Optimal Estimator Design and Properties Analysis for Interconnected Systems with Asymmetric Information Structure

Yan Wang, Junlin Xiong, Zaiyue Yang, Rong Su

Abstract—**This paper studies the optimal state estimation problem for interconnected systems. Each subsystem can obtain its own measurement in real time, while, the measurements transmitted between the subsystems suffer from random delay. The optimal estimator is analytically designed for minimizing the conditional error covariance. The boundedness of the expected error covariance (EEC) is analyzed. In particular, a new condition that is easy to verify is established for the boundedness of EEC. Further, the properties of EEC with respect to the delay probability are studied. We found that there exists a critical probability such that the EEC is bounded if the delay probability is below the critical probability. Also, a lower and upper bound of the critical probability is derived. Finally, the proposed results are applied to a power system, and the effectiveness of the designed methods is illustrated by simulations.**

Index Terms—**Interconnected systems, expected error covariance, subsystems, random delay, optimal state estimation.**

I. INTRODUCTION

State estimation plays an important role in numerous applications such as target tracking [1], control [2], and signal processing [3]. With the development of the wireless network and sensor technologies, the networked state estimation have received considerable attention during past decade. The estimation performance is significantly affected by the network environment.

The network attack is one of the factors having significant impact on the performance of the networked state estimation. The remote state estimation (RSE) under denial-of-service (DoS) attacks was studied by a stochastic game framework in [4]. The nonstationary filtering framework was designed for uncertain fuzzy Markov switching affine systems with deception attacks in [5]. The authors of [6] investigated the distributed dimensionality reduction fusion estimation problem for cyber-physical systems under DoS attacks. The results of [6] are only for a single system with multi-sensors.

The packet drops and network delays are other two major factors affecting the networked state estimation performance. The researchers tried to understand or counteract the effects of the packet drops/delays on the estimation performance. The Kalman filtering with (partial) random packet drops was investigated in [7], [8]. The distributed Kalman filtering with multi-sensors in the presence of packet drops was studied in [9]. The results of [7]–[9] are for Bernoulli packet drops model. The Kalman filtering with Markovian packet drops/delays was studied in [10]–[12]. The authors of [13] focused on the protocol-based filtering of fuzzy Markov affine systems with uncertain packet dropouts. In general, the estimation problems with Markovian packet drops/delays are more complex than the ones with Bernoulli packet drops/delay. However, it is difficult to analytically discuss the estimator properties for the Markovian packet drops/delays cases. The literature [14] studied the state estimation over sensor networks with mixed uncertainties of random delay, packet dropouts and missing measurements. The state estimation problem with multiple packet losses and with the unknown varying delayed measurements were reported in [15] and [16], respectively. Only a single system with one sensor or multi-sensors is considered in [7]–[11], [14]–[16], and the extensions to interconnected systems (ISs) are rarely reported in the literature. Numerous physical systems are modeled as ISs that have attracted lots of research attentions in the last decade [17]–[20]. The distributed optimal estimation problem of IS with local information is studied in [21]. However, the optimal estimator is not explicitly designed, and the obtained condition of the error covariance being bounded is not easy to verify [21].

In this paper, we focus on the optimal state estimator design for ISs with random delays. A condition in term of semidefinite programming is established to ensure the boundedness of the EEC. For the IS, the measurements transmitted between the subsystems suffer from random delays. To reduce the on-line computation and save the storage space, the delayed measurements will be discarded by each subsystems. Under the above-mentioned setup, an optimal state estimator is explicitly designed. Auxiliary equations are defined to analyze the boundedness of the expected error covariance (EEC). In addition, the relationship between the boundedness of EEC and the delay probability is studied. The existence of a critical probability is shown, where the EEC is bounded if the delay probability is less than the critical probability. Also, a lower

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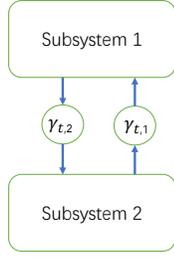
Fig. 1. The information transmission between the subsystems: $\gamma_{t,i} = 1$ means that the corresponding information transmission does not suffer delay, otherwise, $\gamma_{t,i} = 0$.

and upper bound of the critical probability is successfully derived. Finally, the effectiveness of the proposed theories is illustrated on a power system.

Notations: Let $\gamma_{a:b}$ denote the sequence $\gamma_a, \ \gamma_{a+1}, \cdots, \gamma_b$. The probability measure is denoted by $\Pr(\cdot)$. The 2-norm of matrix A is denoted by $||A||$. The spectral radius of matrix A is denoted by $\rho(A)$. The symbol $\underline{\delta}(\cdot)$ represents the minimum eigenvalue of a matrix. The symbols $\otimes$ and $\circ$ are the operators of Kronecker product and Hadamard product, respectively. Let $0_{n \times m}$ denote $n \times m$ zero matrix, and $0_{n \times n}$ is abbreviated as 0_n. Let $1_{n \times m}$ be a $n \times m$ matrix whose all elements are 1, and $1_{n \times n}$ is abbreviated as 1_n. The $n \times n$ unit matrix is denoted by I_n. For a function $f(\cdot)$, define $f^n(\cdot) = f(f^{n-1}(\cdot))$, where $f^1(\cdot) = f(\cdot)$.

II. PROBLEM STATEMENT

Consider an IS composed of two subsystems. The system dynamic is given by

$$\begin{cases} x_{t+1}^1 = A^{11}x_t^1 + A^{12}x_t^2 + \omega_t^1, & \text{(1a)} \\ x_{t+1}^2 = A^{21}x_t^1 + A^{22}x_t^2 + \omega_t^2, & \text{(1b)} \end{cases}$$

where for subsystem i ($i \in \{1,2\}$), $x_t^i \in \mathbb{R}^{n_i}$ and $\omega_t^i \in \mathbb{R}^{n_i}$ are the state and process noise, respectively. The initial state $x_0 = [(x_0^1)^{\mathsf{T}} \ (x_0^2)^{\mathsf{T}}]^{\mathsf{T}}$ is a random vector satisfying $\mathbb{E}(x_0) = 0$ and $\mathbb{E}(x_0^{\mathsf{T}}x_0) = \Sigma_0 \succ 0$. The noise $\omega_t = [(\omega_t^1)^{\mathsf{T}} \ (\omega_t^2)^{\mathsf{T}}]^{\mathsf{T}}$ is an i.i.d. random process with $\mathbb{E}(\omega_t) = 0$ and $\mathbb{E}(\omega_t\omega_t^{\mathsf{T}}) = W \succ 0$.

Each subsystem employs sensors to measure its own subsystem state. The measurement equations are given by

$$y_t^i = C^i x_t^i + v_t^i, \ i \in \{1,2\}, \quad \text{(2)}$$

where $v_t^i \in \mathbb{R}^{m_i}$, ($i \in \{1,2\}$) is the measurement noise, and $v_t = [(v_t^1)^{\mathsf{T}} \ (v_t^2)^{\mathsf{T}}]^{\mathsf{T}}$ is an i.i.d. random process satisfying $\mathbb{E}(v_t) = 0$ and $\mathbb{E}(v_tv_t^{\mathsf{T}}) = V \succ 0$; C^i is the measurement matrix with a proper dimension, for $i \in \{1,2\}$. It is assumed that ω_{t_1} is independent of v_{t_2} for any $t_1, t_2 \geq 0$.

As Fig. 1 shows, subsystem i will transmit the measurement y_t^i to subsystem j through network for $i \neq j$. The communication network between different subsystems suffers from random one step delay (one step delay or no delay). Define the random binary variables $\gamma_{t,1}$ and $\gamma_{t,2}$ to describe the random delay. In particular, $\gamma_{t,i} = 0$ means that the measurement y_t^j transmitted from subsystem j to subsystem i suffers from one step delay, and $\gamma_{t,i} = 1$ indicates that there is no delay, where

