# Efficient multi port-based teleportation schemes

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Abstract—In this manuscript we analyse generalised portbased teleportation (PBT) schemes, allowing for transmitting more than one unknown quantum state (or a composite quantum state) in one go, where the state ends up in several ports at Bob's side. We investigate the efficiency of our scheme discussing both deterministic and probabilistic case, where parties share maximally entangled states. It turns out that the new scheme gives better performance than various variants of the optimal PBT protocol used for the same task. All the results are presented in group-theoretic manner depending on such quantities like dimensions and multiplicities of irreducible representations in the Schur-Weyl duality. The presented analysis was possible by considering the algebra of permutation operators acting on *n* systems distorted by the action of partial transposition acting on more than one subsystem. Considering its action on the n-fold tensor product of the Hilbert space with finite dimension, we present construction of the respective irreducible matrix representations, which are in fact matrix irreducible representations of the Walled Brauer Algebra. I turns out that the introduced formalism, and symmetries beneath it, appears in many aspects of theoretical physics and mathematics - theory of anti ferromagnetism, aspects of gravity theory or in the problem of designing quantum circuits for special task like for example inverting an unknown unitary.

*Index Terms*—quantum information, quantum teleportation, group representation theory, symmetric group, portbased teleportation.

## I. (Multi) Port-based teleportation protocols and their importance

Quantum teleportation is one of the most important primitives in quantum information science. It performs an unknown quantum state transmission between two spatially separated systems. It requires pre-shared entangled resource state and consists of three elements: joint measurement, classical communication and correction operation depending on the result of the measurements. Except quantum teleportation protocol presented by Bennett et al. in [1] we distinguish Knill-Laflamme-Milburn (KLM) scheme [2], based solely on linear optical tools and so-called Port-based Teleportation (PBT) protocols, introduced in [3]. Although, standard teleportation and KLM scheme are of the great importance and have fundamental meaning for the field with range of important applications [4], [5], [6], [7], [8], [9], [10], here we focus on PBT schemes. One of the main reasons of that is the PBT is the only scheme where in the last step the *unitary* correction is absent. The lack of correction in the last step allows for entirely new applications in modern quantum information science and the high amount of its symmetries make it tempting for analysis by representationtheoretic methods. For instance, PBT has found its place

in non-local quantum computations and position-based cryptography [11] resulted in new attacks on the cryptographic primitives, reducing the amount of consumable entanglement from doubly exponential to exponential, communication complexity [12] connecting the field of communication complexity and a Bell inequality violation, theory of universal programmable quantum processor performing computation by teleportation [3], universal simulator for qubit channels [13] improving simulations of the amplitude damping channel and allowing to obtain limitations of the fundamental nature for quantum channels discrimination [14]. Some aspects of PBT play a role in the general theory of construction of universal quantum circuit for inverting general unitary operations [15] as well as theory of storage and retrieval of unitary quantum channels [16].

In the original formulation of PBT scheme, see Figure 1, two parties share a resource state consisting of *N* copies of maximally entangled state  $|\psi^+\rangle$ , each of them called a port. Alice to teleport an unknown state  $\psi_C$  to



Fig. 1. On the left-hand side we present the vanilla scheme for the standard PBT. Two parties share *N* copies EPR pairs  $\Phi_d^+ = |\psi_d^+\rangle \langle \psi_d^+|$ , where  $|\psi_d^+\rangle = (1/\sqrt{d}) \sum_i |ii\rangle$ . Alice to teleport an unknown state  $\psi_C$  applies a joint measurement (the blue trapeze) on the state to be teleported and her half  $A_1 \cdots A_N$  of the resource state, getting a classical outcome *i* transmitted to by by a classical channel. The index *i* indicates port on the Bob's side (red star) on which teleported state appears. On the right-hand side we present basic scheme for multi-port teleportation scheme. Again, two parties share *N* copies EPR pairs  $\Phi_d^+ = |\psi_d^+\rangle \langle \psi_d^+|$ , where  $|\psi_d^+\rangle = (1/\sqrt{d}) \sum_i |ii\rangle$ . Alice to teleport an unknown joint state  $\psi_C = \psi_{C_1C_2\cdots C_k}$ , where  $k \leq \lfloor N/2 \rfloor$ , to Bob performs a global measurement (the blue trapeze) on systems  $C_1 \cdots C_k A_1 \cdots A_N$ , getting a classical outcome  $\mathbf{i} = (i_1, i_2, \dots, i_k)$ . She transmits the outcome  $\mathbf{i}$  via classical communication to Bob. The index  $\mathbf{i}$  indicates on which k ports on the Bob's side the teleported state arrives (red stars). Bob to recover the teleported state has to pick up ports indicated by  $\mathbf{i}$  with the right order.

Bob performs a joint measurement on it and her half of the resource state, communicating the outcome through a classical channel to Bob. It turns out that the outcome received by Bob points to the system in the resource state where the state has been teleported to. We distinguish two versions of PBT protocol - deterministic (dPBT) and *probabilistic* (pPBT). In the first case, after the measurement Alice obtains a classical outcome  $i \in \{1, ..., N\}$ . In this scenario, the unknown state is always teleported, but it decoheres during the process. To learn about the efficiency we compute entanglement fidelity, checking how well we are able to transmit half of the maximally entangled state. From the no go theorem [1] for the deterministic universal processor, we know that we can achieve perfect teleportation only in the asymptotic limit  $N \rightarrow \infty$ . In the second case, the probabilistic one, Alice obtains a classical outcome  $i \in \{0, 1, ..., N\}$ , where index 0 corresponds to an additional measurement  $\Pi_0^{AC}$ indicating the failure of the teleportation process. In all other cases in pPBT, when  $i \in \{1, ..., N\}$ , parties proceed with the procedure getting teleported state perfectly. To learn about efficiency, we compute the average probability of success of such a process. Similarly, as in the deterministic case, the probability is equal to 1 only in the asymptotic limit  $N \to \infty$ . In every case, we can consider also *optimised PBT*, where Alice optimises jointly over the shared state and measurements before she runs the protocol to increase the efficiency, see [17] for further details.

Effective evaluation of the performance of both variants of PBT requires determining all symmetries that occur in the problem and spectral analysis of certain operators. For qubits it has been done in [3], [17] by exploiting representation theory of  $SU(2)^{\otimes N}$ , in particular properties of Clebsch-Gordan (CG) coefficients, together with semidefinite programming. Unfortunately, such methods do not work effectively in a higher dimension, d > 2. It is because in the case of  $SU(d)^{\otimes N}$  there is no closed-form of the CG coefficients and to compute them we need an exponential overhead in N and d.

The first attempt to describe the efficiency of PBT in higher dimensions has been done in [18] by exploiting elements of Temperley-Lieb algebra theory, mostly in its graphical representation. The authors presented closed expressions for entanglement fidelity as well as the probability of success for an arbitrary d and N = 2, 3, 4.

Next, in papers [19], [20], [21], authors develop new mathematical tools allowing for studies of PBT for arbitrary N and d. From a technical point of view, the crucial role is played by the algebra of partially transposed permutation operators and its irreducible components. Or in the other words irreducible representations of the commutant of  $U^{\otimes (n-1)} \otimes \overline{U}$ , where the bar denotes complex conjugation, and U is an element of unitary group  $\mathcal{U}(d)$ . It turns out that basic objects describing all variants of PBT belongs to the mentioned commutant. Knowing the full description of irreducible spaces we can reduce the analysis to every block separately and present entanglement fidelity and the probability of success in terms of parameters describing respective irreducible blocks like multiplicity or dimension. Finally, in paper [22] authors investigated the asymptotic behaviour of PBT schemes which was uncovered in the previous works. Their results required advanced tools coming from connections between representation-theoretic formulas and random matrix theory.

Despite of all the results presented above still, we have many important questions to answer in the field of PBT protocols. Here we focus on the following problem: *What is the most effective way to teleport using PBT-like protocols a state of composite system or several systems, let us say k?* One of the answer could be the following:

- The most obvious one is to run the original PBT with dimension of the port equal to *d<sup>k</sup>*, however the performance of the PBT protocols gets worse with growing local dimension [22], [20].
- We could also keep dimensions of the ports and split the resource state into *k* packages and then run *k* separate PBT procedures independently. Such analysis, together with some aspets of asymptotic discussion of the teleportation protocols analysed here is studied in [23].

In the next sections of this paper, we show that allowing Bob for a mild correction in a form of ports permutation we can find a class of multi-port teleportation protocols (see the right panel of Figure 1), allowing for high performance measured in terms of entanglement fidelity or probability of success. Such class of protocols allows us to transfer the state with higher performance than the respective PBT schemes mentioned above. To obtain the final answers we deliver novel mathematical tools concerning both standard Schur-Weyl duality based on *n*-fold tensor product of unitary transformations,  $U^{\otimes n}$ , as well as, its "skew" version based on the product of type  $U^{\otimes N} \otimes \overline{U}^{\otimes k}$  (where bar denotes complex conjugation). By considering irreducible representations of the commutant of  $U^{\otimes N} \otimes \overline{U}^{\otimes k}$  (with n = N + k), we show its connection with the algebra of partially transposed permutation operators  $\mathcal{A}_{n}^{(k)}(d)$ , composed of all linear combinations of the standard permutation operators deformed by the operation of partial transposition over last *k* subsystems. In fact, our work covers unexplored earlier field of finding irreducible matrix representations of Walled Brauer Algebra [24].

The tool kit presented here is not tailored only for effective description of port-based like teleportation protocols and mentioned kind of symmetries appear in many problems of modern physics and mathematics.

From the perspective of physics, studying quantum systems with such symmetries play an important role in antiferromagnetic systems [25]. In this paper the author considers the spectrum of an integrable antiferromagnetic Hamiltonian of the gl(M|N) spin chain of alternating fundamental and dual representations. In particular, to reduce the complexity of the numerical diagonalisation of the considered Hamiltonian author applies non-trivial tools emerging from the theory of Walled-Brauer Algebra. Here, our new tools possibly enable more analytical approach to the problem or at least further numerical simplifications.

Similar kinds of symmetries have found their place even in some aspects of gravity theories [26], [27] and particle physics [28]. Here, authors by applying elements of the representation theory of (Walled) Brauer Algebras and Schur-Weyl duality focus on diagonalisation of the two-point functions of gauge invariant multimatrix operators. In particular, they describe how labels appearing in diagonal bases are related to respective Casimir operators and irreducible components of the Brauer and Walled Brauer Algebra. It turns out that the depper understanding the spectrum of states from the point of view of the conformal field theory (CFT) yields information about space-time physics via the AdS/CFT duality [29].

Next, our analysis could be applied in the study of the theory of entanglement and positive maps. The first such approach has been made by Werner and Eggeling in seminal paper [30], where the full analysis of tripartite  $U \otimes U \otimes U$  states has been made, concentrating on their positivity after partial transposition property (PPT), which is equivalent to considering  $U \otimes U \otimes \overline{U}$ invariant operators. Our tools can in principle be used for the characterisation of multipartite states after having previously chosen the systems to be transposed. This field, although old, is still under exploration, more in the context of positive and k-positive maps. To support our claim let us consider recent papers by Collins and co-authors [31], [32].

Furthermore, motivated by the recent results on Temperley–Lieb Quantum Channels [33], one can apply methods developed here, together with the above investigations, to the problem of constructive examples of new quantum channels for which the minimum output Rényi entropy is not additive.

The tools described in paper are enough for the full description of the universal  $M \rightarrow N$  quantum cloning machines (where M < N) in the group-theoretic manner. Such approach has been successfully for universal  $1 \rightarrow N$  quantum cloning machines in [34].

Finally, the methods developed here are very similar to the techniques used in abstract harmonic analysis for non-commutative groups, where irreducible representations play a crucial role [35]. This similarity strongly suggests possibility of implementing our mathematical results to some aspect of harmonic analysis in future.

To address at least a part of described above problems we need to diagonalize and investigate properties of some operators representing certain physical quantity. To do so we have to construct an analogue of the celebrated Young-Yamanouchi basis for the symmetric group S(n). This is the only way of investigating operators which are  $U^{\otimes n}$  invariant, thus having non-trivial component only the symmetric part in the Schur-Weyl duality. However, in our case our symmetry is deformed - we have k complex conjugations - the straightforward approach suggested by the Schur-Weyl duality is not enough. Also, combining with the approach based on considering the dual representation to  $\overline{U}$  as it was done [22] is not

enough, since we must have full information about matrix entries of the respective operators on the non-trivial sectors, similarly as it is for  $U^{\otimes n}$  invariant operators, and pre-existing methods simply do not have access to these sectors of the space.

From the perspective of pure mathematics we deliver tools for studying and understanding the Walled Brauer Algebras [24], which is a sub-algebra of the Brauer Algebra [36] on the most friendly level for potential applications - irreducible matrix representation. Namely, the algebra of partially transposed permutation operators studied here is a representation of the Walled Brauer Algebra on the space  $(\mathbb{C}^d)^{\otimes n}$ . Up to our knowledge it is the first result of such kind on this level of generality. We can go even further, and built a bridge between our tools, the above-mentioned physical applications and transposed Jucys-Murphy elements [37], [38] which in their not distorted form generate commutative subalgebra of  $\mathbb{C}[S(n)]$ . This approach opens a new path: the opportunity for studying deformation of the permutation group S(N) within a novel approach to representation theory put forward in [39].

The structure of this paper is the following. In Section II we give summary of all our findings presented in the manuscript. In Section III we rigorously introduce the multi-port-based teleportation schemes and discuss the quantities of interest which are entanglement fidelity and probability of success. Next, in Section IV and discuss briefly the occurring symmetries. We explain the connection with the algebra of partially transposed permutation operators and the necessity of finding its irreducible components. In Section V we introduce the basic notions of the representation theory for the permutation group. We explain how to compute the basic quantities describing irreducible representations such as dimensions and multiplicities. We show how to construct an operator basis in every irreducible component. Schur-Weyl duality and notion of Young's lattice are also shortly explained. Most of the pieces of information are taken from [40]. In Section VI we prove a few results concerning partially transposed permutation operators. The notion of partially reduced irreducible representation (PRIR) in the generalized version concerning previous results is introduced. Using these two we prove certain summation rule for matrix elements of irreducible representations of permutations, which is up to our best knowledge not known in the literature. Finally, we present results on partial traces from the operator basis in every irreducible space of the permutation group. In Section VII we present the main mathematical results of our paper. We construct an operator basis in every irreducible representation of the algebra of partially transposed permutation operators. Next, using this result, we compute matrix elements of a port-based teleportation operator determining the performance of teleportation schemes. We show that this object is diagonal in our basis, allowing us to determine its spectral decomposition. Having all mathematical results, in Section VIII and Section IX, we

describe deterministic and probabilistic MPBT scheme and derive expressions describing their performance. We end up by Section X, where we discuss our results and present possible ways of further exploring the idea of multi-port-based teleportation schemes, for example by simultaneous optimization of the resource state and Alice's measurements.

## II. SUMMARY OF THE MAIN RESULTS

In this paper we present several results concerning twofold aspects. Firstly, we introduce tools relating the characterisation of the structure of the algebra  $\mathcal{A}_n^{(k)}(d)$  to the new technical results for practical calculations in the symmetric group S(n). Secondly, we apply our tools to characterise a class of multi-port based teleportation protocols (MPBT).

Results concerning the symmetric group S(n) and the algebra  $\mathcal{A}_n^{(k)}(d)$ :

- 1) In Proposition 6 we deliver new summation (orthogonality) rule for irreducible representations of the symmetric group S(n), which is motivated by the celebrated Schur orthogonality relations [41]. This summation rule allows us for effective computations and simplifications quantities regarding MPBT protocols, especially when computing matrix elements of MPBT operator describing property of the deterministic scheme. It is also important by itself, giving deeper understanding of connection between matrix elements of a subgroup  $H \subset S(n)$  and the whole group S(n).
- 2) We present effective tools for computing partial traces over an arbitrary number of systems from the irreducible operator basis in every irrep of S(n) emerging from the Schur-Weyl duality. This is contained in Lemma 9 and Corollary 10. Up to our best knowledge these are new results on this level of generality and extending results from [42], [43]. Since these tools allow for effective calculations of partial traces in the group algebra of S(n), which is often the case in quantum information science, they are of the separate interest.
- 3) We show that the algebra  $\mathcal{A}_n^{(k)}(d)$  of partially transposed permutation operators is in fact the matrix representation of the Walled Brauer Algebra on the space  $(\mathbb{C}^d)^{\otimes n}$ . This connection, due to [24], gives us all the ideals of the considered algebra and show how they are nested. In particular we identify the maximal ideal  $\mathcal{M}$  (see Figure 6), which is the main object for further understanding multi-port based teleportation schemes. This identification is implied by the symmetries exhibit in our new teleportation protocols.
- 4) We construct an orthonormal irreducible operator basis in the maximal ideal (Theorem 11). We show how the structure of the irreducible blocks looks like and explain their connection with the irreps of the symmetric groups S(n) and S(n - 2k). In fact, this result gives us a way for constructing irreducible

matrix representations of the Walled Brauer Algebra in the maximal ideal  $\mathcal{M}$  on the space  $(\mathbb{C}^d)^{\otimes n}$ , which is the first result of such kind in the literature. It is analogue of the following basic result regarding representations of S(n) on  $(\mathbb{C}^d)^{\otimes n}$ :

$$E_{ij}^{\lambda} = \frac{d_{\lambda}}{n!} \sum_{\sigma \in S(n)} \phi_{ji}^{\lambda}(\sigma^{-1}) V_{\sigma}, \qquad (1)$$

where  $\lambda$  labels irreps of S(n) of dimension  $d_{\lambda}$ ,  $V_{\sigma}$ is permutation operator, that permutes subsystems in  $(\mathbb{C}^d)^{\otimes n}$  according to permutation  $\sigma \in S(n)$ , and finally numbers  $\phi_{ji}^{\lambda}(\sigma^{-1})$  are matrix elements of irreducible representation of  $\sigma$ . The above formula is actually a general formula that works for any representation of a finite group. However, in our case, we have representation of an algebra, which is not a group algebra, and there is no such general formula. We also construct set of projectors on irreducible blocks of the algebra  $\mathcal{A}_n^{(k)}(d)$  in the maximal ideal  $\mathcal{M}$ , see Definition 15 and Lemma 16.

5) In the considered basis we find *matrix elements* of the basic objects for our study - namely, the permutation operators partially transposed on k systems belonging to maximal ideal  $\mathcal{M}_{i}$  as well as those permutation operators, that are not affected by partial transpose (Lemma 13). Our matrix elements are analogues of matrix elements  $\phi_{ii}^{\lambda}(\sigma)$  of irreps of S(n) in Young-Yamanouchi basis. They are connected with the parameters describing irreps of the symmetric groups S(n) and S(n-2k). This is non-trivial extension of the tools used in the Schur-Weyl duality to the case when one has to deal with symmetry of a different type (partial symmetry)-  $U^{\otimes (n-k)} \otimes \overline{U}^{\otimes k}$ , where the existing tools cannot be applied straightforwardly. It was possible by introducing notion of partially irreducible representations, involving concept of the induced representation and properties of subgroups. These tools allow us for effective calculations of compositions and partial traces of operators exhibiting partial symmetries, see for example Lemma 18, Lemma 19 or Lemma 21, and surely they will find applications far beyond MPBT protocols.

#### **Results concerning multi-port based teleportation:**

- 1) We investigate multi-port based teleportation schemes by identifying all their symmetries and present their connection with the algebra  $\mathcal{A}_n^{(k)}(d)$ , so in fact with matrix representations of the Walled Brauer Algebra. We describe two variants, deterministic and probabilistic one. In particular, we show explicitly how operators, like signal states and measurements, encoding the performance of MPBT decompose in terms of partially transposed permutation operators (Sections III, IV).
- 2) Next, having construction of the irreducible basis in the maximal ideal  $\mathcal{M}$  of the algebra  $\mathcal{A}_n^{(k)}(d)$  we prove Theorem 14 and Theorem 17. In particular these

results show that the MPBT operator encoding properties of our protocols is diagonal in projectors onto irreps of the algebra  $\mathcal{A}_n^{(k)}(d)$  which are known thanks to the first part of the paper. It is important to stress here that adaptation of the pre-existing tools like the Schur-Weyl duality and the dual representation to  $\overline{U}$ , which led to re-computation of some known results in PBT [22], when k = 1, are not enough here. It is due to the fact that to obtain all the results one must have an orthogonal irreducible operator basis in every irreducible sector of the underlying algebra which has been not known previously.

