

Coding Constructions for Efficient Oblivious Transfer from Noisy Channels

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Abstract

We consider oblivious transfer protocols performed over binary symmetric channels in a malicious setting where parties will actively cheat if they can. We provide constructions purely based on coding theory that achieve an explicit positive rate, the essential ingredient being the existence of linear codes whose Schur products are asymptotically good.

1 Introduction

A 1-out-of-2 oblivious transfer is a cryptographic protocol between two players, Alice, who owns two secrets, and Bob, who wishes to acquire one of them. The protocol ensures that one of the secrets is delivered to Bob, while no information about the other secret leaks: furthermore, Alice has no information about which secret Bob selects.

1-out-of-2 oblivious transfer protocols were introduced by Even, Goldreich and Lempel [10], though it was shown that they are equivalent to a variant originally proposed by Rabin [19]. Oblivious transfer has found numerous applications since, notably to multiparty computation [11]. Oblivious transfer was first considered in the case of computationally bounded participants, but Crépeau and Kilian later [8] introduced the idea of unconditionally secure oblivious transfer, by considering the

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situation when the players have access to a noisy channel that they cannot control. It is natural to view the noisy channel as a resource, and to use the number of symbols sent over the channel as measure of the efficiency of the oblivious transfer protocol. The first protocol achieving oblivious transfer of 1-bit secrets from a binary symmetric channel (BSC) [8] required $\Omega(n^{11})$ transmitted bits over the channel to guarantee a probability of protocol failure of 2^{-n} . This was improved by Crépeau in [7], who obtained a protocol that requires only $O(n^3)$ uses of the BSC for a one bit secret, and was later again improved by Crépeau, Morozov and Wolf [9] to $O(n^2 + \varepsilon)$.

The notion of oblivious transfer capacity was introduced in [17] and further developed in [1]. In this setting, the two secrets are not single bits anymore, but strings of bits, and the oblivious transfer rate is defined as the quotient of the number of bits of each secret divided by the number of bits transmitted over the channel: the oblivious transfer capacity is then equal to the supremum of the set of achievable rates. Oblivious transfer is studied in [1] in the semi-honest (or honest but curious) model, meaning that the players do not deviate from the protocol. In this model it is shown that constant rate oblivious transfer is possible for most discrete memoryless channels. Capacities are computed for some channels in [1], and lower bounds provided for others, including the binary symmetric channel, though an actual protocol is not proposed.

Later, Ishai et al. [14] improved the results of Crépeau [7], and are the first to show that constant rate oblivious transfer protocols are possible in a fully malicious setting where both players will actively cheat if they can. The paper [14] also devises a protocol that achieves efficient oblivious transfer of many 1-bit secrets in parallel. The protocols of [14] are quite intricate and call upon a number of cryptographic primitives. In the present paper we again pick up the issue of devising constant rate oblivious transfer protocols and apply a coding-theory approach to the problem. The end result consists of constant rate oblivious transfer protocols that are more direct than that of [14] and that allow us to compute an achievable rate that is of different order of magnitude than what could eventually be derived from [14].¹ Crucial to our protocols is the notion of Schur product of a linear code C . The Schur product (or square) of a linear code $C \subset \mathbb{F}_q^n$ is the linear span of all coordinate-wise products $c * c' = (c_1 c'_1 \cdots c_n c'_n)$ of codewords $c = (c_1, \dots, c_n), c' = (c'_1, \dots, c'_n)$ of C . The central ingredient in our construction is a family of asymptotically good linear codes whose squares are also asymptotically good. In the case of the binary alphabet, the existence of such families of codes is far from immediate and they were not known to exist before the work of Randriambololona [20]. In fact, during the early investigations leading up to this

¹Y. Ishai, personal communication.

work, we realised the usefulness of such codes but were unable to come up with a construction, and inquiries into the matter provided motivation for the paper [20], as is mentioned in its introduction. Codes with good squares also appear indirectly in [14], since they are an essential component of the secret sharing schemes with multiplicative properties that [14] calls upon. The use of codes with good squares is arguably more direct in the present work. In the next section we give an overview of our constructions and outline the structure of the paper.

2 Overview

We assume Alice and Bob have access to two channels: (1) a noiseless channel and (2) a discrete memoryless channel. How unconditional oblivious transfer can be achieved is best understood in the simple case of an erasure channel of erasure probability p , say. First Alice generates a string \mathbf{r} of $2n_0$ random bits and sends them over the noisy channel. Bob will receive approximately $2pn_0$ erasures instead of the original symbols. Bob then separates the index set $[1, 2n_0]$ of the received symbols into two disjoint sets of equal size n_0 , i.e. $[1, 2n_0] = I \cup J$, in such a way that all the erased symbols have their coordinates in one of the two sets (assuming there are no more than n_0 of them, which is typically the case when $p < 1/2$) and communicates the sets I and J noiselessly to Bob. Denoting by \mathbf{r}_I and \mathbf{r}_J the corresponding two n_0 -bit strings derived from the bits initially sent by Alice, she can now use them to send noiselessly to Bob $\mathbf{x} + \mathbf{r}_I$ and $\mathbf{y} + \mathbf{r}_J$ where \mathbf{x} and \mathbf{y} are some n_0 -bit vectors. With this procedure Alice sends to Bob the vectors \mathbf{x} and \mathbf{y} through what amounts to two different channels, one of which is noiseless, the other being in effect an erasure channel. Alice has no way of knowing which of the two channels is the noiseless one, meaning she cannot know which secret Bob will want, and from Bob's side, whatever may be the way he chooses the two sets I and J , at least one of the two vectors \mathbf{x} and \mathbf{y} will be submitted to an erasure channel of erasure probability at least p . We remark now that all that is needed to complete the protocol is to apply standard wiretap-channel techniques to transmit messages through the vectors \mathbf{x} and \mathbf{y} that leak no information to an eavesdropper that would access \mathbf{x} and \mathbf{y} through a channel of erasure probability at least p . The oblivious transfer capacity for this honest but curious setting is explicitly computed in [1] as a function of the error probability p .

We now focus on our central topic, namely the case when the noisy channel is a binary symmetric channel. All known protocols start with the following bit duplication trick, first introduced by Crépeau and Kilian [8] and also used in [7]. Alice again generates a string \mathbf{r} of $2n_0$ random bits, but this time every bit r_i is

sent over the noisy channel as a duplicate couple (r_i, r_i) . We remark that whenever a couple $(0, 1)$ or $(1, 0)$ is received, then either $(0, 0)$ or $(1, 1)$ must have been sent with equal probability $1/2$ at the receiver (Bob's) end. Therefore Bob has no choice than to consider this situation as an erasure, and what duplication achieves is to transform the binary symmetric channel into a mixed channel with errors and erasures. Again, Bob partitions the index set $[1, 2n_0]$ into two n_0 -bit sets I and J , one of which indexes all the erased positions. This again creates two vectors \mathbf{r}_I and \mathbf{r}_J , one of which is received with more noise, namely a mixture of errors and erasures, than the other which is erasure-free. We have effectively created two virtual noisy channels one of which is noisier than the other, and such that Alice does not know which is the noisiest. At this point we make the remark that wire-tap channel techniques are again sufficient to extract from these two channels a semi-honest oblivious transfer protocol. We develop this approach in Section 3, which requires a treatment of the somewhat non-standard mixed error-erasure wiretap channels. The result is a constructive oblivious transfer protocol P_0 for an m -bit secret that is a generalisation of a protocol of [7] for 1-bit secrets, and that achieves the lower bound on the oblivious transfer capacity computed in [1] for a mixed error-erasure channel. The computations of [1] are purely information-theoretic and no explicit schemes were suggested to achieve them. By optimising over the channel parameter we obtain a positive rate $\mathcal{R}_0 = 0.108$ for this first protocol P_0 . For the rate \mathcal{R} of a 1-out-of-2 oblivious transfer protocol (two secrets) we use the following definition, consistent with [1]:

Definition 1. *The rate \mathcal{R} of an oblivious transfer protocol of one out of two m -bit secrets is the ratio of the number of secrets bits, namely $2m$, over the number N of binary symbols transmitted over the channel.*

Protocol P_0 ensures that an honest but curious Bob will have no knowledge on at least one of the two secrets. To measure the possible leakage of information about a secret, we first view the two secrets as uniform and independent random variables in $\{0, 1\}^m$. We write them therefore as X, Y . The protocol P_0 would be ideal if we could state regarding Bob's view that:

$$\begin{aligned} \text{Either } & H(X|Y, \mathcal{O}) = m \\ \text{Or } & H(Y|X, \mathcal{O}) = m \end{aligned}$$

where \mathcal{O} is what Bob observes during the protocol and H is Shannon's entropy. We prove a lower bound on $H(X|Y, \mathcal{O})$ that explicitly states how close (in fractions of bits) the protocol is from the ideal scenario.

Protocol P_0 is only valid in a semi-honest model where Alice does not deviate from her instructions. Alice's goal in cheating is restricted to trying to figure out

the secret that Bob wants (she is not interested in disrupting the protocol, i.e. to make it fail). Contrary to the pure erasure channel case, Alice could actively cheat by transmitting over the binary symmetric channel some falsely duplicated bits r_i under the form $(0, 1)$ or $(1, 0)$, instead of (r_i, r_i) . If $(0, 1)$ is sent over a binary symmetric channel with transition probability $p < 1/2$, then the probability $p^2 + (1 - p)^2$ that $(0, 1)$ or $(1, 0)$ is received (an erasure) is always larger than if $(0, 0)$ or $(1, 1)$ had been transmitted. By sending a few tracker pairs of symbols in this way, a tellingly large number of their indices will end up in the subset, I or J corresponding to the secret that Bob does not want, thus yielding critical information to Alice on which secret Bob is trying to acquire.

A crucial observation made by Crépeau [7] is that the number of falsely duplicated tracker bits r_i that Alice can use can only be a limited portion of the total number of bits transmitted over the noisy channel. This is because these bits have a higher probability of turning up on Bob's side as erasures, and if Bob receives too many erased symbols, contradicting the law of large numbers, he will know that Alice has almost certainly cheated. Hence if one repeats the protocol P_0 many times, say n_0^2 times where n_0 is (as above, up to a multiplicative constant) the number of noisy channel uses for P_0 , this makes the number of channel uses equal to n_0^3 , and the number of corrupt tracker bits that Alice can get away with using without arousing Bob's suspicion, is, by the law of large numbers, not significantly more than the order of $n_0^{3/2}$: this implies that Alice has to be honest for the majority of the n_0^2 P_0 -protocols that are played out, otherwise she will be exposed with probability tending to 1 with n_0 .