$i, j \in \{1,2\}$. The subsystem i will broadcast the value of $\gamma_{t,i}$ to the subsystem j once subsystem i knows the information transmission outcomes. Because the realization of $\gamma_{t,i}$ takes a value of either 0 or 1, it is easy to broadcast the value of $\gamma_{t,i}$. Consider that the delay indicator $\gamma_{t,i}$ ($i \in \{1,2\}$) is a i.i.d. Bernoulli process with

$$\begin{cases} \Pr(\gamma_{t,1} = 1) = 1 - \lambda_1, \ \Pr(\gamma_{t,1} = 0) = \lambda_1, & \text{(3a)} \\ \Pr(\gamma_{t,2} = 1) = 1 - \lambda_2, \ \Pr(\gamma_{t,2} = 0) = \lambda_2, & \text{(3b)} \end{cases}$$

where $0 \leq \lambda_1, \lambda_2 \leq 1$. Due to the random delays, the real time measurements available to subsystem 1 and subsystem 2 are $\{y_t^1, \gamma_{t,1}y_t^2\}$, and $\{y_t^2, \gamma_{t,2}y_t^1\}$, respectively, which are referred to as *asymmetric information* sets. Using the available real time measurements, the estimator 1 and estimator 2 are designed as the Kalman-like filtering form:

$$\begin{cases} \hat{x}_{t|t-1}^1 = A^{11}\hat{x}_{t-1|t-1}^1 + A^{12}\hat{x}_{t-1|t-1}^2, & \text{(4a)} \\ \hat{x}_{t|t}^1 = \hat{x}_{t|t-1}^1 + K_t^{11}\phi_t^1 + \gamma_{t,1}K_t^{12}\phi_t^2, & \text{(4b)} \end{cases}$$

$$\begin{cases} \hat{x}_{t|t-1}^2 = A^{21}\hat{x}_{t-1|t-1}^1 + A^{22}\hat{x}_{t-1|t-1}^2, & \text{(5a)} \\ \hat{x}_{t|t}^2 = \hat{x}_{t|t-1}^2 + \gamma_{t,2}K_t^{21}\phi_t^1 + K_t^{12}\phi_t^2, & \text{(5b)} \end{cases}$$

where $\phi_t^1 = y_t^1 - C^1\hat{x}_{t|t-1}^1$, $\phi_t^2 = y_t^2 - C^2\hat{x}_{t|t-1}^2$, $\hat{x}_{0|0}^1 = 0_{n_i \times 1}$, $\hat{x}_{0|0}^2 = 0_{n_2 \times 1}$; K_t^{ij}, $i,j \in \{1,2\}$, are the gain matrices with proper dimensions. Define $\hat{x}_{t|t} = [(\hat{x}_{t|t}^1)^{\mathsf{T}} \ (\hat{x}_{t|t}^2)^{\mathsf{T}}]^{\mathsf{T}}$ and $\hat{x}_{t|t-1} = [(\hat{x}_{t|t-1}^1)^{\mathsf{T}} \ (\hat{x}_{t|t-1}^1)^{\mathsf{T}}]^{\mathsf{T}}$. The estimator of the IS is written as

$$\begin{cases} \hat{x}_{t|t-1} = A\hat{x}_{t-1|t-1}, \ \hat{x}_{0|0} = 0, & \text{(6a)} \\ \hat{x}_{t|t} = \hat{x}_{t|t-1} + L_t(y_t - C\hat{x}_{t|t-1}), & \text{(6b)} \end{cases}$$

where $A = [A^{ij}]_{i,j \in \{1,2\}}$, $C = \text{diag}\{C^1, \ C^2\}$, and $L_t = \begin{bmatrix} K_t^{11} & \gamma_{t,1}K_t^{12} \\ \gamma_{t,2}K_t^{21} & K_t^{22} \end{bmatrix}$. Denote $x_t = [(x_t^1)^{\mathsf{T}} \ (x_t^2)^{\mathsf{T}}]^{\mathsf{T}}$. The estimation error and prediction error are defined as $e_{t|t} = x_t - \hat{x}_{t|t}$, and $e_{t|t-1} = x_t - \hat{x}_{t|t-1}$, respectively. The conditional error covariances are defined:

$$\begin{cases} P_{t|t} = \mathbb{E}(e_{t|t}e_{t|t}^{\mathsf{T}}|\gamma_t, P_{t|t-1}), & \text{(7a)} \\ P_{t|t-1} = \mathbb{E}(e_{t|t-1}e_{t|t-1}^{\mathsf{T}}|P_{t-1|t-1}), & \text{(7b)} \end{cases}$$

where $\gamma_t = [\gamma_{t,1} \ \gamma_{t,2}]$. Combining (1a)–(1b) and (6a)–(6b), one has

$$\begin{cases} e_{t|t-1} = Ae_{t-1|t-1} + \omega_{t-1}, & \text{(8a)} \\ e_{t|t} = e_{t|t-1} - L_t(Ce_{t|t-1} + v_t), & \text{(8b)} \end{cases}$$

and

$$\begin{cases} P_{t|t-1} = AP_{t-1|t-1}A^{\mathsf{T}} + W, \ P_{0|0} = \Sigma_0, & \text{(9a)} \\ P_{t|t} = (I - L_tC)P_{t|t-1}(I - L_tC)^{\mathsf{T}} + L_tVL_t^{\mathsf{T}}, & \text{(9b)} \end{cases}$$

where $P_{t+1|t}$, $P_{t|t}$ are random matrices induced by the random variables $\gamma_{0:t}$. For the IS (1), we make the following definition and assumption:

Definition 1: A system with parameters (A, C) is detectable with Ω if there exists a K such that $\rho(A - (K \circ \Omega)C) < 1$.

Assumption 1: In this paper, we assume that (A, C) is detectable with $1_{(n_1+n_2) \times (m_1+m_2)}$, and is undetectable with $\text{diag}\{1_{n_1 \times m_1}, 1_{n_2 \times m_2}\}$.

Under Assumption 1, $P_{t|t-1}$ is bounded if $\lambda_1 = \lambda_2 = 0$ and is unbounded if $\lambda_1 = \lambda_2 = 1$, where $P_{t|t-1}$ is viewed as a nonrandom matrix if $\lambda_1, \lambda_2 \in \{0, 1\}$.

III. OPTIMAL ESTIMATOR DESIGN

In this section, the optimal gains of the estimator are analytically derived, and the estimator realization algorithm is presented.

Theorem 1: Consider the system (1), the estimator (6) and the conditional error covariance dynamics (9). Given $P_{t|t-1}$, $\gamma_{t,1}$, $\gamma_{t,2}$, the optimal L_t minimizing $P_{t|t}$ is given by

$$L_t = \gamma_{t,1}\gamma_{t,2}L_t^{[11]} + (1 - \gamma_{t,1})\gamma_{t,2}L_t^{[01]}$$
$$+ \gamma_{t,1}(1 - \gamma_{t,2})L_t^{[10]} + (1 - \gamma_{t,1})(1 - \gamma_{t,2})L_t^{[00]}, \quad (10)$$

where

$$L_t^{[11]} = P_{t|t-1}C^{\mathsf{T}}(V + CP_{t|t-1}C^{\mathsf{T}})^{-1},$$

$$L_t^{[01]} = \left[(N^3 J_t^1 J_t^2 U_t^1 + U_t^2)U_t^3 \quad \begin{bmatrix} 0_{n_1 \times m_2} \\ -J_t^1 J_t^2 \end{bmatrix} \right],$$

$$L_t^{[10]} = \left[\begin{bmatrix} -J_t^3 J_t^4 \\ 0_{n_2 \times m_1} \end{bmatrix} \quad (N^4 J_t^3 J_t^4 U_t^4 + U_t^5)U_t^6 \right],$$

$$L_t^{[00]} = \begin{bmatrix} N^{4\mathsf{T}} U_t^2 U_t^3 & 0_{n_1 \times m_2} \\ 0_{n_2 \times m_1} & N^{3\mathsf{T}} U_t^5 U_t^6 \end{bmatrix},$$

$$N^1 = [I_{m_1} \ 0_{m_1 \times m_2}], \ N^2 = [0_{m_2 \times m_1} \ I_{m_2}],$$

$$N^3 = \begin{bmatrix} 0_{n_1 \times n_2} \\ I_{n_2} \end{bmatrix}, \ N^4 = \begin{bmatrix} I_{n_1} \\ 0_{n_2 \times n_1} \end{bmatrix},$$

$$J_t^1 = N^{3\mathsf{T}}\left(U_t^2 U_t^3 U_t^4 U_t^6 - U_t^5 U_t^6 \right),$$

$$J_t^2 = \left(I - U_t^1 U_t^3 U_t^4 U_t^6 \right)^{-1}, \ J_t^4 = (I - U_t^4 U_t^6 U_t^1 U_t^3)^{-1},$$

$$J_t^3 = N^{4\mathsf{T}}(U_t^5 U_t^6 U_t^1 U_t^3 - U_t^2 U_t^3),$$

$$U_t^1 = N^2 V N^{1\mathsf{T}} + N^2 C P_{t|t-1} C^{\mathsf{T}} N^{1\mathsf{T}},$$

$$U_t^2 = P_{t|t-1}C^{\mathsf{T}} N^{1\mathsf{T}}, \ U_t^5 = P_{t|t-1}C^{\mathsf{T}} N^{2\mathsf{T}},$$

$$U_t^3 = \left(N^1 V N^{1\mathsf{T}} + N^1 C P_{t|t-1} C^{\mathsf{T}} N^{1\mathsf{T}} \right)^{-1},$$

$$U_t^4 = N^1 V N^{2\mathsf{T}} + N^1 C P_{t|t-1} C^{\mathsf{T}} N^{2\mathsf{T}},$$

$$U_t^6 = \left(N^2 V N^{2\mathsf{T}} + N^2 C P_{t|t-1} C^{\mathsf{T}} N^{2\mathsf{T}} \right)^{-1}.$$

Proof: See appendix. ∎

The optimal state estimator of IS (1) with random delay is realized by Algorithm 1. Before presenting Algorithm 1, denote $\bar{N}^1 = (N^4)^{\mathsf{T}}$, $\bar{N}^2 = (N^1)^{\mathsf{T}}$, $\bar{N}^3 = (N^2)^{\mathsf{T}}$, $\bar{N}^4 = (N^3)^{\mathsf{T}}$.

