j}(u_{N_s}). \end{aligned} \tag{25}$$

By simply applying [24, Eq. (2)] and then (11) to (25), we can obtain the second order MGF result as

$$\begin{aligned} MGF_Z(\lambda_1,\lambda_2) &= \sum_{i_{N_s}=1}^{N} \int_0^\infty du_{N_s} p_{i_{N_s}}(u_{N_s})\exp(\lambda_2 u_{N_s}) \sum_{\substack{i_{N_s+1},\ldots,i_N \\ i_{N_s+1}\neq\cdots\neq i_N \\ i_{N_s+1}\neq i_{N_s} \\ \vdots \\ i_N\neq i_{N_s}}}^{1,2,\ldots,N} \prod_{\substack{k=N_s+1 \\ \{i_{N_s+1},\ldots,i_N\}}}^{N} P_{i_k}(u_{N_s}) \\ &\quad\times \sum_{\{i_1,\ldots,i_{N_s-1}\}\in \mathrm{P}_{N_s-1}(I_N-\{i_{N_s}\}-\{i_{N_s+1},\ldots,i_N\})} \prod_{\substack{l=1 \\ \{i_1,\ldots,i_{N_s-1}\}}}^{N_s-1} e_{i_l}(u_{N_s},\lambda_1). \end{aligned} \tag{26}$$

Again, letting $\lambda_1 = -S_1$ and $\lambda_2 = -S_2$, we can obtain the 2-dimensional joint PDF of $Z_1 = \sum_{n=1}^{N_s-1} u_n$ and $Z_2 = u_{N_s}$ by applying the inverse Laplace transform as

$$\begin{aligned} p_Z(z_1,z_2) &= L_{S_1,S_2}^{-1}\left\{\mu_Z(-S_1,-S_2)\right\} \\ &= \sum_{i_{N_s}=1}^{N} \int_0^\infty du_{N_s} p_{i_{N_s}}(u_{N_s}) L_{S_2}^{-1}\left\{\exp(-S_2 u_{N_s})\right\} \sum_{\substack{i_{N_s+1},\ldots,i_N \\ i_{N_s+1}\neq\cdots\neq i_N \\ i_{N_s+1}\neq i_{N_s} \\ \vdots \\ i_N\neq i_{N_s}}}^{1,2,\ldots,N} \prod_{\substack{k=N_s+1 \\ \{i_{N_s+1},\ldots,i_N\}}}^{N} P_{i_k}(u_{N_s}) \\ &\quad\times \sum_{\{i_1,\ldots,i_{N_s-1}\}\in \mathrm{P}_{N_s-1}(I_N-\{i_{N_s}\}-\{i_{N_s+1},\ldots,i_N\})} L_{S_1}^{-1}\left\{ \prod_{\substack{l=1 \\ \{i_1,\ldots,i_{N_s-1}\}}}^{N_s-1} e_{i_l}(u_{N_s},-S_1) \right\}. \end{aligned} \tag{27}$$

Finally, noting that $Z' = Z_1 + Z_2$, we can obtain the target PDF of Z' with the following finite integration

$$p_{Z'}(x) = \int_0^{\frac{x}{N_s}} p_Z(x - z_2, z_2)\, dz_2. \tag{28}$$

■

Theorem 5.2: (Joint PDF of u_m and $\sum_{\substack{n=1\\n\neq m}}^{N_s} u_n$ for $1 < m < N_s - 1$)

Let $X = u_n$ and $Y = \sum_{\substack{n=1\\n\neq m}}^{N_s} u_n$, then the joint PDF of $Z = [X, Y]$ can be obtained as

$$\begin{aligned} p_Z(x,y) &= p_{u_m, \sum\limits_{\substack{n=1\\n\neq m}}^{N_s} u_n}(x,y) \\ &= \int_0^x \int_{(m-1)x}^{y-(N_s-m)z_4} p_{\sum\limits_{n=1}^{m-1} u_n, u_m, \sum\limits_{n=m+1}^{N_s-1} u_n, u_{N_s}}(z_1, x, y-z_1-z_4, z_4)\, dz_1 dz_4. \end{aligned} \tag{29}$$

Proof: For the joint PDF of u_m and $\sum_{\substack{n=1\\n\neq m}}^{N_s} u_n$, as one of original groups is split by u_m, we should consider substituted groups for the split group instead of original groups as shown in Fig. 2. As a result, we will start by obtaining a four order MGF. In this case, the higher dimensional joint PDF can then be used to find the desired 2-dimensional joint PDF of interest by transformation.

Applying the results in [24, Eq. (2)], (9), (11) and (13), we derive in Appendix V the target joint MGF. Let $Z_1 = \sum_{n=1}^{m-1} u_n$, $Z_2 = u_m$, $Z_3 = \sum_{n=m+1}^{N_s-1} u_n$, and $Z_4 = u_{N_s}$, then

$$\begin{aligned} MGF_Z(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = &\sum_{\substack{i_{N_s},\ldots,i_N\\ i_{N_s}\neq\cdots\neq i_N}}^{1,2,\ldots,N} \int_0^\infty du_{N_s} p_{i_{N_s}}(u_{N_s}) \exp(\lambda_4 u_{N_s}) \prod_{\substack{j=N_s+1\\ \{i_{N_s+1},\ldots,i_N\}}}^{N} P_{i_j}(u_{N_s}) \\ &\times \sum_{\substack{i_m=1\\ i_m\neq i_{N_s},\ldots,i_N}}^{N} \int_{u_{N_s}}^\infty du_m p_{i_m}(u_m) \exp(\lambda_2 u_m) \\ &\times \sum_{\{i_{m+1},\ldots,i_{N_s-1}\}\in \mathrm{P}_{N_s-m-1}(I_N-\{i_m\}-\{i_{N_s},\ldots,i_N\})} \prod_{\substack{k=m+1\\ \{i_{m+1},\ldots,i_{N_s-1}\}}}^{N_s-1} \mu_{i_k}(u_{N_s}, u_m, \lambda_3) \\ &\times \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\}-\{i_{N_s},\ldots,i_N\}-\{i_{m+1},\ldots,i_{N_s-1}\})} \prod_{\substack{l=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_l}(u_m,\lambda_1). \end{aligned} \tag{30}$$

Starting from the MGF expressions given above, we apply inverse Laplace transforms in Appendix V in order to derive the following joint PDFs

$$
\begin{aligned}
p_Z(z_1,z_2,z_3,z_4) &= L^{-1}_{S_1,S_2,S_3,S_4}\left\{\mu_Z(-S_1,-S_2,-S_3,-S_4)\right\} \\
&= \sum_{\substack{i_{N_s},\ldots,i_N \\ i_{N_s}\neq\cdots\neq i_N}}^{1,2,\ldots,N} \int_0^{\infty} du_{N_s} p_{i_{N_s}}(u_{N_s}) L^{-1}_{S_4}\left\{\exp(-S_4 u_{N_s})\right\} \prod_{\substack{j=N_s+1 \\ \{i_{N_s+1},\ldots,i_N\}}}^{N} P_{i_j}(u_{N_s}) \\
&\quad \times \sum_{\substack{i_m=1 \\ i_m\neq i_{N_s},\ldots,i_N}}^{N} \int_{u_{N_s}}^{\infty} du_m p_{i_m}(u_m) L^{-1}_{S_2}\left\{\exp(-S_2 u_m)\right\} \\
&\quad \times \sum_{\{i_{m+1},\ldots,i_{N_s-1}\}\in \mathrm{P}_{N_s-m-1}(I_N-\{i_m\}-\{i_{N_s},\ldots,i_N\})} L^{-1}_{S_3}\left\{\prod_{\substack{k=m+1 \\ \{i_{m+1},\ldots,i_{N_s-1}\}}}^{N_s-1} \mu_{i_k}(u_{N_s},u_m,-S_3)\right\} \\
&\quad \times \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\}-\{i_{N_s},\ldots,i_N\}-\{i_{m+1},\ldots,i_{N_s-1}\})} L^{-1}_{S_1}\left\{\prod_{\substack{l=1 \\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_l}(u_m,-S_1)\right\}, \\
&\quad \text{for } z_4<z_2,\ z_1>(m-1)z_2 \text{ and } (N_s-m-1)z_4<z_3<(N_s-m-1)z_2. \qquad (31)
\end{aligned}
$$

■

Note that (29) involves only finite integrations of joint PDFs. Therefore, while a generic closed-form expression is not possible, the desired joint PDF can be easily numerically evaluated with the help of integral tables [29], [30] or using standard mathematical packages, such as Mathematica or Matlab etc.

Theorem 5.3: (Joint PDF of $\sum_{n=1}^{m} u_n$ and $\sum_{n=m+1}^{N_s} u_n$)

Let $X=\sum_{n=1}^{m} u_n$ and $Y=\sum_{n=m+1}^{N_s} u_n$, then we can simply obtain the joint PDF of $Z=[X,Y]$ as

$$
\begin{aligned}
&p_Z(x,y) = p_{\sum_{n=1}^{m} u_n,\ \sum_{n=m+1}^{N_s} u_n}(x,y) \\
&= \int_0^{\frac{y}{N_s-m}} \int_{\frac{y}{N_s-m}}^{\frac{x}{m}} p_{\sum_{n=1}^{m-1} u_n, u_m, \sum_{n=m+1}^{K_s-1} u_n, u_{N_s}}(x-z_2,z_2,y-z_4,z_4)\, dz_2 dz_4,\ \text{for } x>\frac{m}{N_s-m}y. \qquad (32)
\end{aligned}
$$

Proof: Omitted. ■

Note again that only the finite integrations of joint PDFs are involved.

VI. Closed-form expressions for exponential RV case

Now, we focus on obtaining the joint PDFs for i.n.d. exponential RV special cases in a ready-to-use form. The PDF and the CDF of the RVs are given by $p_{i_l}(x) = \frac{1}{\bar{\gamma}_{i_l}} \exp\left(-\frac{x}{\bar{\gamma}_{i_l}}\right)$ and $P_{i_l}(x) = 1 - \exp\left(-\frac{x}{\bar{\gamma}_{i_l}}\right)$ for $\gamma \geq 0$, respectively, where $\bar{\gamma}_{i_l}$ is the average of the l-th RV.

The above novel generic results are quite general and apply to any RVs. We now focus on obtaining the joint PDFs for i.n.d. exponential RV special cases in a ready-to-use form and illustrate in this section some results for the independent non-identical exponential RV special case, where the PDF and the CDF of γ are given by $p_{i_l}(x) = \frac{1}{\bar{\gamma}_{i_l}} \exp\left(-\frac{x}{\bar{\gamma}_{i_l}}\right)$ and $P_{i_l}(x) = 1 - \exp\left(-\frac{x}{\bar{\gamma}_{i_l}}\right)$ for $\gamma \geq 0$, respectively, where $\bar{\gamma}_{i_l}$ is the average of the l-th RV. As shown in Appendix VI, (9), (11) and (13) can be specialized to

i) For special case:

$$c_{i_l}(z_a, \lambda) = \frac{1}{1 - \bar{\gamma}_{i_l}\lambda} \left[1 - \exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)\right], \tag{33}$$

$$e_{i_l}(z_a, \lambda) = \frac{1}{1 - \bar{\gamma}_{i_l}\lambda} \left[\exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)\right], \tag{34}$$

$$\mu_{i_l}(z_a, z_b, \lambda) = \frac{1}{1 - \bar{\gamma}_{i_l}\lambda} \left[\exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_b\right) - \exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)\right]. \tag{35}$$

ii) For general case:

$$\begin{aligned}\prod_{l=n_1}^{n_2} c_{i_l}(z_a, \lambda) &= \frac{1}{\prod\limits_{l=n_1}^{n_2} (1 - \bar{\gamma}_{i_l}\lambda)} \prod_{l=n_1}^{n_2} \left[1 - \exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)\right] \\ &= \sum_{k=n_1}^{n_2} C_{k,n_1,n_2} \left[\frac{1 + \left[\sum\limits_{l=1}^{n_2-n_1+1} \exp(l \cdot z_a \cdot \lambda) \left\{(-1)^l \sum\limits_{j_1=j_0+n_1}^{n_2-l+1} \cdots \sum\limits_{j_l=j_{l-1}+1}^{n_2} \exp\left(-\sum\limits_{m=1}^{l} \frac{z_a}{\bar{\gamma}_{i_{j_m}}}\right)\right\}\right]}{\left(\lambda - \frac{1}{\bar{\gamma}_{i_k}}\right)}\right],\end{aligned} \tag{36}$$

$$\begin{aligned}\prod_{l=n_1}^{n_2} e_{i_l}(z_a, \lambda) &= \frac{1}{\prod\limits_{l=n_1}^{n_2} (1 - \bar{\gamma}_{i_l}\lambda)} \exp\left(\left\{\sum_{l=n_1}^{n_2} \left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right)\right\} z_a\right) \\ &= \sum_{k=n_1}^{n_2} \frac{C_{k,n_1,n_2}}{\left(\lambda - \frac{1}{\bar{\gamma}_{i_k}}\right)} \exp\left(-\sum_{l=n_1}^{n_2} \left(\frac{z_a}{\bar{\gamma}_{i_l}}\right)\right) \exp\left((n_2 - n_1 + 1) z_a \lambda\right),\end{aligned} \tag{37}$$