The following idea is then used by Crépeau [7] to obtain an oblivious transfer protocol secure against malicious participants that would cheat if they could. The treatment of [7] focuses on oblivious transfer of single bit secrets, but it applies just as well to string oblivious transfer. Apply n times the protocol P_0 to intermediate secret pairs x_i, y_j , $i = 1, \dots, n$. The strings $x_i \in \{0, 1\}^m$ are chosen randomly such that $x_1 + \dots + x_n = s$ and the y_i are defined as $y_i = x_i + s + t$, $i = 1, \dots, n$, where s and t are Alice's secrets to be obliviously transferred. Now to foil Alice's tracking strategy, Bob will, for every i , randomly ask for either x_i or y_i , taking care only to ask for an even number of y_i 's if he wishes to eventually acquire s , and an odd number of y_i 's if he wishes to acquire t . We see that summing all the intermediate secrets Bob has acquired yields either s or t according to his wish, and Alice who can only cheat on a fraction of the P_0 protocols, obtains no information on the eventual secret, s or t , obtained by Bob.

The above repetition scheme gives vanishing rates however. The core strategy developed in the present paper is to again repeat n times the protocol P_0 , but to

replace the condition $x_1 + \dots + x_n = s$ by a generalised condition

$$\mathbf{H} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{s} \quad (1)$$

where \mathbf{H} is a suitably chosen binary $r \times n$ matrix, yielding secrets \mathbf{s} and \mathbf{t} of length rm rather than m (every coefficient of \mathbf{s} lives in $\{0, 1\}^m$). It turns out that central among the required properties of \mathbf{H} is that it generates a binary linear code with a square that has a large minimum distance. We develop this approach in Section 4 where we propose first an oblivious transfer protocol P_1 that prevents Alice from cheating but introduces cheating possibilities for Bob (whose goal is to obtain information about the other secret than the one being asked), and then introduce a variation P'_1 of P_1 which prevents both Alice and Bob from cheating. The protocol P'_1 adds to the protocol P_1 a compression function applied to \mathbf{s} and \mathbf{t} . These protocols achieve a positive rate of respectively $\mathcal{R}_1 \approx 0.69 \cdot 10^{-4}$ and $\mathcal{R}'_1 \approx 0.34 \cdot 10^{-4}$.

In Section 5, we introduce generalisations P_2 and P'_2 of protocols P_1 and P'_1 that replace the binary code generated by \mathbf{H} in (1) by a q -ary code, for q a power of 2. This allows us to use algebraic geometry codes with a much improved rate, with the drawback that the protocol P_0 has to be replaced by a less efficient 1-out-of- q semi-honest oblivious transfer protocol. Overall, the rates of the protocols P_2 and P'_2 improve upon P_1 and P'_1 , giving $\mathcal{R}_2 \approx 1/1250$, $\mathcal{R}'_2 \approx 1/2500$.

We finish this overview with a formal definition of oblivious transfer considered in the malicious case.

Definition 2. *Given a noiseless channel with unlimited usage and a binary symmetric channel (BSC), a 1-out-of-2 oblivious transfer protocol of rate \mathcal{R} consists of a two player protocol where one player, Alice, possesses two secrets \mathbf{s} and \mathbf{t} of m bits, and the second player, Bob, asks Alice for one of the two secrets. The protocol uses communication over both channels and should satisfy the following properties:*

1. *The ratio of the number $2m = |\mathbf{s}| + |\mathbf{t}|$ of secrets bits over the total number N of binary symbols transmitted over the noisy channel is \mathcal{R} .*
2. *The protocol is correct, meaning that if Bob follows the protocol, he will obtain the secret he wishes with probability that tends to 1 when N goes to infinity.*

3. Bob, whether he cheats or not, has virtually no information on at least one secret, meaning

$$\begin{aligned} \text{Either } H(S|T, \mathcal{O}) &\geq H(S) - \delta \\ \text{Or } H(T|S, \mathcal{O}) &\geq H(T) - \delta \end{aligned}$$

where S and T are \mathbf{s} and \mathbf{t} viewed as random variables with uniform distribution, \mathcal{O} is what Bob observes during the protocol and δ is a quantity that tends to 0 when N goes to infinity.

4. A cheating Alice who is trying to gain non-trivial information on which secret Bob is asking for will either fail at obtaining anything or be accused of cheating by Bob with probability tending to 1 when N goes to infinity. It may happen that Bob accuses Alice of cheating when she is behaving honestly, but this happens with probability that tends to 0 when N goes to infinity.

We remark that:

- (1) In what follows, we will always assume that the secrets of Alice are two independent uniformly distributed strings. This assumption is made without loss of generality. Indeed, since the noiseless channel is assumed to be available at no cost, Alice may always one-time pad her secrets \mathbf{s} and \mathbf{t} by computing $\mathbf{s} + \mathbf{x}$ and $\mathbf{t} + \mathbf{y}$ and communicating them noiselessly to Bob, for some independent uniform random strings \mathbf{x} and \mathbf{y} . After this, oblivious transfer of the secrets \mathbf{s} and \mathbf{t} is equivalent to oblivious transfer of the random strings \mathbf{x} and \mathbf{y} .
- (2) This condition on secrecy used in [1] is $H(K_{\bar{Z}}|Z, \mathcal{O})$, where Z is a random variable that models the choice of a secret, and $K_{\bar{Z}}$ represents the secret which was not chosen. Our condition is slightly stronger since it assumes the complete knowledge of one secret is given. Deviating from the definition of [1] was required since it makes no sense in the malicious context to model the choice of a secret by a binary random variable. Indeed, we will see that in some instances Bob can try to extract from the protocol some mixture of partial information from both secrets.

Our main results are the protocols P_1, P_2 and P'_1, P'_2 . Protocols P'_1 and P'_2 satisfy Definition 2 with all probabilities that are required to tend to zero doing so subexponentially, i.e. scaling as $\exp(-N^\alpha)$ for $0 < \alpha < 1$. The quantity δ in Point 3. of the definition is also subexponential in N . Protocols P_1 and P_2 are preliminary versions of protocols P'_1 and P'_2 where only Alice is fully malicious while Bob is assumed to be honest-but-curious.

3 A First Binary Oblivious Transfer Protocol

Alice and Bob have access to two channels: (1) a noiseless channel and (2) a binary symmetric channel (BSC) with crossover probability $\varphi < 1/2$. The binary field over $\{0, 1\}$ is denoted by \mathbb{F}_2 , and for $a \in \mathbb{F}_2$, \bar{a} denotes the other element of \mathbb{F}_2 .

The protocol P_0 below is a slight variation of the oblivious transfer proposed by Crépeau in [7], allowing Alice's two secrets to be strings of $m = n_0\epsilon(1 - h(\frac{\varphi^2}{1-\epsilon}))$ bits, instead of 1 bit, where h denotes the binary entropy function, and where we have set $\epsilon = 2\varphi(1 - \varphi)$. The total number of noisy channel uses is $N = 4n_0$.

Protocol P_0 . Alice has two (column) secrets $x, y \in \mathbb{F}_2^m$. Alice and Bob agree on an $((n_0 - k) + m) \times n_0$ binary matrix \mathbf{H}' of the form

$$\mathbf{H}' = \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \end{bmatrix}$$

where \mathbf{H}_0 is the parity check matrix of some (n_0, k) linear code C_0 , which is capacity achieving over a BSC with crossover probability $\frac{\varphi^2}{1-\epsilon}$, and comes with an efficient decoding algorithm, while \mathbf{H}_1 is chosen uniformly at random.

1. Alice generates a string $\mathbf{r} = (r_1, \dots, r_{2n_0})$ of $2n_0$ random bits and sends $2n_0$ pairs (r_i, r_i) of random bits to Bob over the BSC.
2. For every pair of the form (r_i, r_i) or (\bar{r}_i, \bar{r}_i) , Bob decides that the bit r_i or \bar{r}_i is successfully received. He declares an erasure if he receives (r_i, \bar{r}_i) or (\bar{r}_i, r_i) . Bob partitions the indices $[1, 2n_0]$ into two sets: I has size n_0 and contains only indices corresponding to successfully received bits, while J , also of size n_0 , contains the rest of the indices. This is assuming that Bob wants to know the secret x : if instead he prefers the secret y , then he will reverse the roles of I and J . To each set corresponds a string of noisy random bits \mathbf{r}'_I and \mathbf{r}'_J . For \mathbf{r}'_I , the noise comes from Bob accepting (\bar{r}_i, \bar{r}_i) while Alice sent (r_i, r_i) . For \mathbf{r}'_J , the noise also includes erasures. Bob sends both sets of indices I and J to Alice over the noiseless channel. Alice then permutes uniformly at random elements in I and in J and sends the permutations to Bob over the noiseless channel.
3. Alice picks uniformly at random two (column) codewords \mathbf{c}_x and $\mathbf{c}_y \in C_1$ that satisfy respectively

$$\mathbf{H}'\mathbf{c}_x = \begin{bmatrix} \mathbf{0} \\ x \end{bmatrix}, \quad \mathbf{H}'\mathbf{c}_y = \begin{bmatrix} \mathbf{0} \\ y \end{bmatrix}$$

and sends $\mathbf{c}_x + \mathbf{r}_I$ and $\mathbf{c}_y + \mathbf{r}_J$ to Bob over the noiseless channel.

4. Bob computes $(\mathbf{c}_x + \mathbf{r}_I) + \mathbf{r}'_I$ to find \mathbf{c}'_x , a noisy version of \mathbf{c}_x . Bob decodes \mathbf{c}'_x , recovers \mathbf{c}_x , and computes $\mathbf{H}_1 \mathbf{c}_x = x$.

The protocol P_0 requires $2n_0$ uses of the BSC channel for each secret. It provides an oblivious transfer protocol provided that Alice is honest.

Suppose both Alice and Bob are honest. If Alice is honest and sends pairs of the form (r_i, r_i) , Bob receives (r_i, r_i) with probability $(1 - \varphi)^2$ and (\bar{r}_i, \bar{r}_i) with probability φ^2 . He will decide an erasure with probability $2\varphi(1 - \varphi) = \epsilon < 1/2$, and accept a bit with probability $1 - \epsilon$. This can be seen as an instance of an imperfect binary erasure channel (BEC) with erasure probability ϵ : when Bob decides that a random bit is correctly received, there is still a probability φ^2 of getting the wrong random bit.