Remark 1: Compared to [7], [8], [10], [11], the derivation of the optimal estimator gain L_t in this paper is more challenging, because if $\gamma_{t,1}\gamma_{t,2} \neq 1$, the corresponding gains of the estimator must satisfy a certain sparse structure constraint. While, the optimal estimator gain in [7], [8], [10], [11] is directly obtained from standard Kalman filtering formulas. Actually, for any cases of the data reception, the optimal estimator gain in [7], [8], [10], [11] is either a zero matrix or the standard Kalman filtering gain with different measurement matrices.

Algorithm 1 Sub-estimator realization

1. Obtain y_t^i and transmit y_t^i to subsystem j, where $i, j \in \{1, 2\}$ and $i \neq j$.

2. Check whether y_t^j is obtained in real time; broadcast the value of $\gamma_{t,i}$ to subsystem j, and determine the value of $\gamma_{t,j}$.

3. Compute $L_t^{[\gamma_{t,1}\gamma_{t,2}]}$ based on Theorem 1 using $\gamma_{t,1}$, $\gamma_{t,2}$, and $P_{t|t-1}$.

4. Compute $\hat{x}_{t|t-1}^1$ and $\hat{x}_{t|t-1}^2$ by (4a), (5a) using $\hat{x}_{t-1|t-1}^1$ and $\hat{x}_{t-1|t-1}^2$.

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for subsystem 1:

5. Compute $\hat{x}_{t|t}^1 = \hat{x}_{t|t-1}^1 + \bar{N}^1 L_t^{[\gamma_{t,1}\gamma_{t,2}]}\bar{N}^2(y_t^1 - C^1 \hat{x}_{t|t-1}^1)$.

if $\gamma_{t,1} = 1$ **then**

6. Update $\hat{x}_{t|t}^1 = \hat{x}_{t|t}^1 + \bar{N}^1 L_t^{[\gamma_{t,1}\gamma_{t,2}]}\bar{N}^3(y_t^2 - C^2 \hat{x}_{t|t-1}^2)$.

end if

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for subsystem 2:

5. Compute $\hat{x}_{t|t}^2 = \hat{x}_{t|t-1}^2 + \bar{N}^4 L_t^{[\gamma_{t,1}\gamma_{t,2}]}\bar{N}^3(y_t^2 - C^2 \hat{x}_{t|t-1}^2)$.

if $\gamma_{t,2} = 1$ **then**

6. Update $\hat{x}_{t|t}^2 = \hat{x}_{t|t}^2 + \bar{N}^4 L_t^{[\gamma_{t,1}\gamma_{t,2}]}\bar{N}^2(y_t^1 - C^1 \hat{x}_{t|t-1}^1)$.

end if

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7. Transmit $\hat{x}_{t|t}^i$ to subsystem j.

8. Compute $P_{t+1|t}$ based on (9a)–(9b).

9. Let $t = t + 1$ and return to first step.

IV. BOUNDEDNESS ANALYSIS OF EXPECTED ERROR COVARIANCE

In this section, we analyse the boundedness of the EEC through auxiliary matrix functions. For simplicity, hereinafter, we denote $P_t = P_{t|t-1}$. It follows from (9) that

$$\begin{cases} P_{t+1} = A_{L,t} P_t A_{L,t}^{\mathsf{T}} + A L_t V (A L_t)^{\mathsf{T}} + W_t, & (11a) \\ P_1 = A\Sigma_0 A^{\mathsf{T}} + W, & (11b) \end{cases}$$

where $A_{L,t} = A - A L_t C$. In the following, we focus on studying the properties of $\mathbb{E}(P_t)$. According to the definition of P_t, $\mathbb{E}(P_t)$ depends on the distributions of the random variables $\gamma_{t,1}$, $\gamma_{t,2}$. Define the following auxiliary matrix functions

$$a(X, Y) = (A - AXC)Y(A - AXC)^{\mathsf{T}}, \quad (12)$$

$$b(X, Y) = a(X, Y) + AXVX^{\mathsf{T}}A^{\mathsf{T}} + W, \quad (13)$$

$$c(X) = (A - AXC)^{\mathsf{T}}(A - AXC), \quad (14)$$

$$d(X) = (A - AXC) \otimes (A - AXC). \quad (15)$$

Recall Theorem 1. Note that L_t^{γ}, $\gamma \in \{[00], [01], [10], [11]\}$ depends on P_t, and thus we denote L_t^{γ} by $L^{\gamma}[P_t]$, where we take P_t as a variable. Consider the probability distribution of $\gamma_t = [\gamma_{t,1} \ \gamma_{t,2}]$, i.e. (3), we define

$$f(S^1, S^2, S^3, S^4, Y) = \lambda_1\lambda_2 b(S^1, Y) + \lambda_1(1 - \lambda_2)b(S^2, Y)$$
$$+ (1 - \lambda_1)\lambda_2 b(S^3, Y) + (1 - \lambda_1)(1 - \lambda_2)b(S^4, Y), \quad (16)$$

$$g_{\lambda_1\lambda_2}(Y) = f(L^{[00]}[Y], L^{[01]}[Y], L^{[10]}[Y], L^{[11]}[Y], Y). \quad (17)$$

Some useful properties of the the matrix equation (17) are proposed below (Proposition 1 and Lemmas 1–2) and will be used to establish the boundedness condition of EEC.

Proposition 1: Consider (11) and (17). The following equations hold:

$$\mathbb{E}(P_{t+1}|P_t) = g_{\lambda_1\lambda_2}(P_t), \ \mathbb{E}(P_{t+1}) = \mathbb{E}(g_{\lambda_1\lambda_2}(P_t)), \quad (18)$$
$$\mathbb{E}(P_{t+1}) = \mathbb{E}(g_{\lambda_1\lambda_2}(P_t)) \preceq g_{\lambda_1\lambda_2}(\mathbb{E}(P_t)). \quad (19)$$

Proof: See appendix. ∎

Lemma 1: Define the matrix sequence $Y_0, Y_1, Y_2, \cdots$ generated by $Y_{t+1} = g_{\lambda_1\lambda_2}(Y_t)$, $Y_0 = P_0$. One has $\mathbb{E}(P_t) \preceq Y_t$.

Proof: This result directly follows from (19). ∎

Lemma 2: Consider the sequence $Y_0, Y_1, Y_2, \cdots$ defined in Lemma 1. The following result holds: $\lim_{t\to+\infty} Y_t$ is bounded if and only if there exists $X^1, \cdots, X^7$ such that $\rho\left(h(X^1, \cdots, X^7)\right) < 1$, where

$$\begin{aligned}
h(X^1, \cdots, X^7) &= \lambda_1\lambda_2 d(N^4 X^1 N^1 + N^3 X^2 N^2) \\
&+ \lambda_1(1-\lambda_2)d(X^3 N^1 + N^3 X^4 N^2) \\
&+ (1-\lambda_1)\lambda_2 d(N^4 X^4 N^1 + X^6 N^2) \\
&+ (1-\lambda_1)(1-\lambda_2)d(X^7).
\end{aligned} \quad (20)$$

Proof: See appendix. ∎

In the following, we derive a sufficient condition of $\rho\left(h(X^1, \cdots, X^7)\right) < 1$ with the form of semidefinite programming (SDP).

Theorem 2: Consider (11) with (3). Given λ_1, λ_2. If there exists r_1, r_2, r_3, r_4 satisfying $r_1\lambda_1\lambda_2 + r_2\lambda_1(1-\lambda_2) + r_3(1-\lambda_1)\lambda_2 + r_4(1-\lambda_1)(1-\lambda_2) < 1$, where

$$\begin{cases}
r_1 = \arg\min_{r,X,\tilde{X}} \left\{r \geq 0 : \ p_1(r, X, \tilde{X}) \succeq 0\right\}, & (21a) \\
r_2 = \arg\min_{r,X,\tilde{X}} \left\{r \geq 0 : \ p_2(r, X, \tilde{X}) \succeq 0\right\}, & (21b) \\
r_3 = \arg\min_{r,X,\tilde{X}} \left\{r \geq 0 : \ p_3(r, X, \tilde{X}) \succeq 0\right\}, & (21c) \\
r_4 = \arg\min_{r,X} \left\{r \geq 0 : \ p(r, X) \succeq 0\right\}, & (21d)
\end{cases}$$

$$\begin{cases}
p(r, X) = \begin{bmatrix} \sqrt{r}I & A - AXC \\ (A - AXC)^{\mathsf{T}} & \sqrt{r}I \end{bmatrix}, & (22a) \\
p_1(r, X, \tilde{X}) = p(r, N^4 X N^1 + N^3 \tilde{X} N^2), & (22b) \\
p_2(r, X, \tilde{X}) = p(r, X N^1 + N^3 \tilde{X} N^2), & (22c) \\
p_3(r, X, \tilde{X}) = p(r, N^4 X N^1 + \tilde{X} N^2), & (22d)
\end{cases}$$

then $\lim_{t\to+\infty} \mathbb{E}(P_t)$ is bounded.

Proof: See appendix. ∎

Remark 2: Note that $p(r, X) \succeq 0$, $p_1(r, X, \tilde{X}) \succeq 0$, $p_2(r, X, \tilde{X}) \succeq 0$, and $p_3(r, X, \tilde{X}) \succeq 0$ are LMIs. Problems (21a)–(21d) are SDP problems which can be effectively solved by MATLAB toolbox. Thus, the condition established in Theorem 2 can be effectively verified.