3) In the deterministic case we prove Theorem 22 in which we present an explicit expression for entanglement fidelity *F* of the protocol, when parties share N = n - k maximally entangled states of dimension *d* each, and use square-root measurements:

$$F = \frac{1}{d^{N+2k}} \sum_{\alpha \vdash N-k} \left( \sum_{\mu \in \alpha} m_{\mu/\alpha} \sqrt{m_{\mu} d_{\mu}} \right)^2, \qquad (2)$$

where  $m_{\mu}$ ,  $d_{\mu}$  denote multiplicity and dimension of irreducible representations of S(N) respectively in the Schur-Weyl duality, and  $m_{\mu/\alpha}$  denotes number of paths on reduced Young's lattice in which diagram  $\mu$ can be obtained from diagram  $\alpha$  by adding *k* boxes. The efficiency of the new deterministic protocol compared with deterministic PBT when teleporting a composite system is depicted in Figure 2. In this case, we perform significantly better even than the optimal PBT.



Fig. 2. The performance of the deterministic version of our protocol, measured in entanglement fidelity *F*, for various choices of initial parameters which are local dimension *d*, number of ports *N* and number of teleporting particles *k*. One can see that we achieve better performance in teleporting a state of two qubits (d = 2, k = 2) then standard PBT scheme with appropriate port dimension (d = 4, k = 1) as well as the optimal one (OPT).

4) In the probabilistic case we prove Theorem 23 in which we connect probability of success *p* with quan-

tities describing symmetric groups S(N) and S(N - k):

$$p = \frac{k!\binom{N}{k}}{d^{2N}} \sum_{\alpha \vdash N-k} \min_{\mu \in \alpha} \frac{m_{\alpha} d_{\alpha}}{\lambda_{\mu}(\alpha)},$$
(3)

The numbers  $\lambda_{\mu}(\alpha)$  are eigenvalues of MPBT operator and  $m_{\alpha}, d_{\alpha}$  denote multiplicity and dimension of the irrep labelled by  $\alpha$  in the Schur-Weyl duality. The optimal measurements in this case are also derived in the same theorem. The efficiency of the new probabilistic protocol compared with probabilistic PBT when teleporting a composite system is depicted in Figure 3. In this case we outperform the optimal PBT scheme for  $k \geq 3$ . We obtain these results by solving the dual with the primal problem and showing that they coincide, giving us the exact value of the probability and form of the optimal measurements. Exploiting symmetries of the protocol with the mathematical tools developed in this paper, we were able to solve the optimisation problem analytically, which is not the general case in the optimisation theory.



Fig. 3. The performance of the probabilistic version of our protocol, measured in success probability p, for various choices of initial parameters which are local dimension d, number of ports N and number of teleporting particles k. One can see that we start achieving better performance than the corresponding optimal PBT scheme with appropriate port dimension for a state of three qubits (d = 2, k = 3).

## III. QUANTITIES OF INTEREST - ENTANGLEMENT FIDELITY AND PROBABILITY OF SUCCESS

In multi-port based teleportation protocols Alice wishes send to Bob an unknown composite qudit quantum state  $\psi_C = \psi_{C_1C_2\cdots C_k}$ , for  $k \leq \lfloor N/2 \rfloor$ , through N ports, each port given as maximally entangled qudit state  $|\psi^+\rangle = (1/\sqrt{d}) \sum_{i=1}^d |ii\rangle$ , where d stands for the dimension of the underlying local Hilbert space. Both parties share so called resource state of the form  $|\Psi\rangle = \bigotimes_{i=1}^N |\psi^+\rangle_{A_iB_i}$ , see Figure 1. Defining the set

$$\mathcal{I} := \left\{ (i_1, i_2, \dots, i_k) : \forall 1 \le l \le k \ i_l = 1, \dots, N \\ \text{and} \ i_1 \ne i_2 \ne \dots \ne i_k \right\}$$
(4)

consisting of *k*-tuples (not necessarily ordered) denoting ports through which subsystems of the composite state  $\psi_C$  are teleported. For example, having N = 5, k = 2 and  $\mathbf{i} = (5,3)$  means that particle  $\psi_{C_1}$  is on fifth port and  $\psi_{C_2}$  on the third port.

In the next step Alice performs a joint measurement with outcomes **i** from the set  $\mathcal{I}$ . Every effect is described by positive operator valued measure (POVM) satisfying  $\sum_{\mathbf{i}\in\mathcal{I}} \prod_{\mathbf{i}}^{AC} = \mathbf{1}_{AC}$ . Having that we are in the position to describe a teleportation channel  $\mathcal{N}$ , which maps the density operators acting on  $\mathcal{H}_C = \bigotimes_{i=1}^k \mathcal{H}_{C_i}$  to those acting on Bob's side:

$$\mathcal{N}(\psi_{C}) = \sum_{\mathbf{i}\in\mathcal{I}} \operatorname{Tr}_{A\bar{B}_{\mathbf{i}C}} \left[ \sqrt{\Pi_{\mathbf{i}}^{AC}} \left( |\Psi\rangle \langle \Psi|_{AB} \otimes \psi_{C} \right) \sqrt{\Pi_{\mathbf{i}}^{AC}}^{\dagger} \right] = \sum_{\mathbf{i}\in\mathcal{I}} \operatorname{Tr}_{AC} \left[ \Pi_{\mathbf{i}}^{AC} \operatorname{Tr}_{\bar{B}_{\mathbf{i}}} \bigotimes_{j=1}^{N} |\psi\rangle \langle \psi|_{A_{j}B_{j}} \otimes \psi_{C} \right]_{B_{\mathbf{i}} \to \widetilde{B}} = \sum_{\mathbf{i}\in\mathcal{I}} \operatorname{Tr}_{AC} \left[ \Pi_{\mathbf{i}}^{AC} \sigma_{\mathbf{i}}^{A\widetilde{B}} \otimes \psi_{C} \right],$$
(5)

where  $\bar{B}_{\mathbf{i}} = \bar{B}_{i_1}\bar{B}_{i_2}\cdots\bar{B}_{i_k}$  denotes discarded subsystems except those on positions  $i_1, i_2, \ldots, i_k$  and operation  $B_{\mathbf{i}} \rightarrow \tilde{B}$  is assigning  $B_{N+1}\cdots B_{N+k}$  for every index  $\mathbf{i}$  on Bob's side, introduced for the mathematical convenience. The states (signals)  $\sigma_{\mathbf{i}}^{AB}$  or shortly  $\sigma_{\mathbf{i}}$  for  $\mathbf{i} \in \mathcal{I}$ , from (5), are given as

$$\sigma_{\mathbf{i}}^{A\widetilde{B}} := \operatorname{Tr}_{\widetilde{B}_{\mathbf{i}}} \left( P_{A_{1}B_{1}}^{+} \otimes P_{A_{2}B_{2}}^{+} \otimes \cdots \otimes P_{A_{N}B_{N}}^{+} \right)_{B_{\mathbf{i}} \to \widetilde{B}}$$
(6)

$$=\frac{1}{d^{N-k}}\mathbf{1}_{\bar{A}_{\mathbf{i}}}\otimes P^{+}_{A_{\mathbf{i}}\bar{B}}.$$
(7)

In above  $\bar{A}_{i}$  has the same meaning as  $\bar{B}_{i}$ . Then  $P^{+}_{A_{i}\tilde{B}}$  is a tensor product of projectors on maximally entangled sates with respect to subsystems defined by index i and prescription  $B_{i} \rightarrow \tilde{B}$ . For example, when i = (5,3), the notation  $P^{+}_{A_{i}\tilde{B}}$  means  $P^{+}_{A_{5}B_{6}} \otimes P^{+}_{A_{3}B_{7}}$ . For the further reasons, we introduce here the following multi portbased operator given as:

$$\rho := \sum_{\mathbf{i} \in \mathcal{I}} \sigma_{\mathbf{i}}.$$
(8)

In the general case in above sum we have  $k!\binom{N}{k} = \frac{N!}{(N-k)!} = |\mathcal{I}|$  elements. One can see that for k = 1 we reproduce  $|\mathcal{I}| = N$  number of signals from the original PBT scheme. For k = 2 we have  $|\mathcal{I}| = N(N-1)$ , for k = 3 it is  $|\mathcal{I}| = N(N-1)(N-2)$  and so on.

**Deterministic version** In this version of the protocol receiver always accepts state of k of N ports as the teleported states. Since the ideal transmission of states is impossible, we would like to know how well we are able to preform the scheme, possibly as a function of global parameters like number of ports or local dimension. We investigate this by checking how well the teleportation channel  $\mathcal{N}$  transmits quantum correlations. To do so we

compute its entanglement fidelity *F*, teleporting halves of maximally entangled states

$$F = \operatorname{Tr}\left[P_{\widetilde{B}C}^{+}(\mathcal{N} \otimes \mathbf{1}_{D})P_{CD}^{+}\right]$$
  
$$= \sum_{\mathbf{i} \in \mathcal{I}} \operatorname{Tr}\left[P_{\widetilde{B}C}^{+}\Pi_{\mathbf{i}}^{AC}\left(\sigma_{\mathbf{i}}^{A\widetilde{B}} \otimes P_{CD}^{+}\right)\right]$$
  
$$= \frac{1}{d^{2k}} \sum_{\mathbf{i} \in \mathcal{I}} \operatorname{Tr}\left[\Pi_{\mathbf{i}}^{A\widetilde{B}}\sigma_{\mathbf{i}}^{A\widetilde{B}}\right],$$
  
(9)

where  $D = D_1 D_2 \cdots D_k$ . To have explicit answer what is the value of F we need to choose a specific form of POVM operators  $\Pi_i^{AC}$ . As it is explained in previous papers [3], [17], PBT scheme is equivalent to the state discrimination problem, where authors use Nsquare-root measurements for distinguishing an ensemble  $\{1/N, \sigma_i^{AC}\}$ . In our case the situation is similar and the corresponding ensemble is of the form  $\{1/|\mathcal{I}|, \sigma_i^{AC}\}$ with corresponding POVMs:

$$\forall \mathbf{i} \in \mathcal{I} \qquad \Pi_{\mathbf{i}}^{AC} = \frac{1}{\sqrt{\rho}} \sigma_{\mathbf{i}}^{AC} \frac{1}{\sqrt{\rho}} + \Delta, \tag{10}$$

where states  $\sigma_i^{AC}$  are given in (6) and  $\rho$  is the port-based operator from (21). It can be easy seen that operator  $\rho$  is not of the full rank, so inversion on the support is required. Due to this, to every component in (10) an additional term  $\Delta$  of the form

$$\Delta = \frac{1}{|\mathcal{S}_{n,k}|} \left( \mathbf{1}_{(\mathbb{C}^d)^{\otimes n}}^{AC} - \sum_{\mathbf{i}\in\mathcal{I}} \Pi_{\mathbf{i}}^{AC} \right)$$
(11)

is added. This addition ensures that all effects sum up to identity operator on whole space  $(\mathbb{C}^d)^{\otimes n}$ . Such procedure does not change the entanglement fidelity *F* in (9). Our goal is to evaluate the entanglement fidelity from (9) with measurements given in (10). The solution given in terms of group-theoretic parameters is presented in Theorem 22 in Section VIII.

**Probabilistic version** In this scenario transmission sometimes fails, but whenever succeeds then fidelity of the teleported state is maximal F = 1. The teleportation channel in the probabilistic version looks exactly the same as it is in deterministic protocol, however is non-trace preserving. This fact is due to the reason that now Alice has access to  $1 + |\mathcal{I}|$  POVMs, where an additional POVM  $\Pi_0^{AC}$  corresponds to the failure. To evaluate the performance of the scheme we need to calculate the average success probability of teleportation p, where we average over all possible input states. This leads to the following expression (see [17], [23] for detailed calculations):

$$p = \frac{1}{d^{N+k}} \sum_{\mathbf{i} \in \mathcal{I}} \operatorname{Tr} \left( \Pi_{\mathbf{i}}^{AC} \right).$$
(12)

Requirement of the unit fidelity gives strong condition on the form of the measurement applied by Alice. Namely, using argumentation presented in [3], [17], [19] all the POVMs corresponding to the success of teleportation are of the form:

$$\forall \mathbf{i} \in \mathcal{I} \qquad \Pi_{\mathbf{i}}^{AC} = P_{A_{\mathbf{i}}C}^+ \otimes \Theta_{\overline{A}_{\mathbf{i}}}. \tag{13}$$

Having expression for probability of success (12) we ask what is the maximal possible value of p and what is then the optimal form the operators  $\Theta_{\overline{A}_i}$  from (13) ensuring mentioned maximisation. It turns out that this problem can be written as a semidefinite program (SDP). We write down a primal problem whose solution  $p^*$  lower bounds the real value of p, and then we write a dual problem where the solution  $p_*$  is a respective upper bound for the real value of p. In the boxes below we write down explicitly primal and dual problem.

**The primal problem for probabilistic scheme** The goal is to maximise the quantity:

$$p^* = \frac{1}{d^{N+k}} \sum_{\mathbf{i} \in \mathcal{I}} \operatorname{Tr} \Theta_{\overline{A}_{\mathbf{i}}}, \qquad (14)$$

with respect to the following constraints:

(1) 
$$\Theta_{\overline{A}_{i}} \geq 0$$
, (2)  $\sum_{i \in \mathcal{I}} P^{+}_{A_{i}C} \otimes \Theta_{\overline{A}_{i}} \leq \mathbf{1}_{AC}$ . (15)

**The dual problem for probabilistic scheme** The dual problem is to minimise the quantity

$$p_* = \frac{1}{d^{N+k}} \operatorname{Tr} \Omega \tag{16}$$

with respect to the following constraints

(1) 
$$\Omega \ge 0$$
, (2)  $\forall \mathbf{i} \in \mathcal{I} \operatorname{Tr}_{A_{\mathbf{i}}C} \left( P^+_{A_{\mathbf{i}}C} \Omega \right) \ge \mathbf{1}_{N-k}$ ,  
(17)  
where the operator  $\Omega$  acts on  $N + k$  systems, the

identity  $\mathbf{1}_{N-k}$  acts on all systems but  $A_i$  and C.

The solutions of the above primal and the dual problem in terms of group-theoretic parameters are presented in Theorem 23 in Section IX. For potential reader who would like to learn more about the concept of SDP, formulation of the respective primal and dual problem, we refer to book [44].

## IV. Symmetries in multi port-based teleportation

In every variant of (multi) port-based teleportation protocols we distinguish two type of symmetries. One is connected with covariance and invariance with respect to the symmetric group S(n-k), while the second one with invariance with respect to the action of  $U^{\otimes (n-k)} \otimes \overline{U}^{\otimes k}$ . We now shall describe briefly connection of these two types of symmetries with the operators describing analysed teleportation schemes.

Let us take index  $\mathbf{i}_0$  such that  $\mathbf{i}_0 = (N - 2k + 1, N - 2k + 2, ..., N - k)$ , then having n = N + k signal  $\sigma_{\mathbf{i}_0}$  can be written as

$$\sigma_{\mathbf{i}_{0}} = \frac{1}{d^{n-2k}} \mathbf{1}_{\bar{A}_{\mathbf{i}_{0}}} \otimes P_{n-2k+1,n}^{+} \otimes P_{n-2k+2,n-1}^{+} \otimes \cdots \otimes P_{n-k,n-k+1}^{+} \text{ This property with elements describing}$$
$$= \frac{1}{d^{n-k}} \mathbf{1}_{\bar{A}_{\mathbf{i}_{0}}} \otimes V_{n-2k+1,n}^{t_{n}} \otimes V_{n-2k+2,n-1}^{t_{n-1}} \otimes \cdots \otimes V_{n-k,n-k+1}^{t_{n-k+1}} \text{ teleportation protoc}$$
(18) partially transposed

where by  $t_n$ ,  $t_{n-1}$  etc. we denote the partial transpositions with respect to particular subsystem and by  $V_{r,s}$  the permutation operator between system r and s (since now we drop off indices for A, B unless they necessary), for the  $V_{r,s}^{t_s}$  operator is proportional to  $P_{r,s}^+$ . Further we assume whenever it is necessary that permutation operators are properly embedded in whole  $(\mathbb{C}^d)^{\otimes n}$  space so we will write just  $V_{r,s}$  instead of  $V_{r,s} \otimes \mathbf{1}_{\bar{r},\bar{s}}$ . Moreover for the signal  $\sigma_{i_0}$  we introduce simpler notation

$$\sigma_{\mathbf{i}_0} = \frac{1}{d^{n-k}} \mathbf{1} \otimes V_{n-2k+1,n}^{t_n} \otimes V_{n-2k+2,n-1}^{t_{n-1}} \otimes \cdots \otimes V_{n-k,n-k+1}^{t_{n-k+1}}$$
$$\equiv \frac{1}{d^N} V^{(k)},$$
(19)

where

$$V^{(k)} \equiv \mathbf{1} \otimes V^{t_n}_{n-2k+1,n} \otimes V^{t_{n-1}}_{n-2k+2,n-1} \otimes \cdots \otimes V^{t_{n-k+1}}_{n-k,n-k+1},$$
  
(k)  $\equiv t_n \circ t_{n-1} \circ \cdots \circ t_{n-k+1},$  (20)

• denotes composition of maps, and **1** denotes identity acting on the space untouched by tensor product of projectors on maximally entangled states. Form the definition of the signals  $\sigma_i$  in (6) and form of  $\sigma_{i_0}$  from (19) we can deduce that PBT operator  $\rho$  can be written as

$$\rho = \sum_{\mathbf{i}\in\mathcal{I}} \sigma_{\mathbf{i}} = \frac{1}{d^N} \sum_{\tau\in\mathcal{S}_{n,k}} V_{\tau^{-1}} V^{(k)} V_{\tau}, \qquad (21)$$

where sum runs over all permutations  $\tau$  from the coset  $S_{n,k} := \frac{S(n-k)}{S(n-2k)}$ , and  $V_{\tau}$  the permutation operator corresponding to the permutation  $\tau$ . Due to the construction we have that  $|\mathcal{I}| = \left|\frac{S(n-k)}{S(n-2k)}\right|$ . Moreover, it is easy to notice that any signal state satisfies:

$$\forall \ \tau \in \mathcal{S}_{n,k} \qquad V_{\tau} \sigma_{\mathbf{i}} V_{\tau^{-1}} = \sigma_{\tau(\mathbf{i})}. \tag{22}$$

Let us notice that the operator  $\rho$  is invariant with respect to action of any permutation from S(n - k) acting on the first n - k systems:

$$\forall \ \tau \in S(n-k) \qquad V_{\tau} \rho V_{\tau^{-1}} = \rho. \tag{23}$$

In particular, relation (22) and (23) imply covariance of the SRM measurements given in (10) with respect to the coset  $S_{n,k}$ . The same type of covariance we require for POVMs in the probabilistic scheme from (13).

We have also the second kind of symmetries. Notice that all operators  $\sigma_i$  as well PBT operator from (8) are invariant with respect to action of  $U^{\otimes (n-k)} \otimes \overline{U}^{\otimes k}$ , where the bar denotes the complex wise conjugation and Uis an element of unitary group U(d). This observation follows from the structure of the signal states and the fact that every bipartite maximally entangled state  $P_{ij}^+$ , between system *i* and *j*, is  $U \otimes \overline{U}$  invariant.