$$\prod_{l=n_1}^{n_2} \mu_{i_l}(z_a, z_b, \lambda) = \frac{1}{\prod\limits_{l=n_1}^{n_2}(1-\bar{\gamma}_{i_l}\lambda)} \prod_{l=n_1}^{n_2} \left[\exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}} \right) z_a \right) - \exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}} \right) z_b \right) \right]$$

$$= \sum_{k=n_1}^{n_2} C_{k,n_1,n_2} \left[\frac{\exp\left((n_2-n_1+1)\cdot z_a \cdot \lambda\right) \exp\left(-\sum\limits_{l=n_1}^{n_2} \left(\frac{z_a}{\bar{\gamma}_{i_l}} \right) \right)}{\left(\lambda - \frac{1}{\bar{\gamma}_{i_k}} \right)} \right.$$

$$\left. \times \left\{ 1 + \sum_{l=1}^{n_2-n_1+1} \exp\left(l \cdot (z_b - z_a)\cdot \lambda\right) \left\{ (-1)^l \sum_{j_1=j_0+n_1}^{n_2-l+1} \cdots \sum_{j_l=j_{l-1}+1}^{n_2} \exp\left(-\sum_{m=1}^{l} \frac{z_b - z_a}{\bar{\gamma}_{i_{j_m}}} \right) \right\} \right\} \right], \quad (38)$$

where

$$C_{l,n_1,n_2} = \frac{1}{\prod\limits_{l=n_1}^{n_2} (-\bar{\gamma}_{i_l}) F'\left(\frac{1}{\bar{\gamma}_{i_l}} \right)}, \quad (39)$$

$$F'(x) = \left[\sum_{l=1}^{n_2-n_1} (n_2-n_1-l+1)\, x^{n_2-n_1-l} (-1)^l \sum_{j_1=j_0+n_1}^{n_2-l+1} \cdots \sum_{j_l=j_{l-1}+1}^{n_2} \prod_{m=1}^{l} \frac{1}{\bar{\gamma}_{i_{j_m}}} \right] + (n_2-n_1+1)\, x^{n_2-n_1}. \quad (40)$$

After substituting (36), (37) and (38) into the derived expressions of the joint PDF of partial sums of ordered statistics presented in the previous sections, it is easy to derive the following closed-form expressions for the PDFs by applying the classical inverse Laplace transform pair and the property given in [24, Appendix I]. While some of these results have been derived using the successive conditioning approach previously, we list them here for the sake of convenience and completeness in the next page.

1) PDF of $\sum_{n=1}^{N} u_n$:

$$p_Z(z_1) = \sum_{\{i_1,i_2,\ldots,i_N\}\in \mathrm{P}_N(I_N)} \sum_{l=1}^{N} C_{l,1,N} L_{S_1}^{-1}\left\{\frac{1}{\left(-S_1-\frac{1}{\bar{\gamma}_{i_l}}\right)}\right\}, \tag{41}$$

where

$$L_{S_1}^{-1}\left\{\frac{1}{\left(-S_1-\frac{1}{\bar{\gamma}_{i_l}}\right)}\right\} = -\exp\left(-\frac{z_1}{\bar{\gamma}_{i_l}}\right). \tag{42}$$

2) Joint PDF of u_m and $\sum_{\substack{n=1\\n\neq m}}^{N} u_n$:

$$\begin{aligned}
p_Z(z_1,z_2) &= L_{S_1,S_2}^{-1}\left\{\mu_Z(-S_1,-S_2)\right\}\\
&= \sum_{i_m=1}^{N}\int_0^{\infty} du_m \frac{1}{\bar{\gamma}_{i_m}}\exp\left(-\frac{u_m}{\bar{\gamma}_{i_m}}\right) L_{S_1}^{-1}\left\{\exp(-S_1u_m)\right\}\\
&\times \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})} \sum_{\substack{k=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} C_{k,1,m-1}\exp\left(-\sum_{l=1}^{m-1}\left(\frac{u_m}{\bar{\gamma}_{i_l}}\right)\right) \sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}\left(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\}\right)} \sum_{\substack{q=m+1\\ \{i_{m+1},\ldots,i_N\}}}^{N} C_{q,m+1,N} L_{S_2}^{-1}\left\{\frac{\exp\left(-(m-1)u_mS_2\right)}{\left(-S_2-\frac{1}{\bar{\gamma}_{i_k}}\right)\left(-S_2-\frac{1}{\bar{\gamma}_{i_q}}\right)}\right\}\\
&+ \sum_{i_m=1}^{N}\int_0^{\infty} du_m \frac{1}{\bar{\gamma}_{i_m}}\exp\left(-\frac{u_m}{\bar{\gamma}_{i_m}}\right) L_{S_1}^{-1}\left\{\exp(-S_1u_m)\right\} \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})} \sum_{\substack{k=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} C_{k,1,m-1}\exp\left(-\sum_{l=1}^{m-1}\left(\frac{u_m}{\bar{\gamma}_{i_l}}\right)\right)\\
&\times \sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}\left(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\}\right)} \sum_{\substack{q=m+1\\ \{i_{m+1},\ldots,i_N\}}}^{N} C_{q,m+1,N}\left[\sum_{h=1}^{N-m}\left\{(-1)^h \sum_{j_1=j_0+m+1}^{N-h+1}\cdots\sum_{j_h=j_{h-1}+1}^{N}\exp\left(-\sum_{m=1}^{h}\frac{u_m}{\bar{\gamma}_{i_{j_m}}}\right)\right\} L_{S_2}^{-1}\left\{\frac{\exp\left(-(h+m-1)u_mS_2\right)}{\left(-S_2-\frac{1}{\bar{\gamma}_{i_k}}\right)\left(-S_2-\frac{1}{\bar{\gamma}_{i_q}}\right)}\right\}\right],
\end{aligned} \tag{43}$$

where

$$L_{S_1}^{-1}\left\{\exp(-S_1u_m)\right\} = \delta(z_1-u_m), \tag{44}$$

$$L_{S_2}^{-1}\left\{\frac{\exp\left(-(m-1)u_mS_2\right)}{\left(-S_2-\frac{1}{\bar{\gamma}_{i_k}}\right)\left(-S_2-\frac{1}{\bar{\gamma}_{i_q}}\right)}\right\} = \frac{\exp\left(-\left(z_2-(m-1)u_m\right)\left(\frac{1}{\bar{\gamma}_{i_k}}+\frac{1}{\bar{\gamma}_{i_q}}\right)\right)\left\{\exp\left(\frac{z_2-(m-1)u_m}{\bar{\gamma}_{i_q}}\right)-\exp\left(\frac{z_2-(m-1)u_m}{\bar{\gamma}_{i_k}}\right)\right\}U\left(z_2-(m-1)u_m\right)}{\left(\frac{1}{\bar{\gamma}_{i_q}}-\frac{1}{\bar{\gamma}_{i_k}}\right)}, \tag{45}$$

$$L_{S_2}^{-1}\left\{\frac{\exp\left(-\left(h+m-1\right)u_m S_2\right)}{\left(-S_2-\frac{1}{\bar{\gamma}_{i_k}}\right)\left(-S_2-\frac{1}{\bar{\gamma}_{i_q}}\right)}\right\}=\frac{\exp\left(-\left(z_2-\left(h+m-1\right)u_m\right)\left(\frac{1}{\bar{\gamma}_{i_k}}+\frac{1}{\bar{\gamma}_{i_q}}\right)\right)\left\{\exp\left(\frac{z_2-(h+m-1)u_m}{\bar{\gamma}_{i_q}}\right)-\exp\left(\frac{z_2-(h+m-1)u_m}{\bar{\gamma}_{i_k}}\right)\right\}U\left(z_2-\left(h+m-1\right)u_m\right)}{\left(\frac{1}{\bar{\gamma}_{i_q}}-\frac{1}{\bar{\gamma}_{i_k}}\right)}. \tag{46}$$

3) Joint PDF of $\sum_{n=1}^{m} u_n$ and $\sum_{n=m+1}^{N} u_n$:

$$\begin{aligned}
&p_Z\left(z_1,z_2\right)=L_{S_1,S_2}^{-1}\left\{\mu_Z\left(-S_1,-S_2\right)\right\}\\
&=\sum_{i_m=1}^{N}\int_0^{\infty}du_m\frac{1}{\bar{\gamma}_{i_m}}\exp\left(-\frac{u_m}{\bar{\gamma}_{i_m}}\right)\\
&\quad\times\sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\left\{i_m\right\}\right)}\;\sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1}C_{k,1,m-1}\exp\left(-\sum_{l=1}^{m-1}\left(\frac{u_m}{\bar{\gamma}_{i_l}}\right)\right)L_{S_1}^{-1}\left\{\frac{\exp\left(-mu_mS_1\right)}{\left(-S_1-\frac{1}{\bar{\gamma}_{i_k}}\right)}\right\}\sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\left\{i_m\right\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)}\\
&\quad\times\sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N}C_{q,m+1,N}L_{S_2}^{-1}\left\{\frac{1}{\left(-S_2-\frac{1}{\bar{\gamma}_{i_q}}\right)}\right\}\\
&\quad+\sum_{i_m=1}^{N}\int_0^{\infty}du_m\frac{1}{\bar{\gamma}_{i_m}}\exp\left(-\frac{u_m}{\bar{\gamma}_{i_m}}\right)\sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\left\{i_m\right\}\right)}\;\sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1}C_{k,1,m-1}\exp\left(-\sum_{l=1}^{m-1}\left(\frac{u_m}{\bar{\gamma}_{i_l}}\right)\right)L_{S_1}^{-1}\left\{\frac{\exp\left(-mu_mS_1\right)}{\left(-S_1-\frac{1}{\bar{\gamma}_{i_k}}\right)}\right\}\\
&\quad\times\sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\left\{i_m\right\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)}\;\sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N}C_{q,m+1,N}\left[\sum_{h=1}^{N-m}\left\{\left(-1\right)^h\sum_{j_1=j_0+m+1}^{N-h+1}\cdots\sum_{j_h=j_{h-1}+1}^{N}\exp\left(-\sum_{m=1}^{h}\frac{u_m}{\bar{\gamma}_{i_{j_m}}}\right)\right\}L_{S_2}^{-1}\left\{\frac{\exp\left(-hu_mS_2\right)}{\left(-S_2-\frac{1}{\bar{\gamma}_{i_q}}\right)}\right\}\right],
\end{aligned} \tag{47}$$

where

$$L_{S_2}^{-1}\left\{\frac{1}{\left(-S_2-\frac{1}{\bar{\gamma}_{i_q}}\right)}\right\}=-\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right), \tag{48}$$

$$L_{S_1}^{-1}\left\{\frac{\exp\left(-mu_mS_1\right)}{-S_1-\frac{1}{\bar{\gamma}_{i_k}}}\right\}=-\exp\left(-\frac{z_1-mu_m}{\bar{\gamma}_{i_k}}\right)U\left(z_1-mu_m\right), \tag{49}$$

$$L_{S_2}^{-1}\left\{\frac{\exp\left(-hu_mS_2\right)}{-S_2-\frac{1}{\bar{\gamma}_{i_q}}}\right\}=-\exp\left(-\frac{z_2-hu_m}{\bar{\gamma}_{i_q}}\right)U\left(z_2-hu_m\right). \tag{50}$$

4) PDF of $\sum_{n=1}^{N_s} u_n$:

$$
\begin{aligned}
p_Z(z_1,z_2) &= L^{-1}_{S_1,S_2}\left\{\mu_Z(-S_1,-S_2)\right\} \\
&= \sum_{i_{N_s}=1}^{N} \int_0^{\infty} du_{N_s} \frac{1}{\bar{\gamma}_{i_{N_s}}} \exp\left(-\frac{u_{N_s}}{\bar{\gamma}_{i_{N_s}}}\right) L^{-1}_{S_2}\left\{\exp\left(-S_2 u_{N_s}\right)\right\} \sum_{\substack{i_{N_s+1},\ldots,i_N \\ i_{N_s+1}\neq\cdots\neq i_N \\ i_{N_s+1}\neq i_{N_s} \\ \vdots \\ i_N \neq i_{N_s}}}^{1,2,\ldots,N} \prod_{\substack{k=N_s+1 \\ \left\{i_{N_s+1},\ldots,i_N\right\}}}^{N} \left\{1-\exp\left(-\frac{u_{N_s}}{\bar{\gamma}_{i_k}}\right)\right\} \\
&\quad \times \sum_{\left\{i_1,\ldots,i_{N_s-1}\right\}\in \mathrm{P}_{N_s-1}\left(I_N-\left\{i_{N_s}\right\}-\left\{i_{N_s+1},\ldots,i_N\right\}\right)} \prod_{\substack{q=1 \\ \left\{i_1,\ldots,i_{N_s-1}\right\}}}^{N_s-1} C_{q,1,N_s-1} \exp\left(-\sum_{l=1}^{N_s-1}\left(\frac{u_{N_s}}{\bar{\gamma}_{i_l}}\right)\right) L^{-1}_{S_1}\left\{\frac{\exp\left(-\left(N_s-1\right)u_{N_s}S_1\right)}{\left(-S_1-\frac{1}{\bar{\gamma}_{i_q}}\right)}\right\},
\end{aligned}
\tag{51}
$$

where

$$
L^{-1}_{S_2}\left\{\exp\left(-S_2 u_{N_s}\right)\right\} = \delta\left(z_2-u_{N_s}\right), \tag{52}
$$

$$
L^{-1}_{S_1}\left\{\frac{\exp\left(-\left(N_s-1\right)u_{N_s}S_1\right)}{\left(-S_1-\frac{1}{\bar{\gamma}_{i_q}}\right)}\right\} = -\exp\left(-\frac{z_1-\left(N_s-1\right)u_{N_s}}{\bar{\gamma}_{i_q}}\right) U\left(z_1-\left(N_s-1\right)u_{N_s}\right). \tag{53}
$$