Corollary 1: Consider r_1, r_2, r_3 and r_4 defined in Theorem 2. It holds that $r_4 \leq \min(r_2, r_3)$, $r_1 \geq \max(r_2, r_3)$. In addition, if $r_1 = r_4$ and there exists $r_i < 1$, for any $i \in \{1, 2, 3, 4\}$, then $\lim_{t\to+\infty} \mathbb{E}(P_t)$ is bounded for any λ_1, $\lambda_2 \in [0\ 1]$.

Proof: See appendix. ∎

Corollary 2: Consider r_1, r_2, r_3 and r_4 defined in Theorem 2. Assume that the system parameters $(A\ C)$ satisfy Assumption 1. If there exists a X satisfying $c(X) \prec I$, then $r_4 < r_1$, where $c(\cdot)$ is defined in (14).

Proof: See appendix. ∎

Note that $\mathbb{E}(P_t)$ depends on λ_1, λ_2. It is known that if the delay occurs with a bigger probability, then the estimation performance gets worse. This means that $\mathbb{E}(P_t)$ is monotone increasing with respect to λ_1, λ_2. Additionally, if λ_1, λ_2 are big enough, $\mathbb{E}(P_t)$ may became unbounded when the time goes to infinity. The above discussions are concluded as follows.

Lemma 3: Consider (11). For a fixed λ_1, if $\lim_{t\to\infty} \mathbb{E}(P_t)$ is unbounded for $\lambda_2 = 1$, but bounded for $\lambda_2 = 0$, then there exists a critical probability $\lambda_{2,c}$ such that $\lim_{t\to\infty} \mathbb{E}(P_t)$ is bounded for $\lambda_2 \leq \lambda_{2,c}$, and unbounded for $\lambda_2 > \lambda_{2,c}$. Also, for a fixed λ_2, we have the similar results to λ_1.

Proof: See appendix. ∎

Since the accurate critical probability of delay is not easy to obtain, the lower and upper bounds of the critical probability may be useful. A computation method is provided in the following theorem to compute the lower and upper bounds of the critical probability. For ease of notations, we first define

$$q(X^1, X^2) = c(N^4 X^1 N^1 + N^3 X^2 N^2).$$

Theorem 3: Consider (11). For a fixed λ_1, the critical probability $\lambda_{2,c}$ satisfies that if $\lambda_2 \leq \lambda_{2,c}$, $\lim_{t\to\infty} \mathbb{E}(P_t)$ is bounded, otherwise, $\lim_{t\to\infty} \mathbb{E}(P_t)$ is unbounded. A lower and upper bound of $\lambda_{2,c}$ is obtained, i.e. $\underline{\lambda}_{2,c} \leq \lambda_{2,c} \leq \bar{\lambda}_{2,c}$, where

$$\bar{\lambda}_{2,c} = \begin{cases} 1, & \text{if } r_1 = r_4 < 1; \\ \min\left\{\frac{1}{\alpha\lambda_1}, 1\right\},, & \text{otherwise;} \end{cases} \quad (23)$$

$$\underline{\lambda}_{2,c} = \begin{cases} \frac{1 - r_2\lambda_1 - r_4(1-\lambda_1)}{(r_1 - r_2)\lambda_1 + (r_3 - r_4)(1-\lambda_1)}, & \text{if } r_1 \neq r_4; \\ 1, & \text{if } r_1 = r_4 < 1; \\ 0, & \text{if } r_1 = r_4 \geq 1. \end{cases} \quad (24)$$

where $\alpha = \underline{\delta}\left(\min_{X^1, X^2} q(X^1, X^2)\right)$. Similar upper and lower bounds for λ_1^c can be obtained when we fix λ_2.

Proof: See appendix. ∎

Theorem 3 provides a method to compute the upper and lower bounds of the critical probability of delay.

Remark 3 (Extension to large-scale ISs): Consider a large-scale IS composed of n subsystems. We can use a random graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, p)$ to describe the communication between the subsystems, where $\mathcal{V}$ is the nodes set; $\mathcal{E}$ is the possible edges set; $p = [p_1, \ldots, p_{|\mathcal{E}|}]$. At time t, the i-th possible edge in $\mathcal{E}$ occurs in the graph with probability $0 < p_i < 1$. The edge (i, j) occurring in the graph implies that the subsystem i can receive the information of subsystem j in real time. The continuity of $\mathbb{E}(P_t)$ (w.r.t. p) indicates that there should exist a critical plane denoted by p_c, such that the boundedness of $\mathbb{E}(P_t)$ depends on which side of the plane p_c the point p is

on. However, it is not easy to determine the upper and lower bounds of p_c.

V. Applications to Power Systems

In this section, a power system is used as an example to demonstrate the effectiveness of the proposed state estimation methods.
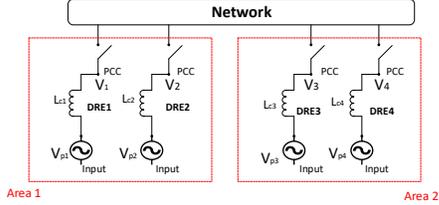

Fig. 2. The model of distributed energy resources (DERs) connecting to the power network [22]. DER 1 and DER 2 are in area 1; DER 3 and DER 4 are in area 2.
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We study the state estimation problem for the distributed energy resources (DERs) connected to the power network that is shown in Fig. 2. In this example, the power network is chosen to be the IEEE 4-bus distribution network (see Fig. 3 of [22]). As Fig. 2 depicts, four DERs are integrated into the main power network at the point common coupling (PCC). The voltages of the PCC are $v_s = [v_1 \ v_2 \ v_3 \ v_4]^\mathsf{T}$, where v_i is the voltage of the ith PCC, and $i \in \{1,2,3,4\}$. Each DER is a voltage source at each bus. The input voltage of the voltage sources is denoted by $v_p = [v_{p1} \ v_{p2} \ v_{p3} \ v_{p4}]^\mathsf{T}$. To maintain the proper operation of DERs, it is required to keep the PCC voltages v_s at reference values v_{ref}. The PCC voltage deviation $x_t = v_s - v_{ref}$ is chosen to be the system state. The DER control effort deviation $u_t = v_p - v_{pref}$ is the control input, where v_{pref} is the reference of the control effort. Following the modeling process provided in [22], [23], the dynamic of x_t is of the following form:

$$\dot{x}_t = Ax_t + Bu_t + n_t, \tag{25}$$

where n_t is a zero-mean Gaussian white noise whose covariance is Q. The system (25) is discretized as:

$$x_{t+1} = A_d x_t + B_d u_t + \omega_t, \tag{26}$$

where $A_d = I + AT_s$, $B_d = T_s B$, and ω_t is the i.i.d. Gaussian process with zero-mean and covariance $W = T_s Q$. Assume the initial state x_0 is a zero mean Gaussian variable with identity covariance. Here, we assume $W = I_4$, and set the sampling time $T_s = 0.05s$. In this example, $\rho(A_d) > 1$, which means that the system (26) under open-loop pattern (there is no control input) is unstable. Here, the controller is of the state feedback form: $u_t = Fx_t$, where F is the gain matrix with proper dimension. The closed-loop system is given by

$$x_{t+1} = A_c x_t + \omega_t, \tag{27}$$

where $A_c = A_d + B_d F$. As Fig. 2 shows, DERs 1–2 are in area 1 and DERs 3–4 are in area 2. Based on different areas,

the system (27) can be written as an IS composed of two subsystems:

$$x_{t+1}^i = A_c^{ii} x_t^i + A_c^{ij} x_t^j + \omega_t^i, \ i,j \in \{1,2\}, \ i \neq j, \tag{28}$$

where x_t^i is the component of the system state corresponding to area i; and $A_c = [A_c^{ij}]_{i,j\in\{1,2\}}$, $x_t^1 = [I_2 \ 0_2]x_t$, $x_t^2 = [0_2 \ I_2]x_t$, $\omega_t^1 = [I_2 \ 0_2]\omega_t$, $\omega_t^2 = [0_2 \ I_2]\omega_t$. To monitor the working status of the power system (28), in each area, the sensors are employed to measure the system state. The measurement equations of area 1 and area 2 are

$$y_t^i = C^i x_t^i + v_t^i, \ i \in \{1,2\}, \tag{29}$$

where $v_t = [v_t^1 \ v_t^2]^\mathsf{T}$ is measurement noise that is the zero-mean Gaussian independent processes with $\mathbb{E}(v_t v_t^\mathsf{T}) = I_3$, and $C^1 = [I_2 \ 0_2]$, $C^2 = [0 \ 0 \ 1 \ 1]$. In area i, the estimator i is installed to estimate the state x_t^i. The measurement y_t^1 is transmitted to estimators 1–2 through network, so does y_t^2. It is assumed that the data y_t^i transmitted to estimator j suffers from random delay (one step delay or no delay), where $i \neq j$ and $i,j \in \{1,2\}$. Assume the delay indicator $\gamma_{t,i}$ is a Bernoulli stochastic process with $\Pr(\gamma_{t,i} = 0) = \lambda_i$, where $i \in \{1,2\}$.