 $\mathcal{L}_{n-k,n-k+1}^{+}$  This property with expression (19) means that basic elements describing the performance of the presented  $\mathcal{L}_{n-k,n-k+1}^{t}$  deleportation protocol belong to the algebra  $\mathcal{A}_{n}^{(k)}(d)$  of (18) partially transposed permutation operators with respect to *k* last subsystems. For k = 1 we reduce to the known case, and standard *d* dimensional port-based teleportation introduced in [19] and the algebra  $\mathcal{A}_n^{(1)}(d)$  discussed in [45], [46]. In this particular case we have  $\mathcal{S}_{n,1} = S(n-1)/S(n-2)$  and all permutations  $\tau$  are of the form of transpositions (a, n) for a = 1, ..., n-1, and |S(n-1)/S(n-2)| = n-1.

The operation of partial transposition changes significantly properties of the operators under consideration, making the resulting set of operators no longer the group algebra of the symmetric group S(n). To see it explicitly, let us consider a swap operator  $V_{i,j}$  interchanging systems on positions i and j. It is obvious that by applying the swap operator twice, we end up with an identity operator. However, applying the partial transposition to  $V_{i,j}$ , the swap operator is mapped to the operator  $dP_{ii}^+$ , which is proportional to maximally entangled state between respective systems. Applying partially transposed swap operator twice, we end up with  $d^2 P_{ij}^+$ , since  $P_{ij}^+ P_{ij}^+ = P_{ij}^+$ . This property makes our further analysis more complex, and direct application of the standard methods from the representation theory of the group algebra of the symmetric group S(n) is insufficient here.

In next sections we introduce notations and definitions and construct irreducible orthonormal basis of the algebra  $\mathcal{A}_n^{(k)}(d)$  and formulate auxiliary lemmas required to spectral analysis of the operator  $\rho$  and describing the performance of the protocol.

#### V. NOTATIONS AND DEFINITIONS

For a given natural number n we can define its partition  $\mu$  in the following way

$$\mu = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(l)})$$
(24)

such that

$$\forall_{1 \le i \le l} \ \mu^{(i)} \in \mathbb{N}, \quad \mu^{(1)} \ge \mu^{(2)} \ge \dots \ge \mu^{(l)} \ge 0,$$
  
 $\sum_{i=1}^{l} \mu^{(i)} = n.$  (25)

The Young frame associated with partition  $\mu$  is the array formed by n boxes with l left-justified rows. The *i*-th row contains exactly  $\mu^{(i)}$  boxes for all i = 1, 2, ..., l. Further, we denote Young diagrams by the Greek letters. The set of all Young diagrams, with up to  $n \in \mathbb{N}$  boxes, is denoted as  $\mathbb{Y}_n$ . The restriction to the set of Young diagrams with no more then d rows is denoted as  $\mathbb{Y}_{n,d}$ . We endow  $\mathbb{Y}_n, \mathbb{Y}_{n,d}$  with a structure of a partially ordered set by setting, for  $\mu = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(l)}) \vdash n$  and  $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(s)}) \vdash n - k$ ,

$$\alpha \leq \mu$$
, (26)

if  $\mu^{(i)} \ge \alpha^{(i)}$  for all i = 1, 2, ..., l. If  $\alpha \le \mu$  we denote by  $\mu/\alpha$  the array, called also a skew shape, obtained by removing from the Young frame  $\mu$  the boxes of the



Fig. 4. Graphics presents construction of a skew shape  $\mu/\alpha$  for Young frames  $\mu = (6,3,3,1)$  and  $\alpha = (3,2,1)$ .

Young frame of  $\alpha$ . We have illustrated this procedure by an example presented in Figure 4. For any  $\alpha, \mu \in \mathbb{Y}_n$  we say that  $\mu$  covers  $\alpha$ , or  $\alpha$  is covered by  $\mu$  if  $\alpha \leq \mu$  and

$$\alpha \leq \nu \leq \mu, \quad \nu \in \mathbb{Y}_n \Rightarrow \nu = \alpha \text{ or } \nu = \mu.$$
 (27)

In other words,  $\mu$  covers  $\alpha$  if and only if  $\alpha \leq \mu$  and  $\mu/\alpha$  consists of at least a single box. Later we use an equivalent symbol  $\mu \in \alpha$  to denote Young diagrams  $\mu \vdash n$  obtained from Young diagrams  $\alpha \vdash n - k$  by adding k boxes. While by the symbol  $\alpha \in \mu$  we denote Young diagrams  $\alpha \vdash n - k$  obtained from Young diagrams  $\mu \vdash n$  by subtracting k boxes. Informally it means that a Young diagram with n - k boxes is contained in a Young diagram and sets  $\mathbb{Y}_n$ ,  $\mathbb{Y}_{n,d}$  we define Young's lattice and its reduced version (see Figure 5). The Young's lattice



Fig. 5. The Young's lattice  $\mathbb{Y}_{6}$ , i.e. with six consecutive layers labelled by permutation groups from S(1) to S(6). By orange and black dashed lines we depict two possible paths from irrep  $\lambda = (1)$  of S(1) to irrep  $\lambda' = (2, 1, 1, 1, 1)$  of S(6). The reduced Young's lattice  $Y_{6,2}$ , i.e. for d = 2is defined by all diagram on the right-hand side of the red line.

arises when we construct subsequent Young diagrams by adding boxes one by one. In this way we obtain subsequent layers of Young diagrams for growing *n*. We connect a diagram with a subsequent diagram by an edge, that is obtained by adding a box. More formally the Young's lattice of  $\mathbb{Y}_n$  is the non-oriented graph with vertex set  $\mathbb{Y}_n$  and an edge from  $\lambda$  to  $\mu$  if and only if  $\lambda$ covers  $\mu$ . The same definition applies for Young's lattice of  $\mathbb{Y}_{n,d}$ , but we remove all Young diagrams with more than *d* rows. A path  $r_{\mu/\alpha}$  in the Young's lattice is a sequence  $r_{\mu/\alpha} = (\mu \equiv \mu_n \vdash n \rightarrow \mu_{n-1} \vdash n - 1 \rightarrow \cdots \rightarrow$   $\alpha \equiv \mu_{n-k} \vdash n-k$ ), for some  $k \in \mathbb{N}$  and k < n. The integer number  $m_{\mu/\alpha}$  is the total lengths of all paths from  $\mu$  to  $\alpha$ .

All irreducible representations (irreps) of S(n) are labelled by Young diagrams with *n* boxes denoted as  $\alpha \vdash_d n$ , where *d* is a natural parameter, meaning we take into account diagrams with at most *d* rows. The dimension  $d_{\alpha}$  of the irrep  $\alpha$  is given by the hook length formula

$$d_{\mu} = \frac{n!}{\prod_{(i,j) \in \mu} h_{\mu}(i,j)},$$
(28)

where  $h_{\mu}(i, j)$  is so called the hook length of the hook with corner at the box (i, j) given as one plus the number of boxes below (i, j) plus number of boxes to the right of (i, j). Multiplicity  $m_{\mu,d}$  of every irrep  $\alpha \vdash_d n$  is characterised by Weyl dimension formula, saying that

$$m_{\mu,d} = \prod_{1 \le i \le j \le d} \frac{\mu_i - \mu_j + j - i}{j - i}.$$
 (29)

Later on we suppress notation to  $\alpha \vdash n$  and  $m_{\mu}$ , having in mind the dependence of the natural parameter *d*, playing later the role of local dimension of the space  $\mathbb{C}^d$ .

Having commuting representations of S(n) and U(d)on  $(\mathbb{C}^d)^{\otimes n}$ , acting by permuting the tensor factors, and multiplication by  $U^{\otimes n}$  respectively, we can decompose the space  $(\mathbb{C}^d)^{\otimes n}$ , using Schur-Weyl duality [47] into direct sum of irreducible subspaces as follows:

$$(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\alpha \vdash_d n} \mathcal{U}_{\alpha} \otimes \mathcal{S}_{\alpha}.$$
 (30)

In the above  $S_{\alpha}$  are representation spaces for the permutation groups S(n), while  $U_{\alpha}$  are representation spaces of U(d). In Schur basis producing the decomposition (30) we can define in every space  $S_{\alpha}$  an orthonormal operator basis  $E_{ij}^{\alpha}$ , for  $i, j = 1, ..., d_{\alpha}$ , separating the multiplicity and representation space of permutations respectively. Namely we have

$$E_{ij}^{\alpha} = \mathbf{1}_{\alpha}^{\mathcal{U}} \otimes |\alpha, i\rangle \langle \alpha, j| \equiv \mathbf{1}_{\alpha}^{\mathcal{U}} \otimes |i\rangle \langle j|_{\alpha}.$$
(31)

We can also use representation of  $E_{ij}^{\alpha}$  on the space  $(\mathbb{C}^d)^{\otimes n}$ , which is of the form

$$E_{ij}^{\alpha} = \frac{d_{\alpha}}{n!} \sum_{\tau \in S(n)} \phi_{ji}^{\alpha}(\tau^{-1}) V_{\tau}, \qquad (32)$$

where  $\phi_{ji}^{\alpha}(\tau^{-1})$  denotes matrix element irreducible representation of the permutation  $\tau^{-1} \in S(n)$ . The operators from (31) have the following properties

$$E_{ij}^{\alpha}E_{kl}^{\beta} = \delta^{\alpha\beta}\delta_{jk}E_{il}^{\alpha}, \quad \text{Tr} \ E_{ij}^{\alpha} = m_{\alpha}\delta_{ij}. \tag{33}$$

Let us observe that operators  $E_{ii}^{\alpha}$  are projectors. Action of the operators  $E_{ij}^{\alpha}$  on an arbitrary permutation operator  $V_{\sigma}$ , from the left and from the right-hand side, for  $\sigma \in S(n)$  is given by

$$E_{ij}^{\alpha}V_{\sigma} = \sum_{k} \varphi_{jk}^{\alpha}(\sigma)E_{ik}^{\alpha}, \quad V_{\sigma}E_{ij}^{\alpha} = \sum_{k} \varphi_{ki}^{\alpha}(\sigma)E_{kj}^{\alpha}.$$
 (34)

Using this basis we can write matrix representation of a given permutation  $\tau^{-1} \in S(n)$ , on every irreducible space labelled by  $\alpha \vdash n$  as

$$\phi^{\alpha}(\tau^{-1}) = \sum_{ij} \phi^{\alpha}_{ij}(\tau^{-1}) E^{\alpha}_{ij}.$$
(35)

Moreover, using this operators we construct Young projectors, the projectors on components  $U_{\alpha} \otimes S_{\alpha}$  from (30):

$$P_{\alpha} = \sum_{i} E_{ii}^{\alpha} = \frac{d_{\alpha}}{n!} \sum_{\tau \in S(n)} \chi^{\alpha}(\tau^{-1}) V_{\tau}.$$
 (36)

The numbers  $\chi^{\alpha}(\tau^{-1}) = \sum_{i} \phi_{ii}^{\alpha}(\tau^{-1})$  are irreducible characters.

Sometimes instead of  $|\alpha, i\rangle$  we write  $|\alpha, i_{\alpha}\rangle$  or just  $|i_{\alpha}\rangle$ . Defining  $c = \sum_{\alpha} m_{\alpha} d_{\alpha}$  any operator  $X \in M(c \times c, \mathbb{C})$  can be written using elements  $\{E_{ij}^{\alpha}\}$  as  $X = \sum_{\alpha} \sum_{i,j=1}^{d_{\alpha}} x_{ij}^{\alpha} E_{ij}^{\alpha}$ . Considering *n*-particle system, by writing  $V_{n-1,n}$  we understand  $\mathbf{1}_{1...n-2} \otimes V_{n-1,n}$ , and similarly for other operators. The operator  $\mathbf{1}_{1...n-2}$  is identity operator on first n-2 particles.

## VI. PRELIMINARY MATHEMATICAL RESULTS

#### A. Partial trace over Young projectors

In this section we present a set of auxiliary lemmas which are crucial for the presentation in the further sections. Introducing notation  $\text{Tr}_{(k)} = \text{Tr}_{n-2k+1,...,n-k}$ , denoting the partial trace over a set  $\{n - 2k + 1, ..., n - k\}$  of particles, and recalling the notation

$$V^{(k)} := V_{n-2k+1,n}^{t_n} V_{n-2k+2,n-1}^{t_{n-1}} \cdots V_{n-k,n-k+1}^{t_{n-k+1}}$$
(37)

we start from formulation of the following:

**Fact 1.** For any operator acting on n - k systems we have the following equality

$$V^{(k)}X \otimes \mathbf{1}_{n-k+1\dots n}V^{(k)} = \operatorname{Tr}_{(k)}(X)V^{(k)},$$
 (38)

where  $\mathbf{1}_{n-k+1...n}$  is the identity operator on k last subsystems, while the operator X on first n - k.

In particular cases, when k = 1, 2, we have respectively

$$V^{(1)}X \otimes \mathbf{1}_{n}V^{(1)} = \operatorname{Tr}_{n-1}(X)V^{(1)},$$
  

$$V^{(2)}X \otimes \mathbf{1}_{n-1,n}V^{(2)} = \operatorname{Tr}_{n-3,n-2}(X)V^{(2)}.$$
(39)

Now we prove Fact 1:

*Proof.* It is enough to show that expression (39) holds for k = 1 and then use the argumentation below iteratively. Using identity  $V^{(1)} = dP_+$ , where  $P_+$  is the projector on maximally entangled state and d is the dimension of local Hilbert space, we write

$$V^{(1)}(X \otimes \mathbf{1}_{n})V^{(1)} = d^{2}(\mathbf{1}_{1,...,n-2} \otimes P_{+}) \left(\sum_{ij} X^{1,...,n-2}_{ij} \otimes e^{(n-1)}_{ij} \otimes \mathbf{1}_{n}\right) (\mathbf{1}_{1,...,n-2} \otimes P_{+})$$
(40)

where  $\{e_{ij}^{(n-1)}\}$  is standard operator basis on subsystem n-1 and  $X_{ij}^{1,...,n-2}$  are operators on subsystems from 1 to n-2. Using explicit form of  $P_+ = (1/d) \sum_{k,l} e_{kl}^{(n-1)} \otimes e_{kl}^{(n)}$  and  $X_{ij}^{(1,...,n-2)}$  we get (41), given at the top of the next page, where  $x_{i_1j_1,...,i_{n-1}j_{n-1}}$  denotes matrix elements of X.

**Fact 2.** Let  $\{|i\rangle_{\alpha}\}_{i=1}^{d_{\alpha}}$  denote a basis in an irrep  $\alpha \vdash n$  of dimension  $d_{\alpha}$ . Then for any operator  $X^{\alpha} = \sum_{ij} x_{ij}^{\alpha} |i\rangle \langle j|_{\alpha}$  acting on the space  $\mathbb{C}^{d_{\alpha}}$ , we have

$$\sum_{\tau \in S(n)} \operatorname{Tr} \left( X^{\alpha} \phi^{\alpha}(\tau^{-1}) \right) V_{\tau} = \frac{n!}{d_{\alpha}} \mathbf{1}^{U}_{\alpha} \otimes X^{\alpha}.$$
(42)

Proof. Inserting expression (35) into (42) we obtain

$$\sum_{\tau \in S(n)} \operatorname{Tr} \left( X^{\alpha} \phi^{\alpha}(\tau^{-1}) \right) V_{\tau} = \sum_{\tau \in S(n)} \operatorname{Tr} \left( X^{\alpha} \sum_{ij} \phi^{\alpha}_{ij}(\tau^{-1}) E^{\alpha}_{ij} \right) V_{\tau}$$
$$= \sum_{ij} \operatorname{Tr} \left( X^{\alpha} E^{\alpha}_{ij} \right) \sum_{\tau \in S(n)} \phi^{\alpha}_{ij}(\tau^{-1}) V_{\tau}$$
$$= \frac{n!}{d_{\alpha}} \sum_{ij} \operatorname{Tr} \left( X^{\alpha} E^{\alpha}_{ij} \right) E^{\alpha}_{ji}.$$
(43)

In Schur basis every operator  $E_{ij}^{\alpha}$  has a form  $\mathbf{1}_{\alpha}^{U} \otimes |i\rangle \langle j|_{\alpha}$ . This allows us to write

$$\frac{n!}{d_{\alpha}} \sum_{ij} \operatorname{Tr} \left( X^{\alpha} E_{ij}^{\alpha} \right) E_{ji}^{\alpha} 
= \frac{n!}{m_{\alpha} d_{\alpha}} \sum_{ij} \operatorname{Tr} \left[ \left( \mathbf{1}_{\alpha}^{U} \otimes X^{\alpha} \right) \left( \mathbf{1}_{\alpha}^{U} \otimes |i\rangle \langle j|_{\alpha} \right) \right] \left( \mathbf{1}_{\alpha}^{U} \otimes |j\rangle \langle i|_{\alpha} \right) 
= \frac{n!}{d_{\alpha}} \mathbf{1}_{\alpha}^{U} \otimes \operatorname{Tr} \left( X^{\alpha} |i\rangle \langle j|_{\alpha} \right) |j\rangle \langle i|_{\alpha} = \frac{n!}{d_{\alpha}} \mathbf{1}_{\alpha}^{U} \otimes X^{\alpha}.$$
(44)

Now, let us introduce the following objects:

 $\widetilde{\phi}^{\mu}(a,n) := d^{\delta_{a,n}} \phi^{\mu}(a,n)$ , and  $\widetilde{V}_{a,n} := d^{\delta_{a,n}} V_{a,n}$ , (45) where  $\phi^{\mu}(a,n)$  is a matrix representation of permutation  $V_{a,n}$  on irrep  $\mu \vdash n$ . Having that we can formulate the following

**Lemma 3.** Let us denote by  $\mathbf{1}^{\mu}_{\gamma}$  the identity on an irrep  $\gamma$  of S(n-1) contained in irrep  $\mu$  of S(n), then we have the following restriction of  $\tilde{\phi}^{\mu}(a,n)$  to irrep  $\beta$  of S(n-1)

$$\left[\sum_{a=1}^{n} \widetilde{\phi}^{\mu}(a,n)\right]_{\beta} = x^{\mu}_{\beta} \mathbf{1}^{\mu}_{\beta} \quad with \quad x^{\mu}_{\beta} = n \frac{m_{\mu} d_{\beta}}{m_{\beta} d_{\mu}}.$$
 (46)

Proof. Consider

$$\sum_{a=1}^{n} \widetilde{\phi}^{\mu}(a,n) = \sum_{a=1}^{n-1} \phi^{\mu}(a,n) + d\mathbf{1}_{\mu}$$
(47)

which is clearly invariant with respect to S(n-1). Hence it admits the decomposition

$$\sum_{a=1}^{n} \widetilde{\phi}(a,n) = \sum_{\gamma \in \mu} x_{\gamma}^{\mu} \mathbf{1}_{\gamma}^{\mu}$$
(48)

for some  $x_{\gamma}^{\mu} \in \mathbb{C}$ . The restriction for chosen irrep  $\beta \in \mu$  reduces the above to

$$\left[\sum_{a=1}^{n} \widetilde{\phi}^{\mu}(a,n)\right]_{\beta} = x_{\beta}^{\mu} \mathbf{1}_{\beta}^{\mu}.$$
(49)

Now our goal is to compute the unknown coefficients  $x_{\gamma}^{\mu}$ . To do so let us first observe that we can write every projector  $P_{\mu}$  in terms of coset elements  $\phi(a, n)$  and permutations from S(n-1). Indeed we have

$$P_{\mu} = \frac{d_{\mu}}{n!} \sum_{\sigma \in S(n)} \operatorname{Tr} \left[ \phi^{\mu}(\sigma^{-1}) \right] V_{\sigma}$$
  
$$= \frac{d_{\mu}}{n!} \sum_{a=1}^{n} \sum_{\tau \in S(n-1)} \operatorname{Tr} \left[ \phi^{\mu}((a,n) \circ \tau^{-1}) \right] V_{a,n} V_{\tau}.$$
(50)