5) Joint PDF of u_m and $\sum_{\substack{n=1 \\ n\neq m}}^{N_s} u_n$ for $1<m<N_s-1$:

$$
\begin{aligned}
&p_Z(z_1,z_2,z_3,z_4) = L^{-1}_{S_1,S_2,S_3,S_4}\left\{\mu_Z(-S_1,-S_2,-S_3,-S_4)\right\} \\
&= \sum_{\substack{i_{N_s},\ldots,i_N \\ i_{N_s}\neq\cdots\neq i_N}}^{1,2,\ldots,N} \int_0^{\infty} du_{N_s} \frac{1}{\bar{\gamma}_{i_{N_s}}} \exp\left(-\frac{u_{N_s}}{\bar{\gamma}_{i_{N_s}}}\right) L^{-1}_{S_4}\left\{\exp\left(-u_{N_s}S_4\right)\right\} \prod_{\substack{j=N_s+1 \\ \left\{i_{N_s+1},\ldots,i_N\right\}}}^{N} \left\{1-\exp\left(-\frac{u_{N_s}}{\bar{\gamma}_{i_j}}\right)\right\} \times \sum_{\substack{i_m=1 \\ i_m\neq i_{N_s},\ldots,i_N}}^{N} \int_{u_{N_s}}^{\infty} du_m \frac{1}{\bar{\gamma}_{i_m}} \exp\left(-\frac{u_m}{\bar{\gamma}_{i_m}}\right) L^{-1}_{S_2}\left\{\exp\left(-u_m S_2\right)\right\} \\
&\quad \times \sum_{\left\{i_{m+1},\ldots,i_{N_s-1}\right\}\in \mathrm{P}_{N_s-m-1}\left(I_N-\left\{i_m\right\}-\left\{i_{N_s},\ldots,i_N\right\}\right)} \sum_{k=m+1}^{N_s-1} C_{k,m+1,N_s-1} \left[L^{-1}_{S_3}\left\{\frac{\exp\left(-\left(N_s-m-1\right)\cdot u_{N_s}\cdot S_3\right)}{\left(-S_3-\frac{1}{\bar{\gamma}_{i_k}}\right)}\right\} \exp\left(-\sum_{l=m+1}^{N_s-1}\left(\frac{u_{N_s}}{\bar{\gamma}_{i_l}}\right)\right)\right. \\
&\quad \left. +\exp\left(-\sum_{l=m+1}^{N_s-1}\left(\frac{u_{N_s}}{\bar{\gamma}_{i_l}}\right)\right) \sum_{l=1}^{N_s-m-1} L^{-1}_{S_3}\left\{\frac{\exp\left(-\left(l\cdot u_m+\left(N_s-m-l-1\right)\cdot u_{N_s}\right)\cdot S_3\right)}{\left(-S_3-\frac{1}{\bar{\gamma}_{i_k}}\right)}\right\} \left\{(-1)^l \sum_{j_1=j_0+m+1}^{N_s-l} \cdots \sum_{j_l=j_{l-1}+1}^{N_s-1} \exp\left(-\sum_{m=1}^{l}\frac{u_m-u_{N_s}}{\bar{\gamma}_{i_{j_m}}}\right)\right\}\right] \\
&\quad \times \sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\left\{i_m\right\}-\left\{i_{N_s},\ldots,i_N\right\}-\left\{i_{m+1},\ldots,i_{N_s-1}\right\}\right)} \sum_{h=1}^{m-1} C_{h,1,m-1} \exp\left(-\sum_{l=1}^{m-1}\left(\frac{u_m}{\bar{\gamma}_{i_l}}\right)\right) L^{-1}_{S_1}\left\{\frac{\exp\left(-\left(m-1\right)u_m S_1\right)}{\left(-S_1-\frac{1}{\bar{\gamma}_{i_h}}\right)}\right\},
\end{aligned}
\tag{54}
$$

March 7, 2022

where

$$L_{S_4}^{-1}\left\{\exp\left(-u_{N_s}S_4\right)\right\} = \delta\left(z_4 - u_{N_s}\right), \tag{55}$$

$$L_{S_2}^{-1}\left\{\exp\left(-u_m S_2\right)\right\} = \delta\left(z_2 - u_m\right), \tag{56}$$

$$L_{S_3}^{-1}\left\{\frac{\exp\left(-\left(N_s - m - 1\right)\cdot u_{N_s}\cdot S_3\right)}{\left(-S_3 - \frac{1}{\bar{\gamma}_{i_k}}\right)}\right\} = -\exp\left(-\frac{z_3 - \left(N_s - m - 1\right)\cdot u_{N_s}}{\bar{\gamma}_{i_k}}\right)U\left(z_3 - \left(N_s - m - 1\right)\cdot u_{N_s}\right), \tag{57}$$

$$L_{S_3}^{-1}\left\{\frac{\exp\left(-\left(l\cdot u_m + \left(N_s - m - l - 1\right)\cdot u_{N_s}\right)\cdot S_3\right)}{\left(-S_3 - \frac{1}{\bar{\gamma}_{i_k}}\right)}\right\} = -\exp\left(-\frac{z_3 - \left(l\cdot u_m + \left(N_s - m - l - 1\right)\cdot u_{N_s}\right)}{\bar{\gamma}_{i_k}}\right)U\left(z_3 - \left(l\cdot u_m + \left(N_s - m - l - 1\right)\cdot u_{N_s}\right)\right), \tag{58}$$

$$L_{S_1}^{-1}\left\{\frac{\exp\left(-\left(m-1\right)u_m S_1\right)}{\left(-S_1 - \frac{1}{\bar{\gamma}_{i_h}}\right)}\right\} = -\exp\left(-\frac{z_1 - \left(m-1\right)\cdot u_m}{\bar{\gamma}_{i_h}}\right)U\left(z_1 - \left(m-1\right)\cdot u_m\right), \tag{59}$$

$$\prod_{k=n_1}^{n_2}\left(1 - \exp\left(-\frac{u_{N_s}}{\bar{\gamma}_{i_k}}\right)\right) = 1 + \sum_{k=1}^{n_2-n_1+1}\left(-1\right)^k \sum_{j_1=j_0+n_1}^{n_2-k+1}\cdots\sum_{j_k=j_{k-1}+1}^{n_2}\exp\left(-\sum_{m=1}^{k}\frac{u_{N_s}}{\bar{\gamma}_{i_{j_m}}}\right). \tag{60}$$

VII. Application Example

The above derived joint PDFs of partial sums of ordered statistics can be applied to the performance analysis of various wireless communication systems. In this section, we discuss several selected application examples.

A. Derivation of the Capture Probability of GSC RAKE receiver over i.n.d. Rayleigh fading conditions

Recently, we presented the exact performance analyses of the capture probability on GSC RAKE receivers in [23]. For analytical simplification, the fading was assumed both independent and identically distributed from path to path. However, the average SNR of each path (or branch) is different for most practical channel models, especially for wide-band SS signals since the average fading power may vary from one path to the other. For example, experimental measurements indicate that the radio channel is characterized by an exponentially decaying multipath intensity profile (MIP) for indoor office buildings [31] as well as urban [32] and suburban areas [33]. Based on this motivation in mind, with the help of our derived results in Sec V, we can extend our previous result (a closed-form formula of the capture probability on GSC RAKE receivers) by maintaining the assumption of independence among the diversity paths but relaxing the identically distributed assumption.

Let u_i be the order statistics obtained by arranging N $(N \geq 2)$ nonnegative i.n.d. RVs, $\{\gamma_{i_l}\}_{i_l=1}^{N}$, in decreasing order of magnitude such that $u_1 \geq u_2 \geq \cdots \geq u_N$. Based on the system model and definition in [23], the capture probability can be written as

$$\text{Prob}_{GSC-capture} = \Pr\left[\frac{\sum\limits_{n=1}^{m} u_n}{\sum\limits_{n=1}^{N} u_n} > T\right], \tag{61}$$

where $0 < T < 1$ and $m < N$. If we assume $Z = [Z_1, Z_2]$, $Z_1 = \sum_{n=1}^{m} u_n$ and $Z_2 = \sum_{n=m+1}^{N} u_n$, then (61) can be calculated in terms of the 2-dimensional joint PDF of Z_1 and Z_2 easily as

$$\text{Prob}_{GSC-capture} = \Pr\left[\frac{Z_1}{Z_1 + Z_2} > T\right] = \int_0^{\infty} \int_0^{\left(\frac{1-T}{T}\right)z_1} p_Z\left(z_1, z_2\right) dz_2 dz_1. \tag{62}$$

The joint PDF of $\sum_{n=1}^{m} u_n$ and $\sum_{n=m+1}^{N} u_n$, $p_Z\left(z_1, z_2\right)$ can be derived with the help of our extended approach in this paper. More specifically, inserting (47) into (62), the closed-form expression for i.n.d. Rayleigh fading conditions is shown at the top of the next page (refer to Appendix-VII for details).

$$
\begin{aligned}
&\text{Prob}_{GSC-capture}\\
&=\sum_{i_m=1}^{N}\frac{1}{\bar{\gamma}_{i_m}}\sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\left\{i_m\right\}\right)}\sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1}C_{k,1,m-1}\sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\left\{i_m\right\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)}\sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N}C_{q,m+1,N}\\
&\quad\times\left[\frac{1}{\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)}\right]\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)dz_2dz_1\\
&\quad-\sum_{i_m=1}^{N}\frac{1}{\bar{\gamma}_{i_m}}\sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\left\{i_m\right\}\right)}\sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1}C_{k,1,m-1}\sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\left\{i_m\right\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)}\sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N}C_{q,m+1,N}\\
&\quad\times\left[\frac{1}{\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)}\right]\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)\right)\frac{z_1}{m}\right)dz_2dz_1\\
&\quad+\sum_{i_m=1}^{N}\frac{1}{\bar{\gamma}_{i_m}}\sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\left\{i_m\right\}\right)}\sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1}C_{k,1,m-1}\sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\left\{i_m\right\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)}\sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N}C_{q,m+1,N}\\
&\quad\times\left[\sum_{h=1}^{N-m}\left\{(-1)^h\sum_{j_1=j_0+m+1}^{N-h+1}\cdots\sum_{j_h=j_{h-1}+1}^{N}\left(\frac{1}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)}\right)\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)dz_2dz_1\right\}\right]\\
&\quad-\sum_{i_m=1}^{N}\frac{1}{\bar{\gamma}_{i_m}}\sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\left\{i_m\right\}\right)}\sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1}C_{k,1,m-1}\sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\left\{i_m\right\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)}\sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N}C_{q,m+1,N}\\
&\quad\times\left[\sum_{h=1}^{N-m}(-1)^h\sum_{j_1=j_0+m+1}^{N-h+1}\cdots\sum_{j_h=j_{h-1}+1}^{N}\left(\frac{1}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)}\right)\right.\\
&\qquad\left.\times\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right)U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)dz_2dz_1\right]\\
&\quad+\sum_{i_m=1}^{N}\frac{1}{\bar{\gamma}_{i_m}}\sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\left\{i_m\right\}\right)}\sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1}C_{k,1,m-1}\sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\left\{i_m\right\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)}\sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N}C_{q,m+1,N}\\
&\quad\times\left[\sum_{h=1}^{N-m}(-1)^h\sum_{j_1=j_0+m+1}^{N-h+1}\cdots\sum_{j_h=j_{h-1}+1}^{N}\left(\frac{1}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)}\right)\right.\\
&\qquad\left.\times\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1\right]\\
&\quad-\sum_{i_m=1}^{N}\frac{1}{\bar{\gamma}_{i_m}}\sum_{\left\{i_1,\ldots,i_{m-1}\right\}\in \mathrm{P}_{m-1}\left(I_N-\left\{i_m\right\}\right)}\sum_{\substack{k=1\\ \left\{i_1,\ldots,i_{m-1}\right\}}}^{m-1}C_{k,1,m-1}\sum_{\left\{i_{m+1},\ldots,i_N\right\}\in \mathrm{P}_{N-m}\left(I_N-\left\{i_m\right\}-\left\{i_1,\ldots,i_{m-1}\right\}\right)}\sum_{\substack{q=m+1\\ \left\{i_{m+1},\ldots,i_N\right\}}}^{N}C_{q,m+1,N}\\
&\quad\times\left[\sum_{h=1}^{N-m}(-1)^h\sum_{j_1=j_0+m+1}^{N-h+1}\cdots\sum_{j_h=j_{h-1}+1}^{N}\left(\frac{1}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)}\right)\right.\\
&\qquad\left.\times\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_1}{m}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1\right].
\end{aligned}
\tag{63}
$$