- Since Bob accepts a pair of random bits with probability $1 - \epsilon$ ($\epsilon < 1/2$), he should receive on average $2(1 - \epsilon)n_0$ non-erased symbols, and I can be assumed of size n_0 .
- The string \mathbf{r}'_I is \mathbf{r}_I affected by an additive noise, that is $\mathbf{r}'_I = \mathbf{r}_I + \mathbf{e}$, where \mathbf{e} is an error vector, which contains a 1 whenever Alice sent (r_i, r_i) and Bob received (\bar{r}_i, \bar{r}_i) . If Alice is honest, a bit flip happens with probability φ^2 , thus the proportion of bit flips among the bits that are not erased is $\frac{2n_0\varphi^2}{2n_0(1-\epsilon)}$, yielding, by restricting over the non-erased bits, a binary symmetric channel (BSC) with crossover probability $\frac{\varphi^2}{(1-\epsilon)}$. It is enough that the chosen error capability of the code allows an honest Bob to recover \mathbf{c}_x . However, by choosing a code which is capacity achieving, n_0 is minimized, and m is maximized, as shown below, while discussing the optimization of \mathcal{R}_0 , the rate of P_0 . Polar codes [2] provide examples of capacity achieving codes for the BSC which furthermore come with an efficient decoding algorithm.

Suppose Bob is dishonest. We now check that Bob, even if he is dishonest and deviates from the protocol by putting indices of erased positions in both sets I and J , will not recover any information about at least one of the two secrets, that is, he cannot recover information involving both \mathbf{c}_x and \mathbf{c}_y . Bob gets roughly $2n_0(1 - \epsilon)$ bits (the rest being erased), he can thus partition the $2n_0$ bits into two groups

in any way he wants, where I will have some bits erased, some not, and the rest will be in J . Bob will receive some \mathbf{c}'_x and \mathbf{c}'_y , which are noisy versions of \mathbf{c}_x and \mathbf{c}_y respectively, with noise depending on the choice of I and J . These noises can be seen as the result of a transmission through a channel between Alice and Bob that behaves as a mixture of an erasure channel and a binary symmetric channel. Concretely, the noisiest of the two channels will have an average proportion of erased symbols that is at least ϵ , and its non-erased symbols are all submitted to a binary symmetric channel of transition probability $\varphi^2/(1-\epsilon)$, as they were before the partition into I and J , since Bob has no way of differentiating symbols in error from error-free symbols.

For the purposes of the present study, let us call a *binary symmetric channel with erasures* of parameters $e \in [1, n_0]$ and $0 \leq p < 1/2$, where e is an integer, a channel which acts on strings of n_0 bits in the following way:

- it erases e coordinates chosen uniformly among all possible $\binom{n_0}{e}$ patterns (in the protocol P_0 , Alice permutes the indices in I and J uniformly at random),
- it applies a binary symmetric channel of transition probability p to the remaining $n_0 - e$ symbols.

We will rely on the following result:

Lemma 3. *Let C be a binary code of length n and rate R , and let C_X be a random variable with values in C and uniform distribution. Let C_X be submitted to a binary symmetric channel with erasures of parameters e and p , and let Z be the output variable. Define the conditional min-entropy of C_X given $Z = z$ by*

$$H_\infty(C_X|Z = z) = -\log \max_{c \in C} P(C_X = c|Z = z).$$

Then, for all $\alpha > 0$, with probability that tends to 1 exponentially in n , a vector z is received such that

$$H_\infty(C_X|Z = z) \geq n[R - (1 - e/n)(1 - h(p)) - \alpha].$$

The proof of Lemma 3 is given in the Appendix.

In the case under study, the value of e may vary, but the probability that e/n_0 falls significantly below $\epsilon = 2\varphi(1-\varphi)$, i.e. is separated from ϵ by a constant, is exponentially small in n_0 . To obtain a uniformly distributed secret from the transmitted codeword \mathbf{c}_x or \mathbf{c}_y in the protocol P_0 , it suffices to hash it to a sufficiently smaller string, which is exactly the purpose of the multiplication by \mathbf{H}_1 . Since the

set of multiplications by \mathbf{H}_1 makes up a universal family of hash functions, we will invoke the Leftover Hash Lemma [12, Lemma 4.5.1] [3, Theorem 3] to evaluate how close the protocol P_0 is from the ideal scenario

$$\text{Either } H(X|Y, \mathcal{O}) = m \quad (2)$$

$$\text{Or } H(Y|X, \mathcal{O}) = m \quad (3)$$

where we view the two secrets x and y as uniform and independent random variables X, Y in $\{0, 1\}^m$ and \mathcal{O} is what Bob observes during protocol P_0 . The nature of protocol P_0 is such that $H(X|Y, \mathcal{O}) = H(X|\mathcal{O})$, because X and Y are really transmitted over two independent channels. Without loss of generality we assume that X is transmitted over the noisiest of the two channels. We have the following lower bound on $H(X|\mathcal{O})$:

Theorem 4. *Suppose protocol P_0 is implemented with some (n_0, k) linear code C_0 of rate $R_0 = k/n_0$. For any $\varepsilon > 0$, whenever the length m of the secret satisfies $m \leq n_0[R_0 - (1 - \varepsilon)(1 - h(\frac{\varepsilon^2}{1-\varepsilon}))] - \varepsilon$, then $H(X|\mathcal{O}) \geq m - f_0(\varepsilon, m)$, for $f_0(\varepsilon, m)$ exponentially small in m .*

Proof. What is observed by Bob is a noisy version z of a codeword c sent through a binary symmetric channel with erasures of parameters e and $p = \varepsilon^2/(1 - \varepsilon)$, with e/n_0 arbitrarily close to ε . Lemma 3 claims that with probability tending to 1 (exponentially in n_0 , meaning with probability $1 - \exp(-n_0)$), Bob observes $\omega = z$ such that the min-entropy $H_\infty(c|\mathcal{O} = \omega)$ of the transmitted codeword c is at least $n_0[R_0 - (1 - e/n_0)(1 - h(p))] - \alpha n_0$, with α arbitrarily small. We then invoke Theorem 3 of [3] and the fact that the Renyi entropy is never less than the min-entropy to claim that, since $X = \mathbf{H}_1 c$, we have

$$H(X|\mathcal{O} = \omega, \mathbf{H}_1) = H(\mathbf{H}_1 c|Z = z, \mathbf{H}_1) \geq m - 2^{m - n_0[R_0 - (1 - e/n_0)(1 - h(p))] + \alpha n_0 - \log_2 \ln 2}.$$

Since $m \leq n_0[R_0 - (1 - \varepsilon)(1 - h(\frac{\varepsilon^2}{1-\varepsilon}))] - \varepsilon$, then $m - n_0[R_0 - (1 - e/n_0)(1 - h(p))] + \alpha n_0 - \log_2 \ln 2 \leq -4\beta n_0$ for some $\beta = \beta(\varepsilon) > 0$, which shows that we are already close to m in a way which is exponential in n_0 , given \mathbf{H}_1 . Next, we remove the dependency on \mathbf{H}_1 . We just showed that on average over \mathbf{H}_1 ,

$$H(X|\mathcal{O} = \omega, \mathbf{H}_1) \geq m - 2^{-4\beta n_0}. \quad (4)$$

Suppose now that we were to be unlucky and choose \mathbf{H}_1 in the set of \mathcal{H}_1 of “bad” matrices h (that may depend on ω) such that $H(X|\mathcal{O} = \omega, \mathbf{H}_1 = h) \leq m - 2^{-2\beta n_0}$.

Since

$$\begin{aligned}
H(X|\mathcal{O} = \omega, \mathbf{H}_1) &= \sum_{h \in \mathcal{H}_1 \cup \bar{\mathcal{H}}_1} P(\mathbf{H}_1 = h) H(X|\mathcal{O} = \omega, \mathbf{H}_1 = h) \\
&\leq \sum_{h \in \mathcal{H}_1} P(\mathbf{H}_1 = h)(m - 2^{-2\beta n_0}) + \sum_{h \in \bar{\mathcal{H}}_1} P(\mathbf{H}_1 = h)m \\
&= - \sum_{h \in \mathcal{H}_1} P(\mathbf{H}_1 = h)2^{-2\beta n_0} + \sum_{h \in \mathcal{H}_1 \cup \bar{\mathcal{H}}_1} P(\mathbf{H}_1 = h)m \\
&= m - 2^{-2\beta n_0} \sum_{h \in \mathcal{H}_1} P(\mathbf{H}_1 = h),
\end{aligned}$$

we upper bound the quantity $H(X|\mathcal{O} = \omega, \mathbf{H}_1)$ by

$$H(X|\mathcal{O} = \omega, \mathbf{H}_1) \leq m - 2^{-2\beta n_0} P_u$$

where P_u is the probability to choose $\mathbf{H}_1 = h$ in \mathcal{H}_1 . Together with (4) the above inequality implies that $P_u \leq 2^{-2\beta n_0}$. Therefore with probability $1 - P_u \geq 1 - 1/2^{2\beta n_0}$ over the choice of the random matrix \mathbf{H}_1 , we have

$$H(X|\mathcal{O} = \omega) \geq m - 2^{-2\beta n_0}.$$

Now Lemma 3 does not exclude the existence of a “bad” event $\omega \in \Omega_1$, for which we cannot guarantee (4). But we can write

$$\begin{aligned}
H(X|\mathcal{O}) &= \sum_{\omega \in \Omega_1 \cup \bar{\Omega}_1} P(\mathcal{O} = \omega) H(X|\mathcal{O} = \omega) \\
&\geq \sum_{\omega \in \bar{\Omega}_1} P(\mathcal{O} = \omega)(m - 2^{-2\beta n_0}) \\
&= (m - 2^{-2\beta n_0})(1 - 2^{-\gamma n_0})
\end{aligned}$$

where $2^{-\gamma n_0}$ is the probability of a bad event ω .

We make the final remark that the above computation assumed that e/n_0 is arbitrarily close to ϵ . Of course, the number of erasures can deviate significantly from the average: but this happens with probability exponentially small in n_0 , so that again this rare event can only diminish $H(X|\mathcal{O})$ by a quantity exponentially small in n_0 . \square

Corollary 5. *If C_0 is capacity-achieving on the erasureless channel, meaning the code C_0 has a vanishing decoding error probability for a BSC of parameter $p = \varphi^2/(1-\epsilon)$ and a rate R_0 arbitrarily close to $1 - h(\frac{\varphi^2}{1-\epsilon})$, and if $m \leq n_0\epsilon[1 - h(\frac{\varphi^2}{1-\epsilon}) - \epsilon]$, then Bob can only obtain a vanishingly small number of bits of information on one of the two secrets.*

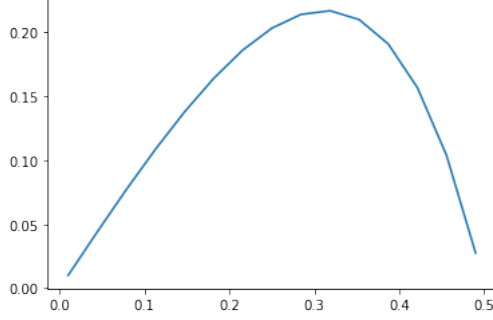


Figure 1: $\epsilon(1 - h(\varphi^2/(1 - \epsilon)))$ is shown as a function of the erasure probability ϵ of the imperfect BEC.