Since every representation is a homomorphism we have  $\phi^{\mu}((a, n) \circ \tau^{-1}) = \phi^{\mu}((a, n))\phi^{\mu}(\tau^{-1})$ . Moreover, because  $\tau \in S(n-1)$  and  $\mu \vdash n$ , representation  $\phi^{\mu}(\tau^{-1})$  has to  $\tau$  be block diagonal in  $\alpha \vdash n-1$ :

$$\phi^{\mu}(\tau^{-1}) = \bigoplus_{\alpha \in \mu} \phi^{\alpha}(\tau^{-1}), \tag{51}$$

where the symbol  $\alpha \in \mu$  denotes all Young frames obtained from  $\mu$  by removing a single box. Denoting by  $\mathbf{1}_{\mu}, \mathbf{1}_{\alpha}$  identities on irreps  $\mu \vdash n$  and  $\alpha \vdash n - 1$  respectively, for which  $\alpha \in \mu$  holds, we write  $\mathbf{1}_{\mu} = \bigoplus_{\alpha \in \mu} \mathbf{1}_{\alpha}$ . Applying this identity together with (51) to equation (50) we rewrite as

$$P_{\mu} = \frac{d_{\mu}}{n!} \sum_{a=1}^{n} V_{a,n} \sum_{\alpha \in \mu} \sum_{\tau \in S(n-1)} \operatorname{Tr}\left( \left[ \phi^{\mu}(a,n) \right]_{\alpha} \phi^{\alpha}(\tau^{-1}) \right) V_{\tau},$$
(52)

where  $[\phi^{\mu}(a, n)]_{\alpha} \equiv \mathbf{1}_{\alpha}\phi^{\mu}(a, n)\mathbf{1}_{\alpha}$ . Using Fact 2 to expression (52) we have

$$P_{\mu} = \frac{1}{n} \sum_{a=1}^{n} V_{a,n} \sum_{\alpha \in \mu} \frac{d_{\mu}}{d_{\alpha}} \left( \mathbf{1}_{\alpha}^{U} \otimes [\phi^{\mu}(a,n)]_{\alpha} \right).$$
(53)

Having (53) and definitions (45), together with (49), and  $\operatorname{Tr}_n V(a, n) = d^{\delta_{a,n}} \mathbf{1}_{1...n-1}$ , we write

$$\operatorname{Tr}_{n-1} P_{\mu} = \frac{1}{n} \sum_{\beta \in \mu} \frac{d_{\mu}}{d_{\beta}} \mathbf{1}_{\beta}^{U} \otimes \left[ \sum_{a=1}^{n} \widetilde{\phi}^{\mu}(a, n) \right]_{\beta} = \frac{1}{n} \sum_{\beta \in \mu} \frac{d_{\mu}}{d_{\beta}} x_{\beta}^{\mu} \mathbf{1}_{\beta}^{U} \otimes \mathbf{1}_{\beta}^{S} = \frac{1}{n} \sum_{\beta \in \mu} \frac{d_{\mu}}{d_{\beta}} x_{\beta}^{\mu} P_{\beta}.$$
(54)

Using (99), property  $\text{Tr}(P_{\beta}P_{\mu}) = m_{\mu}d_{\beta}$ , for  $\mu \vdash n, \beta \vdash n-1$ , and

$$\operatorname{Tr} \left( P_{\beta} P_{\mu} \right) = \frac{1}{n} \sum_{\gamma \in \mu} \frac{d_{\mu}}{d_{\gamma}} x_{\gamma}^{\mu} \operatorname{Tr} \left( P_{\gamma} P_{\beta} \right)$$
  
$$= \frac{1}{n} \frac{d_{\mu}}{d_{\beta}} x_{\beta}^{\mu} d_{\beta} m_{\beta} = \frac{1}{n} d_{\mu} m_{\beta} x_{\beta}^{\mu}$$
(55)

we deduce that

$$x^{\mu}_{\beta} = n \frac{m_{\mu} d_{\beta}}{m_{\beta} d_{\mu}}.$$
(56)

This finishes the proof.

$$V^{(1)}(X \otimes \mathbf{1}_{n})V^{(1)} = \sum_{kl} \left( \mathbf{1}_{1,\dots,n-2} \otimes e_{kl}^{(n-1)} \otimes e_{kl}^{(n)} \right) \left( \sum_{ij} X_{ij}^{1,\dots,n-2} \otimes e_{ij}^{(n-1)} \otimes \mathbf{1}_{n} \right) \sum_{pq} \left( \mathbf{1}_{1,\dots,n-2} \otimes e_{pq}^{(n-1)} \otimes e_{pq}^{(n)} \right)$$
(41)  
$$= \sum_{kq} \left( \sum_{\substack{i_{1},\dots,i_{n-2},p, \\ j_{1},\dots,j_{n-2},p}} x_{i_{1}j_{1},\dots,i_{n-2}j_{n-2},pp} e_{i_{1}j_{1}}^{(1)} \otimes \dots \otimes e_{i_{n-2}j_{n-2}}^{(n-2)} \right) \otimes e_{kq}^{(n-1)} \otimes e_{kq}^{(n)}$$
$$= \sum_{kq} \left( \sum_{p} X_{pp}^{1,\dots,n-2} \right) \otimes e_{kq}^{(n-1)} \otimes e_{kq}^{(n)}$$
$$= \operatorname{Tr}_{n-1}(X) V^{(1)}$$

B. A new summation rule for irreducible representations and PRIR (Partially Reduced Irreducible Representation) notation

Let  $H \subset S(n)$  be an arbitrary subgroup of S(n) with transversal  $T = \{\tau_k : k = 1, ..., \frac{n!}{|H|}\}$ , i.e. we have

$$S(n) = \bigcup_{k=1}^{\frac{n!}{|H|}} \tau_k H.$$
(57)

For the further purposes, we can also introduce simplified notation

**Notation 4.** Let us take  $\mu \vdash n$  and  $\alpha \vdash n - k$ , for k < n. By index  $r_{\mu/\alpha}$  we denote a path on Young's lattice from diagram  $\mu$  to  $\alpha$ . This path is uniquely determined by choosing a chain of covered young frames from  $\mu$  to  $\alpha$ , differencing by one box in each step:

$$r_{\mu/\alpha} = (\mu, \mu_{n-1}, \dots, \mu_{n-k+1}, \alpha)$$
 (58)

and

$$\mu \ni \mu_{n-1} \ni \dots \ni \mu_{n-k+1} \ni \alpha.$$
(59)

Consider an arbitrary unitary irrep  $\phi^{\mu}$  of S(n). It can be always unitarily transformed to *PRIR*  $\phi^{\mu}_{R}$ , such that

$$\forall \varkappa \in H \quad \phi_R^{\mu}(\varkappa) = \bigoplus_{\alpha \in \mu, r_{\mu/\alpha}} \varphi^{\alpha, r_{\mu/\alpha}}(\varkappa) \equiv \bigoplus_{r_{\mu/\alpha}} \varphi^{r_{\mu/\alpha}}(\varkappa),$$
(60)

where  $\alpha$  labels the type of if a irrep of H and  $r_{\mu/\alpha}$  denotes path on Young's lattice from  $\mu$  to  $\alpha$ . It means that element  $\varphi^{\alpha,r_{\mu/\alpha}}(\varkappa)$  is repeated  $|\mathcal{R}_{\mu/\alpha}| = m_{\mu/\alpha}$  times, where  $\mathcal{R}_{\mu/\alpha}$  is the set composed of all paths  $r_{\mu/\alpha}$  from  $\mu$  to  $\alpha$ . Whenever it is clear from the context we write just  $\varphi^{\alpha}(\varkappa)$  instead of  $\varphi^{\alpha,r_{\mu/\alpha}}(\varkappa)$ . Diagonal blocks in the decomposition (60) are labelled and in fact ordered by the two indices  $\alpha, r_{\mu/\alpha}$ . The *PRIR* representation of S(n), reduced to the subgroup *H*, has block diagonal form of completely reduced representation, which in matrix notation takes the form

$$\forall \varkappa \in H \quad (\phi_R^{\mu})_{i_{\alpha} j_{\beta}}^{r_{\mu/\alpha} \widetilde{r}_{\mu/\beta}}(\varkappa) = \delta^{r_{\mu/\alpha} \widetilde{r}_{\mu/\beta}} \varphi_{i_{\alpha} j_{\alpha}}^{\alpha}(\varkappa), \qquad (61)$$

where indices  $i_{\alpha}, j_{\alpha}$  run from 1 to dimension of the irrep  $\alpha$ , and  $\delta^{r_{\mu/\alpha}\tilde{r}_{\mu/\beta}} = \delta^{\mu\nu}\delta^{\mu_{n-1}\nu_{n-1}}\cdots\delta^{\alpha\beta}$ . The above considerations allow us to introduce the following

**Notation 5.** Every basis index  $i_{\mu}$ , where  $\mu \vdash n$ , can be written uniquely using a path on Young's lattice as

$$i_{\mu} \equiv (r_{\mu/\alpha}, l_{\alpha}), \qquad \alpha \in \mu,$$
 (62)

and  $l_{\alpha}$  denotes now index running only within the range of the irrep  $\alpha$ . The indices  $i_{\mu}$ ,  $l_{\alpha}$  are of the same type as  $r_{\mu/\alpha}$ , but with trivial last element, i.e. a single box Young diagram. Equation (62) defines the division of the chosen path on Young's lattice from diagram  $\mu$  to single box diagram, through a diagram  $\alpha$ . By writing  $\delta_{i_{\mu}j_{\nu}}$ , where  $\mu \vdash n$  and  $\alpha \vdash n - k$ , we understand the following

$$\delta_{i_{\mu}j_{\nu}} = \delta^{r_{\mu/\alpha}\bar{r}_{\nu/\beta}}\delta_{l_{\alpha}l_{\beta}'} = \delta_{\mu\nu}\delta_{\mu_{n-1}\nu_{n-1}}\cdots\delta_{\mu_{n-k+1}\nu_{n-k+1}}\delta_{\alpha\beta}\delta_{l_{\alpha}l_{\beta}'}.$$
(63)

Similarly as in [19], [21] the block structure of this reduced representation allows to introduce such a block indexation for the *PRIR*  $\phi_R^\mu$  of *S*(*n*), which gives

$$\forall \sigma \in S(n) \quad \phi_R^{\mu}(\sigma) = \left( (\phi_R^{\mu})_{i_{\alpha} \quad j_{\beta}}^{r_{\mu/\alpha}, \tilde{r}_{\mu/\beta}}(\sigma) \right), \qquad (64)$$

where the matrices on the diagonal  $(\phi_R^{\mu})_{i_{\alpha}}^{r_{\mu/\alpha},\tilde{r}_{\mu/\beta}}(\sigma)$  are of dimension of corresponding irrep  $\varphi^{\alpha}$  of S(n-1). The off diagonal blocks need not to be square.

Now we formulate the main result of this subsection, the generalized version of *PRIR* orthogonality relation. The following proposition plays the central role in investigating matrix elements of MPBT operator in irreducible orthonormal operator basis presented later in Section VII.

**Proposition 6.** Let  $H \subset S(n)$  be an arbitrary subgroup of S(n) with transversal  $T = \{\tau_k : k = 1, ..., \frac{n!}{|H|}\}$ , In the PRIR notation  $\phi_R^{\mu}$  of S(n) satisfy the following bilinear sum rule

n

$$\sum_{k=1}^{|\widetilde{H}|} \sum_{k_{\beta}=1}^{d_{\beta}} (\phi_{R}^{\mu})_{i_{\alpha}}^{r_{\mu/\alpha},\widetilde{r}_{\mu/\beta}} (\tau_{k}^{-1}) (\phi_{R}^{\nu})_{k_{\beta}}^{r_{\nu/\beta},\widetilde{r}_{\nu/\gamma}} (\tau_{k})$$
$$= \frac{n!}{|H|} \frac{d_{\beta}}{d_{\mu}} \delta^{r_{\mu/\alpha},\widetilde{r}_{\nu/\gamma}} \delta_{i_{\alpha}j_{\gamma}}, \forall r_{\mu/\alpha}, \widetilde{r}_{\mu/\beta}, r_{\nu/\beta}, \widetilde{r}_{\nu/\gamma} \quad (65)$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$  are irreps of H contained in the irrep  $\mu$  of S(n), and |H| denotes cardinality of the subgroup H.

*Proof.* The proof is based on the classical orthogonality relations for irreps, which in PRIR notation takes a form

$$\sum_{g \in G} (\phi_R^{\mu})_{i_{\alpha}}^{r_{\mu/\alpha}\tilde{r}_{\mu/\beta}} (g^{-1}) (\phi_R^{\nu})_{k_{\beta}}^{r_{\nu/\beta}\tilde{r}_{\nu/\gamma}} (g) = \frac{|G|}{d_{\mu}} \delta^{\tilde{r}_{\mu/\beta}r_{\nu/\beta}} \delta_{i_{\alpha}j_{\gamma}},$$
(66)

where |G| denotes cardianlity of the group *G*. It means, that even if  $\alpha = \gamma$ , i.e. these representations are of the same type, but  $\tilde{r}_{\mu/\beta} \neq r_{\nu/\beta}$ , the *RHS* of the above equation is equal to zero. Next part of the proof follows from the proof of Proposition 29 in paper [21].

## C. Properties of irreducible operator basis and Young projectors under partial trace

For further purposes, namely for effective computations of performance of our teleportation schemes, we prove here how irreducible operator basis given in (32) or (31), and Young projectors from (36) behave under taking a partial trace over last k systems. Our formulas are generalisations of attempts to similar problem made in [42]. We start considerations from calculating the partial trace from operators (32) over last system. In all lemmas presented below we use PRIR representation described in Subsection VI-B.

**Lemma 7.** For irreducible operator basis  $E_{kl}^{\mu}$ , where  $\mu \vdash n$ , introduced in (32), the partial trace over last system equals to

$$\operatorname{Tr}_{n} E_{i_{\beta}j_{\beta'}}^{\beta\beta'}(\mu) = \sum_{\alpha \in \mu} \frac{m_{\mu}}{m_{\alpha}} E_{i_{\alpha}j_{\alpha}}^{\alpha} \delta_{\alpha\beta} \delta_{\alpha\beta'} = \frac{m_{\mu}}{m_{\beta}} E_{i_{\beta}j_{\beta}}^{\beta} \delta_{\beta\beta'}.$$
 (67)

*Proof.* Similarly as it was done for Young projectors  $P_{\mu}$  in (50), we can rewrite  $E_{ij}^{\mu}$  as

$$E_{kl}^{\mu} = \frac{d_{\mu}}{n!} \sum_{\sigma \in S(n)} \phi_{lk}^{\mu}(\sigma^{-1}) V_{\sigma}$$
  
=  $\frac{d_{\mu}}{n!} \sum_{a=1}^{n} \sum_{\tau \in S(n-1)} \phi_{lk}^{\mu}((a,n) \circ \tau^{-1}) V_{a,n} V_{\tau}.$  (68)

Observing that  $\phi_{lk}^{\mu}(\sigma^{-1}) = \text{Tr}(|k\rangle\langle l|\phi^{\mu}(\sigma^{-1}))$ , where  $|k\rangle, |l\rangle$  are basis vector in irrep  $\mu$ , we can write in PRIR notation  $k = k_{\mu} = (\beta, i_{\beta})$  and  $l = l_{\mu} = (\beta', j_{\beta'})$  having

$$E_{i_{\beta}j_{\beta'}}^{\beta\beta'}(\mu) = \frac{d_{\mu}}{n!} \sum_{a=1}^{n} V_{a,n} \sum_{\tau \in S(n-1)} \operatorname{Tr}\left[ |\beta, i_{\beta}\rangle \langle \beta', j_{\beta'} | \phi^{\mu}(a, n) \phi^{\mu}(\tau^{-1}) \right] V_{\tau}.$$
 (69)

Since  $\tau \in S(n-1)$  and  $\mu \vdash n$ , we can apply directly decomposition from (60) writing

$$E_{i_{\beta}j_{\beta'}}^{\beta\beta'}(\mu) = \frac{d_{\mu}}{n!} \sum_{a=1}^{n} V_{a,n} \sum_{\alpha \in \mu} \sum_{\tau \in S(n-1)} \operatorname{Tr}\left(\left[|\beta, i_{\beta}\rangle\langle\beta', j_{\beta'}|\phi^{\mu}(a, n)\right]_{\alpha} \phi^{\alpha}(\tau^{-1})\right) V_{\tau}.$$
 (70)

In the above, by  $\left[ |\beta, i_{\beta}\rangle \langle \beta', j_{\beta'} | \phi^{\mu}(a, n) \right]_{\alpha}$  we denote the restriction to irrep  $\alpha$ . Applying Fact 2 to the above expression we write

$$E_{i_{\beta}j_{\beta'}}^{\beta\beta'}(\mu) = \frac{d_{\mu}}{n!} \sum_{a=1}^{n} V_{a,n} \sum_{\alpha \in \mu} \frac{(n-1)!}{d_{\alpha}} \mathbf{1}_{\alpha}^{U} \otimes \left[ |\beta, i_{\beta}\rangle \langle \beta', j_{\beta'}| \phi^{\mu}(a,n) \right]_{\alpha}.$$
(71)

Taking the partial trace over last system, and having in mind the definition of  $\tilde{\phi}^{\mu}(a, n-1)$  from (45), we have:

$$\begin{aligned} \operatorname{Tr}_{n} E_{i_{\beta}j_{\beta'}}^{\beta\beta'}(\mu) \\ &= \frac{d_{\mu}}{n!} \operatorname{Tr}_{n} \left( \sum_{a=1}^{n} V_{a,n} \right) \sum_{\alpha \in \mu} \frac{(n-1)!}{d_{\alpha}} \mathbf{1}_{\alpha}^{U} \otimes \left[ |\beta, i_{\beta}\rangle \langle \beta', j_{\beta'}| \phi^{\mu}(a,n) \right]_{\alpha} \\ &= \frac{d_{\mu}}{n!} \sum_{a=1}^{n} \left( \sum_{\alpha \in \mu} \frac{(n-1)!}{d_{\alpha}} \mathbf{1}_{\alpha}^{U} \otimes \left[ |\beta, i_{\beta}\rangle \langle \beta', j_{\beta'}| \tilde{\phi}^{\mu}(a,n) \right]_{\alpha} \right). \end{aligned}$$

$$(72)$$

From the proof of Lemma 3 we know that the object  $[\sum_{a=1}^{n} \tilde{\phi}^{\mu}(a, n)]_{\beta}$  is invariant with respect to S(n-1). Together with property  $\mathbf{1}_{\mu}^{\alpha} |\beta, i_{\beta}\rangle\langle\beta', j_{\beta'}|\mathbf{1}_{\mu}^{\alpha} = \delta_{\alpha\beta}\delta_{\alpha,\beta'}|\alpha, i_{\alpha}\rangle\langle\alpha, j_{\alpha}|$ , we have

$$\begin{aligned} \operatorname{Ir}_{n} E_{i_{\beta}j_{\beta'}}^{\beta\beta'}(\mu) \\ &= \frac{d_{\mu}}{n!} \sum_{\alpha \in \mu} \frac{(n-1)!}{d_{\alpha}} x_{\alpha}^{\mu} \mathbf{1}_{\alpha}^{U} \otimes |\alpha, i_{\alpha}\rangle \langle \alpha, j_{\alpha}|\delta_{\alpha\beta}\delta_{\alpha\beta'} \\ &= \frac{1}{n} \frac{d_{\mu}}{d_{\beta}} x_{\beta}^{\mu} \mathbf{1}_{\alpha}^{U} \otimes |\beta, i_{\beta}\rangle \langle \beta, j_{\beta}|\delta_{\beta\beta'} = \frac{m_{\mu}}{m_{\beta}} E_{i_{\beta}j_{\beta}}^{\beta}\delta_{\beta\beta'}. \end{aligned}$$
(73)

In the last step we use explicit form of coefficients  $x^{\mu}_{\beta}$  given in Lemma 3 and expression (31).

**Corollary 8.** From Lemma 9 we see that taking a partial trace over n-th subsystem we destroys all the coherences between block labelled by different  $\beta \vdash n-1$ .

For further purpose of having explicit connection with the structure of multi-port teleportation scheme, let us assume that now  $\mu \vdash n - k$ , such that 2k < n. Having that and extended notion of *PRIR*, we are in position to present the second main result of this section.