Case 1 (the stable closed-loop system): The controller gain is chosen such that $\rho(A_c) < 1$, which means that the system (28) with the chosen F is stable. Then, our proposed method is applied for the state estimation of the system (28). The estimation performance is evaluated by the trace of the expected error covariance $\mathbb{E}(P_t)$. The trace of $\mathbb{E}(P_t)$ with respect to different delay probabilities (λ_1, λ_2) is approximately computed by averaging 1000 Monte-Carlo simulations, see Fig. 3. From Fig. 3, we can see that (i) the estimation performance gets worse under larger delay probabilities; (ii) the expected error covariance $\mathbb{E}(P_t)$ is bounded even though $\lambda_1 = \lambda_2 = 1$ (Note that $\lambda_1 = \lambda_2 = 1$ means that the delay always happens); (iii) under different delay probabilities, the expected error covariance $\mathbb{E}(P_t)$ is within the same order of magnitude as the error covariance of standard Kalman filtering. The simulations illustrate that our estimators can well monitor the working state of the power system (28), even though the data transmitted between different areas suffer from random delays.
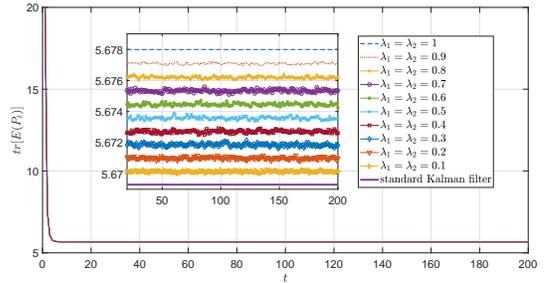

Fig. 3. The value of $\mathrm{tr}[\mathbb{E}(P_t)]$ under different λ_1, λ_2.

Case 2 (the unstable closed-loop system): It is known that to estimate the states of unstable systems is more challenging than to estimate the states of stable systems. Hence, to further illustrate the effectiveness of our estimators, we also consider

the unstable system case. The controller gain is chosen such that $\rho(A_c) > 1$. Hence, the system (28) with the chosen controller gain is unstable. In this case, the values of $\mathrm{tr}[\mathbb{E}(P_t)]$ with different (λ_1, λ_2) are presented in Figs. 4–5.
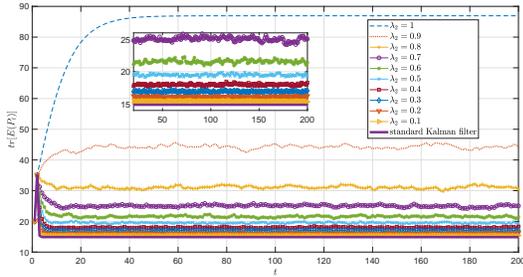

Fig. 4. $\lambda_1 = 1$, the value of $\mathrm{tr}[\mathbb{E}(P_t)]$ under different λ_2
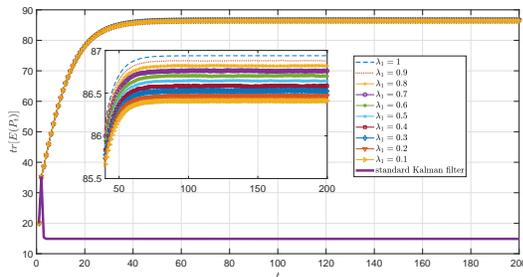

Fig. 5. $\lambda_2 = 1$, the value of $\mathrm{tr}[\mathbb{E}(P_t)]$ under different λ_1

Figs. 4–5 show that the expected error covariance $\mathbb{E}(P_t)$ (with any delay probabilities) is within the same order of magnitude as the error covariance of standard Kalman filtering. This implies that even when the power systems (28) become unstable, the working states are also well monitored by our estimator under random delay. In addition, comparing Fig. 4 and Fig. 5, we find that for the power system with the parameters given in this case, the estimation performance is more sensitive to λ_2 than to λ_1.

VI. Conclusion

This paper studied the optimal estimator design problem for IS with random delay. Due to the random delay occurring among the subsystems, the information available to different subsystem may be different. This type of IS is called IS with asymmetric information structure. An optimal estimator has been analytically designed. The estimator realization algorithm for each subsystem was developed. In addition, some useful properties of the estimation performance were obtained. Finally, the proposed estimator was applied to a power system. The simulations showed that the designed estimator is effective and is of good performance.

In this work, we assume the delay indicator obeying Bernoulli process. In the further work, we may extend this work to the Markov chain delay indicator case.

References

[1] W. Li, K. Xiong, Y. Jia, and J. Du, "Distributed kalman filter for multi-target tracking systems with coupled measurements," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 10, pp. 6599–6604, 2020.

[2] C. Danielson, S. A. Bortoff, and A. Chakrabarty, "Extremum seeking control with an adaptive gain based on gradient estimation error," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2022.

[3] S. Ding, Z. Ke, Z. Yue, C. Song, and L. Lu, "Noncontact multiphysiological signals estimation via visible and infrared facial features fusion," *IEEE Transactions on Instrumentation and Measurement*, vol. 71, pp. 1–13, 2022.

[4] K. Ding, S. Dey, D. E. Quevedo, and L. Shi, "Stochastic game in remote estimation under DoS attacks," *IEEE Control System Letters*, vol. 1, no. 1, pp. 146–151, 2017.

[5] J. Cheng, Y. Wu, Z.-G. Wu, and H. Yan, "Nonstationary filtering for fuzzy markov switching affine systems with quantization effects and deception attacks," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2022.

[6] B. Chen, D. W. C. Ho, W.-A. Zhang, and L. Yu, "Distributed dimensionality reduction fusion estimation for cyber-physical systems under DoS attacks," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 49, no. 2, pp. 455–468, 2019.

[7] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, 2004.

[8] X. Liu and A. Goldsmith, "Kalman filtering with partial observation losses," in *Proceedings of IEEE Conference on Decision and Control*, 2004, pp. 4180–4186.

[9] J. Zhou, G. Gu, and X. Chen, "Distributed Kalman filtering over wireless sensor networks in the presence of data packet drops," *IEEE Transactions on Automatic Control*, vol. 64, no. 4, pp. 1603–1610, 2019.

[10] M. Huang and S. Dey, "Stability of kalman filtering with markovian packet losses," *Automatica*, vol. 43, no. 4, pp. 598–607, 2007.

[11] K. You, M. Fu, and L. Xie, "Mean square stability for kalman filtering with markovian packet losses," *Automatica*, vol. 47, no. 12, pp. 2647–2657, 2011.

[12] C. Han, H. Zhang, and M. Fu, "Optimal filtering for networked systems with markovian communication delays," *Automatica*, vol. 49, no. 10, pp. 3097–3104, 2013.

[13] J. Cheng, Y. Wu, H. Yan, Z.-G. Wu, and K. Shi, "Protocol-based filtering for fuzzy markov affine systems with switching chain," *Automatica*, vol. 141, p. 110321, 2022.

[14] J. Ma and S. Sun, "Optimal linear estimators for systems with random sensor delays, multiple packet dropouts and uncertain observations," *IEEE Transactions on Signal Processing*, vol. 59, no. 11, pp. 5181–5192, 2011.

[15] H. Ren, R. Lu, J. Xiong, Y. Wu, and P. Shi, "Optimal filtered and smoothed estimators for discrete-time linear systems with multiple packet dropouts under markovian communication constraints," *IEEE Transactions on Cybernetics*, vol. 50, no. 9, pp. 4169–4181, 2019.

[16] M. Pohjola and H. Koivo, "Measurement delay estimation for kalman filter in networked control systems," *IFAC Proceedings Volumes*, vol. 41, no. 2, pp. 4192–4197, 2008.

[17] Y. Wang, J. Xiong, and D. W. Ho, "Decentralized control scheme for large-scale systems defined over a graph in presence of communication delays and random missing measurements," *Automatica*, vol. 98, pp. 190–200, 2018.

[18] T. Souxes, I.-M. Granitsas, and C. Vournas, "Effect of stochasticity on voltage stability support provided by wind farms: Application to the hellenic interconnected system," *Electric Power Systems Research*, vol. 170, pp. 48–56, 2019.

[19] Y. Wang, J. Xiong, and D. W. Ho, "Globally optimal state-feedback lqg control for large-scale systems with communication delays and correlated subsystem process noises," *IEEE Transactions on Automatic Control*, vol. 64, no. 10, pp. 4196–4201, 2019.

[20] Y. Wang, R. Su, and B. Wang, "Optimal control of interconnected systems with time-correlated noises: Application to vehicle platoon," *Automatica*, vol. 137, p. 110018, 2022.

[21] Y. Wang, J. Xiong, and D. W. Ho, "Distributed lmmse estimation for large-scale systems based on local information," *IEEE Transactions on Cybernetics*, 2021.

[22] H. Li, L. Lai, and H. V. Poor, "Multicast routing for decentralized control of cyber physical systems with an application in smart grid," *IEEE Journal ON Selected Areas in Communications*, vol. 30, no. 6, pp. 1097–1107, 2012.