The closed-form expressions of integral parts in the expression presented in (63) can be derived as

i) The first integral part:

$$\int_0^\infty \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) dz_2 dz_1 = \bar{\gamma}_{i_q}\bar{\gamma}_{i_k} - \frac{\bar{\gamma}_{i_q}}{\left(\frac{1}{T\cdot\bar{\gamma}_{i_q}} + \frac{1}{\bar{\gamma}_{i_k}} - \frac{1}{\bar{\gamma}_{i_q}}\right)}. \quad (64)$$

ii) The second integral part:

$$\begin{aligned}&\int_0^\infty \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) \exp\left(-\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)\right)\frac{z_1}{m}\right) dz_2 dz_1 \\ &= \frac{\bar{\gamma}_{i_q}}{\left(\sum\limits_{l=1}^{m}\left(\frac{1}{m\cdot\bar{\gamma}_{i_l}}\right)\right)} - \frac{\bar{\gamma}_{i_q}}{\left(\sum\limits_{l=1}^{m}\left(\frac{1}{m\cdot\bar{\gamma}_{i_l}}\right) + \frac{1-T}{T\cdot\bar{\gamma}_{i_q}}\right)}.\end{aligned} \quad (65)$$

iii) The third integral part:

$$\begin{aligned}&\int_0^\infty \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) U\left(\frac{z_1}{m} - \frac{z_2}{h}\right) dz_2 dz_1 \\ &= \bar{\gamma}_{i_q}\bar{\gamma}_{i_k} U\left(\frac{1}{m} - \frac{1-T}{T\cdot h}\right) - \frac{\bar{\gamma}_{i_q}}{\left(\frac{1-T}{\bar{\gamma}_{i_q}T} + \frac{1}{\bar{\gamma}_{i_k}}\right)} U\left(\frac{1}{m} - \frac{1-T}{T\cdot h}\right) \\ &\quad + \bar{\gamma}_{i_q}\bar{\gamma}_{i_k}\left[1 - U\left(\frac{1}{m} - \frac{1-T}{T\cdot h}\right)\right] - \frac{\bar{\gamma}_{i_q}}{\left(\frac{h}{\bar{\gamma}_{i_q}m} + \frac{1}{\bar{\gamma}_{i_k}}\right)}\left[1 - U\left(\frac{1}{m} - \frac{1-T}{T\cdot h}\right)\right].\end{aligned} \quad (66)$$

iv) The forth integral part:

$$\begin{aligned}&\int_0^\infty \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right) U\left(\frac{z_1}{m} - \frac{z_2}{h}\right) dz_2 dz_1 \\ &= \frac{\bar{\gamma}_{i_k}h}{\left(\sum\limits_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum\limits_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)} U\left(\frac{1}{m} - \frac{1-T}{T\cdot h}\right) \\ &\quad - \frac{h}{\left(\sum\limits_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum\limits_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)\left\{\left(\sum\limits_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum\limits_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)\frac{1-T}{T\cdot h} + \frac{1}{\bar{\gamma}_{i_k}}\right\}} U\left(\frac{1}{m} - \frac{1-T}{T\cdot h}\right) \\ &\quad + \frac{\bar{\gamma}_{i_k}h}{\left(\sum\limits_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum\limits_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)}\left[1 - U\left(\frac{1}{m} - \frac{1-T}{T\cdot h}\right)\right] \\ &\quad - \frac{h}{\left(\sum\limits_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum\limits_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)\left\{\left(\sum\limits_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum\limits_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)\frac{1}{m} + \frac{1}{\bar{\gamma}_{i_k}}\right\}}\left[1 - U\left(\frac{1}{m} - \frac{1-T}{T\cdot h}\right)\right].\end{aligned} \quad (67)$$

v) The fifth integral part:

$$\begin{aligned}&\int_0^\infty\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1\\&=\frac{\bar{\gamma}_{i_q}}{\left(\frac{h}{m\cdot\bar{\gamma}_{i_q}}+\frac{1}{\bar{\gamma}_{i_k}}\right)}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)-\frac{\bar{\gamma}_{i_q}}{\left(\frac{1-T}{T\cdot\bar{\gamma}_{i_q}}+\frac{1}{\bar{\gamma}_{i_k}}\right)}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right).\end{aligned}\tag{68}$$

vi) The sixth integral part:

$$\begin{aligned}&\int_0^\infty\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_1}{m}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1\\&=\frac{m\cdot\bar{\gamma}_{i_q}}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)\right)}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)\\&\quad-\frac{m\cdot\bar{\gamma}_{i_q}}{\left\{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)+\frac{m(1-T)}{T\cdot\bar{\gamma}_{i_q}}\right\}}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right).\end{aligned}\tag{69}$$

B. Finger Replacement Schemes for RAKE Receivers in the Soft Handover Region over i.n.d. fading channels

Recently, new finger replacement techniques for RAKE reception in the soft handover (SHO) region [34] has been proposed and analyzed over independent and identical fading (i.i.d.) channel. The proposed schemes are basically based on the block comparison among groups of resolvable paths from different base stations and lead to the reduction of complexity while offering commensurate performance. If we let $Y=\sum_{i=1}^{L_c-L_s}u_i$, $W_1=\sum_{i=L_c-L_s+1}^{L_c}u_i$ and $W_n=\sum_{i=1}^{L_s}v_i^n$ (for $n=2,\ldots,N$), where u_i ($i=1,2,\ldots,L_1$) and v_i^n ($i=1,2,\ldots,L_n$) are the order statistics obtained by arranging L_n nonnegative i.n.d. path SNRs corresponding to the nth base station ($2\leq n\leq N$) in descending order, then the RAKE combiner output SNR with GSC is given by $Y+\max_n W_n$. Y and W_1 are dependent but Y and W_n are independent. In practice, the i.i.d. fading assumption on the diversity paths is not always realistic due to, for example, the different adjacent multipath routes with the same path loss. Although non-identical fading is important for practical implementation, [34] have only investigated the non-uniform power delay profile case only through computer simulation due to the high analysis complexity. The major difficulty in this problem is to derive the joint statistics of ordered exponential variates over non-identical fading assumptions, which can be obtained by applying Theorem 5.1 and 5.3 of section V. Due to space limitation, the analytical details are omitted in this work.

C. Outage Probability of GSC RAKE Receivers Over i.n.d. Rayleigh Fading Channel subject to self-interference

Recently, the outage probability of GSC RAKE receivers subject to self-interference over independent and identically distributed Rayleigh fading channels has been investigated in [23]. Let γ_i be the SNR of the i-th diversity path and u_i ($i = 1, 2, \ldots, N$) be the order statistics obtained by arranging N $(N \geq 2)$ nonnegative i.n.d. RVs, $\{\gamma_i\}_{i=1}^{N}$, in decreasing order of magnitude such that $u_1 \geq u_2 \geq \cdots \geq u_N$. Then, the outage probability, denoted by $\mathrm{P}_{\mathrm{Out}}$, is then defined as [23],

$$\mathrm{P}_{\mathrm{Out}} = \Pr\left[\frac{\sum\limits_{n=1}^{m} u_n}{1 + \alpha \sum\limits_{n=m+1}^{N} u_n} < T\right], \tag{70}$$

where T $(0 \leq T)$ is the outage threshold and α $(0 \leq \alpha \leq 1)$ is the self-interference cancellation coefficient (in practice, each path may have the different value of α). The closed-form expression for this outage probability over i.i.d. Rayleigh fading paths has been derived and compared to that of partial RAKE receivers. However, the average signal-to-noise ratio (SNR) of each path (or branch) is different for most practical channel models, especially for wide-band spread spectrum signals. As results, to evaluate the outage probability over i.n.d. fading channel subject to self-interference, the major difficulty is to derive the joint PDF of $\sum\limits_{n=1}^{m} u_n$ and $\sum\limits_{n=m+1}^{N} u_n$ for i.n.d. case. Fortunately, the target joint PDF can be obtained with the help of Theorem 4.2 in Section IV.

APPENDICES

APPENDIX I

DERIVATION OF J_m

In this appendix, we derive Eq. (9). At first, we derive special case $N = 3$ and then we extend this result to general case for arbitrary N and m.

A. Special Case

Let us first consider $N = 3$ and $m = 3$ case as

$$\sum_{\substack{i_1,i_2,i_3\\ i_1\neq i_2\neq i_3}}^{1,2,3} \int_0^\infty du_1 p_{i_1}\left(u_1\right)\exp\left(\lambda u_1\right) \int_0^{u_1} du_2 p_{i_2}\left(u_2\right)\exp\left(\lambda u_2\right) \int_0^{u_2} du_3 p_{i_3}\left(u_3\right)\exp\left(\lambda u_3\right). \tag{71}$$

In here, we can rewrite (71) as

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,i_3\\ i_1\neq i_2\neq i_3}}^{1,2,3} \int_0^\infty du_1 p_{i_1}\left(u_1\right)\exp\left(\lambda u_1\right) \int_0^{u_1} du_2 p_{i_2}\left(u_2\right)\exp\left(\lambda u_2\right) \int_0^{u_2} du_3 p_{i_3}\left(u_3\right)\exp\left(\lambda u_3\right)\\
&=\sum_{i_1=1}^{3} \int_0^\infty du_1 p_{i_1}\left(u_1\right)\exp\left(\lambda u_1\right) \sum_{\substack{i_2,i_3\\ i_2\neq i_3\\ i_2\neq i_1\\ i_3\neq i_1}}^{1,2,3} \int_0^{u_1} du_2 p_{i_2}\left(u_2\right)\exp\left(\lambda u_2\right) \int_0^{u_2} du_3 p_{i_3}\left(u_3\right)\exp\left(\lambda u_3\right).
\end{aligned} \tag{72}$$

To simply (72), we consider $i_1 = 1, 2, 3$ separately.

i) for $i_1 = 1$

In this case, we can obtain the following result by deploying (72) as

$$\begin{aligned}
&\sum_{\substack{i_2,i_3\\ i_2\neq i_3\\ i_2\neq i_1\\ i_3\neq i_1}}^{1,2,3} \int_0^{u_1} du_2 p_{i_2}\left(u_2\right)\exp\left(\lambda u_2\right) \int_0^{u_2} du_3 p_{i_3}\left(u_3\right)\exp\left(\lambda u_3\right)\\
&=\int_0^{u_1} du_2 p_2\left(u_2\right)\exp\left(\lambda u_2\right) \int_0^{u_2} du_3 p_3\left(u_3\right)\exp\left(\lambda u_3\right) + \int_0^{u_1} du_2 p_3\left(u_2\right)\exp\left(\lambda u_2\right) \int_0^{u_2} du_3 p_2\left(u_3\right)\exp\left(\lambda u_3\right).
\end{aligned} \tag{73}$$

In (73), noting that $p_n(u_m)\exp(\lambda u_m) = {c_n}'(u_m,\lambda)$, after applying integration by part similar to [24], we can obtain the following result

$$\begin{aligned}
&\int_0^{u_1} du_2 p_2(u_2)\exp(\lambda u_2)\int_0^{u_2} du_3 p_3(u_3)\exp(\lambda u_3)\\
&=\int_0^{u_1} du_2 {c_2}'(u_2,\lambda)\, c_3(u_2,\lambda)\\
&=c_2(u_1,\lambda)\, c_3(u_1,\lambda)-\int_0^{u_1} du_2 c_2(u_2,\lambda)\, {c_3}'(u_2,\lambda)\\
&=c_2(u_1,\lambda)\, c_3(u_1,\lambda)-\int_0^{u_1} du_2 p_3(u_2)\exp(\lambda u_2)\int_0^{u_2} du_3 p_2(u_3)\exp(\lambda u_3).
\end{aligned} \tag{74}$$