**Lemma 9.** For basis operators  $E_{kl}^{\mu}$  in the irreducible representation labelled by  $\mu \vdash n - k$ , we have the following equality:

$$\operatorname{Tr}_{(k)} E_{i_{\beta}}^{r_{\mu/\beta}} {}^{\tilde{r}_{\mu/\beta'}}_{j_{\beta'}} = \frac{m_{\mu}}{m_{\beta}} E_{i_{\beta}j_{\beta}}^{\beta} \delta_{r_{\mu/\beta}\tilde{r}_{\mu/\beta'}}$$
(74)

where we use simplified notation  $\operatorname{Tr}_{(k)} = \operatorname{Tr}_{n-2k+1,\dots,n-k}$ .

*Proof.* To prove the above statement we use iteratively Lemma 7. Let us write explicitly indices  $k = k_{\mu}$ ,  $l = l_{\mu}$  in PRIR notation:

$$\begin{aligned} \kappa_{\mu} &= (\mu_{n-k-1}, r_{\mu_{n-k-1}}) = (\mu_{n-k-1}, \dots, \mu_{n-2k+1}, \mu_{n-2k}, \iota_{\mu_{n-2k}}) \\ &= (\mu_{n-k-1}, \dots, \mu_{n-2k+1}, \beta, i_{\beta}), \\ l_{\mu} &= (\mu'_{n-k-1}, s_{\mu'_{n-k-1}}) = (\mu'_{n-k-1}, \dots, \mu'_{n-2k+1}, \mu'_{n-2k}, \iota_{\mu'_{n-2k}}) \\ &= (\mu'_{n-k-1}, \dots, \mu'_{n-2k+1}, \beta', j_{\beta'}), \end{aligned}$$

$$(75)$$

where we put  $\beta = \mu_{n-2k}$ ,  $\beta' = \mu'_{n-2k}$  for simpler notation. Each lower index denotes a proper layer on the reduced Young's lattice, starting from the highest layer labelled by the number n - k. In the first step we compute the partial trace over (n - k)-th system getting

$$\operatorname{Tr}_{n-k} E_{r_{\mu_{n-k-1}}}^{\mu_{n-k-1}} \sum_{\mu'_{n-k-1}}^{\mu'_{n-k-1}} (\mu_{n-k}) = \frac{m_{\mu_{n-k}}}{m_{\mu_{n-k-1}}} E_{r_{\mu_{n-k-1}}}^{\mu_{n-k-1}} \delta_{\mu_{n-k-1}} \mu'_{n-k-1}.$$
(76)

This procedure reduced paths in (75) to

$$r_{\mu_{n-k-1}} = (\mu_{n-k-2}, q_{\mu_{n-k-2}}) = (\mu_{n-k-2}, \mu_{n-k-3}, \dots, \mu_{n-2k+1}, \beta, i_{\beta}),$$
(77)

$$s_{\mu_{n-k-1}} = (\mu'_{n-k-2}, p_{\mu'_{n-k-2}}) = (\mu'_{n-k-2}, \mu'_{n-k-3}, \dots, \mu'_{n-2k+1}, \beta', j_{\beta'}),$$
(78)

where  $\beta = \mu_{n-2k}$ ,  $\beta' = \mu'_{\mu-2k}$ . Now computing the trace from (76) over (n - k - 1)-th particle we write

$$\delta_{\mu_{n-k-1}\mu'_{n-k-1}} \frac{m_{\mu_{n-k}}}{m_{\mu_{n-k-1}}} \operatorname{Tr}_{n-k-1} E_{q_{\mu_{n-k-2}}p_{\mu'_{n-k-2}}}^{\mu_{n-k-2}}(\mu_{n-k-1}) = \delta_{\mu_{n-k-1}\mu'_{n-k-1}} \delta_{\mu_{n-k-2}\mu'_{n-k-2}} \frac{m_{\mu_{n-k}}}{m_{\mu_{n-k-1}}} \frac{m_{\mu_{n-k-2}}}{m_{\mu_{n-k-2}}} E_{q_{\mu_{n-k-2}}p_{\mu_{n-k-2}}}^{\mu_{n-k-2}}(79)$$

Continuing the above procedure, up to last system in  $\text{Tr}_{(k)}$  we obtain expression (80), displayed at the top of the following page since in the last line we used definition of  $r_{\mu/\alpha}$  and suppressed indices labelling layers on reduced Bratelli diagram. This finishes the proof.  $\Box$ 

Then Lemma 9 implies the following statement about the Young projector:

**Corollary 10.** Let  $P_{\mu}$  be a Young projector on irrep labelled by  $\mu \vdash n - k$ , then

$$\operatorname{Tr}_{(k)} P_{\mu} = \sum_{\beta \in \mu} m_{\mu/\beta} \frac{m_{\mu}}{m_{\beta}} P_{\beta}$$
(81)

where we use simplified notation  $\operatorname{Tr}_{(k)} = \operatorname{Tr}_{n-2k+1,\dots,n-k}$ .

Indeed, knowing that  $P_{\mu} = \sum_{k} E_{ii}^{\mu}$ , we write in PRIR basis

$$\operatorname{Tr}_{(k)} P_{\mu} = \sum_{k_{\mu}} E_{k_{\mu}k_{\mu}}^{\mu} = \sum_{\beta \in \mu} \sum_{r_{\mu/\beta}} \sum_{i_{\beta}} \operatorname{Tr}_{(k)} E_{i_{\beta}}^{r_{\mu/\beta}r_{\mu/\beta}}$$
$$= \sum_{\beta \in \mu} \sum_{r_{\mu/\beta}} \sum_{i_{\beta}} \frac{m_{\mu}}{m_{\beta}} E_{i_{\beta}i_{\beta}}^{\beta} = \sum_{\beta \in \mu} \sum_{r_{\mu/\beta}} \sum_{i_{\beta}} \frac{m_{\mu}}{m_{\beta}} E_{i_{\beta}i_{\beta}}^{\beta} \quad (82)$$
$$= \sum_{\beta \in \mu} \sum_{r_{\mu/\beta}} \frac{m_{\mu}}{m_{\beta}} P_{\beta} = \sum_{\beta \in \mu} m_{\mu/\beta} \frac{m_{\mu}}{m_{\beta}} P_{\beta}.$$

## VII. The Commutant Structure of $U^{\otimes (n-k)} \otimes \overline{U}^{\otimes k}$ Transformations and MPBT operator

) In this section we deliver an orthonormal basis for the commutant of  $U^{\otimes (n-k)} \otimes \overline{U}^{\otimes k}$ , or equivalently for ) the algebra  $\mathcal{A}_n^{(k)}(d)$ . Being more strict, we introduce an irreducible basis for an two-sided ideal  $\mathcal{M}$  generated by the element  $V^{(k)}$  and elements of the algebra  $\mathcal{A}_n^{(k)}(d)$ :

$$\mathcal{M} = \{ V_{\tau} V^{(k)} V_{\tau'}^{\dagger} \mid \tau, \tau' \in S(n-k) \}.$$
(83)

For our problem full description of  $\mathcal{M}$ , together with irreducible representation is enough since all basic objects describing MPBT scheme belong to this ideal, see for example definition of MPBT operator from (21). In the most general case the algebra  $\mathcal{A}_n^{(k)}(d)$  contains also two-sided ideals generated by the elements  $V^{(k')}$ , for k' < k, and elements of the algebra  $\mathcal{A}_n^{(k)}(d)$ . We have the following chain of inclusions

$$\mathcal{M} \equiv \mathcal{M}^{(k)} \subset \mathcal{M}^{(k-1)} \subset \dots \subset \mathcal{M}^{(1)} \subset \mathcal{M}^{(0)} \equiv \mathcal{A}_n^{(k)}(d).$$
(84)

The irreducible basis fir the ideals with k' < k will be studied elsewhere, since we do not use objects from the outside of the ideal  $\mathcal{M}$ . In Figure 6 we present nested structure of  $\mathcal{A}_5^{(2)}(d)$  for d > 3, together with labelling subsequent blocks within them. Having expressions for partial trace over an arbitrary number of particles from irreducible basis operators of the symmetric group we are in the position to formulate the main result, namely we have:

**Theorem 11.** The orthonormal operator basis of the commutant of  $U^{\otimes (n-k)} \otimes \overline{U}^{\otimes k}$  in the maximal ideal  $\mathcal{M}$  is given by the following set of operators

$$F_{i_{\mu}}^{r_{\mu/\alpha}r_{\nu/\alpha}} = \frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}} E_{i_{\mu}} {}^{r_{\mu/\alpha}}_{1_{\alpha}} V^{(k)} E_{1_{\alpha}}^{r_{\nu/\alpha}}_{1_{\alpha}} {}^{j_{\nu}}$$
(85)

satisfying the following composition rule

$$F_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}}F_{k_{\mu'} l_{\nu'}}^{r_{\mu'/\beta}r_{\nu'/\beta}} = \delta^{r_{\nu/\alpha}r_{\mu'/\beta}}\delta_{j_{\nu}k_{\mu'}}F_{i_{\mu} l_{\nu'}}^{r_{\mu/\alpha}r_{\nu'/\alpha}}$$
(86)

where  $m_{\mu}$ ,  $m_{\nu}$  and  $m_{\alpha}$  are multiplicities of respective irreps of S(n-k) and S(n-2k) in the Schur-Weyl duality.

*Proof.* The proof contains two main steps:

Showing that operators are orthonormal, i.e.

$$F_{i_{\mu}j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}}F_{k_{\mu'}l_{\nu'}}^{r_{\mu'/\beta}r_{\nu'/\beta}} = \delta^{r_{\nu/\alpha}r_{\mu'/\beta}}\delta_{j_{\nu}k_{\mu'}}F_{i_{\mu}l_{\nu'}}^{r_{\mu/\alpha}r_{\nu'/\alpha}}.$$
 (87)

Indeed, writing explicitly the above composition and using orthogonality relation for operators  $E^{\mu}_{i_{\mu}j_{\mu}}$ , we have

$$\delta_{\mu_{n-k-1}\mu'_{n-k-1}}\delta_{\mu_{n-k-2}\mu'_{n-k-2}} \times \cdots \times \delta_{\mu_{n-2k+1}\mu'_{n-2k+1}}\delta_{\mu_{n-2k}\mu'_{n-2k}}\frac{m_{\mu_{n-k}}}{m_{\mu_{n-k-1}}}\frac{m_{\mu_{n-k-1}}}{m_{\mu_{n-k-2}}} \times \cdots \times \frac{m_{\mu_{n-2k+1}}}{m_{\mu_{n-2k}}}E^{\mu_{n-2k}}_{i_{\mu_{n-2k}}j_{\mu_{n-2k}}}E^{\mu_{n-2k}}_{i_{\mu_{n-2k}}j_{\mu_{n-2k}}}E^{\mu_{n-2k}}_{i_{\mu_{n-2k}}j_{\mu_{n-2k}}}=\delta^{r_{\mu/\beta}\tilde{r}_{\mu/\beta'}}\frac{m_{\mu}}{m_{\beta}}E^{\beta}_{i_{\beta}j_{\beta'}}$$
(80)



Fig. 6. Graphic presents the interior structure of the algebra  $\mathcal{A}_6^{(2)}(d)$ , with the nested structure of the ideals  $\mathcal{M}^{(0)}, \mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ , for  $d \geq 4$ (only with this requirement all the Young frames occur in the decomposition). In particular, we focus on the interior block structure of the ideal  $\mathcal{M}^{(2)}$ , on which objects describing multi-port teleportation schemes are defined. The middle figure represents the process of the induction by adding two boxes to two allowed starting Young frames which are (2) and (1,1). In this case we have nested structure of three layers. The most right figure presents process of the reduction from irreps labelled by Young frames of 4 boxes to irreps labelled to Young frames of two boxes. We present here the process of the reduction for the two most left upper blocks.

$$F_{i_{\mu}}^{r_{\mu},\alpha r_{\nu},\alpha}F_{k_{\mu'}}^{r_{\mu'},\beta r_{\nu'},\beta} = \frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}} \frac{m_{\beta}}{\sqrt{m_{\mu'}m_{\nu'}}} E_{i_{\mu}}^{r_{\mu/\alpha}}V^{(k)}E_{1_{\alpha}}^{r_{\nu/\alpha}} _{j_{\nu}} \\ = \delta^{\nu\mu'}\delta_{j_{\nu}k_{\mu'}}\frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}} \frac{m_{\beta}}{\sqrt{m_{\mu'}m_{\nu'}}} E_{i_{\mu}}^{r_{\mu'},\beta}V^{(k)}E_{1_{\beta}}^{r_{\nu'},\beta} _{l_{\nu'}} \\ \times E_{1_{\alpha}}^{r_{\nu'},\alpha}r_{\mu',\beta}^{r_{\mu'},\beta}V^{(k)}E_{1_{\beta}}^{r_{\nu'},\beta} _{l_{\nu'}}. (88)$$

Now, applying Fact 1 to operator  $E_{1_{\alpha} \ 1_{\beta}}^{r_{\nu/\alpha}r_{\nu'/\beta}}$ , together with Lemma 9, we reduce to

$$F_{i_{\mu}}^{r_{\mu}\alpha r_{\nu'\alpha}}F_{k_{\mu'}}^{r_{\mu'}\beta r_{\nu'}\beta} = \delta^{\nu\mu'}\delta^{\alpha\beta}\delta^{r_{\nu'\alpha}r_{\mu'}\beta}\delta_{j_{\nu}k_{\mu'}}\frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\mu'}}}\frac{m_{\alpha}}{\sqrt{m_{\mu'}m_{\nu'}}}\frac{m_{\mu'}}{m_{\alpha}} \\ = \delta^{r_{\nu'\alpha}r_{\mu'}\beta}\delta_{j_{\nu}k_{\mu'}} \frac{m_{\alpha}}{m_{\alpha}}\frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu'}}} \\ \times E_{i_{\mu}}\frac{r_{\mu'\alpha}}{r_{\alpha}}E_{1_{\alpha}1_{\alpha}}^{\alpha}V^{(k)}E_{1_{\alpha}}^{r_{\nu'}\alpha}l_{\nu'}}.$$
(89)

Showing that element V<sup>(k)</sup> generating the ideal *M*, see (83) can be expressed as a linear combination of basis elements F<sup>rµ/α<sup>r</sup>ν/α</sup><sub>iµ</sub> j<sub>ν</sub>. Indeed, we have

$$V^{(k)} = \sum_{\mu,\nu \vdash n-k} P_{\mu} V^{(k)} P_{\nu} = \sum_{\mu,\nu \vdash n-k} \sum_{i_{\mu},j_{\nu}} E^{\mu}_{i_{\mu}i_{\mu}} V^{(k)} E^{\nu}_{j_{\nu}j_{\nu}},$$
(90)

since  $\mathbf{1} = \sum_{\mu} P_{\mu}$  together with (36). Writing indices  $i_{\mu}, j_{\nu}$  in PRIR notation, according to Notation 5 we get

$$V^{(k)} = \sum_{\mu,\nu} \sum_{r_{\mu/\alpha},\widetilde{r}_{\nu/\beta}} \sum_{l_{\alpha},l_{\beta}'} E_{l_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}} V^{(k)} E_{l_{\beta}'}^{r_{\nu/\beta}r_{\nu/\beta}} \\ = \sum_{\mu,\nu} \sum_{r_{\mu/\alpha},\widetilde{r}_{\nu/\beta}} \sum_{l_{\alpha},l_{\beta}'} E_{l_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}} E_{1_{\alpha}l_{\alpha}}^{\alpha} V^{(k)} E_{l_{\beta}'1_{\beta}}^{\beta} E_{1_{\beta}}^{\widetilde{r}_{\nu/\beta}\widetilde{r}_{\nu/\beta}}.$$

$$\tag{91}$$

Having  $[E_{1_{\alpha}l_{\alpha}}^{\alpha}, V^{(k)}] = [E_{l'_{\beta}1_{\beta}}^{\beta}, V^{(k)}] = 0$  and or-thogonality relation  $E_{1_{\alpha}l_{\alpha}}^{\alpha}E_{l'_{\beta}1_{\beta}}^{\beta} = \delta^{\alpha\beta}\delta_{l_{\alpha}l'_{\beta}}E_{1_{\alpha}1_{\alpha}}^{\alpha}$  we reduce (91) to

$$V^{(k)} = \sum_{\mu,\nu} \sum_{r_{\mu/\alpha},\tilde{r}_{\nu/\alpha}} \sum_{l_{\alpha}} E_{l_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}} V^{(k)} E_{1_{\alpha}}^{\tilde{r}_{\nu/\alpha}} I_{\alpha}^{\tilde{r}_{\nu/\alpha}}$$

$$= \sum_{\mu,\nu} \sum_{r_{\mu/\alpha},\tilde{r}_{\nu/\alpha}} \sum_{l_{\alpha}} \frac{\sqrt{m_{\mu}m_{\nu}}}{m_{\alpha}} \left( \frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}} E_{l_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}} V^{(k)} E_{1_{\alpha}}^{\tilde{r}_{\nu/\alpha}} I_{\alpha}^{\tilde{r}_{\nu/\alpha}} \right)$$

$$= \sum_{\mu,\nu} \sum_{r_{\mu/\alpha},\tilde{r}_{\nu/\alpha}} \sum_{l_{\alpha}} \frac{\sqrt{m_{\mu}m_{\nu}}}{m_{\alpha}} F_{l_{\alpha}} \frac{r_{\mu/\alpha}}{r_{\mu/\alpha}} \tilde{r}_{\nu/\alpha}} I_{\alpha}.$$
(92)

In the above we use representation of  $F_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}}$  in full PRIR basis:

$$F_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}} \rightarrow F_{k_{\beta} r_{\mu/\beta} r_{\nu/\gamma} k'_{\gamma}}^{r_{\mu/\alpha} r_{\nu/\alpha}}$$
(93)

since  $i_{\mu} = (r_{\mu/\beta}, k_{\beta})$  and  $j_{\nu} = (r_{\nu/\gamma}, k'_{\gamma})$ . This finishes the proof.

Next we focus on the relations analogous to (34) for the basis elements  $F_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}}$  and operators  $V^{(k)}$ ,  $V_{\tau}$ , where Finally observing that  $E_{i_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}} E_{1_{\alpha}1_{\alpha}}^{\alpha} = E_{i_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}}$  we get  $\tau \in S_{n,k} \equiv \frac{S(n-k)}{S(n-2k)}$ . To have all required tools let us first rewrite expressions from (34) in PRIR notation, but for a specific choice of indices and partitions  $\mu \vdash n - k$  and to (99) we write:  $\alpha \vdash n - 2k$ :

$$\forall \tau \in S(n-k) \qquad V_{\tau} E_{i_{\mu}}^{r_{\mu/\alpha}} = \sum_{l_{\mu}} \phi_{l_{\mu}i_{\mu}}^{\mu}(\tau) E_{l_{\mu}}^{r_{\mu/\alpha}}_{1_{\alpha}}$$
  
$$\forall \tau \in S(n-k) \qquad E_{1_{\alpha}}^{r_{\nu/\alpha}}{}_{j_{\nu}} V_{\tau^{-1}} = \sum_{k_{\nu}} \phi_{j_{\nu}k_{\nu}}^{\nu}(\tau^{-1}) E_{1_{\alpha}}^{r_{\nu/\alpha}}{}_{k_{\nu}}$$
(94)

where  $\phi_{l_{\mu}i_{\mu}}^{\mu}(\tau), \phi_{j_{\nu}k_{\nu}}^{\nu}(\tau^{-1})$  are the matrix elements of  $V_{\tau}, V_{\tau^{-1}}$  in irreducible basis expressed in the PRIR notation, see (35) and Section VI. Having the above we are in position to prove the following

**Lemma 12.** Let us take basis operators for the ideal  $\mathcal{M}$  given through Theorem 11, together with (93). Then for the operator  $V^{(k)}$  defined in (37) and an arbitrary permutation operator  $V_{\tau}$ , for  $\tau \in S(n - k)$ , the following relations hold:

$$F_{k_{\beta}} r_{\mu/\beta} r_{\nu/\gamma} r_{\nu/\gamma} l_{\gamma} V^{(k)} = \sum_{\mu'} \sum_{r_{\mu'/\gamma}} \frac{\sqrt{m_{\nu} m_{\mu'}}}{m_{\gamma}} F_{k_{\beta}} r_{\mu/\beta} r_{\mu'/\gamma} r_{\mu'/\gamma} l_{\gamma} \delta^{r_{\nu/\alpha} r_{\nu/\gamma}}$$
(95)

and

$$F_{k_{\beta}} r_{\mu/\beta} r_{\nu/\gamma} r_{\nu/\gamma} l_{\gamma} V_{\tau} = \sum_{k_{\nu}} \phi_{j_{\nu}k_{\nu}}^{\nu}(\tau) F_{i_{\mu}}^{r_{\mu/\alpha}r_{\nu/\alpha}}$$
(96)

where  $\phi_{j_{\nu}k_{\nu}}^{\nu}(\tau)$  are the matrix elements of  $V_{\tau}$  in the irreducible basis expressed in the PRIR notation introduced in Section VI.