Optimization of the length m and of the rate R_0 . Finally, m is maximized by maximizing R_0 , that is by having Alice use a capacity achieving code for the relevant BSC channel, for which Corollary 5 has just told us that we may set

$$m = n_0 \epsilon (1 - h(\frac{\varphi^2}{1 - \epsilon}) - \varepsilon),$$

for an arbitrarily small positive ε . This gives us an oblivious transfer rate (see Definition 1) arbitrarily close to

$$\mathcal{R}_0 = \frac{m}{2n_0} = \frac{n_0 \epsilon (1 - h(\frac{\varphi^2}{1 - \epsilon}))}{2n_0} = \varphi(1 - \varphi) \left(1 - h\left(\frac{\varphi^2}{1 - 2\varphi(1 - \varphi)}\right) \right).$$

This is the lower bound on the oblivious transfer capacity found by Alshwede and Csiszár in [1, Example 1]. What this shows is therefore that we incur no penalty on the achievable oblivious transfer rate by assuming a possibly malicious Bob as opposed to the honest but curious Bob of [1].

The optimal value of m is obtained when $\varphi \approx 0.198$, $\epsilon \approx 0.31$, for which $\epsilon(1 - h(\frac{\varphi^2}{1 - \epsilon})) \approx 0.216$ (see Figure 1) and

$$\mathcal{R}_0 \approx \frac{0.216n_0}{2n_0} = 0.108.$$

Suppose Alice is dishonest. She might send a pair of the form (\bar{r}_i, r_i) . Now Bob will receive (\bar{r}_i, r_i) or (r_i, \bar{r}_i) , that is an erasure, with probability $\varphi^2 + (1 - \varphi)^2 = 1 - \epsilon > 1/2$, in which case he will put the index i in the more “noisy” set J . Since it is more likely that a symbol of the form (r_i, \bar{r}_i) stays an erasure (rather than being changed into the valid symbol (r_i, r_i) or (\bar{r}_i, \bar{r}_i)), by tracking which set

contains the most indices on which she cheated, she can guess which is more likely to be I or J . For this reason, the protocol P_0 is only valid in a semi-honest model where Alice is assumed not to deviate from the protocol.

To obtain a protocol valid against a malicious Alice that will send falsely duplicated pairs of the form (\bar{r}_i, r_i) , we may repeat $n = n_0^2$ times the protocol P_0 , as in [7] and as sketched in Section 2. If Alice is honest, the first time, she sends $(r_{1,1}r_{1,1}), \dots, (r_{1,2n_0}r_{1,2n_0})$, the second time, she sends $(r_{2,1}r_{2,1}), \dots, (r_{2,2n_0}r_{2,2n_0}), \dots$. Let Z be the random variable counting the number of valid (i.e. non-erased) bits that Bob should receive. It is binomially distributed, with mean $E[Z] = 2nn_0(1 - \epsilon)$, and standard deviation $\sigma = \sqrt{2nn_0\epsilon(1 - \epsilon)}$.

Suppose now Alice is dishonest, and that she cheats by sending M falsely duplicated pairs out of the total $2nn_0$ transmitted pairs for all the n iterations of protocol P_0 . Every time she cheats and sends $(\bar{r}_{i,j}r_{i,j})$, Bob will declare an erasure with probability $1 - \epsilon$, therefore the average number of valid bits that Bob will get is

$$E[Z] = (2nn_0 - M)(1 - \epsilon) + M\epsilon = 2nn_0(1 - \epsilon) - M(1 - 2\epsilon), \quad (5)$$

with $\epsilon < 1/2$. If Alice cheats at least once on every one of the n instances of the protocol P_0 (so that $M \geq n$), then for $n = n_0^2$, the typical value of Z will deviate from $2nn_0(1 - \epsilon)$ by a quantity that is much too close to n_0^2 than the standard deviation of Z , that behaves as $n_0^{3/2}$, allows. This tells Bob that Alice is cheating.

A bit more specifically, Bob will set a *threshold* τ to be equal to

$$\tau = 2nn_0(1 - \epsilon - \eta) \quad \text{with} \quad \eta = \frac{1}{4n_0}(1 - 2\epsilon), \quad (6)$$

which is exactly midway between the expected number $2nn_0(1 - \epsilon)$ of unerased symbols he should receive if Alice does not try to cheat and the expected number (5) of unerased symbols he should receive if Alice cheats sufficiently many times ($M = n$) to access his secret. Bob will declare that Alice cheats if the number of unerased symbols that he receives falls below the threshold τ . The Chernov-Hoeffding inequality tells us that the probability that Alice succeeds in cheating without being accused scales as $\exp(-\eta^2 2nn_0)$, and similarly that the probability that Bob wrongly accuses Alice of cheating is also $\exp(-\eta^2 2nn_0)$. With $n = n_0^2$, we have $\exp(-\eta^2 2nn_0) = \exp(-n_0)$.

More generally, we note that Alice's cheating will be almost surely noticed whenever the number M of corrupted bits she sends satisfies

$$M \gg \sqrt{nn_0}. \quad (7)$$

Conversely, whenever the order of magnitude of M stays below $\sqrt{nn_0}$, she gets away with her behaviour. We notice in particular that to prevent Alice from cheating on at least one bit for every instance of P_0 , the number n of times the protocol P_0 must be repeated has to satisfy $n \gg n_0$.

In the next section we further exploit this strategy of repeating n times P_0 to devise an efficient oblivious transfer protocol secure against a cheating Alice.

4 A Positive Rate Binary Oblivious Transfer Protocol

4.1 A protocol that defeats Alice's cheating strategy

We recall the definition of a Schur product of codes over the finite field \mathbb{F}_q , for q a prime power. Schur products were possibly explicitly first used in Coding Theory for decoding applications [18] and later came under attention in cryptographic contexts, in part because of their relevance to secret sharing and multiparty computation. For details on applications see the introduction of [5] and for a survey of their properties see [21].

Definition 6. *Given a q -ary linear code C of length n , for q a prime power, the Schur product (or square) of C , denoted \hat{C} , is defined as the linear span of all componentwise products $c * c'$ of code vectors c, c' of C , i.e.*

$$\hat{C} = \langle c * c', c, c' \in C \rangle$$

with

$$c * c' = (c_1 c'_1, \dots, c_n c'_n).$$

We will denote the length, the dimension, and the minimum Hamming distance of the code C and the code \hat{C} respectively by $[n, r, d]$ and $[n, \hat{r}, \hat{d}]$.

For the moment we restrict ourselves to $q = 2$. Let now \mathbf{H} be an $r \times n$ binary matrix of rank r whose i th row is denoted by H_i , so that

$$\mathbf{H} = \begin{bmatrix} H_1 \\ \vdots \\ H_r \end{bmatrix}$$

satisfying

$$H_i H_j^\top = \delta_{ij}, \quad i, j = 1, \dots, r. \quad (8)$$

The protocol P_1 described below provides an oblivious transfer between Alice and Bob, assuming this time that Bob is honest (but not Alice). We will from now on use the letter m to denote the secret size in protocol P_0 : the secret size in protocol P_1 will be equal to rm , for an integer r equal to the dimension of a binary linear code C that we now introduce.

Protocol P_1 . Alice and Bob agree on an $r \times n$ binary matrix \mathbf{H} satisfying (8), which forms the generator matrix of an $[n, r, d]$ code C . The dimension r of C and the minimum distance \hat{d} of the square \hat{C} should both be linear in n .

Alice has two secrets

$$\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_r \end{bmatrix}$$

with coefficients s_i, t_i in \mathbb{F}_2^m .

1. Alice then picks uniformly at random a vector $\mathbf{x} = [x_1, \dots, x_n]$, $x_i \in \mathbb{F}_2^m$, such that

$$\mathbf{H}\mathbf{x}^\top = \mathbf{s}$$

and computes

$$\mathbf{y} = \mathbf{x} + \sum_{j=1}^r (s_j + t_j) H_j = [y_1, \dots, y_n], \quad y_i \in \mathbb{F}_2^m, \quad (9)$$

such that $\mathbf{H}\mathbf{y}^\top = \mathbf{t}$.

2. Bob computes a binary vector $\mathbf{u} = [u_1, \dots, u_n] \in \mathbb{F}_2^n$ which is orthogonal to \hat{C} .
3. If Bob wants the secret \mathbf{s} (respectively \mathbf{t}), then for every coefficient u_ℓ of \mathbf{u} , Bob asks Alice through the protocol P_0 for the string
 - $x_\ell \in \mathbb{F}_2^m$ (respectively y_ℓ) if $u_\ell = 0$,
 - $y_\ell \in \mathbb{F}_2^m$ (respectively x_ℓ) if $u_\ell = 1$.
4. After n rounds of the protocol P_0 , Alice has sent Bob the requested n -tuple $\mathbf{v} = [v_1, \dots, v_n]$ which may be expressed as

$$\mathbf{v} = \mathbf{x} * (\mathbf{1} + \mathbf{u}) + \mathbf{y} * \mathbf{u} \quad \text{if Bob requested } \mathbf{s} \quad (10)$$

$$\mathbf{v} = \mathbf{x} * \mathbf{u} + \mathbf{y} * (\mathbf{1} + \mathbf{u}) \quad \text{if Bob requested } \mathbf{t} \quad (11)$$

5. Once Bob gets \mathbf{v} , he computes $\mathbf{H}\mathbf{v}^\top$ to recover \mathbf{s} (or \mathbf{t}).

We first remark that in the simple case when the matrix \mathbf{H} is a single row made up of the all-one vector, $\mathbf{H} = [1, 1, \dots, 1]$, then the protocol P_1 reduces to the string version of Crépeau's oblivious transfer protocol sketched in Section 2. We now check that in the general case, protocol P_1 does what is required of it when the players do not try to deviate. We need to check that Bob indeed recovers \mathbf{s} or \mathbf{t} and obtains no information on the other secret.

Suppose both Alice and Bob are honest.