Using (74) in (73) and then some manipulation, we can show

$$\begin{aligned}
&\sum_{\substack{i_2,i_3\\ i_2\neq i_3\\ i_2\neq i_1\\ i_3\neq i_1}}^{1,2,3}\int_0^{u_1} du_2 p_{i_2}(u_2)\exp(\lambda u_2)\int_0^{u_2} du_3 p_{i_3}(u_3)\exp(\lambda u_3)\\
&=c_2(u_1,\lambda)\, c_3(u_1,\lambda)-\int_0^{u_1} du_2 p_3(u_2)\exp(\lambda u_2)\int_0^{u_2} du_3 p_2(u_3)\exp(\lambda u_3)\\
&\quad+\int_0^{u_1} du_2 p_3(u_2)\exp(\lambda u_2)\int_0^{u_2} du_3 p_2(u_3)\exp(\lambda u_3) \qquad (75)\\
&=c_2(u_1,\lambda)\, c_3(u_1,\lambda). \qquad (76)
\end{aligned}$$

ii) for $i_1=2$

In this case, we can obtain the following result by deploying (72) as

$$\begin{aligned}
&\sum_{\substack{i_2,i_3\\ i_2\neq i_3\\ i_2\neq i_1\\ i_3\neq i_1}}^{1,2,3}\int_0^{u_1} du_2 p_{i_2}(u_2)\exp(\lambda u_2)\int_0^{u_2} du_3 p_{i_3}(u_3)\exp(\lambda u_3)\\
&=\int_0^{u_1} du_2 p_1(u_2)\exp(\lambda u_2)\int_0^{u_2} du_3 p_3(u_3)\exp(\lambda u_3)+\int_0^{u_1} du_2 p_3(u_2)\exp(\lambda u_2)\int_0^{u_2} du_3 p_1(u_3)\exp(\lambda u_3).
\end{aligned} \tag{77}$$

With (77), by applying similar approach like I-A-i), we can show the following result

$$\sum_{\substack{i_2,i_3\\ i_2\neq i_3\\ i_2\neq i_1\\ i_3\neq i_1}}^{1,2,3} \int_0^{u_1} du_2 p_{i_2}\left(u_2\right)\exp\left(\lambda u_2\right)\int_0^{u_2} du_3 p_{i_3}\left(u_3\right)\exp\left(\lambda u_3\right)$$
$$= c_1\left(u_1,\lambda\right)c_3\left(u_1,\lambda\right). \tag{78}$$

iii) for $i_1 = 3$

In this case, we can also obtain the following result by deploying (72) as

$$\sum_{\substack{i_2,i_3\\ i_2\neq i_3\\ i_2\neq i_1\\ i_3\neq i_1}}^{1,2,3} \int_0^{u_1} du_2 p_{i_2}\left(u_2\right)\exp\left(\lambda u_2\right)\int_0^{u_2} du_3 p_{i_3}\left(u_3\right)\exp\left(\lambda u_3\right)$$
$$= \int_0^{u_1} du_2 p_1\left(u_2\right)\exp(\lambda u_2)\int_0^{u_2} du_3 p_2\left(u_3\right)\exp(\lambda u_3) + \int_0^{u_1} du_2 p_2\left(u_2\right)\exp(\lambda u_2)\int_0^{u_2} du_3 p_1\left(u_3\right)\exp(\lambda u_3). \tag{79}$$

With (79), by applying similar approach like I-A-i) and I-A-ii), we can show the following result

$$\sum_{\substack{i_2,i_3\\ i_2\neq i_3\\ i_2\neq i_1\\ i_3\neq i_1}}^{1,2,3} \int_0^{u_1} du_2 p_{i_2}\left(u_2\right)\exp\left(\lambda u_2\right)\int_0^{u_2} du_3 p_{i_3}\left(u_3\right)\exp\left(\lambda u_3\right)$$
$$= c_1\left(u_1,\lambda\right)c_2\left(u_1,\lambda\right). \tag{80}$$

From results (75), (78), and (80), we can finally simplify (71) as

$$\sum_{\substack{i_2,i_3\\ i_2\neq i_3\\ i_2\neq i_1\\ i_3\neq i_1}}^{1,2,3} \int_0^{u_1} du_2 p_{i_2}\left(u_2\right)\exp\left(\lambda u_2\right)\int_0^{u_2} du_3 p_{i_3}\left(u_3\right)\exp\left(\lambda u_3\right)$$
$$= \sum_{\{i_2,i_3\}\in \mathrm{P}_2(I_3-\{i_1\})} \prod_{\substack{l=1\\ \{i_2,i_3\}}}^{2} c_{i_l}\left(u_1,\lambda\right) \tag{81}$$

B. General Case

With arbitrary N and m, we can re-write (71) as

$$J_m = \sum_{\substack{i_m, i_{m+1}, \ldots, i_N \\ i_m \neq i_{m+1} \neq \cdots \neq i_N \\ i_m \neq i_1, i_2, \ldots, i_{m-1} \\ i_{m+1} \neq i_1, i_2, \ldots, i_{m-1} \\ \vdots \\ i_N \neq i_1, i_2, \ldots, i_{m-1}}}^{1,2,\ldots,N} \int_0^{u_{m-1}} du_m p_{i_m}(u_m) \exp(\lambda u_m) \int_0^{u_m} du_{m+1} p_{i_{m+1}}(u_{m+1}) \exp(\lambda u_{m+1}) \cdots \int_0^{u_{N-1}} du_N p_{i_N}(u_N) \exp(\lambda u_N). \tag{82}$$

By applying the process presented in I-A to (82) similarly, the (81) can be generalized to arbitrary N and m, which leads to the result in Eq. (9) as

$$J_m = \sum_{\{i_m, i_{m+1}, \ldots, i_N\} \in \mathrm{P}_{N-m+1}(I_N - \{i_1, i_2, \ldots, i_{m-1}\})} \prod_{\substack{l=m \\ \{i_m, i_{m+1}, \ldots, i_N\}}}^{N} c_{i_l}(u_{m-1}, \lambda). \tag{83}$$

Appendix II

Derivation of J'_m

In this appendix, we derive Eq. (11). At first, we similarly derive special case $N=3$ and $m=3$ and then we extend this result to general case for arbitrary N and m.

A. Special Case

Let us first consider $N=3$ and $m=3$ case as

$$\sum_{\substack{i_1, i_2, i_3 \\ i_1 \neq i_2 \neq i_3}}^{1,2,3} \int_{u_4}^{\infty} du_3 p_{i_3}(u_3) \exp(\lambda u_3) \int_{u_3}^{\infty} du_2 p_{i_2}(u_2) \exp(\lambda u_2) \int_{u_2}^{\infty} du_1 p_{i_1}(u_1) \exp(\lambda u_1). \tag{84}$$

In here, similar to I-A, after deploying (84) and then some manipulation with the help of integral by part based on $p_n(u_m)\exp(\lambda u_m) = -{e_n}'(\lambda u_m)$, we can finally simplify (84) as

$$\sum_{\substack{i_1, i_2, i_3 \\ i_1 \neq i_2 \neq i_3}}^{1,2,3} \int_{u_4}^{\infty} du_3 p_{i_3}(u_3) \exp(\lambda u_3) \int_{u_3}^{\infty} du_2 p_{i_2}(u_2) \exp(\lambda u_2) \int_{u_2}^{\infty} du_1 p_{i_1}(u_1) \exp(\lambda u_1) = e_1(u_4, \lambda) e_2(u_4, \lambda) e_3(u_4, \lambda). \tag{85}$$

B. General Case

With arbitrary N and m, we can re-write (84) as

$$J'_m = \sum_{\substack{i_1,i_2,\ldots,i_m \\ i_1\neq i_2\neq\cdots\neq i_m \\ i_1\neq i_{m+1},i_{m+2},\ldots,i_N \\ i_2\neq i_{m+1},i_{m+2},\ldots,i_N \\ \vdots \\ i_m\neq i_{m+1},i_{m+2},\ldots,i_N}}^{1,2,\ldots,N} \int_{u_{m+1}}^{\infty} du_m p_{i_m}(u_m)\exp(\lambda u_m) \int_{u_m}^{\infty} du_{m-1} p_{i_{m-1}}(u_{m-1})\exp(\lambda u_{m-1}) \cdots \int_{u_2}^{\infty} du_1 p_{i_1}(u_1)\exp(\lambda u_1). \tag{86}$$

By applying the process presented in II-A to (86) similar to I, the (85) can be generalized to arbitrary N and m, which leads to the result in Eq. (11) as the closed-form

$$J'_m = \sum_{\{i_1,i_2,\ldots,i_m\}\in \mathrm{P}_m(I_N-\{i_{m+1},i_{m+2},\ldots,i_N\})} \prod_{\substack{l=1 \\ \{i_1,i_2,\ldots,i_m\}}}^{m} e_{i_l}(u_{m+1},\lambda). \tag{87}$$

Appendix III

Derivation of $J''_{a,b}$

In this appendix, we show the derivation of Eq.(13). Similar to the derivation progress of (9) and (11), we first derive special case $N=3$ and $m=3$ and then we extend this result to general case for arbitrary N and m.

A. Special Case

Let us first consider $N=3$ and $m=3$ case as

$$\sum_{\substack{i_1,i_2,i_3 \\ i_1\neq i_2\neq i_3}}^{1,2,3} \int_{u_5}^{u_1} du_4 p_{i_3}(u_4)\exp(\lambda u_4) \int_{u_4}^{u_1} du_3 p_{i_2}(u_3)\exp(\lambda u_3) \int_{u_3}^{u_1} du_2 p_{i_1}(u_2)\exp(\lambda u_2). \tag{88}$$

In here, by deploying (88), (88) can be re-written as

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,i_3\\ i_1\neq i_2\neq i_3}}^{1,2,3}\int_{u_5}^{u_1} du_4 p_{i_3}(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_{i_2}(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_{i_1}(u_2)\exp(\lambda u_2)\\
&=\int_{u_5}^{u_1} du_4 p_1(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_2(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_3(u_2)\exp(\lambda u_2)\\
&\quad+\int_{u_5}^{u_1} du_4 p_1(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_3(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_2(u_2)\exp(\lambda u_2)\\
&\quad+\int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_1(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_3(u_2)\exp(\lambda u_2)\\
&\quad+\int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_3(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_2(u_2)\exp(\lambda u_2)\\
&\quad+\int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_1(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_2(u_2)\exp(\lambda u_2)\\
&\quad+\int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_2(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_1(u_2)\exp(\lambda u_2). \qquad (89)
\end{aligned}$$

In (89), using similar manipulations with (74) to the ones used in the previous Appendices I and II, the first, the second and the third multiple integral terms can be also re-written as, respectively

$$\begin{aligned}
&\int_{u_5}^{u_1} du_4 p_1(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_2(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_3(u_2)\exp(\lambda u_2)\\
&=\int_{u_5}^{u_1} du_4 p_1(u_4)\exp(\lambda u_4)\left\{c_2(u_4,\lambda)c_3(u_4,\lambda)-c_2(u_4,\lambda)c_3(u_1,\lambda)+\int_{u_4}^{u_1} du_3 p_3(u_3)\exp(\lambda u_3)c_2(u_3,\lambda)\right\}, \qquad (90)
\end{aligned}$$

$$\begin{aligned}
&\int_{u_5}^{u_1} du_4 p_1(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_3(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_2(u_2)\exp(\lambda u_2)\\
&=\int_{u_5}^{u_1} du_4 p_1(u_4)\exp(\lambda u_4)\left\{c_3(u_1,\lambda)c_2(u_1,\lambda)-c_3(u_4,\lambda)c_2(u_1,\lambda)-\int_{u_4}^{u_1} du_3 p_3(u_3)\exp(\lambda u_3)c_2(u_3,\lambda)\right\}, \qquad (91)
\end{aligned}$$

$$\begin{aligned}
&\int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_1(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_3(u_2)\exp(\lambda u_2)\\
&=\int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\left\{c_1(u_4,\lambda)c_3(u_4,\lambda)-c_1(u_4,\lambda)c_3(u_1,\lambda)+\int_{u_4}^{u_1} du_3 p_3(u_3)\exp(\lambda u_3)c_1(u_3,\lambda)\right\}. \qquad (92)
\end{aligned}$$

Similarly in (89), the 4-th, 5-th and the final multiple integral terms can be also re-written as respectively

$$\begin{aligned}
&\int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_3(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_1(u_2)\exp(\lambda u_2)\\
&=\int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\left\{c_3(u_1,\lambda)\,c_1(u_1,\lambda)-c_3(u_4,\lambda)\,c_1(u_1,\lambda)-\int_{u_4}^{u_1} du_3 p_3(u_3)\exp(\lambda u_3)\,c_1(u_3,\lambda)\right\},
\end{aligned} \tag{93}$$

$$\begin{aligned}
&\int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_1(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_2(u_2)\exp(\lambda u_2)\\
&=\int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\left\{c_1(u_4,\lambda)\,c_2(u_4,\lambda)-c_1(u_4,\lambda)\,c_2(u_1,\lambda)+\int_{u_4}^{u_1} du_3 p_2(u_3)\exp(\lambda u_3)\,c_1(u_3,\lambda)\right\},
\end{aligned} \tag{94}$$

$$\begin{aligned}
&\int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_2(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_1(u_2)\exp(\lambda u_2)\\
&=\int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\left\{c_2(u_1,\lambda)\,c_1(u_1,\lambda)-c_2(u_4,\lambda)\,c_1(u_1,\lambda)-\int_{u_4}^{u_1} du_3 p_2(u_3)\exp(\lambda u_3)\,c_1(u_3,\lambda)\right\}.
\end{aligned} \tag{95}$$

Using all the above results from (90) to (95) in (89) and then after some manipulations similar to the one used in previous Appendices I and II, (89) can be simplified as