[23] M. M. Rana, L. Li, and S. W. Su, "Distributed state estimation over unreliable communication networks with an application to smart grids," *IEEE Transactions on Green Communications and Networking*, vol. 1, no. 1, pp. 89–96, 2017.

[24] K. J. Åström, *Introduction to Stochastic Control Theory*. Courier Corporation, 2012.

VII. APPENDIX

A. The proof of Theorem 1

The matrices $L_t^{[11]}$, U_t^3, U_t^6, J_t^2, J_t^4 are defined by the "inverse" operator. Firstly, we show that these matrices are well-defined. It follows from $V + CP_{t|t-1}C^\mathsf{T} \succeq V \succ 0$ that $(V + CP_{t|t-1}C^\mathsf{T})^{-1}$ is well-defined. Thus, $L_t^{[11]}$ is well-defined. Since both N^1 and N^2 are row full rank, $N^1 V N^{1^\mathsf{T}} + N^1 C P_{t|t-1} C^\mathsf{T} N^{1^\mathsf{T}} \succeq N^1 V N^{1^\mathsf{T}} \succ 0$, $N^2 V N^{2^\mathsf{T}} + N^2 C P_{t|t-1} C^\mathsf{T} N^{2^\mathsf{T}} \succeq N^2 V N^{2^\mathsf{T}} \succ 0$. Thus, U_t^3 and U_t^6 are well-defined. Based on the Schur complement decomposition, it follows from

$$V + C P_{t|t-1} C^\mathsf{T} = \begin{bmatrix} (U_t^3)^{-1} & U_t^4 \\ U_t^1 & (U_t^6)^{-1} \end{bmatrix} \succ 0$$

that $U_t^1 U_t^3 U_t^4 \prec (U_t^6)^{-1}$, and $U_t^4 U_t^6 U_t^1 \prec (U_t^3)^{-1}$. As a result, $I - U_t^1 U_t^3 U_t^4 U_t^6 \succ 0$, $I - U_t^4 U_t^6 U_t^1 U_t^3 \succ 0$. This shows that J_t^2 and J_t^4 are well-defined.

Now, we start to prove that the optimal L_t is of the form (10). For ease of notation, define $R_t^1 = \begin{bmatrix} K_t^{11} \\ K_t^{21} \end{bmatrix}$, $R_t^2 = \begin{bmatrix} K_t^{12} \\ K_t^{22} \end{bmatrix}$.

(**Case 1**) If $\gamma_{t,1} = 1$, $\gamma_{t,2} = 1$, then the optimal $L_t = L_t^{[11]}$ is the gain of the standard Kalman filtering [24].

(**Case 2**) If $\gamma_{t,1} = 0$, $\gamma_{t,2} = 1$, then L_t has the form $L_t = R_t^1 N^1 + N^3 K_t^{22} N^2$. Inserting $L_t = R_t^1 N^1 + N^3 K_t^{22} N^2$ into (9b), we have

$$\begin{aligned} P_{t|t} &= (I - (R_t^1 N^1 + N^3 K_t^{22} N^2)C)P_{t|t-1} \\ &\quad \times (I - (R_t^1 N^1 + N^3 K_t^{22} N^2)C)^\mathsf{T} \\ &\quad + (R_t^1 N^1 + N^3 K_t^{22} N^2)V(R_t^1 N^1 + N^3 K_t^{22} N^2)^\mathsf{T}. \end{aligned}$$

Taking R_t^1 as a variable, $\mathrm{tr}(P_{t|t})$ has the form $\mathrm{tr}(P_{t|t}) = \mathrm{tr}\left(R_t^1 (U_t^3)^{-1} R_t^{1^\mathsf{T}} + r_{t,R}\right)$, where $r_{t,R}$ is a linear function of R_t^1. Using the formula $\mathrm{tr}(AXBX^\mathsf{T}) = \mathrm{vec}^\mathsf{T}(X)(B^\mathsf{T} \otimes A)\mathrm{vec}(X)$, one has $\mathrm{tr}(P_{t|t}) = \mathrm{vec}^\mathsf{T}(R_t^1)((U_t^3)^{-1} \otimes I)\mathrm{vec}(R_t^1) + \bar{r}_{t,R}$, where $\bar{r}_{t,R}$ is a linear function of $\mathrm{vec}(R_t^1)$. Thus, $\frac{\partial^2 \mathrm{tr}(P_{t|t})}{\partial^2 \mathrm{vec}(R_t^1)} = (U_t^3)^{-1} \otimes I \succ 0$. Similarly, taking K_t^{22} as a variable, we have $\frac{\partial^2 \mathrm{tr}(P_{t|t})}{\partial^2 \mathrm{vec}(K_t^{22})} = (U_t^6)^{-1} \otimes N^{3^\mathsf{T}} N^3 = (U_t^6)^{-1} \otimes I \succ 0$. Thus, $\mathrm{tr}(P_{t|t})$ is convex with respect to both R_t^1 and K_t^{22}. The optimal R_t^1 and K_t^{22} are given by solving $\frac{\partial \mathrm{tr}(P_{t|t})}{\partial R_t^1} = 0$ and $\frac{\partial \mathrm{tr}(P_{t|t})}{\partial K_t^{22}} = 0$. That is

$$\frac{\partial \mathrm{tr}(P_{t|t})}{\partial R_t^1} = R_t^1 (U_t^3)^{-1} + N^3 K_t^{22} U_t^1 + U_t^2 = 0$$

$$\frac{\partial \mathrm{tr}(P_{t|t})}{\partial K_t^{22}} = K_t^{22} (U_t^6)^{-1} + N^{3^\mathsf{T}} R_t^1 U_t^4 + N^{3^\mathsf{T}} U_t^5 = 0,$$

which gives

$$R_t^{1*} = -(N^3 J_t^1 J_t^2 U_t^1 + U_t^2)U_t^3, \quad K_t^{22*} = J_t^1 J_t^2.$$

As a result, if $\gamma_{t,1} = 0$, $\gamma_{t,2} = 1$, then the optimal $L_t = R_t^{1*} N^1 + N^3 K_t^{22*} N^2 = L_t^{[01]}$.

(**Case 3**) If $\gamma_{t,1} = 1$, $\gamma_{t,2} = 0$, then L_t is of the form $L_t = N^4 K_t^{11} N^1 + R_t^2 N^2$. Following from the the derivation similar to the one of **Case 2**, we have the optimal K_t^{11} and

R_t^2 are given by

$$K_t^{11*} = J_t^3 J_t^4, \quad R_t^{2*} = -(N^4 J_t^3 J_t^4 U_t^4 + U_t^5)U_t^6.$$

Thus, if $\gamma_{t,1} = 1$, $\gamma_{t,2} = 0$, then the optimal $L_t = N^4 K_t^{11*} N^1 + R_t^{2*} N^2 = L_t^{[10]}$.

(**Case 4**) If $\gamma_{t,1} = 0$, $\gamma_{t,2} = 0$, then $L_t = N^4 K_t^{11} N^1 + N^3 K_t^{22} N^2$. Similarly, we obtain that the optimal K_t^{11} and K_t^{22} are of the form

$$K_t^{11*} = -N^{4^\mathsf{T}}(N^3 K_t^{22} U_t^1 + U_t^2)U_t^3 = -N^{4^\mathsf{T}} U_t^2 U_t^3,$$

$$K_t^{22*} = -N^{3^\mathsf{T}}(N^4 K_t^{11} U_t^4 + U_t^5)U_t^6 = -N^{3^\mathsf{T}} U_t^5 U_t^6.$$

Hence, if $\gamma_{t,1} = 0$, $\gamma_{t,2} = 0$, the optimal $L_t = N^4 K_t^{11*} N^1 + N^3 K_t^{22*} N^2 = L_t^{[00]}$. The proof is completed.

B. The proof of Proposition 1

The equations (18) are obvious. We focus on proving (19). For any $Y^1, Y^2 \succeq 0$, let $Y = \alpha Y^1 + (1-\alpha)Y^2$. One has

$$\begin{aligned} &g_{\lambda_1 \lambda_2}(Y) \\ &= f(L^{[00]}[Y], L^{[01]}[Y], L^{[10]}[Y], L^{[11]}[Y], Y) \\ &= f(L^{[00]}[Y], L^{[01]}[Y], L^{[10]}[Y], L^{[11]}[Y], \alpha Y^1 + (1-\alpha)Y^2) \\ &= \alpha f(L^{[00]}[Y], L^{[01]}[Y], L^{[10]}[Y], L^{[11]}[Y], Y^1) \\ &\quad + (1-\alpha)f(L^{[00]}[Y], L^{[01]}[Y], L^{[10]}[Y], L^{[11]}[Y], Y^2) \\ &\succeq \min_{S^1 \in \Lambda^{[00]}, S^2 \in \Lambda^{[01]}, S^3 \in \Lambda^{[10]}, S^4 \in \Lambda^{[11]}} \alpha f(S^1, S^2, S^3, S^4, Y^1) \\ &\quad + \min_{S^1 \in \Lambda^{[00]}, S^2 \in \Lambda^{[01]}, S^3 \in \Lambda^{[10]}, S^4 \in \Lambda^{[11]}} (1-\alpha) \\ &\quad \times f(S^1, S^2, S^3, S^4, Y^2) \\ &= \alpha f(L^{[00]}[Y^1], L^{[01]}[Y^1], L^{[10]}[Y^1], L^{[11]}[Y^1], Y^1) \\ &\quad + (1-\alpha)f(L^{[00]}[Y^2], L^{[01]}[Y^2], L^{[10]}[Y^2], L^{[11]}[Y^2], Y^2) \\ &= \alpha g_{\lambda_1 \lambda_2}(Y^1) + (1-\alpha)g_{\lambda_1 \lambda_2}(Y^2), \end{aligned}$$