- As needed in Step 1 of P_1 , the vector $\mathbf{y} = \mathbf{x} + \sum_{j=1}^r (s_j + t_j)H_j = [y_1, \dots, y_n]$, $y_i \in \mathbb{F}_2^m$, satisfies

$$\mathbf{H}\mathbf{y}^\top = \mathbf{t}.$$

Indeed, from (8), $\mathbf{H}H_j^\top = \mathbf{e}_j$, where \mathbf{e}_j is the weight 1 column vector with 1 at the j th position, and

$$\mathbf{H}\mathbf{y}^\top = \mathbf{H}\mathbf{x}^\top + \sum_{j=1}^r (s_j + t_j)\mathbf{H}H_j^\top = \mathbf{s} + \sum_{j=1}^r (s_j + t_j)\mathbf{e}_j = \mathbf{s} + (\mathbf{s} + \mathbf{t}) = \mathbf{t}.$$

- To show that Bob can recover \mathbf{s} or \mathbf{t} by computing $\mathbf{H}\mathbf{v}^\top$ in Step 5 of P_1 , we first remark that \mathbf{v} can be expressed as a Schur product

$$\mathbf{v} = \mathbf{x} + \mathbf{u} * (\mathbf{x} + \mathbf{y}) = [x_1 + u_1(x_1 + y_1), \dots, x_n + u_n(x_n + y_n)],$$

if Bob wants \mathbf{s} , or

$$\mathbf{v} = \mathbf{x} + (\mathbf{u} + \mathbf{1}) * (\mathbf{x} + \mathbf{y}) = [x_1 + (u_1 + 1)(x_1 + y_1), \dots, x_n + (u_n + 1)(x_n + y_n)],$$

if Bob wants \mathbf{t} , according to (10) and (11). Now Bob gets \mathbf{v} , and computes

$$\mathbf{H}\mathbf{v}^\top = \mathbf{s} + \mathbf{H}(\mathbf{u} * (\mathbf{x} + \mathbf{y}))^\top = \mathbf{s} + \mathbf{H}(\mathbf{u} * (\sum_{j=1}^r (s_j + t_j)H_j))^\top,$$

since $\mathbf{y} = \mathbf{x} + \sum_{j=1}^r (s_j + t_j)H_j$, and

$$\mathbf{H}(\mathbf{u} * (\sum_{j=1}^r (s_j + t_j)H_j))^\top = \mathbf{H}(\sum_{j=1}^r (s_j + t_j)(\mathbf{u} * H_j))^\top = \sum_{j=1}^r (s_j + t_j)\mathbf{H}(\mathbf{u} * H_j)^\top.$$

Now the i th row of the vector $\mathbf{H}(\mathbf{u} * H_j)^\top$ is

$$H_i(\mathbf{u} * H_j)^\top = \sum_{k=1}^n H_{ik}u_k H_{jk} = \mathbf{u}(H_i * H_j)^\top.$$

Since \mathbf{u} is orthogonal to \hat{C} , $\mathbf{u}(H_i * H_j)^\top = 0$, and as desired, Bob gets \mathbf{s} . Now if $\mathbf{u} + \mathbf{1}$ is used instead of \mathbf{u} , we have

$$\mathbf{H}((\mathbf{u} + \mathbf{1}) * (\mathbf{x} + \mathbf{y}))^\top = \mathbf{H}(\mathbf{u} * (\mathbf{x} + \mathbf{y}) + \mathbf{x} + \mathbf{y})^\top = \mathbf{H}(\mathbf{x} + \mathbf{y})^\top$$

and Bob gets \mathbf{t} .

Additionally, we remark that Bob has obtained \mathbf{s} (say) and $\mathbf{v}_1 \in (\mathbb{F}_2^m)^n$ given by (10) and that protocol P_0 guarantees that he essentially has no information on the coefficients of the other n -tuple \mathbf{v}_2 given by (11). More precisely, the argument above shows that in an idealised version of protocol P_0 , where (2) and (3) hold, Bob has no information from \mathbf{v}_2 in the sense that, given \mathbf{v}_1 and \mathbf{s} , all possible values for $\mathbf{t} = \mathbf{H}\mathbf{v}_2^\top$ are equally likely, in other words Bob has no information on \mathbf{t} . Let us now prove that Bob has almost no knowledge on \mathbf{t} , even when given \mathbf{v}_1 (which implies knowledge of \mathbf{s}), and the actual output of protocol P_0 . Given \mathbf{v}_1 , (9) proves that, for every fixed vector \mathbf{u} chosen by Bob, the vector \mathbf{v}_2 lives in a code whose codewords are in one-to-one correspondence with the values of \mathbf{t} . Let us denote by V_2 the random variable equal to v_2 with distribution conditioned by the knowledge of \mathbf{v}_1 , (which does not depend on the actual value of \mathbf{v}_1). We have that protocol P_0 transforms every coordinate $(V_2)_\ell$ of V_2 , $\ell = 1 \dots n$, into an n_0 -tuple Z of $\{0, 1, *\}^{n_0}$ (where $*$ denotes an erasure), in a way that is memoryless and without feedback. In other words, the distribution of Z_ℓ conditional on $(V_2)_\ell$ is the same as the distribution of Z_ℓ conditional on $(V_2)_1, \dots, (V_2)_\ell$, and the distribution of $(V_2)_\ell$ conditional on $(V_2)_1, \dots, (V_2)_{\ell-1}$ and $Z_1, \dots, Z_{\ell-1}$ is the same as the distribution of $(V_2)_\ell$ conditional on $(V_2)_1, \dots, (V_2)_{\ell-1}$ alone. These properties are well-known (e.g. [6, Ch. 7]) to imply that

$$I(V_2, Z) \leq \sum_{\ell=1}^n I((V_2)_\ell, Z_\ell).$$

From Theorem 4 we have $I((V_2)_\ell, Z_\ell) \leq f_0(m, \varepsilon)$, from which we get, since $H(V_2) = H(T) = rm$,

$$H(T|\mathcal{O}) \geq H(V_2|\mathcal{O}) \geq rm - nf_0(m, \varepsilon),$$

where T is \mathbf{t} viewed as a random variable with uniform distribution and \mathcal{O} is Bob's view of the whole protocol. In other words, the amount of information leaked in the whole process is at most $nf_0(m, \varepsilon)$. We have thus proved the following:

Corollary 7. *Suppose that protocol P_1 is implemented using an $[n, r, d]$ code C satisfying the requirements, and used by an honest Alice who owns two secrets \mathbf{s} and \mathbf{t} of length r (linear in n), i.e. in $(\mathbb{F}_2^m)^r$, and an honest Bob. Then*

$$H(T|S, \mathcal{O}) \geq rm - nf_0(m, \varepsilon),$$

where ε and $f_0(m, \varepsilon)$ are as in Theorem 4 and in particular $f_0(m, \varepsilon)$ is exponentially small in m .

Suppose Alice is dishonest. In Step 3 of P_1 , Bob asks Alice for either x_ℓ or y_ℓ via protocol P_0 . If Alice is honest, she has no information on whether Bob is asking for x_ℓ or for y_ℓ . Now we know that Alice can cheat in protocol P_0 , and might guess whether Bob is asking for x_ℓ or y_ℓ : however, among the n iterations of P_0 , Alice can only cheat up to M times while staying under the radar, as long as (from (7)) M stays below a linear function of $\sqrt{(\# \text{ channel uses for } P_0)n}$. Keeping the notation of Section 3, take n linear in n_0^2 (say), so that M cannot exceed a quantity linear in $\sqrt{n_0 n} = n_0^{3/2} = n^{3/4}$.

Obtaining information on whether Bob asks for x_ℓ or for y_ℓ is equivalent to obtaining information on the ℓ -th coefficient u_ℓ of the vector $\mathbf{u} = (u_1, \dots, u_n)$. Now \mathbf{u} is randomly chosen in \hat{C}^\perp , therefore the $M \approx n^{3/4}$ coefficients seen by Alice will be distributed uniformly at random in $\{0, 1\}^M$ as long as M is less or equal to the dual minimum Hamming distance of \hat{C}^\perp (see [15], Ch.5. §5. Theorem 8), which is the minimum Hamming distance of \hat{C} set to be linear in n , assuming for the moment that such codes exist. Therefore, Alice cannot differentiate with only M values of \mathbf{u} whether Bob is asking for \mathbf{s} or for \mathbf{t} : Alice's cheating strategy is foiled.

To be more specific, Alice can only gain something from her cheating attempt if she cheats on M instances of P_0 with M exceeding the Hamming distance of \hat{C} , i.e. $M \geq cn$ for some constant c . As discussed at the end of Section 3, Bob will set a threshold τ as in (6), with the value η being adjusted to $\eta = \frac{c}{4n_0}(1 - 2\epsilon)$, so that τ sits exactly between the expected number of unerased symbols he should receive when Alice does not try to cheat, and the expected number of symbols he will receive when Alice chooses $M = cn$. Again, the probability that either Alice cheats successfully without being caught and the probability that Bob wrongly accuses Alice of cheating both scale like $\exp(-n_0)$.

We are left to show that codes C with all the required properties exist. The code C should have positive rate, i.e. its dimension r should be a linear function of n , so that the oblivious transfer protocol has positive rate. As we have just seen, the minimum distance \hat{d} should be large enough. We note that the protocol would still work with a code C such that \hat{d} is $o(n)$. However, whatever the value of n viewed as a function of n_0 , we will always need $\hat{d} \gg n^{1/2}$ which exceeds what one obtains with straightforward constructions. In [20], H. Randriambololona showed the existence of asymptotically good Schur codes, that is, with both dimension r and product minimum Hamming distance \hat{d} linear in n . These codes will therefore suit our purposes. To be complete, we just need to show that we may incorporate the extra requirement (8). We do this below.

Existence of a suitable code. We will show that punctured subcodes of the codes of [20] satisfy all the requirements of protocol P_1 . Recall that if $C \subset \mathbb{F}_q^n$ is a linear code, and if $I \subset \{1, \dots, n\}$ is a subset of coordinate positions, then the punctured code C^I on the subset I is the set of vectors of length $n - |I|$

$$\mathbf{x}^I := (x_i)_{i \in \{1, \dots, n\} \setminus I}$$

obtained from all codewords of $\mathbf{x} = (x_1, \dots, x_n)$ of C . If d is the minimum distance of C , and if the number $|I|$ of punctured positions is $< d$, then the dimension of the punctured code C^I equals the dimension of C , and the minimum distance of C^I is at least $d - |I|$. We rely on the following lemma, that we state in a general q -ary case since we shall require it in non-binary form in the next section. Let us say that the vectors H_1, H_2, \dots, H_r in \mathbb{F}_q^n make up an *orthonormal basis* of C if they satisfy (8). We shall use the notation $(A | B)$ for the scalar product AB^\top of vectors A and B .