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,i_3\\ i_1\neq i_2\neq i_3}}^{1,2,3}\int_{u_5}^{u_1} du_4 p_{i_3}(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_{i_2}(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_{i_1}(u_2)\exp(\lambda u_2)\\
&=\int_{u_5}^{u_1} du_4 p_1(u_4)\exp(\lambda u_4)\{c_2(u_1,\lambda)-c_2(u_4,\lambda)\}\{c_3(u_1,\lambda)-c_3(u_4,\lambda)\}\\
&\quad+\int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\{c_1(u_1,\lambda)-c_1(u_4,\lambda)\}\{c_3(u_1,\lambda)-c_3(u_4,\lambda)\}\\
&\quad+\int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\{c_1(u_1,\lambda)-c_1(u_4,\lambda)\}\{c_2(u_1,\lambda)-c_2(u_4,\lambda)\}.
\end{aligned} \tag{96}$$

In (96), after applying (74) to the first integral terms and then some manipulations, it can be simply re-written as

$$\begin{aligned}
&\int_{u_5}^{u_1} du_4 p_1(u_4)\exp(\lambda u_4)\{c_2(u_1,\lambda)-c_2(u_4,\lambda)\}\{c_3(u_1,\lambda)-c_3(u_4,\lambda)\} \\
&= -c_1(u_5,\lambda)\{c_2(u_1,\lambda)-c_2(u_5,\lambda)\}\{c_3(u_1,\lambda)-c_3(u_5,\lambda)\} \\
&\quad + \int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\, c_1(u_4,\lambda)\{c_3(u_1,\lambda)-c_3(u_4,\lambda)\} \\
&\quad + \int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\, c_1(u_4,\lambda)\{c_2(u_1,\lambda)-c_2(u_4,\lambda)\}.
\end{aligned} \tag{97}$$

Using (97) in (96), (96) can be simplified as

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,i_3 \\ i_1\neq i_2\neq i_3}}^{1,2,3} \int_{u_5}^{u_1} du_4 p_{i_3}(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_{i_2}(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_{i_1}(u_2)\exp(\lambda u_2) \\
&= -c_1(u_5,\lambda)\{c_2(u_1,\lambda)-c_2(u_5,\lambda)\}\{c_3(u_1,\lambda)-c_3(u_5,\lambda)\} \\
&\quad + \int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\, c_1(u_1,\lambda)\{c_3(u_1,\lambda)-c_3(u_4,\lambda)\} \\
&\quad + \int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\, c_1(u_1,\lambda)\{c_2(u_1,\lambda)-c_2(u_4,\lambda)\}.
\end{aligned} \tag{98}$$

In (98), after applying (74) to the first integral terms with the help of similar manipulations used in (97), the first integral terms in (98) can be simply re-written as

$$\begin{aligned}
&\int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\, c_1(u_1,\lambda)\{c_3(u_1,\lambda)-c_3(u_4,\lambda)\} \\
&= -c_1(u_1,\lambda)\, c_2(u_5,\lambda)\{c_3(u_1,\lambda)-c_3(u_5,\lambda)\} \\
&\quad + \int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\, c_1(u_1,\lambda)\, c_2(u_4,\lambda).
\end{aligned} \tag{99}$$

Now, using (99), after adding (99) and the second integral term in (98), we can obtain the following result

$$\begin{aligned}
&\int_{u_5}^{u_1} du_4 p_2(u_4)\exp(\lambda u_4)\, c_1(u_4,\lambda)\{c_3(u_1,\lambda)-c_3(u_4,\lambda)\} \\
&\quad+\int_{u_5}^{u_1} du_4 p_3(u_4)\exp(\lambda u_4)\, c_1(u_4,\lambda)\{c_2(u_1,\lambda)-c_2(u_4,\lambda)\} \\
&= -c_1(u_1,\lambda)\, c_2(u_5,\lambda)\{c_3(u_1,\lambda)-c_3(u_5,\lambda)\} \\
&\quad+\{c_3(u_1,\lambda)-c_3(u_5,\lambda)\}\, c_1(u_1,\lambda)\, c_2(u_1,\lambda) \\
&= c_1(u_1,\lambda)\{c_2(u_1,\lambda)-c_2(u_5,\lambda)\}\{c_3(u_1,\lambda)-c_3(u_5,\lambda)\}. \qquad (100)
\end{aligned}$$

Finally, using (100) in (98), (98) can be re-written as

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,i_3\\ i_1\neq i_2\neq i_3}}^{1,2,3} \int_{u_5}^{u_1} du_4 p_{i_3}(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_{i_2}(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_{i_1}(u_2)\exp(\lambda u_2) \\
&= c_1(u_1,\lambda)\{c_2(u_1,\lambda)-c_2(u_5,\lambda)\}\{c_3(u_1,\lambda)-c_3(u_5,\lambda)\} \\
&\quad -c_1(u_5,\lambda)\{c_2(u_1,\lambda)-c_2(u_5,\lambda)\}\{c_3(u_1,\lambda)-c_3(u_5,\lambda)\}. \qquad (101)
\end{aligned}$$

By simplifying (101), we can obtain the final closed-form for special case $N=3$ and $m=3$ as

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,i_3\\ i_1\neq i_2\neq i_3}}^{1,2,3} \int_{u_5}^{u_1} du_4 p_{i_3}(u_4)\exp(\lambda u_4)\int_{u_4}^{u_1} du_3 p_{i_2}(u_3)\exp(\lambda u_3)\int_{u_3}^{u_1} du_2 p_{i_1}(u_2)\exp(\lambda u_2) \\
&= \{c_1(u_1,\lambda)-c_1(u_5,\lambda)\}\{c_2(u_1,\lambda)-c_2(u_5,\lambda)\}\{c_3(u_1,\lambda)-c_3(u_5,\lambda)\} \qquad (102) \\
&= \mu_1(u_5,u_1,\lambda)\,\mu_2(u_5,u_1,\lambda)\,\mu_3(u_5,u_1,\lambda). \qquad (103)
\end{aligned}$$

B. General Case

With arbitrary N and m, we can also re-write (88) as

$$\begin{aligned}
J'_{a,b} \;=\; &\sum_{\substack{i_{a+1},\ldots,i_{b-1}\\ i_{a+1}\neq i_{a+2}\neq\cdots\neq i_{b-1}\\ i_{a+1}\neq i_1,\cdots,i_a,i_b,\ldots,i_N\\ i_{a+2}\neq i_1,\cdots,i_a,i_b,\ldots,i_N\\ \vdots\\ i_{b-1}\neq i_1,\cdots,i_a,i_b,\ldots,i_N}}^{1,2,\ldots,N} \int_{u_b}^{u_a} du_{b-1} p_{i_{b-1}}(u_{b-1})\exp(\lambda u_{b-1})\int_{u_{b-1}}^{u_a} du_{b-2} p_{i_{b-2}}(u_{b-2})\exp(\lambda u_{b-2}) \\
&\cdots\int_{u_{a+2}}^{u_a} du_{a+1} p_{i_{a+1}}(u_{a+1})\exp(\lambda u_{a+1}). \qquad (104)
\end{aligned}$$

By applying the similar process presented in I and II, the (102) can be generalized to arbitrary N and m, which leads to the result in Eq. (13) as the closed-form

$$J'_{a,b} = \sum_{\{i_{a+1},\ldots,i_{b-1}\}\in \mathrm{P}_{b-a+1}(I_N-\{i_1,\cdots,i_a,i_b,\ldots,i_N\})} \prod_{\substack{l=a+1\\ \{i_{a+1},\ldots,i_{b-1}\}}}^{b-1} \mu_{i_l}\left(u_b,u_a,\lambda\right). \tag{105}$$

APPENDIX IV

DERIVATION OF (20)

Starting with (19), with the help of integral solution, (19) can be simply re-written as

$$\begin{aligned}
&MGF_Z\left(\lambda_1,\lambda_2\right)\\
=&\sum_{i_m=1}^{N}\int_0^\infty du_m p_{i_m}\left(u_m\right)\exp\left(\lambda_1 u_m\right)\\
&\times \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})}\int_{u_m}^\infty du_{m-1}p_{i_{m-1}}\left(u_{m-1}\right)\exp(\lambda_1 u_{m-1})\cdots\int_{u_2}^\infty du_1 p_{i_1}\left(u_1\right)\exp(\lambda_1 u_1)\\
&\times \sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})}\int_0^{u_m} du_{m+1}p_{i_{m+1}}\left(u_{m+1}\right)\exp(\lambda_2 u_{m+1})\cdots\int_0^{u_{N-1}} du_N p_{i_N}\left(u_N\right)\exp(\lambda_2 u_N).
\end{aligned} \tag{106}$$

In (106), by simply applying (9) and (11), we can easily obtain each of the following results

$$\begin{aligned}
&\sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})}\int_0^{u_m} du_{m+1}p_{i_{m+1}}\left(u_{m+1}\right)\exp(\lambda_2 u_{m+1})\cdots\int_0^{u_{N-1}} du_N p_{i_N}\left(u_N\right)\exp(\lambda_2 u_N)\\
&= \sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})} \prod_{\substack{l=m+1\\ \{i_{m+1},\ldots,i_N\}}}^{N} c_{i_l}\left(u_m,\lambda_2\right),
\end{aligned} \tag{107}$$

$$\begin{aligned}
&\sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})}\int_{u_m}^\infty du_{m-1}p_{i_{m-1}}\left(u_{m-1}\right)\exp\left(\lambda_1 u_{m-1}\right)\cdots\int_{u_2}^\infty du_1 p_{i_1}\left(u_1\right)\exp\left(\lambda_1 u_1\right)\\
&= \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})} \prod_{\substack{k=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_k}\left(u_m,\lambda_1\right).
\end{aligned} \tag{108}$$

By inserting (107) and (108) in order into (19), we can obtain the second order MGF of $Z_1 = \sum_{n=1}^{m} u_n$ and $Z_2 = \sum_{n=m+1}^{N} u_n$ as

$$\begin{aligned} MGF_Z\left(\lambda_1,\lambda_2\right) &= \sum_{i_m=1}^{N}\int_0^{\infty} du_m p_{i_m}\left(u_m\right)\exp\left(\lambda_1 u_m\right) \\ &\times \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})} \prod_{\substack{k=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_k}\left(u_m,\lambda_1\right) \\ &\times \sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})} \prod_{\substack{l=m+1\\ \{i_{m+1},\ldots,i_N\}}}^{N} c_{i_l}\left(u_m,\lambda_2\right). \end{aligned} \tag{109}$$

APPENDIX V

DERIVATION OF THE JOINT PDF OF u_m AND $\sum_{\substack{n=1\\ n\neq m}}^{N_s} u_n$ FOR $1<m<N_s-1$ AMONG N ORDERED RVS

In this Appendix, we derive the joint PDF of u_m and $\sum_{\substack{n=1\\ n\neq m}}^{N_s} u_n$ among N ordered RVs by considering $1<m<N_s-1$.

Let $Z_1=\sum_{n=1}^{m-1}u_n$, $Z_2=u_m$, $Z_3=\sum_{n=m+1}^{N_s-1}u_n$ and $Z_4=u_{N_s}$. The 4-dimensional MGF of $Z=[Z_1,Z_2,Z_3,Z_4]$ is given by the expectation

$$\begin{aligned} & MGF_Z\left(\lambda_1,\lambda_2,\lambda_3,\lambda_4\right)=E\left\{\exp\left(\lambda_1 Z_1+\lambda_2 Z_2+\lambda_3 Z_3+\lambda_4 Z_4\right)\right\} \\ &= \sum_{\substack{i_1,i_2,\cdots,i_N\\ i_1\neq i_2\neq\cdots\neq i_N}}^{1,2,\cdots,N} \int_0^{\infty} du_1 p_{i_1}\left(u_1\right)\exp\left(\lambda_1 u_1\right)\cdots \int_0^{u_{m-2}} du_{m-1}p_{i_{m-1}}\left(u_{m-1}\right)\exp\left(\lambda_1 u_{m-1}\right) \\ &\times \int_0^{u_{m-1}} du_m p_{i_m}\left(u_m\right)\exp\left(\lambda_2 u_m\right) \\ &\times \int_0^{u_m} du_{m+1}p_{i_{m+1}}\left(u_{m+1}\right)\exp\left(\lambda_3 u_{m+1}\right)\cdots \int_0^{u_{N_s-2}} du_{N_s-1}p_{i_{N_s-1}}\left(u_{N_s-1}\right)\exp\left(\lambda_3 u_{N_s-1}\right) \\ &\times \int_0^{u_{N_s-1}} du_{N_s}p_{i_{N_s}}\left(u_{N_s}\right)\exp\left(\lambda_4 u_{N_s}\right)\prod_{j=N_s+1}^{N} P_{i_j}\left(u_{N_s}\right). \end{aligned} \tag{110}$$

With the help of integral solution presented in [24], (9), (11) and (13), we can easily obtain the