where $\Lambda^{[00]} = \left\{ \begin{bmatrix} X^{11} & 0 \\ 0 & X^{22} \end{bmatrix} : X^{11} \in R^{n_1 \times m_1}, X^{22} \in R^{n_2 \times m_2} \right\}$, $\Lambda^{[01]} = \left\{ \begin{bmatrix} X^{11} & 0 \\ X^{21} & X^{22} \end{bmatrix} : X^{11} \in R^{n_1 \times m_1} X^{21} \in R^{n_2 \times m_1}, X^{22} \in R^{n_2 \times m_2} \right\}$, $\Lambda^{[10]} = \left\{ \begin{bmatrix} X^{11} & X^{12} \\ 0 & X^{22} \end{bmatrix} : X^{11} \in R^{n_1 \times m_1} X^{12} \in R^{n_1 \times m_2}, X^{22} \in R^{n_2 \times m_2} \right\}$, $\Lambda^{[11]} = \mathbb{R}^{n \times m}$.

This shows that $g_{\lambda_1 \lambda_2}(Y)$ is concave. Using Jensen's Inequality, one has $\mathbb{E}(g_{\lambda_1 \lambda_2}(Y)) \preceq g_{\lambda_1 \lambda_2}(\mathbb{E}(Y))$. This completes the proof.

C. The proof of Lemma 2

(*Sufficiency:*) Construct a sequence $\tilde{Y}_0, \tilde{Y}_1, \tilde{Y}_2, \cdots$ by

$$\begin{aligned} \tilde{Y}_{t+1} = f\big(&N^4 X^1 N^1 + N^3 X^2 N^2, X^3 N^1 + N^3 X^4 N^2, \\ &N^4 X^4 N^1 + X^6 N^2, X^7, \tilde{Y}_t\big), \tilde{Y}_0 = P_0. \quad (30) \end{aligned}$$

Using the vectorization and Kronecker products, we can obtain that $\mathrm{vec}(\tilde{Y}_{t+1}) = h(X^1, \cdots, X^7)\mathrm{vec}(\tilde{Y}_t) + Q$, where Q is a bounded vector that does not depends on $\tilde{Y}_t$. If

$\rho\Big(h(X^1,\cdots,X^7)\Big) < 1$, $\lim_{t\to+\infty}\text{vec}(\tilde{Y}_t)$ is bounded, i.e. $\lim_{t\to+\infty}\tilde{Y}_t$ is bounded. It follows from

$$g_{\lambda_1\lambda_2}(Y) = \min_{X^1,\ldots,X^7} f\Big(N^4X^1N^1 + N^3X^2N^2, X^3N^1$$
$$+ N^3X^4N^2, N^4X^4N^1 + X^6N^2, X^7, Y\Big), \quad (31)$$

that $Y_t \preceq \tilde{Y}_t$. Hence, $\lim_{t\to+\infty}Y_t$ is bounded.

(*Necessity:*) From (31), the necessity is obvious. The proof is completed.

D. The proof of Theorem 2

Using Schur complement for (21a) and (22b), we have there exists X^1, X^2 such that $\Big(A - A(N^4X^1N^1 + N^3X^2N^2))C\Big)\Big(A - A(N^4X^1N^1 + N^3X^2N^2))C\Big)^\mathsf{T} \preceq r_1 I$, which means that $||A - A(N^4X^1N^1 + N^3X^2N^2))C|| \le \sqrt{r_1}$. It follows from the formula $||A \otimes B|| = ||A|| \times ||B||$ that $||d(N^4X^1N^1 + N^3X^2N^2)|| \le r_1$. Similarly, there exists $X^3,\ldots,X^7$ satisfying $||d(X^3N^1 + N^3X^4N^2)|| \le r_2$, $||d(N^4X^4N^1 + X^6N^2)|| \le r_3$, $||d(X^7)|| \le r_4$. As a result, there exists $X^1,\ldots,X^7$ such that $\rho\Big(h(X^1,\cdots,X^7)\Big) \le \lambda_1\lambda_2||d(N^4X^1N^1 + N^3X^2N^2)|| + \lambda_1(1-\lambda_2)||d(X^3N^1 + N^3X^4N^2)|| + (1-\lambda_1)\lambda_2||d(N^4X^4N^1 + X^6N^2)|| + (1-\lambda_1)(1-\lambda_2)||d(X^7)|| \le r_1\lambda_1\lambda_2 + r_2\lambda_1(1-\lambda_2) + r_3(1-\lambda_1)\lambda_2 + r_4(1-\lambda_1)(1-\lambda_2) < 1$. It follows from Lemma 2 that $\lim_{t\to+\infty}Y_t$ is bounded. From Lemma 1, we know $\lim_{t\to+\infty}\mathbb{E}(P_t) \preceq \lim_{t\to+\infty}Y_t$. Thus, $\lim_{t\to+\infty}\mathbb{E}(P_t)$ is bounded. The proof is completed.

E. The proof of Corollary 1

Consider (21c). The optimal $X, \tilde{X}$ to the following optimization problem

$$\min_{r,X,\tilde{X}}\Big\{r \ge 0 : p_3(r,X,\tilde{X}) \succeq 0\Big\} \quad (32)$$

is denoted by $X^{o3}, \tilde{X}^{o3}$. Then, we have $p_3(r_3, X^{o3}, \tilde{X}^{o3}) = p(r_3, N^4X^{o3}N^1 + \tilde{X}^{o3}N^2) \succeq 0$. This implies that $r = r_3$, $X = N^4X^{o3}N^1 + \tilde{X}^{o3}N^2$ is a solution to $p(r,X) \succeq 0$. It follows from the definition of r_4 (see (21d)) that $r_4 \le r_3$. Similarly, we can prove $r_4 \le r_2$, $r_2 \le r_1$, $r_3 \le r_1$. If $r_1 = r_4$, it follows from $r_4 \le r_3$, $r_4 \le r_2$, $r_2 \le r_1$, $r_3 \le r_1$ that $r_1 = r_2 = r_3 = r_4$. According to Theorem 2, if $r_1 = r_2 = r_3 = r_4 < 1$, $\lim_{t\to+\infty}\mathbb{E}(P_t)$ is bounded for any λ_1, λ_2. The proof is completed.

F. The proof of Corollary 2

From Corollary 1, we know that 1) $r_4 \le r_1$, and 2) if $r_1 = r_4 < 1$, then $\lim_{t\to+\infty}\mathbb{E}(P_t)$ is bounded for any λ_1, $\lambda_2 \in [0\ 1]$. Now, we start to prove $r_1 \ne r_4$ by contradiction. According to the definition of r_4, one has $r_4 < 1$ if there exists X such that $c(X) \prec I$. Thus, $\lim_{t\to+\infty}\mathbb{E}(P_t)$ is bounded for any λ_1, λ_2 if $c(X) \prec I$ and $r_1 = r_4$ holds. However, under Assumption 1, $\lim_{t\to+\infty}\mathbb{E}(P_t)$ is unbounded for $\lambda_1 = \lambda_2 = 1$,

because (A,C) is undetectable with $\text{diag}\{1_{n_1\times m_1}, 1_{n_2\times m_2}\}$. This is a contradiction. As a result, $r_1 \ne r_4$ if there exists X such that $c(X) \prec I$. The proof is completed.

G. The proof of Lemma 3

To prove Lemma 3, we need Proposition 1. It is known that Lemma 3 holds if $\mathbb{E}(P_t)$ is monotone increasing with respect to λ_1, λ_2. From Proposition 1, one has $\mathbb{E}(P_{t+1}) = \mathbb{E}(g_{\lambda_1\lambda_2}(P_t))$. Hence, we only need to prove that $g_{\lambda_1\lambda_2}(Y)$ is monotone increasing with respect to λ_1, λ_2 for any $Y \succ 0$. In particular, we should show that

- If $0 \le \lambda_1 \le 1$ is fixed and $0 \le \lambda_2^{[1]} \le \lambda_2^{[2]} \le 1$, then $g_{\lambda_1\lambda_2^{[1]}}(Y) \preceq g_{\lambda_1\lambda_2^{[2]}}(Y)$.
- If $0 \le \lambda_2 \le 1$ is fixed and $0 \le \lambda_1^{[1]} \le \lambda_1^{[2]} \le 1$, then $g_{\lambda_1^{[1]}\lambda_2}(Y) \preceq g_{\lambda_1^{[2]}\lambda_2}(Y)$.