Lemma 8. *Let q be a power of 2 and let \mathbb{F}_q be the associated finite field. Let $C \subset \mathbb{F}_q^n$ be a linear code of dimension r and minimum distance $d > r$. Then there exists a subset $I \subset \{1, \dots, n\}$ of at most r coordinate positions, such that puncturing C on the set I yields a code of dimension r that has an orthonormal basis.*

Proof. Note that the condition $d > r$ ensures that puncturing on at most r positions does not decrease the code dimension.

We start with a systematic generating matrix of C : denoting by B_1, \dots, B_r its rows, we have, for $j = 1, 2, \dots, r$, that the j -th coordinate B_{ij} of B_i equals $B_{ij} = \delta_{ij}$.

Let $I_1 = \emptyset$ if $(B_1 | B_1) \neq 0$ and $I_1 = \{1\}$ if $(B_1 | B_1) = 0$. In both cases we therefore have $(B_1^{I_1} | B_1^{I_1}) \neq 0$, and because q is a power of 2 every element in \mathbb{F}_q is a square, and therefore we have that a non-zero multiple of B_1 , that we name H_1 , satisfies $(H_1^{I_1} | H_1^{I_1}) = 1$.

Next, suppose by induction that we have found ℓ codewords H_1, \dots, H_ℓ of C , $1 \leq \ell \leq r - 1$ and a subset $I \subset \{1, \dots, \ell\}$ such that

1. for every $1 \leq i, j \leq \ell$, $(H_i^I | H_j^I) = \delta_{ij}$
2. for every $i = 1, \dots, \ell$, for every $i + 1 \leq j \leq r$, the j -th coordinate of H_i satisfies $H_{ij} = 0$.

We show that we can add a codeword $H_{\ell+1}$ to H_1, \dots, H_ℓ and possibly add coordinate $\ell + 1$ to I , while keeping properties 1 and 2 above satisfied. This will prove

the Lemma by induction. To this end consider the linear combination

$$\Lambda = B_{\ell+1} + \sum_{i=1}^{\ell} \lambda_i H_i.$$

There clearly is a choice of $\lambda_1, \dots, \lambda_\ell \in \mathbb{F}_q$ that makes Λ^I orthogonal to H_1^I, \dots, H_ℓ^I . If $(\Lambda^I | \Lambda^I) \neq 0$ leave I unchanged, otherwise adjoin the element $\ell + 1$ to it. Property 2 ensures that the orthogonality relations $(\Lambda^I | H_i^I) = 0, i = 1 \dots \ell$ are unchanged. The required code vector is $H_{\ell+1} = \lambda \Lambda$ where λ is chosen so that $\lambda^2 (\Lambda^I | \Lambda^I) = 1$. \square

Now suppose the code C has dimension r and square distance $\hat{d} > r$. We always have $d \geq \hat{d}$ (consider $\mathbf{x} * \mathbf{x}$ where \mathbf{x} is a minimum weight codeword of C) so that Lemma 8 applies and we obtain a punctured code of C that has an orthonormal basis, that has dimension r and square distance at least $\hat{d} - r$, since it should be clear that puncturing and taking the square yields the same code as taking the square and then puncturing. In particular if we start from the codes of [20] that are guaranteed to have \hat{d} at least equal to a linear function of n , we may first take a subcode to ensure a dimension that stays linear in n but satisfies $r < \hat{d}$, and then puncturing will yield a code with square minimum distance that still behaves as a linear function of n . Actual rates are computed at the end of this section.

Protocol P_1 works under the assumption that Bob is honest. Now Bob may cheat in Step 2, and ask for any mixture of x_ℓ and y_ℓ of his choice, that may differ from (10) and (11), in an attempt to obtain some mixture of the two secrets, e.g. some bits of \mathbf{s} and some bits of \mathbf{t} , or some sums of the bits of \mathbf{s} and \mathbf{t} . The modified protocol P'_1 described next makes sure he cannot do anything of the kind.

4.2 Defeating Bob's cheating strategies

The protocol P'_1 below simply adds a compression function to the secrets \mathbf{s} and \mathbf{t} of protocol P_1 . The compression function is revealed to Bob only after protocol P_1 has been performed.

Protocol P'_1 . Consider the setting of Protocol P_1 . Alice has two secrets

$$\tilde{\mathbf{s}} = \begin{bmatrix} \tilde{s}_1 \\ \vdots \\ \tilde{s}_u \end{bmatrix}, \quad \tilde{\mathbf{t}} = \begin{bmatrix} \tilde{t}_1 \\ \vdots \\ \tilde{t}_u \end{bmatrix}$$

with coefficients in \mathbb{F}_2^m , and $u = r(\frac{1}{2} - \delta)$.

1. Alice picks uniformly at random two matrices, M_s and M_t both of dimension $r(\frac{1}{2} - \delta) \times r$, with coefficients in \mathbb{F}_2 , and two r -dimensional vectors $\mathbf{s}, \mathbf{t} \in (\mathbb{F}_2^m)^r$ such that

$$M_s \mathbf{s} = \tilde{\mathbf{s}}, \quad M_t \mathbf{t} = \tilde{\mathbf{t}}.$$

2. Alice and Bob perform Protocol P_1 with \mathbf{s} and \mathbf{t} computed above, so that Bob gets either \mathbf{s} or \mathbf{t} .
3. Finally Alice sends Bob the matrices M_s and M_t , and Bob computes

$$M_s \mathbf{s} = \tilde{\mathbf{s}} \quad (\text{or } M_t \mathbf{t} = \tilde{\mathbf{t}})$$

to get the secret he wanted.

Suppose Bob is dishonest. In Step 3 of P_1 , Bob asks Alice either x_ℓ or y_ℓ via the protocol P_0 . He could cheat by asking for some choice of x_ℓ and y_ℓ that does not correspond to (10) or (11). As a result, in Step 5, Bob will get some vector \mathbf{v} whose components v_ℓ are either x_ℓ or y_ℓ . Specifically, from (9) we have that Bob gets exactly a vector

$$\mathbf{v} = \mathbf{x} + (\mathbf{s} + \mathbf{t})^\top \mathbf{H} * \mathbf{u}$$

where \mathbf{u} is an arbitrary row-vector chosen by Bob, the column vector $\mathbf{s} + \mathbf{t}$ is chosen by Alice independently of \mathbf{x} (since \mathbf{t} can be any quantity independent of \mathbf{s}), and $\mathbf{H} * \mathbf{u}$ can be taken to be the matrix deduced from \mathbf{H} by replacing its ℓ th column by the zero column whenever $u_\ell = 0$. With this convention we have $((\mathbf{s} + \mathbf{t})\mathbf{H}) * \mathbf{u} = (\mathbf{s} + \mathbf{t})(\mathbf{H} * \mathbf{u})$.

Consider now $\mathbf{H}\mathbf{v}^\top$. We have

$$\begin{aligned} \mathbf{H}\mathbf{v}^\top &= \mathbf{H}\mathbf{x}^\top + \mathbf{H}(\mathbf{H} * \mathbf{u})^\top (\mathbf{s} + \mathbf{t}) \\ &= \mathbf{s} + V(\mathbf{s} + \mathbf{t}) \\ &= (\mathbf{I} + V)\mathbf{s} + V\mathbf{t} \end{aligned}$$

where V is the $r \times r$ matrix $\mathbf{H}(\mathbf{H} * \mathbf{u})^\top$ over \mathbb{F}_2 . We will not attempt to characterize the set of possible matrices V that arise in this way and simply assume that V can be any binary $r \times r$ matrix. Let us also write

$$\mathbf{H}\mathbf{v}^\top = U\mathbf{s} + V\mathbf{t} \tag{12}$$

and remark that knowledge of $\mathbf{s} + \mathbf{t}$ gives us $(\mathbf{s} + \mathbf{t})^\top \mathbf{H} = \mathbf{x} + \mathbf{y}$ and enables us to turn \mathbf{v} into its complement vector, i.e. with coordinate y_ℓ for every $v_\ell = x_\ell$ and with coordinate x_ℓ for every $v_\ell = y_\ell$. Therefore, for any fixed \mathbf{u} , there is a bijection between the couples $(\mathbf{x}, \mathbf{s} + \mathbf{t})$ and $(\mathbf{v}, \mathbf{s} + \mathbf{t})$, both of which live in $(\mathbb{F}_2^m)^{n+r}$. We also remark that $\mathbf{s} + \mathbf{t}$ and $\mathbf{s} + V(\mathbf{s} + \mathbf{t})$ gives us \mathbf{s} and therefore \mathbf{t} : therefore the map $(\mathbf{s}, \mathbf{t}) \mapsto (\mathbf{s} + \mathbf{t}, U\mathbf{s} + V\mathbf{t})$ is one-to-one. The conclusion is that knowledge of \mathbf{v} gives us $U\mathbf{s} + V\mathbf{t}$, and no additional knowledge on $\mathbf{s} + \mathbf{t}$, hence no additional knowledge on the couple (\mathbf{s}, \mathbf{t}) . Henceforth we forget all properties of (U, V) stemming from their particular structure, except for this last fact.

We now prove that:

Proposition 9. *One of the following holds:*

- either $\text{rank } V \leq r/2$, in which case Bob has no information on $\tilde{\mathbf{t}}$, even when he is given $\tilde{\mathbf{s}}$, meaning precisely that

$$H(\tilde{T}|\tilde{S}, \mathcal{O}) \geq rm \left(\frac{1}{2} - \delta \right) - f_1(r, \delta) - nf_0(\varepsilon, m)$$

where $f_1(r, \delta)$ is exponentially small in r , and where $f_0(\varepsilon, m)$ is from Theorem 4.

- or $\text{rank } V > r/2$, in which case Bob has no information on $\tilde{\mathbf{s}}$, even when he is given $\tilde{\mathbf{t}}$, which means

$$H(\tilde{S}|\tilde{T}, \mathcal{O}) \geq rm \left(\frac{1}{2} - \delta \right) - f_1(r, \delta) - nf_0(\varepsilon, m).$$

We will rely on the following classical lemma (e.g. [15, Ch. 14, exercise 8]):

Lemma 10. *Let B, C be $a \times b$ and $a \times c$ binary matrices respectively, and let $A = [B|C]$ be the $a \times (b + c)$ matrix that is obtained from concatenating B and C . Suppose $0 \leq b < a < b + c$. Let B be a fixed matrix of rank b , and let C be chosen randomly and uniformly among all binary $a \times c$ matrices. Then $P(\text{rank } A < a) \leq 1/2^{b+c-a}$.*

Proof of Proposition 9.