4-dimensional MGF of Z_1, Z_2, Z_3 and Z_4 as

$$\begin{aligned}
&MGF_Z\left(\lambda_1,\lambda_2,\lambda_3,\lambda_4\right)\\
&=\sum_{\substack{i_{N_s},\ldots,i_N\\ i_{N_s}\neq\cdots\neq i_N}}^{1,2,\ldots,N}\int_0^{\infty} du_{N_s} p_{i_{N_s}}\left(u_{N_s}\right)\exp\left(\lambda_4 u_{N_s}\right)\prod_{\substack{j=N_s+1\\ \{i_{N_s+1},\ldots,i_N\}}}^{N} P_{i_j}\left(u_{N_s}\right)\\
&\quad\times\sum_{\substack{i_m=1\\ i_m\neq i_{N_s},\ldots,i_N}}^{N}\int_{u_{N_s}}^{\infty} du_m p_{i_m}\left(u_m\right)\exp\left(\lambda_2 u_m\right)\\
&\quad\times\sum_{\{i_{m+1},\ldots,i_{N_s-1}\}\in \mathrm{P}_{N_s-m-1}(I_N-\{i_m\}-\{i_{N_s},\ldots,i_N\})}\prod_{\substack{k=m+1\\ \{i_{m+1},\ldots,i_{N_s-1}\}}}^{N_s-1}\mu_{i_k}\left(u_{N_s},u_m,\lambda_3\right)\\
&\quad\times\sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\}-\{i_{N_s},\ldots,i_N\}-\{i_{m+1},\ldots,i_{N_s-1}\})}\prod_{\substack{l=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_l}\left(u_m,\lambda_1\right). \qquad (111)
\end{aligned}$$

Having a MGF expression given in (111), we are now in the position to derive the 4-dimensional joint PDF of $Z_1=\sum_{n=1}^{m-1}u_n$, $Z_2=u_m$, $Z_3=\sum_{n=m+1}^{N_s-1}u_n$ and $Z_4=u_{N_s}$. Letting $\lambda_1=-S_1$, $\lambda_2=-S_2$, $\lambda_3=-S_3$, and $\lambda_4=-S_4$ we can derive the 4-dimensional PDF of Z_1, Z_2, Z_3 and Z_4 by applying an inverse Laplace transform yielding

$$\begin{aligned}
&p_Z\left(z_1,z_2,z_3,z_4\right)=\mathcal{L}^{-1}_{S_1,S_2,S_3,S_4}\left\{MGF_Z\left(-S_1,-S_2,-S_3,-S_4\right)\right\}\\
&=\sum_{\substack{i_{N_s},\ldots,i_N\\ i_{N_s}\neq\cdots\neq i_N}}^{1,2,\ldots,N}\int_0^{\infty} du_{N_s} p_{i_{N_s}}\left(u_{N_s}\right)L^{-1}_{S_4}\left\{\exp\left(-S_4 u_{N_s}\right)\right\}\prod_{\substack{j=N_s+1\\ \{i_{N_s+1},\ldots,i_N\}}}^{N} P_{i_j}\left(u_{N_s}\right)\\
&\quad\times\sum_{\substack{i_m=1\\ i_m\neq i_{N_s},\ldots,i_N}}^{N}\int_{u_{N_s}}^{\infty} du_m p_{i_m}\left(u_m\right)L^{-1}_{S_2}\left\{\exp\left(-S_2 u_m\right)\right\}\\
&\quad\times\sum_{\{i_{m+1},\ldots,i_{N_s-1}\}\in \mathrm{P}_{N_s-m-1}(I_N-\{i_m\}-\{i_{N_s},\ldots,i_N\})}L^{-1}_{S_3}\left\{\prod_{\substack{k=m+1\\ \{i_{m+1},\ldots,i_{N_s-1}\}}}^{N_s-1}\mu_{i_k}\left(u_{N_s},u_m,-S_3\right)\right\}\\
&\quad\times\sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\}-\{i_{N_s},\ldots,i_N\}-\{i_{m+1},\ldots,i_{N_s-1}\})}L^{-1}_{S_1}\left\{\prod_{\substack{l=1\\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} e_{i_l}\left(u_m,-S_1\right)\right\}. \qquad (112)
\end{aligned}$$

With this 4-dimensional joint PDF, letting $X=Z_2$ and $Y=Z_1+Z_3+Z_4$ we can obtain the 2-dimensional joint PDF of $Z'=[X,Y]$ by integrating over z_1 and z_4 yielding

$$p_{Z'}\left(x,y\right)=\int_0^{x}\int_{(m-1)x}^{y-(N_s-m)z_4} p_Z\left(z_1,x,y-z_4,z_4\right)dz_1 dz_4, \qquad (113)$$

or equivalently we can obtain the 2-dimensional joint PDF of $Z' = [X, Y]$ by integrating over z_3 and z_4 giving

$$p_{Z'}(x,y) = \int_0^x \int_{(N_s-m-1)z_4}^{(N_s-m-1)x} p_Z(y - z_3 - z_4, x, z_3, z_4)\, dz_3 dz_4. \tag{114}$$

APPENDIX VI

DERIVATION OF MULTIPLE PRODUCT OF COMMON FUNCTIONS

In VI-ii), (36), (37), and (38) have the form of multiple product of (33), (34), and (35), respectively. Therefore, to apply an inverse LT for deriving final PDF closed-form expressions from MGF expressions, a multiple product expression needs to be converted to a summation expression of λ function. In this appendix, we derive simple summation expressions of λ function from multiple product expressions. To derive them, the following four formulas should be converted to a summation expression.

i) $\frac{1}{\prod\limits_{l}\left(1-\bar{\gamma}_{i_l}\lambda\right)}$

At first, we derive special case for a) the multiple product from 1 to n and then we extend this result to general case for b) the multiple product from arbitrary n_1 to n_2.

For case a), we need to convert the following multiple product from 1 to n to a summation expression.

$$\frac{1}{\prod\limits_{l=1}^{n}\left(1-\bar{\gamma}_{i_l}\lambda\right)}. \tag{115}$$

With (115), after deploying the multiple product term and then rearrange and simplify them, the multiple product term can be converted to the summation expression of just λ as

$$\frac{1}{\prod\limits_{l=1}^{n}\left(1-\bar{\gamma}_{i_l}\lambda\right)} = \sum_{l=1}^{n} \frac{C_{l,1,n}}{\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right)}, \tag{116}$$

where $j_0 = 0$,

$$C_{l,1,n} = \frac{1}{\prod\limits_{l=1}^{n} (-\bar{\gamma}_{i_l}) F'\left(\frac{1}{\bar{\gamma}_{i_l}}\right)}, \tag{117}$$

$$F'(x) = \left[\sum_{l=1}^{n-1} (n-l)\, x^{n-1-l} (-1)^l \sum_{j_1=j_0+1}^{n-l+1} \cdots \sum_{j_l=j_{l-1}+1}^{n} \prod_{m=1}^{l} \frac{1}{\bar{\gamma}_{i_{j_m}}}\right] + (n)\, x^{n-1}. \tag{118}$$

For the case of the multiple product from arbitrary n_1 to n_2, after applying the same derivation progress as (116), we can obtain the final result as

$$\frac{1}{\prod\limits_{l=n_1}^{n_2} (1-\bar{\gamma}_{i_l}\lambda)} = \sum_{l=n_1}^{n_2} \frac{C_{l,n_1,n_2}}{\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right)}, \tag{119}$$

where

$$C_{l,n_1,n2} = \frac{1}{\prod\limits_{l=n_1}^{n_2} (-\bar{\gamma}_{i_l}) F'\left(\frac{1}{\bar{\gamma}_{i_l}}\right)}, \tag{120}$$

$$\begin{aligned} F'(x) \quad &= \left[\sum_{l=1}^{n_2-n_1} (n_2-n_1-l+1)\, x^{n_2-n_1-l} (-1)^l \sum_{j_1=j_0+n_1}^{n_2-l+1} \cdots \sum_{j_l=j_{l-1}+1}^{n_2} \prod_{m=1}^{l} \frac{1}{\bar{\gamma}_{i_{j_m}}}\right] \\ &\quad + (n_2-n_1+1)\, x^{n_2-n_1}. \end{aligned} \tag{121}$$

ii) $\prod\limits_{l} \left[1 - \exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)\right]$

Similar to VI-i), at first, we derive special case for a) the multiple product from 1 to n and then we extend this result to general case for b) the multiple product from arbitrary n_1 to n_2.

For case a), after deploying the multiple product term of exponential function from 1 to n and then simplify them, the multiple product term can be converted to the summation expression of λ as

$$\begin{aligned} &\prod_{l=1}^{n} \left[1 - \exp\left(\left(\lambda - \frac{1}{\bar{\gamma}_{i_l}}\right) z_a\right)\right] \\ &= 1 + \left[\sum_{l=1}^{n} \exp\left(l \cdot z_a \cdot \lambda\right) \left\{(-1)^l \sum_{j_1=j_0+1}^{n-l+1} \cdots \sum_{j_l=j_{l-1}+1}^{n} \exp\left(-\sum_{m=1}^{l} \frac{z_a}{\bar{\gamma}_{i_{j_m}}}\right)\right\}\right], \end{aligned} \tag{122}$$

where $j_0 = 0$.

For case b), after applying the same derivation progress as (122), the multiple product from arbitrary n_1 to n_2 can be obtained as

$$\begin{aligned}&\prod_{l=n_1}^{n_2}\left[1-\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_a\right)\right]\\&=1+\left[\sum_{l=1}^{n_2-n_1+1}\exp\left(l\cdot z_a\cdot\lambda\right)\left\{(-1)^l\sum_{j_1=j_0+n_1}^{n_2-l+1}\cdots\sum_{j_l=j_{l-1}+1}^{n_2}\exp\left(-\sum_{m=1}^{l}\frac{z_a}{\bar{\gamma}_{i_{j_m}}}\right)\right\}\right].\end{aligned}\quad(123)$$

iii) $\prod_l\left[\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_a\right)-\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_b\right)\right]$

Similar to VI-i) and ii), especially, using the similar manipulation used in VI-i) and ii) in (123), the final simple summation expression from arbitrary n_1 to n_2 can be obtained as

$$\begin{aligned}&\prod_{l=n_1}^{n_2}\left[\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_a\right)-\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_b\right)\right]\\&=\prod_{l=n_1}^{n_2}\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_a\right)\prod_{l=n_1}^{n_2}\left[1-\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)(z_b-z_a)\right)\right]\\&=\exp\left((n_2-n_1+1)\cdot\lambda\cdot z_a\right)\exp\left(-\sum_{l=n_1}^{n_2}\frac{z_a}{\bar{\gamma}_{i_l}}\right)\\&\quad\times\left[1+\sum_{l=n_1}^{n_2-n_1+1}\exp\left(l\cdot(z_b-z_a)\cdot\lambda\right)\left\{(-1)^l\sum_{j_1=j_0+n_1}^{n_2-l+1}\cdots\sum_{j_l=j_{l-1}+1}^{n_2}\exp\left(-\sum_{m=1}^{l}\frac{z_b-z_a}{\bar{\gamma}_{i_{j_m}}}\right)\right\}\right].\end{aligned}\quad(124)$$

iv) $\prod_l\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_a\right)$

In this case, with the help of the property of exponential multiplication, we can easily derive the summation expression from the multiple product expression from arbitrary n_1 to n_2, respectively, as

$$\begin{aligned}\prod_{l=n_1}^{n_2}\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_a\right)&=\exp\left(\left\{\sum_{l=n_1}^{n_2}\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)\right\}z_a\right)\\&=\exp\left(\left\{-\sum_{l=n_1}^{n_2}\left(\frac{z_a}{\bar{\gamma}_{i_l}}\right)\right\}\right)\exp\left((n_2-n_1+1)\,z_a\lambda\right).\end{aligned}\quad(125)$$

Based on the above results, we can now obtain the summation expressions of (33), (34), and (35) for arbitrary n_1 to n_2. With (33), (34), and (35), we can write the multiple product of (33), (34), and (35) for arbitrary n_1 to n_2 respectively as

$$\prod_{l=n_1}^{n_2} c_{i_l}\left(z_a,\lambda\right)=\frac{1}{\prod\limits_{l=n_1}^{n_2}\left(1-\bar{\gamma}_{i_l}\lambda\right)}\prod_{l=n_1}^{n_2}\left[1-\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_a\right)\right], \tag{126}$$

$$\prod_{l=n_1}^{n_2} e_{i_l}\left(z_a,\lambda\right)=\frac{1}{\prod\limits_{l=n_1}^{n_2}\left(1-\bar{\gamma}_{i_l}\lambda\right)}\prod_{l=n_1}^{n_2}\left[\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_a\right)\right], \tag{127}$$

$$\prod_{l=n_1}^{n_2} \mu_{i_l}\left(z_a,z_b,\lambda\right)=\frac{1}{\prod\limits_{l=n_1}^{n_2}\left(1-\bar{\gamma}_{i_l}\lambda\right)}\prod_{l=n_1}^{n_2}\left[\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_a\right)-\exp\left(\left(\lambda-\frac{1}{\bar{\gamma}_{i_l}}\right)z_b\right)\right]. \tag{128}$$

For the summation expression of the multiple product of (33) for arbitrary n_1 to n_2, using (119) and (123) in (126), we can obtain the final summation closed-form expression (36).

For the summation expression of the multiple product of (34) for arbitrary n_1 to n_2, using (119) and (125) in (127), we can obtain the final summation closed-form expression (37).