*Proof.* First let us calculate action of  $F_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}}$  on  $V^{(k)}$ . Using expression (93) we have

$$F_{k_{\beta}} {}^{r_{\mu/\alpha}}_{r_{\nu/\gamma}} {}^{r_{\nu/\alpha}}_{l_{\gamma}} V^{(k)} = \frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}} E^{r_{\mu/\beta}r_{\mu/\alpha}}_{k_{\beta}} V^{(k)} E^{r_{\nu/\alpha}r_{\nu/\gamma}}_{1_{\alpha}} V^{(k)}_{1_{\alpha}} {}^{r_{\nu/\gamma}}_{l_{\gamma}} V^{(k)}$$
$$= \sqrt{\frac{m_{\nu}}{m_{\mu}}} \delta^{r_{\nu/\alpha}r_{\nu/\gamma}} E^{r_{\mu/\beta}r_{\mu/\gamma}}_{k_{\beta}} E^{\gamma}_{1_{\gamma}} l_{\gamma}} V^{(k)},$$
(97)

where in the second equality we used Fact 1 and Lemma 9. Now decomposing identity acting on n - k systems in PRIR basis

$$\mathbf{1} = \sum_{\mu' \vdash n-k} P_{\mu'} = \sum_{\mu'} \sum_{r_{\mu'/\alpha'}} \sum_{s_{\alpha'}} E_{s_{\alpha'}}^{r_{\mu'/\alpha'}r_{\mu'/\alpha'}} \qquad \alpha' \vdash n - 2k,$$
(98)

and multiplying by it the right hand side of (97) we have

$$F_{k_{\beta}} r_{\mu/\beta} r_{\nu/\gamma} l_{\gamma} V^{(k)} = \sqrt{\frac{m_{\nu}}{m_{\mu}}} \sum_{\mu'} \sum_{r_{\mu'/\alpha'}} \sum_{s_{\alpha'}} E_{k_{\beta}}^{r_{\mu/\beta}r_{\mu/\gamma}} V^{(k)} E_{l_{\gamma}l_{\gamma}}^{\gamma} E_{s_{\alpha'}}^{r_{\mu'/\alpha'}r_{\mu'/\alpha'}} \delta^{r_{\nu/\alpha}r_{\nu/\gamma}}$$
(99)

since  $\begin{bmatrix} E_{1_{\gamma}l_{\gamma}}^{\gamma} V^{(k)} \end{bmatrix} = 0$ . Moreover we have  $E_{1_{\gamma}l_{\gamma}}^{\gamma} E_{s_{\alpha'}}^{r_{\mu'/\alpha'}r_{\mu'/\alpha'}} = \delta^{\alpha'\gamma} \delta_{s_{\alpha'}l_{\gamma}} E_{1_{\gamma}l_{\gamma}}^{r_{\mu'/\gamma}r_{\mu'/\gamma}}$ . Substituting

$$F_{k_{\beta}} r_{\mu/\beta} r_{\nu/\gamma} r_{\nu/\gamma} l_{\gamma} V^{(k)} = \sqrt{\frac{m_{\nu}}{m_{\mu}}} \sum_{\mu'} \sum_{r_{\mu'/\gamma}} E_{k_{\beta}}^{r_{\mu/\beta}r_{\mu/\gamma}} V^{(k)} E_{1_{\gamma}}^{r_{\mu'/\gamma}r_{\mu'/\gamma}} \delta^{r_{\nu/\alpha}r_{\nu/\gamma}}$$

$$= \sum_{\mu'} \sum_{r_{\mu'/\gamma}} \frac{\sqrt{m_{\nu}m_{\mu'}}}{m_{\gamma}} F_{k_{\beta}} r_{\mu/\beta} r_{\mu'/\gamma} l_{\gamma} \delta^{r_{\nu/\alpha}r_{\nu/\gamma}}.$$
(100)

This proves expression (95). To prove equation (96) we use directly (94) with (85):

$$F_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}}V_{\tau} = \frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}}E_{i_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}}V^{(k)}E_{1_{\alpha} j_{\nu}}^{r_{\nu/\alpha}}V_{\tau}$$
$$= \sum_{k_{\nu}}\phi_{j_{\nu}k_{\nu}}^{\nu}(\tau)\frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}}E_{i_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}}V^{(k)}E_{1_{\alpha} k_{\nu}}^{r_{\nu/\alpha}}$$
$$= \sum_{k_{\nu}}\phi_{j_{\nu}k_{\nu}}^{\nu}(\tau)E_{i_{\mu} k_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}}.$$
(101)

This finishes the proof.

Analogously we can evaluate expressions (95), (96) for action from the right-hand side. For the further purposes we write explicitly such action on  $V_{\tau}$ , for  $\tau \in S(n - k)$ :

$$V_{\tau}F_{i_{\mu}j_{\nu}j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}} = \sum_{k_{\mu}}\phi_{k_{\mu}i_{\mu}}^{\mu}(\tau)F_{k_{\mu}j_{\nu}j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}}.$$
 (102)

Using the second part of the proof of Theorem 11 we can formulate the following

**Lemma 13.** The operator  $V^{(k)}$  defined in (37) and an arbitrary permutation operator  $V_{\tau}$ , for  $\tau \in S(n-k)$  in the operator basis from Theorem 11 have matrix elements equal to:

$$\left(V^{(k)}\right)_{k\beta} \frac{r_{\mu/\alpha} r_{\nu/\alpha}}{r_{\mu/\beta} r_{\nu/\gamma} l_{\gamma}} = \delta_{k\beta} l_{\gamma} \delta^{r_{\mu/\alpha} r_{\mu/\beta}} \delta^{r_{\nu/\alpha} r_{\nu/\gamma}} \frac{\sqrt{m_{\mu} m_{\nu}}}{m_{\alpha}},$$

$$(103)$$

and

$$(V_{\tau})_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}} = \delta^{r_{\mu/\alpha}r_{\nu/\alpha}}\delta_{i_{\mu}j_{\nu}}\sqrt{\frac{m_{\mu}}{m_{\nu}}}\sum_{k_{\mu}}\phi^{\mu}_{k_{\mu}i_{\mu}}(\tau), \qquad (104)$$

where  $m_{\mu}, m_{\nu}, m_{\alpha}$  are multiplicities of respective irreducible representations in the Schur-Weyl duality, and  $\phi^{\mu}_{k_{\mu}i_{\mu}}(\tau)$  are the matrix elements of  $V_{\tau}$  in the irreducible basis expressed in the PRIR notation introduced in Section VI.

*Proof.* To prove the statement of the lemma we have to compute overlap of  $V^{(k)}$  with  $F_{i\mu}^{r_{\mu/\alpha}r_{\nu/\alpha}}$  written in PRIR basis:

$$\left( V^{(k)} \right)_{k_{\beta} r_{\mu/\beta} r_{\nu/\gamma} l_{\gamma}}^{r_{\mu/\alpha} r_{\nu/\alpha}} = \frac{1}{m_{\alpha}} \operatorname{Tr} \left[ V^{(k)} F_{k_{\beta} r_{\mu/\beta} r_{\nu/\gamma} l_{\gamma}}^{r_{\mu/\alpha} r_{\nu/\alpha}} \right]$$

$$= \frac{1}{\sqrt{m_{\mu}m_{\nu}}} \operatorname{Tr} \left[ V^{(k)} E_{k_{\beta} - 1_{\alpha}}^{r_{\mu/\beta} r_{\mu/\alpha}} V^{(k)} E_{1_{\alpha} - l_{\gamma}}^{r_{\nu/\alpha} r_{\nu/\gamma}} \right]$$

$$(105)$$

Applying Fact 1 and Lemma 9 we reduce to

$$\left( V^{(k)} \right)_{k_{\beta} r_{\mu/\beta} r_{\nu/\gamma} l_{\gamma}}^{r_{\mu/\alpha} r_{\nu/\alpha}} = \delta^{r_{\mu/\alpha} r_{\mu/\beta}} \frac{1}{\sqrt{m_{\mu} m_{\nu}}} \frac{m_{\mu}}{m_{\alpha}} \operatorname{Tr} \left[ E^{\alpha}_{k_{\alpha} l_{\alpha}} V^{(k)} E^{r_{\nu/\alpha}}_{l_{\alpha}} \right]_{\alpha}$$

$$= \delta^{r_{\mu/\alpha} r_{\mu/\beta}} \frac{1}{m_{\alpha}} \sqrt{\frac{m_{\mu}}{m_{\nu}}} \operatorname{Tr} \left[ E^{\alpha}_{k_{\alpha} l_{\alpha}} E^{r_{\nu/\alpha} r_{\nu/\gamma}}_{l_{\alpha} l_{\gamma}} \right],$$

$$(106)$$

since only the operator  $V^{(k)}$  acts non-trivially on last k systems. Now, let us observe that the operator  $E_{k_{\alpha}1_{\alpha}}^{\alpha}$  acts on first n - 2k systems, while the operator  $E_{1_{\beta} \ l_{\nu}}^{r_{\nu}/\beta}$  on n-k, so

In the second equality we applied Lemma 9, while in fourth we used property from (33). Now we evaluate the matrix elements of  $V_{\tau}$ . Using expression (102) we write

$$(V_{\tau})_{i_{\mu}}^{r_{\mu/\alpha}r_{\nu/\alpha}} = \frac{1}{m_{\alpha}} \operatorname{Tr} \left[ V_{\tau} F_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}} \right]$$

$$= \frac{1}{m_{\alpha}} \sum_{k_{\mu}} \phi_{k_{\mu}i_{\mu}}^{\mu}(\tau) \operatorname{Tr} \left[ F_{k_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}} \right]$$

$$= \frac{1}{\sqrt{m_{\mu}m_{\nu}}} \sum_{k_{\mu}} \phi_{k_{\mu}i_{\mu}}^{\mu}(\tau) \operatorname{Tr} \left[ E_{k_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}} V^{(k)} E_{1_{\alpha}}^{r_{\nu/\alpha}} \right]$$

$$= \frac{1}{\sqrt{m_{\mu}m_{\nu}}} \sum_{k_{\mu}} \phi_{k_{\mu}i_{\mu}}^{\mu}(\tau) \operatorname{Tr} \left[ E_{k_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}} E_{1_{\alpha}}^{r_{\nu/\alpha}} \right]$$

$$= \delta^{r_{\mu/\alpha}r_{\nu/\alpha}} \frac{1}{\sqrt{m_{\mu}m_{\nu}}} \sum_{k_{\mu}} \phi_{k_{\mu}i_{\mu}}^{\mu}(\tau) \operatorname{Tr} \left( E_{k_{\mu}j_{\mu}}^{\mu} \right) .$$

$$(108)$$

Knowing that  $\operatorname{Tr}\left(E_{k_{\mu}j_{\mu}}^{\mu}\right) = \delta_{k_{\mu}j_{\mu}}m_{\mu} = \delta^{\mu\nu}\delta_{k_{\mu}j_{\nu}}m_{\mu}$  we simplify to

$$(V_{\tau})_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\alpha}} = \delta^{r_{\mu/\alpha}r_{\nu/\alpha}}\delta_{k_{\mu}j_{\nu}}\sqrt{\frac{m_{\mu}}{m_{\nu}}}\sum_{k_{\mu}}\phi^{\mu}_{k_{\mu}i_{\mu}}(\tau).$$
(109)

This finishes the proof.

Having description of the basis elements in the ideal  $\mathcal{M}$  and action properties we are ready to calculate matrix elements of the multi-port teleportation operator (21).

Theorem 14. The matrix elements of the MPBT operator (21), with number of ports N and local dimension d, in operator basis from Theorem 11 are of the form

$$(\rho)_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\beta}} = \frac{k!\binom{N}{k}}{d^{N}} \frac{m_{\mu}}{m_{\alpha}} \frac{d_{\alpha}}{d_{\mu}} \delta^{r_{\mu/\alpha}r_{\nu/\beta}} \delta_{i_{\mu}j_{\nu}}.$$
 (110)

The numbers  $m_{\mu}$ ,  $m_{\alpha}$  and  $d_{\mu}$ ,  $d_{\alpha}$  denote respective multiplicities and dimensions of the irrpes in the Schur-Weyl duality, The led by  $\alpha \vdash n - 2k$  and  $\mu \vdash n - k$ , such that  $\mu \in \alpha$ .

Proof. The proof proceeds similarly as the proof of Lemma 13, namely we compute

$$\begin{aligned} (\rho)_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\beta}} \\ &= \frac{1}{m_{\alpha}} \operatorname{Tr} \left[ \rho E_{i_{\mu}}^{r_{\mu/\alpha}r_{\nu/\alpha}} \right] \\ &= \frac{1}{d^{N}} \frac{1}{\sqrt{m_{\mu}m_{\nu}}} \sum_{\tau \in \mathcal{S}_{n,k}} \operatorname{Tr} \left[ V^{(k)} V_{\tau} E_{i_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}} V^{(k)} E_{1_{\alpha} j_{\nu}}^{r_{\nu/\alpha}} V_{\tau^{-1}} \right], \end{aligned}$$

$$(111)$$

et

$$\sum_{\tau \in \mathcal{S}_{n,k}} \sum_{l_{\mu}} \sum_{k_{\nu}} \phi_{l_{\mu} l_{\mu}}^{\mu}(\tau) \phi_{j_{\nu} k_{\nu}}^{\nu}(\tau^{-1}) \times \operatorname{Tr} \left[ V^{(k)} E_{l_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}} V^{(k)} E_{1_{\alpha}} \frac{r_{\nu/\alpha}}{k_{\nu}} \right].$$
(112)

Using Fact 1 we write the following chain of equalities:

$$\operatorname{Tr}\left[V^{(k)}E_{l_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}} V^{(k)}E_{1_{\alpha}}^{r_{\nu/\alpha}} k_{\nu}\right] = \operatorname{Tr}\left[\operatorname{Tr}_{(k)}\left(E_{l_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}}\right) V^{(k)}E_{1_{\alpha}}^{r_{\nu/\alpha}} k_{\nu}\right] = \operatorname{Tr}\left[\operatorname{Tr}_{(k)}\left(E_{l_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}}\right) E_{1_{\alpha}}^{r_{\nu/\alpha}} k_{\nu}\right] = \operatorname{Tr}\left[\operatorname{Tr}_{(k)}\left(E_{l_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}}\right) \operatorname{Tr}_{(k)}\left(E_{1_{\alpha}}^{r_{\nu/\alpha}} k_{\nu}\right)\right],$$
(113)

where  $\operatorname{Tr}_{(k)} = \operatorname{Tr}_{n-2k+1,\dots,n-k}$ . Expanding rest of the indices in PRIR notation, i.e.  $l_{\mu} = (s_{\mu/\beta}, p_{\beta}), k_{\nu} =$  $(s_{\nu/\beta'}, q_{\beta'})$  and applying Lemma 9 we have

$$\operatorname{Tr}_{(k)}\left(E_{p_{\beta}}^{s_{\mu/\beta}r_{\mu/\alpha}}\right) = \delta^{s_{\mu/\beta}r_{\mu/\alpha}}\frac{m_{\mu}}{m_{\alpha}}E_{p_{\alpha}1_{\alpha}}^{\alpha}$$

$$\operatorname{Tr}_{(k)}\left(E_{1_{\alpha}}^{r_{\nu/\alpha}s_{\nu/\beta'}}\right) = \delta^{r_{\nu/\alpha}s_{\nu/\beta'}}\frac{m_{\nu}}{m_{\alpha}}E_{1_{\alpha}q_{\alpha}}^{\alpha}.$$
(114)

Now, we substitute the above into (112) writing as follows

In the above we use orthonormality relation, together wit the trace property (33), so  $\operatorname{Tr}\left(E_{p_{\alpha}1_{\alpha}}^{\alpha}E_{1_{\alpha}q_{\alpha}}^{\alpha}\right) = \operatorname{Tr}E_{p_{\alpha}q_{\alpha}}^{\alpha} =$ 

 $m_{\alpha}\delta_{p_{\alpha}q_{\alpha}}$ . Finally applying summation rule from Proposition 6 we arrive at

$$(\rho)_{i_{\mu} j_{\nu}}^{r_{\mu/\alpha}r_{\nu/\beta}} = \frac{1}{d^{N}} |\mathcal{S}_{n,k}| \frac{m_{\mu}}{m_{\alpha}} \frac{d_{\alpha}}{d_{\mu}} = \frac{k! \binom{N}{k}}{d^{N}} \frac{m_{\mu}}{m_{\alpha}} \frac{d_{\alpha}}{d_{\mu}} \delta^{r_{\mu/\alpha}r_{\nu/\beta}} \delta_{i_{\mu}j_{\nu}}.$$
(117)
This finishes the proof.

This finishes the proof.

Let us check the consequences of Theorem 14. Expression (110) tells us that multi-port teleportation operator  $\rho$  is diagonal in the operator basis given in Theorem 11. It means  $\rho$  can expressed as

$$\rho = \frac{k!\binom{N}{k}}{d^n} \sum_{\alpha} \sum_{\mu \in \alpha} \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} \frac{m_{\mu}}{m_{\alpha}} \frac{d_{\alpha}}{d_{\mu}} F_{k_{\mu}}^{r_{\mu/\alpha}r_{\mu/\alpha}}$$

$$= \sum_{\alpha} \sum_{\mu \in \alpha} \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} \lambda_{\mu}(\alpha) F_{k_{\mu}}^{r_{\mu/\alpha}r_{\mu/\alpha}}$$
(118)

where we introduced the quantity

$$\lambda_{\mu}(\alpha) \equiv \frac{k!\binom{N}{k}}{d^{N}} \frac{m_{\mu}}{m_{\alpha}} \frac{d_{\alpha}}{d_{\mu}}.$$
(119)

Now we can formulate the following

**Definition 15.** Having basis elements from (85) of Theorem 11, we define the following operators

$$\forall \alpha \ \forall \mu \in \alpha \quad F_{\mu}(\alpha) \equiv \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} F_{k_{\mu}}^{r_{\mu/\alpha}r_{\mu/\alpha}} k_{\mu}.$$
 (120)

Having the above definition we prove:

**Lemma 16.** Operators  $F_{\mu}(\alpha)$  for  $\alpha \vdash N - k$  and  $\mu \in \alpha$  are projectors and span identity  $\mathbf{1}_{\mathcal{M}}$  on the ideal  $\mathcal{M}$ .

*Proof.* First let us check that operators  $F_{\mu}(\alpha)$  given through Definition 15 are indeed orthonormal projectors. Indeed using (86) we have

$$F_{\mu}(\alpha)F_{\nu}(\beta) = \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} \sum_{r_{\nu/\beta}} \sum_{l_{\nu}} F_{k_{\mu}}^{r_{\mu/\alpha}r_{\mu/\alpha}} F_{l_{\nu}}^{r_{\nu/\beta}r_{\nu/\beta}} \\ = \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} \sum_{r_{\nu/\beta}} \sum_{l_{\nu}} \delta^{r_{\mu/\alpha}r_{\nu/\beta}} \delta_{k_{\mu}l_{\nu}} F_{k_{\mu}}^{r_{\mu/\alpha}r_{\nu/\beta}} \\ = \delta^{\mu\nu} \delta^{\alpha\beta} \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} \sum_{r_{\nu/\beta}} \sum_{l_{\nu}} \delta^{r_{\mu/\alpha}r_{\nu/\beta}} \delta_{k_{\mu}l_{\nu}} F_{k_{\mu}}^{r_{\mu/\alpha}r_{\nu/\beta}} \\ = \delta^{\mu\nu} \delta^{\alpha\beta} \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} F_{k_{\mu}}^{r_{\mu/\alpha}r_{\mu/\alpha}} \\ = \delta^{\mu\nu} \delta^{\alpha\beta} F_{\mu}(\alpha),$$
(121)

since for fixed  $\mu$ ,  $\nu$  and  $\alpha$ ,  $\beta$  we use the property  $\delta^{r_{\mu/\alpha}r_{\nu/\beta}} \equiv \delta^{\mu\nu}\delta^{\alpha\beta}\delta^{r_{\mu/\alpha}r_{\nu/\beta}}$ , see Notation 4.