1. $\text{rank } V \leq r/2$. Suppose that Bob knows \mathbf{s} which is stronger than Bob knowing $\tilde{\mathbf{s}}$, that is

$$H(\tilde{T}|\tilde{S}, \mathcal{O}) \geq H(\tilde{T}|S, \mathcal{O}).$$

We first suppose that Bob's observation \mathcal{O} reduces to \mathbf{v} , i.e. the collection of n binary m -tuples v_ℓ that are either x_ℓ or y_ℓ , whichever he has requested when executing the ℓ -th instance of protocol P_0 . Then between $U\mathbf{s} + V\mathbf{t}$ and \mathbf{s} , Bob obtains the fixed quantity $V\mathbf{t}$ and for him \mathbf{t} is uniformly distributed among vectors of the form $\boldsymbol{\tau} + (\text{Ker } V)^m$, where we have identified V with a linear map $\mathbb{F}_2^r \rightarrow \mathbb{F}_2^r$ and where $\boldsymbol{\tau}$ is any fixed preimage of $V\mathbf{t}$. The secret $\tilde{\mathbf{t}}$ may therefore be any quantity in $M_t(\boldsymbol{\tau} + (\text{Ker } V)^m)$. Let \mathcal{M} be the set of matrices M_t such that $M_t(\text{Ker } V)$ is the full image space $\mathbb{F}_2^{r(\frac{1}{2}-\delta)}$. Since we have supposed $\text{rank } V \leq r/2$, we have $\dim \text{Ker } V \geq r/2$, and there must exist at least $r/2$ linearly independent vectors in $\text{Ker } V$. Consider the images by M_t of these $r/2$ vectors: they make up the columns of a uniform random $r(1/2 - \delta) \times r/2$ matrix, which by Lemma 10, is of full-rank $r(1/2 - \delta)$ with probability at least $1 - 2^{-r\delta}$. In other words we have $P(M_t \in \mathcal{M}) \geq 1 - 2^{-r\delta}$. We therefore have:

$$\begin{aligned}
H(\tilde{T}|S, \mathcal{O}) &= H(M_t T|S, \mathcal{O}) \\
&= P(M_t \in \mathcal{M})H(\tilde{T}|S, \mathcal{O}, M_t \in \mathcal{M}) \\
&\quad + P(M_t \notin \mathcal{M})H(\tilde{T}|S, \mathcal{O}, M_t \notin \mathcal{M}) \\
&\geq (1 - \frac{1}{2^{r\delta}})H(\tilde{T}|S, \mathcal{O}, M_t \in \mathcal{M}) = (1 - \frac{1}{2^{r\delta}})mr(\frac{1}{2} - \delta) \\
&\geq m(\frac{r}{2} - \delta) - \frac{mr}{2^{r\delta}} \\
H(\tilde{T}|S, \mathcal{O}) &\geq m(\frac{r}{2} - \delta) - f_1(r, \delta). \tag{13}
\end{aligned}$$

2. $\text{rank } V > r/2$. Bob again obtains $\mathbf{z} = U\mathbf{s} + V\mathbf{t}$, and we suppose this time that he is given $\tilde{\mathbf{t}} = M_t\mathbf{t}$. Our goal is to show that Bob obtains no information on $\tilde{\mathbf{s}} = M_s\mathbf{s}$. First consider that the possible values of \mathbf{t} given $\tilde{\mathbf{t}}$ are $\boldsymbol{\tau} + (\text{Ker } M_t)^m$, for some fixed $\boldsymbol{\tau}$ such that $M_t\boldsymbol{\tau} = \tilde{\mathbf{t}}$. The possible values of $V\mathbf{t}$ are $V\boldsymbol{\tau} + (V \text{Ker } M_t)^m$. We have $\dim(V \text{Ker } M_t) = \dim \text{Ker } M_t - \dim(\text{Ker } V \cap \text{Ker } M_t)$.

Now the kernel $\text{Ker } M_t$ is a random subspace of \mathbb{F}_2^r of dimension at least $r(\frac{1}{2} + \delta)$. Since we have supposed $\text{rank } V > r/2$ we have $\dim \text{Ker } V < r/2$. Choose a basis of $\text{Ker } V$, and add arbitrary vectors of \mathbb{F}_2^r so as to obtain a basis B of some subspace $\langle B \rangle$ of \mathbb{F}_2^r of dimension $r/2$ that contains $\text{Ker } V$. Applying Lemma 10 we obtain that $\langle B \rangle + \text{Ker } M_t$ is of full rank r with probability at least $1 - 1/2^{r\delta}$.

In this case we have $\dim(\langle B \rangle \cap \text{Ker } M_t) = \dim(\text{Ker } M_t) + \dim \langle B \rangle - r = \dim(\text{Ker } M_t) - r/2$. Since $\text{Ker } V \subset \langle B \rangle$ we also have $\dim(\text{Ker } V \cap \text{Ker } M_t) \leq \dim(\text{Ker } M_t) - r/2$. Therefore,

$$\dim(V \text{Ker } M_t) = \dim \text{Ker } M_t - \dim(\text{Ker } V \cap \text{Ker } M_t) \geq r/2.$$

Now we have that the set $\{\mathbf{s}, U\mathbf{s} + V\mathbf{t} = \mathbf{z}\}$ is an \mathbb{F}_2^m expansion of a translate of an \mathbb{F}_2 -vector space of dimension at least $\dim(V \text{Ker } M_t)$. As before, the image

under the random matrix M_s of a fixed subspace of dimension at least $r/2$ has maximum dimension $r(1/2 - \delta)$ with probability at least $1 - 2^{-r\delta}$. So with probability $(1 - 2^{-r\delta})^2$ both random matrices M_t and M_s behave as desired and \tilde{s} can be any vector in $(\mathbb{F}_2^m)^{r(\frac{1}{2}-\delta)}$ with uniform probability. Therefore similarly to (13) we obtain

$$H(\tilde{S}|T, \mathcal{O}) \geq m(\frac{r}{2} - \delta) - f_1(r, \delta). \quad (14)$$

The estimates (13) and (14) have been obtained with the assumption that Bob's observation reduces to \mathbf{v} . In the actual protocol, for every execution of protocol P_0 , Bob obtains v_ℓ , which is equal to one of the two secrets x_ℓ or y_ℓ , plus $f_0(m, \varepsilon)$ bits of information on the other secret, as guaranteed by Theorem 4. By the same argument as that preceding Corollary 7, consisting of viewing all the individual instances of Protocol P_0 as the successive instantiations of a discrete memoryless channel without feedback, we have that Bob obtains at most $nf_0(m, \varepsilon)$ additional bits of information, hence the expressions in Proposition 9. \square

Rate of the oblivious transfer protocol P'_1 . The protocol P'_1 is instantiated with two secrets of length $u = r(\frac{1}{2} - \delta)$, where r is the length of the secrets in the protocol P_1 and δ is a positive number that can be taken arbitrarily close to 0. Thus, the rate of P'_1 can be arbitrarily close to

$$\mathcal{R}'_1 = \frac{\mathcal{R}_1}{2}.$$

Now the protocol P_1 requires n uses of the protocol P_0 , where n is the length of the $[n, r, d]$ code C . The total length of a secret is r times the length of a secret of P_0 . Therefore the overall rate of the protocol P_1 is

$$\mathcal{R}_1 = R\mathcal{R}_0$$

where \mathcal{R}_0 is the rate of P_0 and $R = r/n$ is the rate of the code C . From Section 3 we have that \mathcal{R}_0 can be taken arbitrarily close to the limiting value $\mathcal{R}_0 = 0.108$, and from [20] we have an infinite family of linear codes of length n and square minimum distance $\hat{d} \geq n/1575$ and dimension $> \hat{d}$. From the discussion at the end of Section 4.1 we get that a punctured version of the codes of [20] with a rate R arbitrarily close to $1/1575$ will satisfy all the conditions of protocol P_1 . We get therefore the achievable rate:

$$\mathcal{R}_1 = \frac{0.108}{1575} \approx 0.69 \cdot 10^{-4}$$

and hence $\mathcal{R}'_1 \approx 0.34 \cdot 10^{-4}$.

4.3 Summary and Comments

Let N denote the total number of bits sent over the noisy channel during protocol P'_1 . We have $N = 4n_0n$, where the number of bits sent by Alice over the noisy channel is $4n_0$ for each instance of protocol P_0 , and n is the number of times protocol P_0 is repeated. We have set $n = n_0^2$ to be specific, but this is somewhat arbitrary, and any value $n = n_0^\alpha$, $\alpha > 1$ would yield similar asymptotic guarantees. These guarantees are the following:

1. When Bob follows the protocol precisely, he obtains the secret he wishes for with probability at least $1 - \exp(-N^\alpha)$ for some $0 < \alpha < 1$. This is obtained by using polar codes in protocol P_0 that achieve the capacity of the least noisy (in effect erasureless) channel. We recall that this family of codes is constructive and can be decoded in quasi-linear time, with a probability of a decoding error that is guaranteed to be subexponential in the blocklength, i.e. $\exp(-n_0^\beta)$, see [23, Theorem 1]. The probability of a decoding error on at least one of the instances of P_0 scales therefore as $\exp(-N^\alpha)$.
2. Theorem 4 and Proposition 9 guarantee that whatever Bob does, on at least one of the two secrets he obtains at most a vanishing number of bits of information, that behaves like $\exp(-cm^\alpha)$ where m is the secret size, $0 < \alpha < 1$, and c is a constant that is determined by how close we are to the limiting rate R'_1 computed in the last subsection. In other words, c is dependent on the values of ε and δ in Theorem 4 and Proposition 9.
3. The protocol is protected against a cheating Alice in the following sense: Bob will declare Alice a cheater if he receives a total number of erased symbols that exceeds a certain threshold τ . If Alice cheats so as to uncover which of the two secrets Bob is trying to obtain, she will almost surely be accused, i.e. with probability $1 - \exp(-n_0) = 1 - \exp(-N^{1/3})$. It may happen that an honest Alice will be wrongly accused of cheating by Bob, but this happens with a vanishingly small probability that scales as $\exp(-n_0) = \exp(-N^{1/3})$.

In the next section we improve upon the rate R'_1 computed in Section 4.2 by modifying protocol P_1 , so as to allow us to replace the binary code C by a q -ary one: codes with large square distances are easier to construct in the q -ary case and better rates are obtained. The techniques require to measure leakage of information to Bob and the probabilities of Alice successfully cheating without being found out, or of Alice being wrongly accused of cheating are unchanged, and we will compute the new limiting rates without explicitly mentioning that we need to be ε and δ away from them as in Theorem 4 and Proposition 9.