Finally, for the summation expression of the multiple product of (35) for arbitrary n_1 to n_2, using (119) and (124) in (128), we can obtain the final summation closed-form expression (38).

APPENDIX VII

CAPTURE PROBABILITY OF GSC RAKE RECEIVERS

A. Joint PDF

Starting from (47), we can re-write the joint PDF (47) as

$$
\begin{aligned}
& p_Z(z_1, z_2) \\
&= \sum_{i_m=1}^{N} \frac{1}{\bar{\gamma}_{i_m}} \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})} \sum_{\substack{k=1 \\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} C_{k,1,m-1} \sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})} \sum_{\substack{q=m+1 \\ \{i_{m+1},\ldots,i_N\}}}^{N} C_{q,m+1,N} \\
&\quad \times \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \int_0^{\frac{z_1}{m}} du_m \exp\left(-\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right) u_m\right) \\
&\quad + \sum_{i_m=1}^{N} \frac{1}{\bar{\gamma}_{i_m}} \sum_{\{i_1,\ldots,i_{m-1}\}\in \mathrm{P}_{m-1}(I_N-\{i_m\})} \sum_{\substack{k=1 \\ \{i_1,\ldots,i_{m-1}\}}}^{m-1} C_{k,1,m-1} \sum_{\{i_{m+1},\ldots,i_N\}\in \mathrm{P}_{N-m}(I_N-\{i_m\}-\{i_1,\ldots,i_{m-1}\})} \sum_{\substack{q=m+1 \\ \{i_{m+1},\ldots,i_N\}}}^{N} C_{q,m+1,N} \\
&\quad \times \left[\sum_{h=1}^{N-m} (-1)^h \sum_{j_1=j_0+m+1}^{N-h+1} \cdots \sum_{j_h=j_{h-1}+1}^{N} \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) \right. \\
&\qquad \left. \times \int_0^{\infty} du_m \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right) + \sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}} - \frac{h}{\bar{\gamma}_{i_q}}\right) u_m\right) U(z_1 - m u_m)\, U(z_2 - h u_m) \right].
\end{aligned}
\tag{129}
$$

In (129), there are two integral expressions and the first integral part can be directly derived as the following closed form expression

$$
\int_0^{\frac{z_1}{m}} du_m \exp\left(-\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right) u_m\right) = \frac{1 - \exp\left(-\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_1}{m}\right)}{\left(\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right) - \frac{m}{\bar{\gamma}_{i_k}}\right)}. \tag{130}
$$

However, for the second integral part, we need to consider two cases separately based on the valid integral region of z_1, z_2, and u_m as

$$\begin{aligned}
&\int_0^{\infty} du_m \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)u_m\right)U\left(z_1-mu_m\right)U\left(z_2-hu_m\right)\\
&=\int_0^{\frac{z_2}{h}} du_m \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)u_m\right)U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\\
&\quad+\int_0^{\frac{z_1}{m}} du_m \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)u_m\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right].
\end{aligned} \tag{131}$$

With simplified (131), we can get the following closed-form expressions, respectively, as

$$\begin{aligned}
&\int_0^{\frac{z_2}{h}} du_m \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)u_m\right)U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\\
&=\frac{1-\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_2}{h}\right)}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)}U\left(\frac{z_1}{m}-\frac{z_2}{h}\right),
\end{aligned} \tag{132}$$

and

$$\begin{aligned}
&\int_0^{\frac{z_1}{m}} du_m \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)u_m\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]\\
&=\frac{1-\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_1}{m}\right)}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}-\frac{h}{\bar{\gamma}_{i_q}}\right)}\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right].
\end{aligned} \tag{133}$$

B. Capture Probability

Starting from (62), inserting the closed-form expression of (47) presented in VII-A into (62), the closed-form expression for i.n.d. Rayleigh fading conditions can be written in (63). In (63), there are six double-integral expressions. For the first and second cases, we can directly obtain the closed-from expression as shown in (64) and (65). However, for others, we need to carefully consider the valid integral region respectively as

iii) The third integral expression:

In this case, for valid integration, we need to consider two cases separately. If $\frac{h}{m} \geq \frac{1-T}{T}$, then $z_2 \leq \frac{1-T}{T}z_1$ and $\frac{1}{m} \geq \frac{1-T}{T\cdot h}$. If $\frac{h}{m} < \frac{1-T}{T}$, then $z_2 \leq \frac{h}{m}z_1$ and $\frac{1}{m} < \frac{1-T}{T\cdot h}$. As a result, we can re-write the third integral expression as

$$\begin{aligned}
&\int_0^\infty \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) U\left(\frac{z_1}{m}-\frac{z_2}{h}\right) dz_2 dz_1 \\
&= \int_0^\infty \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right) dz_2 dz_1 \\
&\quad + \int_0^\infty \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \int_0^{\left(\frac{h}{m}\right)z_1} \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) \left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right] dz_2 dz_1.
\end{aligned} \tag{134}$$

From (134), we can directly derive the closed-form expressions as

$$\begin{aligned}
&\int_0^\infty \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right) dz_2 dz_1 \\
&= \bar{\gamma}_{i_q}\bar{\gamma}_{i_k} U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right) - \frac{\bar{\gamma}_{i_q}}{\left(\frac{1-T}{\bar{\gamma}_{i_q}T}+\frac{1}{\bar{\gamma}_{i_k}}\right)} U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right),
\end{aligned} \tag{135}$$

and

$$\begin{aligned}
&\int_0^\infty \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \int_0^{\left(\frac{h}{m}\right)z_1} \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) \left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right] dz_2 dz_1 \\
&= \bar{\gamma}_{i_q}\bar{\gamma}_{i_k} \left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right] - \frac{\bar{\gamma}_{i_q}}{\left(\frac{h}{\bar{\gamma}_{i_q}m}+\frac{1}{\bar{\gamma}_{i_k}}\right)} \left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right].
\end{aligned} \tag{136}$$

iv) The forth integral expression:

In this case, similar to the case iii), we also need to consider two cases separately. As a result, we can re-write the forth integral expression as

$$\begin{aligned}
&\int_0^\infty \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right) U\left(\frac{z_1}{m}-\frac{z_2}{h}\right) dz_2 dz_1 \\
&= \int_0^\infty \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right) U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right) dz_2 dz_1 \\
&\quad + \int_0^\infty \exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right) \int_0^{\left(\frac{h}{m}\right)z_1} \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right) \left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right] dz_2 dz_1.
\end{aligned} \tag{137}$$

With (137), we can also directly derive the closed-form expressions as

$$\int_0^{\infty}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right)U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)dz_2dz_1$$
$$=\frac{\bar{\gamma}_{i_k}h}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)}U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)$$
$$-\frac{h}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\left\{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{1-T}{T\cdot h}+\frac{1}{\bar{\gamma}_{i_k}}\right\}}U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right), \tag{138}$$

and

$$\int_0^{\infty}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\int_0^{\left(\frac{h}{m}\right)z_1}\exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{z_2}{h}\right)\left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right]dz_2dz_1$$
$$=\frac{\bar{\gamma}_{i_k}h}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)}\left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right]$$
$$-\frac{h}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\left\{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{m}{\bar{\gamma}_{i_k}}\right)\frac{1}{m}+\frac{1}{\bar{\gamma}_{i_k}}\right\}}\left[1-U\left(\frac{1}{m}-\frac{1-T}{T\cdot h}\right)\right]. \tag{139}$$

v) The fifth integral expression:

In this case, we need to consider two cases separately for valid integration. If $\frac{1-T}{T}\geq\frac{h}{m}$, then $\frac{h}{m}z_1<z_2\leq\frac{1-T}{T}z_1$ and $\frac{1-T}{T\cdot h}\geq\frac{1}{m}$. If $\frac{1-T}{T}<\frac{h}{m}$, then there is no valid overlap integration region. As a result, we can re-write the third integral expression as

$$\int_0^{\infty}\int_0^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1$$
$$=\int_0^{\infty}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\int_{\left(\frac{h}{m}\right)z_1}^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)dz_2dz_1. \tag{140}$$

With (140), we can also directly derive the closed-form expressions as

$$\int_0^{\infty}\exp\left(-\frac{z_1}{\bar{\gamma}_{i_k}}\right)\int_{\left(\frac{h}{m}\right)z_1}^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)dz_2dz_1$$
$$=\frac{\bar{\gamma}_{i_q}}{\left(\frac{h}{m\cdot\bar{\gamma}_{i_q}}+\frac{1}{\bar{\gamma}_{i_k}}\right)}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)-\frac{\bar{\gamma}_{i_q}}{\left(\frac{1-T}{T\cdot\bar{\gamma}_{i_q}}+\frac{1}{\bar{\gamma}_{i_k}}\right)}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right). \tag{141}$$

vi) The sixth integral expression:

In this case, similar to the case v), we also need to consider two cases separately. As a result, we can re-write the forth integral expression as

$$\int_0^\infty \int_0^{\left(\frac{1-T}{T}\right)z_1} \exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right) \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_1}{m}\right)\left[1-U\left(\frac{z_1}{m}-\frac{z_2}{h}\right)\right]dz_2dz_1$$
$$=\int_0^\infty \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_1}{m}\right)\int_{\left(\frac{h}{m}\right)z_1}^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)dz_2dz_1. \quad (142)$$

With (142), we can also directly derive the closed-form expressions as

$$\int_0^\infty \exp\left(-\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)\frac{z_1}{m}\right)\int_{\left(\frac{h}{m}\right)z_1}^{\left(\frac{1-T}{T}\right)z_1}\exp\left(-\frac{z_2}{\bar{\gamma}_{i_q}}\right)U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)dz_2dz_1$$
$$=\frac{m\cdot\bar{\gamma}_{i_q}}{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)\right)}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right)$$
$$-\frac{m\cdot\bar{\gamma}_{i_q}}{\left\{\left(\sum_{m=1}^{h}\left(\frac{1}{\bar{\gamma}_{i_{j_m}}}\right)+\sum_{l=1}^{m}\left(\frac{1}{\bar{\gamma}_{i_l}}\right)-\frac{h}{\bar{\gamma}_{i_q}}\right)+\frac{m(1-T)}{T\cdot\bar{\gamma}_{i_q}}\right\}}U\left(\frac{1-T}{T\cdot h}-\frac{1}{m}\right). \quad (143)$$

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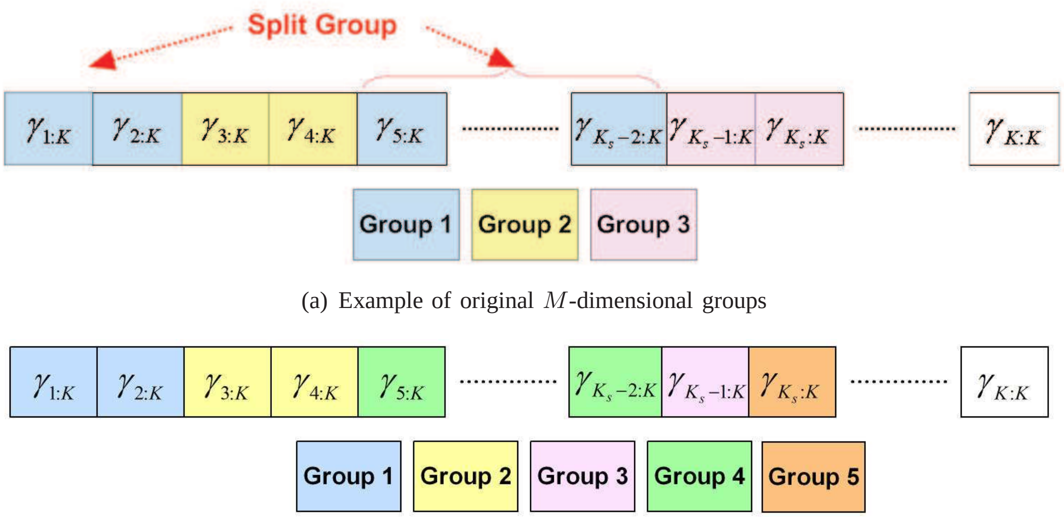

(a) Example of original M-dimensional groups

(b) Example of substituted split groups

Fig. 1. Examples for 3-dimensional joint PDF with split groups.

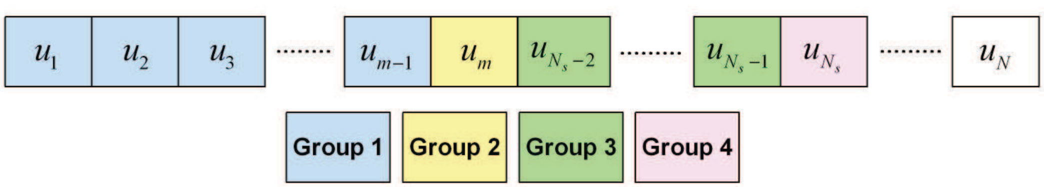

Fig. 2. Joint MGF of u_m and $\sum_{\substack{n=1 \\ n\neq m}}^{N_s} u_n$ for $1 < m < N_s - 1$.