To prove  $\sum_{\alpha} \sum_{\mu \in \alpha} F_{\mu}(\alpha) = \mathbf{1}_{\mathcal{M}}$  we must show that  $\forall x \in \mathcal{M} \text{ we have } x \sum_{\alpha} \sum_{\mu \in \alpha} F_{\mu}(\alpha) = \sum_{\alpha} \sum_{\mu \in \alpha} F_{\mu}(\alpha) x = x.$ Expanding x in the operator basis from Theorem 11

$$x = \sum_{\alpha',\beta'} \sum_{\mu' \in \alpha'} \sum_{\nu' \in \beta'} \sum_{i_{\mu'}, j_{\nu'}} x_{i_{\mu'}, j_{\nu'}}^{r_{\mu'/\alpha'}r_{\nu'/\beta'}} F_{i_{\mu'}, j_{\nu'}}^{r_{\mu'/\alpha'}r_{\nu'/\beta'}},$$
$$x_{i_{\mu'}, j_{\nu'}}^{r_{\mu'/\alpha'}r_{\nu'/\beta'}} \in \mathbb{C}, \quad (122)$$

and using expression (86) we get the statement. 

Finally thanks to Lemma 16 and decomposition (118), together with (119) we formulate spectral theorem for the multi-port teleportation operator (the multiplicities given below come from Lemma 20):

**Theorem 17.** The MPBT operator given through (21) has the following spectral decomposition

$$\rho = \sum_{\alpha} \sum_{\mu \in \alpha} \lambda_{\mu}(\alpha) F_{\mu}(\alpha), \qquad (123)$$

where eigenprojectors  $F_{\mu}(\alpha)$  are given in Definition 15 with corresponding eigenvalues  $\lambda_{\mu}(\alpha)$  from (119) with multiplicities  $m_{\mu/\alpha}m_{\alpha}d_{\mu}$ .

Checking that indeed we have  $\rho F_{\mu}(\alpha) = \lambda_{\mu}(\alpha)F_{\mu}(\alpha)$ , follows directly from orthonormality property of operators  $F_{\mu}(\alpha)$  proven in Lemma 16.

At the end of this section we prove two additionally lemmas on projectors  $F_{\mu}(\alpha)$  given in Definition 15. Defining symbol the  $\text{Tr}_{(2k)} \equiv \text{Tr}_{n-2k+1,\dots,n}$  which is a partial trace operation with respect to last 2k systems we have the following

**Lemma 18.** For a partially transposed permutation operator  $V^{(k)}$  from (37) and operators  $F_{\mu}(\alpha)$  given through Definition 15 the following holds:

$$\forall \alpha \vdash n - 2k \quad \forall \mu \in \alpha \quad \operatorname{Tr}_{(2k)} \left[ V^{(k)} F_{\mu}(\alpha) \right] = m_{\mu/\alpha} \frac{m_{\mu}}{m_{\alpha}} P_{\alpha},$$
(124)

where the numbers  $m_{\mu}$ ,  $m_{\alpha}$  denote respective multiplicities in the Schur-Weyl duality, while  $P_{\alpha}$  is a Young projector on n - 12k particles.

*Proof.* Using definition of the operator  $F_{\mu}(\alpha)$  and expression (85) we write

$$\sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} \operatorname{Tr}_{(2k)} \left[ V^{(k)} F_{k_{\mu}}^{r_{\mu/\alpha}r_{\mu/\alpha}} \right] \\ = \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} \frac{m_{\alpha}}{m_{\mu}} \operatorname{Tr}_{(2k)} \left[ V^{(k)} E_{i_{\mu}}^{r_{\mu/\alpha}} V^{(k)} E_{1_{\alpha}}^{r_{\mu/\alpha}} _{k_{\mu}} \right].$$
(125)

Using Fact 1, Lemma 9 and  $i_{\mu} = (s_{\mu/\beta}, i_{\beta})$  to operator  $E_{k_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}} = E_{k_{\beta}} \frac{r_{\mu/\beta}}{1_{\alpha}} r_{\mu/\alpha}$  we simplify the above equation to

$$\sum_{r_{\mu/\alpha}} \sum_{k_{\alpha}} \operatorname{Tr}_{(2k)} \left[ E_{k_{\alpha}1_{\alpha}}^{\alpha} V^{(k)} E_{1_{\alpha} k_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}} \right]$$

$$= \sum_{r_{\mu/\alpha}} \sum_{k_{\alpha}} \operatorname{Tr}_{(k)} \left[ E_{k_{\alpha}1_{\alpha}}^{\alpha} E_{1_{\alpha} k_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}} \right]$$

$$= \sum_{r_{\mu/\alpha}} \sum_{k_{\alpha}} \operatorname{Tr}_{(k)} \left[ E_{k_{\alpha} k_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}} \right]$$

$$= m_{\mu/\alpha} \frac{m_{\mu}}{m_{\alpha}} \sum_{k_{\alpha}} E_{k_{\alpha}k_{\alpha}}^{\alpha}$$

$$= m_{\mu/\alpha} \frac{m_{\mu}}{m_{\alpha}} P_{\alpha},$$
(126)

where in the last equality we used the definition of projectors  $P_{\alpha}$  given in (36). 

Further, below the proof of Lemma 21 we discuss alternative proof method of the above lemma.

**Lemma 19.** For operators  $F_{\mu}(\alpha)$  given through Definition 15 the following holds:

$$\forall \alpha \vdash n - 2k \quad \forall \mu \in \alpha \quad \operatorname{Tr}_{(k)} \left( F_{\mu}(\alpha) \right) = m_{\mu/\alpha} \frac{m_{\alpha}}{m_{\mu}} P_{\mu},$$
(127)

where the numbers  $m_{\mu}, m_{\alpha}$  denote respective multiplicities of the irrpes in the Schur-Weyl duality,  $m_{\mu/\alpha}$  denotes number of paths on reduced Young's lattice in which diagram  $\mu$  can be obtained from diagram  $\alpha$ , while  $P_{\mu}$  is a Young projector on n - k particles.

*Proof.* The proof is based on the straightforward calculations and observations made in the proof of Lemma 18. Using Definition 15 we have

$$\operatorname{Tr}_{(k)}(F_{\mu}(\alpha)) = \frac{m_{\alpha}}{m_{\mu}} \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} \operatorname{Tr}_{(k)}\left(E_{k_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}} V^{(k)} E_{1_{\alpha}}^{r_{\mu/\alpha}} k_{\mu}\right)$$
$$= \frac{m_{\alpha}}{m_{\mu}} \sum_{r_{\mu/\alpha}} \sum_{k_{\mu}} E_{k_{\mu}} \frac{r_{\mu/\alpha}}{1_{\alpha}} E_{1_{\alpha}}^{r_{\mu/\alpha}} k_{\mu}$$
$$= m_{\mu/\alpha} \frac{m_{\alpha}}{m_{\mu}} \sum_{k_{\mu}} E_{k_{\mu}k_{\mu}}^{\mu} = m_{\mu/\alpha} \frac{m_{\alpha}}{m_{\mu}} P_{\mu}$$
(128)

where in the last equality we used the definition of projectors  $P_{\mu}$  given in (36).

**Lemma 20.** For operators  $F_{\mu}(\alpha)$  given through Definition 15 the following holds:

$$\forall \alpha \vdash n - 2k \quad \forall \mu \in \alpha \quad \operatorname{Tr} \left( F_{\mu}(\alpha) \right) = m_{\mu/\alpha} m_{\alpha} d_{\mu}, \quad (129)$$

where the numbers  $m_{\mu}$ ,  $m_{\alpha}$  denote respective multiplicities of irreps in the Schur-Weyl duality,  $d_{\mu}$  stands for the dimension of the irrep  $\mu$ ,  $m_{\mu/\alpha}$  denotes number of paths on reduced Young's lattice in which diagram  $\mu$  can be obtained from diagram  $\alpha$ .

*Proof.* To compute the trace from  $F_{\mu}(\alpha)$  is enough to compute the trace from the right-hand side of (127) of Lemma 19, knowing that  $\text{Tr } P_{\mu} = m_{\mu}d_{\mu}$ .

**Lemma 21.** For operators  $F_{\mu}(\alpha)$  given through Definition 15 and operator  $V^{(k)}$  defined in (37), the following holds:

$$V^{(k)}F_{\mu}(\alpha) = V^{(k)}P_{\alpha}P_{\mu}.$$
(130)

*Proof.* First let us write explicitly the left-hand side of (130) using Definition 15 and Lemma 9:

$$V^{(k)}F_{\mu}(\alpha) = V^{(k)}\sum_{\substack{r_{\mu/\alpha} \\ m_{\mu}}} \sum_{\substack{r_{\mu/\alpha} \\ r_{\mu/\alpha}}} \sum_{\substack{r_{\mu}} \\ \sum_{i_{\beta}}} \sum_{i_{\beta}} V^{(k)} E^{r_{\mu/\beta}r_{\mu/\alpha}}_{i_{\beta}} V^{(k)} E^{r_{\mu/\alpha}r_{\mu/\beta}}_{1_{\alpha} \quad i_{\beta}}$$

$$= V^{(k)}\sum_{\substack{r_{\mu/\alpha} \\ r_{\mu/\alpha}}} \sum_{\substack{i_{\alpha} \\ i_{\alpha}}} E^{\alpha}_{i_{\alpha}1_{\alpha}} E^{r_{\mu/\alpha}r_{\mu/\alpha}}_{1_{\alpha} \quad i_{\alpha}}$$

$$= V^{(k)}\sum_{\substack{r_{\mu/\alpha} \\ r_{\mu/\alpha}}} \sum_{i_{\alpha}} E^{r_{\mu/\alpha}r_{\mu/\alpha}}_{i_{\alpha} \quad i_{\alpha}}.$$
(131)

Now, writing composition  $P_{\alpha}P_{\mu}$  in PRIR basis we get:

$$V^{(k)}P_{\alpha}P_{\mu} = V^{(k)}\sum_{i_{\alpha}} E^{\alpha}_{i_{\alpha}i_{\alpha}}\sum_{r_{\mu/\beta}}\sum_{j_{\beta}} E^{r_{\mu/\beta}r_{\mu/\beta}}_{j_{\beta}}$$
$$= V^{(k)}\sum_{r_{\mu/\alpha}}\sum_{i_{\alpha}} E^{r_{\mu/\alpha}r_{\mu/\alpha}}_{i_{\alpha}}$$
(132)

since  $E_{i_{\alpha}i_{\alpha}}^{\alpha} E_{j_{\beta}}^{r_{\mu/\beta}r_{\mu/\beta}} = \delta^{\alpha\beta} \delta_{i_{\alpha}j_{\beta}} E_{i_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}}$ . Now observing that right-hand sides of (131) and (132) coincide we finish the proof.

One can observe that having (130) we can prove the statement of Lemma 18 applying directly Corollary 10 to projector  $P_{\mu}$ . Indeed we have

$$\operatorname{Tr}_{(2k)}\left(V^{(k)}F_{\mu}(\alpha)\right) = \operatorname{Tr}_{(2k)}\left(V^{(k)}P_{\alpha}P_{\mu}\right)$$
$$= \operatorname{Tr}_{(k)}\left(P_{\alpha}P_{\mu}\right) = m_{\mu/\alpha}\frac{m_{\mu}}{m_{\alpha}}P_{\alpha},$$
(133)

where  $\text{Tr}_{(2k)} \equiv \text{Tr}_{n-2k+1,...,n}$  and  $\text{Tr}_{(k)} = \text{Tr}_{n-2k+1,...,n-k}$ .

## VIII. ENTANGLEMENT FIDELITY IN DETERMINISTIC VERSION OF THE PROTOCOL

Having description of the deterministic version of MPBT from Section III and mathematical tools developed in Section VII, especially the spectral decomposition of the operator  $\rho$ , given in Theorem 17, we can formulate the following:

**Theorem 22.** The entanglement fidelity in the deterministic multi-port teleportation with N ports and local dimension d is given as

$$F = \frac{1}{d^{N+2k}} \sum_{\alpha \vdash N-k} \left( \sum_{\mu \in \alpha} m_{\mu/\alpha} \sqrt{m_{\mu} d_{\mu}} \right)^2, \qquad (134)$$

where  $m_{\mu}$ ,  $d_{\mu}$  denote multiplicity and dimension of irreducible representations of S(N) respectively, and  $m_{\mu/\alpha}$  denotes number of paths on reduced Young's lattice in which diagram  $\mu$  can be obtained from diagram  $\alpha$  by adding k boxes.

*Proof.* In the first step of the proof we apply the covariance property (22) and (23) to equation (9) describing the entanglement fidelity and obtain the following expression:

$$F = \frac{1}{d^{2k}} \sum_{\mathbf{i}\in\mathcal{I}} \operatorname{Tr}\left(\Pi_{\mathbf{i}}^{A\widetilde{B}}\sigma_{\mathbf{i}}^{A\widetilde{B}}\right) = \frac{|\mathcal{S}_{n,k}|}{d^{2k}} \operatorname{Tr}\left(\Pi_{\mathbf{i}_{0}}^{A\widetilde{B}}\sigma_{\mathbf{i}_{0}}^{A\widetilde{B}}\right)$$
$$= \frac{k!\binom{N}{k}}{d^{2k}} \operatorname{Tr}\left(\frac{1}{\sqrt{\rho}}\sigma_{\mathbf{i}_{0}}^{A\widetilde{B}}\frac{1}{\sqrt{\rho}}\sigma_{\mathbf{i}_{0}}^{A\widetilde{B}}\right),$$
(135)

where  $\sigma_{i_0}^{A\bar{B}}$  is defined in (19). In the second equality we used the covariance property of signals  $\sigma_i^{A\bar{B}}$  and invariance of  $\rho$  with respect to the coset  $S_{n,k}$ . Using

spectral decomposition of the operator  $\rho$  presented in Theorem 17 we expand equation (135) to:

$$F = \frac{k!}{d^{2k}} \binom{N}{k} \operatorname{Tr} \left( \Pi_{\mathbf{i}_{0}}^{A\widetilde{B}} \sigma_{\mathbf{i}_{0}}^{A\widetilde{B}} \right)$$
  
$$= \frac{k!}{d^{2N}} \binom{N}{k} \sum_{\alpha,\beta \vdash N-k} \sum_{\mu \in \alpha} \sum_{\nu \in \beta} \frac{1}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{1}{\sqrt{\lambda_{\nu}(\beta)}} \times \operatorname{Tr} \left( F_{\mu}(\alpha) V^{(k)} F_{\nu}(\beta) V^{(k)} \right).$$
  
(136)

Now applying Lemma 21 we can rid of the operators  $F_{\mu}(\alpha)$ 

$$F = \frac{k!}{d^{2N}} \binom{N}{k} \sum_{\alpha,\beta \vdash N-k} \sum_{\mu \in \alpha} \sum_{\nu \in \beta} \frac{1}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{1}{\sqrt{\lambda_{\nu}(\beta)}}$$
$$\operatorname{Tr} \left( F_{\mu}(\alpha) V^{(k)} F_{\nu}(\beta) V^{(k)} \right)$$
$$= \frac{k!}{d^{2N}} \binom{N}{k} \sum_{\alpha,\beta \vdash N-k} \sum_{\mu \in \alpha} \sum_{\nu \in \beta} \frac{1}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{1}{\sqrt{\lambda_{\nu}(\beta)}}$$
$$\operatorname{Tr} \left( P_{\mu} P_{\alpha} V^{(k)} P_{\nu} P_{\beta} V^{(k)} \right).$$
(137)

Observing  $\left[P_{\beta}, V^{(k)}\right] = 0$ , we can apply Fact 1 together with Corollary 10 to  $V^{(k)}P_{\nu}V^{(k)}$ , getting

$$F = \frac{k!}{d^{2N}} \binom{N}{k} \sum_{\alpha,\beta \vdash N-k} \sum_{\mu \in \alpha} \sum_{\nu \in \beta} \frac{1}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{1}{\sqrt{\lambda_{\nu}(\beta)}} \\ \times \sum_{\beta' \in \nu} m_{\nu/\beta'} \frac{m_{\nu}}{m_{\beta'}} \operatorname{Tr} \left( P_{\mu} P_{\alpha} P_{\beta} P_{\beta'} V^{(k)} \right) \\ = \frac{k!}{d^{2N}} \binom{N}{k} \sum_{\alpha \vdash N-k} \sum_{\mu,\nu \in \alpha} \frac{1}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{1}{\sqrt{\lambda_{\nu}(\alpha)}} m_{\nu/\alpha} \frac{m_{\nu}}{m_{\alpha}} \\ \times \operatorname{Tr} \left( P_{\mu} P_{\alpha} \operatorname{Tr}_{(k)} V^{(k)} \right) \\ = \frac{k!}{d^{2N}} \binom{N}{k} \sum_{\alpha \vdash N-k} \sum_{\mu,\nu \in \alpha} \frac{1}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{1}{\sqrt{\lambda_{\nu}(\alpha)}} m_{\nu/\alpha} \frac{m_{\nu}}{m_{\alpha}} \\ \times \operatorname{Tr} \left( P_{\mu} P_{\alpha} \right).$$
(138)

Again applying Corollary 10, this time to projector  $P_{\mu}$ , together with Tr  $P_{\alpha} = m_{\alpha}d_{\alpha}$ , we have

$$F = \frac{k!}{d^{2N}} \binom{N}{k} \sum_{\alpha \vdash N-k} \sum_{\mu,\nu \in \alpha} \frac{1}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{1}{\sqrt{\lambda_{\nu}(\alpha)}} m_{\nu/\alpha} m_{\mu/\alpha} m_{\mu} m_{\nu} \frac{d_{\alpha}}{m_{\alpha}}$$
(1)

Using explicit expression for eigenvalues  $\lambda_{\mu}(\alpha), \lambda_{\nu}(\alpha)$  given in (119) we have

$$F = \frac{k!}{d^{2N+2k}} \binom{N}{k} \frac{d^N}{k! \binom{N}{k}} \sum_{\alpha \vdash N-k} \sum_{\mu,\nu \in \alpha} m_{\mu/\alpha} m_{\nu/\alpha} \\ \times \sqrt{\frac{m_{\alpha}d_{\mu}}{m_{\mu}d_{\alpha}}} \sqrt{\frac{m_{\alpha}d_{\nu}}{m_{\nu}d_{\alpha}}} \frac{m_{\mu}m_{\nu}}{m_{\alpha}} d_{\alpha}$$
$$= \frac{1}{d^{N+2k}} \sum_{\alpha \vdash N-k} \sum_{\mu,\nu \in \alpha} m_{\mu/\alpha} \sqrt{m_{\mu}d_{\mu}} m_{\nu/\alpha} \sqrt{m_{\nu}d_{\nu}}$$
$$= \frac{1}{d^{N+2k}} \sum_{\alpha \vdash N-k} \left( \sum_{\mu \in \alpha} m_{\mu/\alpha} \sqrt{m_{\mu}d_{\mu}} \right)^2.$$
(140)

This finishes the proof.

An alternative proof of Theorem 22 is presented in Appendix A. One can see that by setting k = 1 to (134) we reproduce known expression for entanglement fidelity in ordinary port-based teleportation [19]. Indeed, in this case always  $m_{\mu/\alpha} = 1$ , for any  $\mu \in \alpha$ , since we can move only by one layer on reduced Young's lattice. The expression from (134) is plotted in Figure 2 for different number of ports *N* as well local dimension *d* and number of teleported states *k*. We see that our deterministic scheme performs significantly better than standard PBT protocol, even in the optimal scheme, with respective dimension of the port.