From the definition of $g_{\lambda_1\lambda_2}(Y)$, one has

$$g_{\lambda_1\lambda_2^{[1]}}(Y) - g_{\lambda_1\lambda_2^{[2]}}(Y)$$
$$= \lambda_1(\lambda_2^{[1]} - \lambda_2^{[2]})b(L_t^{00}[Y], Y) + \lambda_1(\lambda_2^{[2]} - \lambda_2^{[1]})b(L^{[01]}[Y], Y)$$
$$+ (1-\lambda_1)(\lambda_2^{[1]} - \lambda_2^{[2]})b(L^{[10]}[Y], Y)$$
$$+ (1-\lambda_1)(\lambda_2^{[2]} - \lambda_2^{[1]})b(L^{[11]}[Y], Y)$$
$$= (1-\lambda_1)(\lambda_2^{[2]} - \lambda_2^{[1]})\Big[b(L^{[11]}[Y], Y) - b(L^{[10]}[Y], Y)\Big]$$
$$+ \lambda_1(\lambda_2^{[2]} - \lambda_2^{[1]})\Big[b(L^{[01]}[Y], Y) - b(L^{[00]}[Y], Y)\Big].$$

According to the definitions of $b(X,Y)$, and $L^\gamma[Y]$, $\gamma \in \{[00], [01], [10], [11]\}$ (see after equation (15)), one has $b(L^{[00]}[Y], Y) = \min_{X\in\Lambda^{[00]}}b(X,Y)$, $b(L^{[01]}[Y], Y) = \min_{X\in\Lambda^{[01]}}b(X,Y)$, $b(L^{[10]}[Y], Y) = \min_{X\in\Lambda^{[10]}}b(X,Y)$, $b(L^{[11]}[Y], Y) = \min_{X\in\Lambda^{[11]}}b(X,Y)$, where $\Lambda^{[00]}$, $\Lambda^{[01]}$, $\Lambda^{[10]}$, $\Lambda^{[11]}$ are defined in the proof of Proposition 1. It follows from $\Lambda^{[11]} \supseteq \Lambda^{[10]}$, $\Lambda^{[01]} \supseteq \Lambda^{[00]}$ that $b(L^{[11]}[Y], Y) \preceq b(L^{[10]}[Y], Y)$, $b(L^{[01]}[Y], Y) \preceq b(L_t^{[00]}[Y], Y)$. As a result, $g_{\lambda_1\lambda_2^{[1]}} - g_{\lambda_1\lambda_2^{[2]}} \preceq 0$. Similarly, we can obtain that $g_{\lambda_1^{[1]}\lambda_2} - g_{\lambda_1^{[2]}\lambda_2} \preceq 0$. This completes the proof.

H. The proof of Theorem 3

Based on Lemma 3, we need to show two points: (***Point 1:***) $\lim_{t\to\infty}\mathbb{E}(P_t)$ is bounded if $\lambda_2 \le \underline{\lambda}_{2,c}$; (***Point 2:***) $\lim_{t\to\infty}\mathbb{E}(P_t)$ is unbounded if $\lambda_2 > \bar{\lambda}_{2,c}$.

Point 1: For the case $r_1 = r_4 \ge 1$, $\underline{\lambda}_{2,c} = 0$ is obtained directly from $0 \le \lambda_2 \le 1$. For the case $r_1 = r_4 < 1$, according to Corollary 1, $\lim_{t\to\infty}\mathbb{E}(P_t)$ is bounded when $\lambda_2 = 1$. Thus, $\underline{\lambda}_{2,c} = \bar{\lambda}_{2,c} = 1$ if $r_1 = r_4 < 1$. For the case $r_1 \ne r_4$, according to (24), $\lambda_2 \le \underline{\lambda}_{2,c}$ becomes

$$\lambda_2 \le \frac{1 - r_2\lambda_1 - r_4(1-\lambda_1)}{(r_1-r_2)\lambda_1 + (r_3-r_4)(1-\lambda_1)}, \quad (33)$$

Since $r_4 \le \min(r_2, r_3)$, $r_1 \ge \max(r_2, r_3)$, and $r_1 \ne r_4$, one has $r_1 \ne r_2$ or $r_3 \ne r_4$ holds. Thus, $(r_1-r_2)\lambda_1 + (r_3-r_4)(1-\lambda_1) > 0$. Applying (33) to Theorem 2 gives that $\lim_{t\to\infty}\mathbb{E}(P_t)$ is bounded.

Point 2: Define $s_{\lambda_1\lambda_2}(Y) = \lambda_1\lambda_2 b(L^{[00]}[Y], Y)$. The proof is divided into two steps. (**Step 1:**) to prove that $\mathbb{E}[P_t] \succeq s_{\lambda_1\lambda_2}^t(P_0)$, where the definition of the composite function $f^t(\cdot)$ is defined in "Notations" paragraph. (**Step 2:**) to prove that $\lim_{t\to\infty} s_{\lambda_1\lambda_2}^t(P_0)$ is unbounded for $\lambda_2 > \bar{\lambda}_{2,c}$.

(**Step 1:**) Denote $z_\gamma(Y) = b(L^\gamma[Y], Y)$, $\gamma \in \{[00], [01], [10], [11]\}$, and define

$$z_{\gamma_{t-1}, \gamma_{t-2}, \ldots, \gamma_0}(\cdot) = z_{\gamma_{t-1}}(z_{\gamma_{t-2}}(\ldots z_{\gamma_0}(\cdot))). \quad (34)$$

It follows from the mathematical expectation formula, (11) and (13) that

$$\mathbb{E}[P_t] = \sum_{\gamma_0, \ldots, \gamma_{t-1} \in \{11, 01, 10, 00\}} \Pr(\gamma_0, \cdots, \gamma_{t-1})$$
$$\times z_{\gamma_{t-1}, \gamma_{t-2}, \ldots, \gamma_0}(P_0). \quad (35)$$

Because $\gamma_t = [\gamma_{t,1} \ \gamma_{t,2}]$ is an i.i.d. process and satisfies (3), one has

$$\Pr(\gamma_0, \cdots, \gamma_{t-1}) = \Pr(\gamma_0)\Pr(\gamma_1)\cdots\Pr(\gamma_{t-1}), \quad (36)$$

where for any $i \in \{0, \ldots, t-1\}$,

$\Pr(\gamma_i = [11]) = (1-\lambda_1)(1-\lambda_2), \Pr(\gamma_i = [01]) = (1-\lambda_1)\lambda_2,$
$\Pr(\gamma_i = [10]) = \lambda_1(1-\lambda_2), \Pr(\gamma_i = [00]) = \lambda_1\lambda_2.$

It is known that $s_{\lambda_1\lambda_2}^t(P_0) = (\lambda_1\lambda_2)^t z_{[00]}^t(P_0)$. From (35) and (36), we can obtain $\mathbb{E}[P_t] = h_{\lambda_1\lambda_2}^t(P_0) + \Gamma$, where $\Gamma \succeq 0$, because $z_{\gamma_{t-1}\gamma_{t-2}\ldots\gamma_0}(P_0) \succeq 0$ holds for any $\gamma_0, \ldots, \gamma_{t-1} \in \{[11], [01], [10], [00]\}$. Thus, we have $s_{\lambda_1\lambda_2}^t(P_0) \preceq \mathbb{E}[P_t]$.

(**Step 2:**) If $\lambda_2 > \bar{\lambda}_{2,c}$, then $\lambda_1\lambda_2\underline{\delta}\left(\min_{X^1, X^2} q(X^1, X^2)\right) > 1$. Denote

$$l(Y) = \lambda_1\lambda_2 a(L^{[00]}[Y], Y),$$
$$\hat{l}(X^1, X^2, Y) = \lambda_1\lambda_2 a(N^4 X^1 N^1 + N^3 X^2 N^2, Y).$$

From $s_{\lambda_1\lambda_2}(Y) \succ l(Y) = \min_{X^1, X^2} \hat{l}(X^1, X^2, Y)$, and the fact that both $s_{\lambda_1\lambda_2}(Y)$ and $\min_{X_1, X_2} \hat{l}(X^1, X^2, Y)$ are monotonically increasing with respect to Y, one has $s_{\lambda_1\lambda_2}^t(P_0) \succ \breve{l}^t(P_0)$, where $\breve{l}(P_0) = \min_{X^1, X^2} \hat{l}(X^1, X^2, P_0)$.

It is known that $\text{tr}\left(\breve{l}(P_0)\right) = \text{tr}\left(\min_{X^1, X^2} \lambda_1\lambda_2 q(X^1, X^2) P_0\right)$, where $q(X^1, X^2)$ is defined before Theorem 3. Because $\lambda_1\lambda_2\underline{\delta}\left(\min_{X^1, X^2} q(X^1, X^2)\right) > 1$, one can obtain $\text{tr}\left(\breve{l}^t(P_0)\right) \to \infty$ when $t \to \infty$. Recall that $s_{\lambda_1\lambda_2}^t(P_0) \succ \breve{l}^t(P_0)$. Thus, $\lim_{t\to\infty} s_{\lambda_1\lambda_2}^t(P_0)$ is unbounded for $\lambda_2 > \bar{\lambda}_{2,c}$. The proof is completed.

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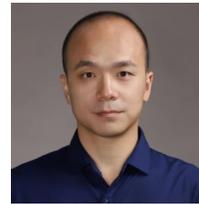
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