## IX. PROBABILITY OF SUCCESS IN PROBABILISTIC VERSION OF THE PROTOCOL

Having description of the probabilistic version of MPBT scheme from Section III we are in position to solve SDP programs and evaluate optimal probability of success p when the parties share maximally entangled states. Namely, we have the following:

**Theorem 23.** The average probability of success in the probabilistic multi-port teleportation with N ports and local dimension d is given as

$$p = \frac{k!\binom{N}{k}}{d^{2N}} \sum_{\alpha \vdash N-k} \min_{\mu \in \alpha} \frac{m_{\alpha} d_{\alpha}}{\lambda_{\mu}(\alpha)},$$
 (141)

with optimal measurements of the form

$$\forall \mathbf{i} \in \mathcal{I} \qquad \Pi_{\mathbf{i}}^{AC} = \frac{k! \binom{N}{k}}{d^{2N}} P_{A_{\mathbf{i}}C}^{+} \otimes \sum_{\alpha \vdash N-k} P_{\alpha} \min_{\mu \in \alpha} \frac{1}{\lambda_{\mu}(\alpha)}.$$
(142)

Numbers  $\lambda_{\mu}(\alpha)$  are eigenvalues of  $\rho$  and are given in (119) and  $m_{\alpha}, d_{\alpha}$  denote multiplicity and dimension of the irrep labelled by  $\alpha$ .

*Proof.* The solution of optimisation tasks, so proof of the above theorem, is based solely on methods and tools delivered in Section VI and Section VII. We start from solving the primal problem. Due to symmetry in our scheme we assume that  $\forall \mathbf{i} \in \mathcal{I} \quad \Theta_{\overline{A}_{\mathbf{i}}} = \sum_{\alpha \vdash N-k} x_{\alpha} P_{\alpha}$  (139) with  $x_{\alpha} \geq 0$  to satisfy constraint (1) from (15). Operators

 $P_{\alpha}$  are Young projectors acting on subsystems defined by the symbol  $\overline{A}_{i}$ . To satisfy constraint (2) from (15) we write for every irreducible block  $\alpha$ :

$$\sum_{\mathbf{i}\in\mathcal{I}} P_{A_{\mathbf{i}}C}^{+} \otimes \Theta_{\overline{A}_{\mathbf{i}}}(\alpha) = \frac{x_{\alpha}}{d^{k}} \sum_{\tau\in\mathcal{S}_{n,k}} V_{\tau^{-1}} V^{(k)} \otimes P_{\alpha} V_{\tau}$$

$$= d^{N-k} x_{\alpha} \rho(\alpha) \leq P_{\alpha}.$$
(143)

In the above expression we use fact that for operator  $\rho$  from (21) and projection  $P_{\alpha}$  we have  $\rho(\alpha) = P_{\alpha}\rho P_{\alpha}$ . Now, to satisfy inequality 143 it is enough to require:

$$\forall \alpha \quad x_{\alpha} \le d^{k-N} \min_{\mu \in \alpha} \frac{1}{\lambda_{\mu}(\alpha)}, \tag{144}$$

where numbers  $\lambda_{\mu}(\alpha)$  are eigenvalues of  $\rho$  and are given in (119). Using assumption of covariance of measurements  $\forall \tau \in S_{n,k}$   $V_{\tau}\Pi_{\mathbf{i}}V_{\tau^{-1}} = \Pi_{\tau(\mathbf{i})}$  it is enough to work with the index  $\mathbf{i}_0$  only. Having that and border solution for  $x_{\alpha}$  from (144), we calculate the quantity  $p^*$ from (14):

$$p^{*} = \frac{1}{d^{N+k}} \sum_{\mathbf{i}\in\mathcal{I}} \operatorname{Tr}\left(\sum_{\alpha\vdash N-k} x_{\alpha} P_{\alpha}\right) = \frac{k!\binom{N}{k}}{d^{N+k}} \sum_{\alpha} x_{\alpha} \operatorname{Tr} P_{\alpha}$$
$$= \frac{k!\binom{N}{k}}{d^{2N}} \min_{\mu\in\alpha} \frac{m_{\alpha} d_{\alpha}}{\lambda_{\mu}(\alpha)},$$
(145)

since Tr  $P_{\alpha} = m_{\alpha}d_{\alpha}$ . For showing optimality of  $p^*$  we need to solve the dual problem from (16) and (17). We assume the following form of the operator  $\Omega$  in (16):

$$\Omega = \sum_{\alpha \vdash N-k} x_{\mu^*}(\alpha) F_{\mu^*}(\alpha), \qquad x_{\mu^*}(\alpha) = d^k \frac{1}{m_{\mu^*/\alpha}} \frac{m_\alpha}{m_{\mu^*}}.$$
(146)

The symbol  $\mu^*$  means that we are looking for such  $\mu \in \alpha$ which minimizes the quantity  $p_*$  from (16). Operators  $F_{\mu^*}(\alpha)$  are eigenprojectors of  $\rho$  given through Definition 15 and Theorem 17, symbol  $m_{\mu^*/\alpha}$  denotes number of paths on reduced Young's lattice in which diagram  $\mu^*$ can be obtained from diagram  $\alpha$ . Finally  $m_{\mu^*}, m_{\alpha}$  denote respective multiplicities of irreps. Since we are looking for any feasible solution to bound exact average probability of success p from the below we are allowed for such kind of assumptions. The first constraint from (17) is automatically satisfied due to assumed form of  $\Omega$ in (146). To check the second condition we need to compute

$$\operatorname{Tr}_{(2k)}\left(P_{A_{\mathbf{i}C}}^{+}\Omega\right) = \operatorname{Tr}_{(2k)}\left(P_{A_{\mathbf{i}0}C}^{+}\Omega\right) = \frac{1}{d^{k}}\operatorname{Tr}_{(2k)}\left(V^{(k)}\Omega\right),$$
(147)

where we used covariance property of  $P_{A_iC}^+$  and covariance of  $\Omega$  with respect to the elements from the coset  $S_{n,k}$ . Writing explicitly  $\Omega$  and using Lemma 18 we have

$$\frac{1}{d^{k}}\operatorname{Tr}_{(2k)}\left(V^{(k)}\Omega\right) = \sum_{\alpha} \frac{1}{m_{\mu^{*}/\alpha}} \frac{m_{\alpha}}{m_{\mu^{*}}} \operatorname{Tr}_{(2k)}\left(V^{(k)}F_{\mu^{*}}(\alpha)\right)$$

$$= \sum_{\alpha} P_{\alpha} = \mathbf{1},$$
(148)

so we satisfy the second constraint from (17) with equality. Now we are in position to compute  $p_*$  from (17):

$$p_* = \frac{1}{d^{N+k}} \operatorname{Tr} \Omega = \frac{1}{d^N} \sum_{\alpha} \frac{1}{m_{\mu^*/\alpha}} \frac{m_{\alpha}}{m_{\mu^*}} \operatorname{Tr} \left( F_{\mu^*}(\alpha) \right)$$
$$= \frac{1}{d^N} \sum_{\alpha} \frac{m_{\alpha}^2 d_{\mu^*}}{m_{\mu^*}} = \frac{k! \binom{N}{k}}{d^{2N}} \min_{\mu \in \alpha} \frac{m_{\alpha} d_{\alpha}}{\lambda_{\mu}(\alpha)}.$$
(149)

In third equality we use Lemma 20, in fourth we used the definition of the symbol  $\mu^*$  and form of  $\lambda_{\mu}(\alpha)$  from (119). From expressions (145) and (149) we see that  $p^* = p_*$ . We conclude that exact value of the average success probability indeed is given through expression (141) with corresponding measurements (142) presented in Theorem 23.

#### X. DISCUSSION

In this paper, we deliver analysis of the the multiport based teleportation schemes, which are non-trivial generalisation of the famous port-based teleportation protocol. These schemes allow for teleporting several unknown quantum states (or a composite quantum state) in one go so that the states end up in the respective number of ports on Bob's side. This protocol offers much better performance than the original PBT at the price of requiring corrections on the receiver's side which are permutations of the ports where the teleported states arrive. We discuss the deterministic protocol where the transmission always happens, but the teleported state is distorted, and the probabilistic case, where we have to accept the probability of failure, but whenever the protocol succeeds the teleportation is perfect. In both cases, we calculate parameters describing the performance of discussed schemes, like entanglement fidelity (see Theorem 22) and the probability of success (see Theorem 23). Expressions, except the global parameters such as the number of ports N and local dimension d, depend on purely group-theoretical quantities like for example dimensions and multiplicities of irreducible representations of the permutation group. The whole analysis is possible due to the rigorous description of the algebra of partially transposed permutation operators provided in this paper. In particular, we deliver the matrix operator basis in irreducible spaces on which respective operators describing teleportation protocol are supported (see Theorem 14, Theorem 17). The developed formalism applied to the considered problem allows to reduce calculations from the natural representation space to every irreducible block separately, simplifying it significantly. Moreover, symmetries occurring in the protocol allow us to solve semidefinite programming problems in an analytical way, which is not granted in general in SDP problems, see Section IX.

The methods presented in this paper may be applied to solve some related problems, but require further development of the formalism. The first one is the construction of the optimized version of the multi-port schemes. In this case, we have to find the operation  $O_A$ which Alice has to apply to her part of the resource state before she runs the protocol. Clearly in this case the resource state is no longer in the form of product of the maximally entangled pairs. The second problem is to understand the scaling of the entanglement fidelity and probability of success in the number of ports N, the number of teleported particles k and local dimension d. To answer this question one needs to adapt the analysis presented in [22] and examine the asymptotic behavior of the quantity  $m_{\mu/\alpha}$  appearing in our analysis (see for example Theorem 22). The third problem is to understand multi-port recycling schemes as a generalization of ideas in [48]. We would like to know how much the resource state degrades after the teleportation procedure and is there, in principle, the possibility of exploiting the resource state again.

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#### Appendix

Using spectral decomposition of the operator  $\rho$  presented in Theorem 17 we expand equation (135) to:

. . . . .

$$F = \frac{k!}{d^{2k}} {N \choose k} \operatorname{Tr} \left( \Pi_{\mathbf{i}_0}^{A\widetilde{B}} \sigma_{\mathbf{i}_0}^{A\widetilde{B}} \right)$$
  
$$= \frac{k!}{d^{2N}} {N \choose k} \sum_{\alpha,\beta \vdash N-k} \sum_{\mu \in \alpha} \sum_{\nu \in \beta} \frac{1}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{1}{\sqrt{\lambda_{\nu}(\beta)}}$$
(150)  
$$\times \operatorname{Tr} \left( F_{\mu}(\alpha) V^{(k)} F_{\nu}(\beta) V^{(k)} \right).$$

Now we have to compute the trace from the composition  $F_{\mu}(\alpha)V^{(k)}F_{\nu}(\beta)V^{(k)}$  between partially transposed permutation operator defined in (37) and eigenprojectors  $F_{\mu}(\alpha)$  presented in Definition 15. Numbers  $\lambda_{\mu}(\alpha)$  denote respective eigenvalues of multi-port teleportation operator given in (119). Using explicit form of eigenprojectors from Definition 15, expression (85) for basis operator in Theorem 11, together with Fact 1 and Lemma 9 we can write the following chain of equalities, displayed at the top of the following page, where the simplified form (152) follows from  $\left[E_{l_{\alpha}1_{\alpha}}^{\alpha}V^{(k)}\right] = 0$  and  $E_{l_{\alpha}1_{\alpha}}^{\alpha}E_{1_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}} = E_{l_{\alpha}}^{r_{\mu/\alpha}r_{\mu/\alpha}}$ , where we applied definition of  $P_{\alpha} = \sum_{l_{\alpha}} E_{l_{\alpha}l_{\alpha}}^{\alpha}$ , orthogonality relation  $P_{\alpha}P_{\beta} = \delta^{\alpha\beta}P_{\alpha}$  and

finally Tr  $P_{\alpha} = m_{\alpha}d_{\alpha}$ . The symbol  $m_{\mu/\alpha}$  denotes number of paths on reduced Young's lattice in which frame  $\mu$  can be obtained from frame  $\alpha$  by adding k boxes. Substituting final form of (152) to (150) we have

$$F = \frac{k!}{d^{2N+2k}} \binom{N}{k} \sum_{\alpha \vdash N-k} \sum_{\mu,\nu \in \alpha} \frac{d_{\alpha}}{m_{\alpha}} \frac{m_{\mu}}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{m_{\nu}}{\sqrt{\lambda_{\nu}(\beta)}} m_{\mu/\alpha} m_{\nu/\alpha}$$
(153)

Inserting explicit form of eigenvalues  $\lambda_{\nu}(\alpha)$ ,  $\lambda_{\mu}(\alpha)$  given in (119), we reduce to:

$$F = \frac{k!}{d^{2N+2k}} {\binom{N}{k}} \sum_{\alpha \vdash N - k} \sum_{\mu, \nu \in \alpha} \frac{d_{\alpha}}{m_{\alpha}} \frac{m_{\mu}}{\sqrt{\lambda_{\mu}(\alpha)}} \frac{m_{\nu}}{\sqrt{\lambda_{\nu}(\beta)}} m_{\mu/\alpha} m_{\nu/\alpha}$$

$$= \frac{k!}{d^{2N+2k}} {\binom{N}{k}} \frac{d^{N}}{k! {\binom{N}{k}}} \sum_{\alpha \vdash N - k} \sum_{\mu, \nu \in \alpha} m_{\mu/\alpha} m_{\mu/\alpha} m_{\nu/\alpha}$$

$$\sqrt{\frac{m_{\alpha} d_{\mu}}{m_{\mu} d_{\alpha}}} \sqrt{\frac{m_{\alpha} d_{\nu}}{m_{\nu} d_{\alpha}}} \frac{m_{\mu} m_{\nu}}{m_{\alpha}} d_{\alpha}$$

$$= \frac{1}{d^{N+2k}} \sum_{\alpha \vdash N - k} \sum_{\mu, \nu \in \alpha} m_{\mu/\alpha} \sqrt{m_{\mu} d_{\mu}} m_{\nu/\alpha} \sqrt{m_{\nu} d_{\nu}}$$

$$= \frac{1}{d^{N+2k}} \sum_{\alpha \vdash N - k} \left(\sum_{\mu \in \alpha} m_{\mu/\alpha} \sqrt{m_{\mu} d_{\mu}}\right)^{2}.$$
(154)

This finishes the proof.

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$$\operatorname{Tr}\left(V^{(k)}F_{\mu}(\alpha)V^{(k)}F_{\nu}(\beta)\right) = \sum_{\substack{r_{\mu/\alpha'}r_{\nu/\beta}}}\sum_{\substack{k_{\mu}j_{\nu}}}\operatorname{Tr}\left(V^{(k)}F^{r_{\mu/\alpha}r_{\mu/\alpha}}_{k_{\mu}}V^{(k)}F^{r_{\nu/\beta}r_{\nu/\beta}}_{j_{\nu}}\right) \\
= \sum_{\substack{r_{\mu/\alpha'}r_{\nu/\beta}}}\sum_{\substack{k_{\mu}j_{\nu}}}\frac{m_{\alpha}m_{\beta}}{m_{\mu}m_{\nu}}\operatorname{Tr}\left(V^{(k)}E^{r_{\mu/\alpha'}}_{k_{\mu}}1^{\alpha}V^{(k)}E^{r_{\mu/\alpha'}}_{1_{\alpha}}V^{(k)}E^{r_{\nu/\beta}}_{j_{\nu}1_{\beta}}V^{(k)}E^{r_{\nu/\beta'}}_{1_{\beta}j_{\nu}}\right) \\
= \sum_{\substack{r_{\mu/\alpha'}r_{\nu/\beta}}}\sum_{\substack{r_{\mu/\alpha'}r_{\nu/\beta'}}}\sum_{\substack{l_{\alpha'},s_{\beta'}}}\frac{m_{\alpha}m_{\beta}}{m_{\mu}m_{\nu}}\operatorname{Tr}\left(V^{(k)}E^{r_{\mu/\alpha'}r_{\mu/\alpha}}_{l_{\alpha'}}1^{\alpha}V^{(k)}E^{r_{\mu/\alpha'}r_{\mu/\alpha'}}_{1_{\alpha}}V^{(k)}E^{r_{\nu/\beta'}r_{\nu/\beta}}_{1_{\alpha}l_{\alpha'}}V^{(k)}E^{r_{\nu/\beta'}r_{\nu/\beta}}_{s_{\beta'}}V^{(k)}E^{r_{\nu/\beta'}r_{\nu/\beta'}}_{1_{\beta}}S^{r_{\beta'}}_{\beta'}\right)$$
(151)
$$=\sum_{\substack{r_{\mu/\alpha'}r_{\nu/\beta}}}\sum_{\substack{r_{\mu/\alpha'}r_{\nu/\beta'}}}\sum_{\substack{l_{\alpha'},s_{\beta'}}}\delta^{r_{\mu/\alpha'}r_{\mu/\alpha'}}\delta^{r_{\nu/\beta}r_{\nu/\beta'}}\operatorname{Tr}\left(E^{\alpha}_{l_{\alpha}1_{\alpha}}V^{(k)}E^{r_{\mu/\alpha'}r_{\mu/\alpha}}_{1_{\alpha}l_{\alpha}}E^{\beta}_{s_{\beta}1_{\beta}}V^{(k)}E^{r_{\nu/\beta'}r_{\nu/\beta}}_{1_{\beta}}S^{r_{\beta'}}_{\beta}\right)$$

which simplifies to

$$\operatorname{Tr}\left(V^{(k)}F_{\mu}(\alpha)V^{(k)}F_{\nu}(\beta)\right) = \sum_{\substack{r_{\mu/\alpha},r_{\nu/\beta}}}\sum_{\substack{l_{\alpha},s_{\beta}}}\operatorname{Tr}\left(V^{(k)}E^{r_{\mu/\alpha},r_{\mu/\alpha}}_{l_{\alpha}}V^{(k)}E^{r_{\nu/\beta},r_{\nu/\beta}}_{s_{\beta}}\right)$$
$$= \sum_{\substack{r_{\mu/\alpha},r_{\nu/\beta}}}\sum_{\substack{l_{\alpha},s_{\beta}}}\frac{m_{\mu}}{m_{\alpha}}\operatorname{Tr}\left(E^{\alpha}_{l_{\alpha}l_{\alpha}}V^{(k)}E^{r_{\nu/\beta},r_{\nu/\beta}}_{s_{\beta}}\right) = \sum_{\substack{r_{\mu/\alpha},r_{\nu/\beta}}}\sum_{\substack{k_{\alpha},j_{\beta}}}\frac{m_{\mu}}{m_{\alpha}}\operatorname{Tr}\left(E^{\alpha}_{k_{\alpha}k_{\alpha}}E^{\beta}_{j_{\beta}}\right)$$
$$= \sum_{\substack{r_{\mu/\alpha},r_{\nu/\beta}}}\sum_{\substack{k_{\alpha},j_{\beta}}}\frac{m_{\mu}}{m_{\alpha}}\frac{m_{\nu}}{m_{\beta}}\operatorname{Tr}\left(E^{\alpha}_{l_{\alpha}l_{\alpha}}E^{\beta}_{s_{\beta}s_{\beta}}\right) = \frac{m_{\mu}}{m_{\alpha}}\frac{m_{\nu}}{m_{\beta}}m_{\mu/\alpha}m_{\nu/\beta}\operatorname{Tr}(P_{\alpha}P_{\beta})$$
$$= \delta^{\alpha\beta}\frac{m_{\mu}m_{\nu}}{m_{\alpha}^{2}}m_{\mu/\alpha}m_{\nu/\beta}\operatorname{Tr}P_{\alpha} = \delta^{\alpha\beta}m_{\mu/\alpha}m_{\nu/\alpha}m_{\mu}m_{\nu}\frac{d_{\alpha}}{m_{\alpha}}.$$
(152)

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