5 A Positive Rate q -ary Oblivious Transfer Protocol

Let q be a power of 2. Protocol P_2 below is a q -ary variant of protocol P_1 : it relies upon a version of protocol P_0 that is a 1-out-of- q oblivious transfer protocol, that, like P_0 does not protect against Alice's cheating strategy. Let us denote P_0^q this protocol: we shall show later in this section how to transform the original protocol P_0 into its q -ary version P_0^q . The matrix \mathbf{H} is this time a matrix over the field on q elements, it is the generating matrix of a code C , with otherwise the same requirements as in the binary case, namely that the rows of \mathbf{H} make up an orthonormal basis of C and the square distance \hat{d} of C grows linearly in n . The integer m is now the length of the q secrets in protocol P_0^q .

Protocol P_2 . Consider the same setting as Protocol P_1 , over \mathbb{F}_q instead of \mathbb{F}_2 , that is the $r \times n$ generating matrix \mathbf{H} of the code C has coefficients in \mathbb{F}_q , and the two secrets \mathbf{s} and \mathbf{t} have length r , with coefficients in \mathbb{F}_q^m .

1. As in protocol P_1 , Alice picks uniformly at random a vector $\mathbf{x} \in (\mathbb{F}_q^m)^n$ such that

$$\mathbf{H}\mathbf{x}^\top = \mathbf{s}.$$

She then computes the $q-1$ vectors \mathbf{y}_i , $i = 2, \dots, q-1$, with coefficients in \mathbb{F}_q^m , given by

$$\mathbf{y}_i = \mathbf{x} + \lambda_i \sum_{j=1}^r (s_j + t_j) H_j,$$

where λ_i runs through every non-zero element of \mathbb{F}_q . We set $\lambda_1 = 1$, so that \mathbf{y}_1 coincides with \mathbf{y} in protocol P_1 . We write $\mathbf{y}_i = [y_{i1}, y_{i2}, \dots, y_{in}]$.

2. Bob computes a q -ary vector $\mathbf{u} = [u_1, \dots, u_n]$ which is orthogonal to \hat{C} .
3. If Bob wants the secret \mathbf{s} , then he asks Alice through the protocol P_0^q for the vector $\mathbf{x} + \mathbf{u} * (\mathbf{x} + \mathbf{y}_1)$. Equivalently, whenever $u_\ell = \lambda_i$, Bob asks for the string $y_{i\ell}$. If Bob wants the secret \mathbf{t} instead, he asks for the vector $\mathbf{y}_1 + \mathbf{u} * (\mathbf{x} + \mathbf{y}_1)$.
4. After n rounds of the protocol P_0^q , Alice has sent Bob the requested n -tuple $\mathbf{v} = [v_1, \dots, v_n]$ which has again exactly the expression given by (10) and (11).
5. Once Bob gets \mathbf{v} , he computes $\mathbf{H}\mathbf{v}^\top$ to recover \mathbf{s} or \mathbf{t} .

Protocol P'_2 .

The protocol is obtained from protocol P_2 in exactly the same way as protocol P'_1 is obtained from P_1 , with the two secrets $\tilde{\mathbf{s}}$ and $\tilde{\mathbf{t}}$ of length $u = r(\frac{1}{2} - \delta)$ having their coefficients in \mathbb{F}_q^m .

We first argue that Bob will indeed recover the secrets by computing $\mathbf{H}\mathbf{v}^\top$. The proof from Protocol P_1 carries through quite straightforwardly. As previously, we have that $\mathbf{x} + \mathbf{y}_1$ belongs to the code C , and since \mathbf{u} is orthogonal to \hat{C} , we have that the scalar product of a row of \mathbf{H} with $\mathbf{u} * (\mathbf{x} + \mathbf{y}_1)$ is equal to the scalar product of \mathbf{u} with the $*$ -product of two vectors of C , hence equals zero. Therefore $\mathbf{H}\mathbf{v}^\top$ is always equal to either $\mathbf{H}\mathbf{x}^\top$ or $\mathbf{H}\mathbf{y}_1^\top$. We have $\mathbf{H}\mathbf{x}^\top = \mathbf{s}$ by choice of \mathbf{x} and $\mathbf{H}\mathbf{y}_1^\top = \mathbf{t}$ through the orthonormal property of the rows H_i of \mathbf{H} .

A 1-out-of- q oblivious transfer P_0^q . Next, a 1-out-of- q oblivious transfer protocol is needed for Bob to obtain every v_i , $i = 1, \dots, n$. It may be obtained by $q - 1$ applications of the 1-out-of-2 oblivious transfer protocol P_0 , as in [4]. If x_1, x_2, \dots, x_q are the secrets to be transferred, Alice chooses $q - 1$ random strings r_1, r_2, \dots, r_{q-1} uniformly among all strings such that $r_1 + r_2 + \dots + r_{q-1} = x_q$. Then protocol P_0 is applied to the $q - 1$ pairs of secrets

$$(x_1, r_1), (x_2 + r_1, r_2), \dots, (x_i + r_1 + \dots + r_{i-1}, r_i), \dots, (x_{q-1} + r_1 + \dots + r_{q-2}, r_{q-1}).$$

We see that if Bob asks for the first term $x_i + r_1 + \dots + r_{i-1}$ of the i -th pair, he loses all chance of obtaining r_i , and the subsequent x_j , $j > i$. He can obtain x_i by asking for the second term, r_j , of the preceding pairs for $j = 1 \dots i - 1$.

Construction of the required code C . We need a code with an orthonormal basis and a large square distance \hat{d} . From Lemma 8 and the discussion just afterwards, such a code will be obtained as soon as we have a code of dimension $k = r$ and large square distance \hat{d} with $k < \hat{d}$. We also want this code to have the largest possible dimension, so we try to obtain the code with the largest possible square distance \hat{d} satisfying $k \geq \hat{d}$, from which we will then take a subcode so as to have $k < \hat{d}$.

We turn to algebraic geometry codes (see e.g. [13, 22]). Consider codes $C(D, G)$ of length n over \mathbb{F}_q , defined to be the image of the linear evaluation map $ev : L(G) \rightarrow \mathbb{F}_q^n$, $f \mapsto ev(f) = (f(P_1), \dots, f(P_n))$, where $D = P_1 + \dots + P_n$ is a divisor on an algebraic curve \mathcal{X} for the rational points P_1, \dots, P_n , $\mathbb{F}(\mathcal{X})$ is the function

field of the curve \mathcal{X} , and $L(G) = \{f \in \mathbb{F}(\mathcal{X})^*, \sum_{P \in \mathcal{X}} \nu_P(f)P + G \geq 0\} \cup \{0\}$, and G some other divisor whose support is disjoint from D .

As observed in [21, Lemma 14], we have

$$\hat{C}(D, G) = C(D, G) * C(D, G) \subset C(D, 2G). \quad (15)$$

Indeed, for $c, c' \in C(D, G)$, we have

$$c * c' = ev(f) * ev(f') = (f(P_1)f'(P_1), \dots, f(P_n)f'(P_n)) = ev(ff'),$$

and $ff' \in \mathbb{F}(\mathcal{X})$ with

$$\sum_{P \in \mathcal{X}} \nu_P(ff')P + 2G = \left(\sum_{P \in \mathcal{X}} \nu_P(f)P + G \right) + \left(\sum_{P \in \mathcal{X}} \nu_P(f')P + G \right) \geq 0$$

showing that $ff' \in L(2G)$.

Now the parameters of the evaluation code $C(D, G)$ are known to satisfy, when the degree $\deg G$ of the divisor G is strictly less than n , [22, Cor. II.2.3]

$$d \geq n - \deg G \quad \text{and} \quad k \geq \deg G + 1 - g$$

where g is the genus of the algebraic curve. From (15) we therefore also have, as long as $2 \deg G < n$,

$$\hat{d} \geq n - 2 \deg G.$$

To have $k = \hat{d}$ we shall therefore aim for a divisor G of degree satisfying

$$n = 3 \deg G + 1 - g. \quad (16)$$

Rate of the oblivious transfer Protocols P'_2 . Let us first establish the rate R of the code C . We will obtain the best result for $q = 16$. The Tsfasman-Vladut-Zink bound [13, 22] tells us that we may choose curves with genus g such that $n \rightarrow \infty$ and n/g is arbitrarily close to $\sqrt{q} - 1 = 3$. Choosing $\deg G$ as in (16) gives us $\frac{1}{n} \deg G \rightarrow 4/9$ and $k \geq \deg G + 1 - g$ gives us a rate at least $1/9$ for the code. The actual code used in the protocol is possibly a punctured version, but since the rate can only increase by puncturing up to the minimum distance, we may guarantee a rate arbitrarily close to $R = 1/9$ for the code C .

Now the overall rate of the protocol P_2 is

$$\begin{aligned}\mathcal{R}_2 &= \frac{2rm}{n\# \text{ channel uses for } P_0^q} \\ &= \frac{2rm}{n(q-1)\# \text{ channel uses for } P_0} \\ &= \frac{1}{q-1}R\mathcal{R}_0\end{aligned}$$

where \mathcal{R}_0 is the rate of protocol P_0 . Hence,

$$\mathcal{R}_2 = \frac{1}{9 \times 15} 0.108 = 0.8 \cdot 10^{-3}.$$

The rate \mathcal{R}'_2 of the protocol P'_2 is $\mathcal{R}'_2 = \mathcal{R}_2/2$, hence

$$\mathcal{R}'_2 = 0.4 \cdot 10^{-3}.$$

6 Concluding comments

Binary codes C with large rate and such that \hat{C} has a large minimum distance would of course yield improved rates for protocol P'_1 . How large can these rates be is a very intriguing question.

Apart from exhibiting codes C with some extraordinary \hat{C} behaviour, improving upon the rates of this paper probably involves some alternative approaches to the problem, or other ways of using the potential of q -ary codes.

As discussed in Section 4.3, the probabilities of something going wrong (Bob does not get his secret, Alice cheats, Bob falsely accuses Alice of cheating) are subexponential in the total number N of transmitted bits. An interesting avenue of research would be to find explicit achievable rates that guarantee an exponential behaviour for these failure probabilities.

Appendix: Proof of Lemma 3

Recall that the channel randomly introduces e erasures and a number of errors on the non-erased symbols. Let us first modify slightly the channel by assuming

a fixed number w of errors, chosen uniformly among the $\binom{n-e}{w}$ possible choices, instead of binomially distributed errors.

Build a bipartite graph, consisting of the 2^{nR} codewords as vertices on the left, and on the right all possible ternary strings over the alphabet $\{0, 1, *\}$ with e erasures (an erasure is represented by the $*$ symbol). thus the right hand side of the graph has

$$\binom{n}{e} 2^{n-e}$$

vertices. We put an edge connecting a codeword, i.e. a left vertex, to a ternary string if the ternary string can be obtained from the codeword by erasing e coefficients, and flipping w others. There are thus $\binom{n}{e} \binom{n-e}{w}$ outgoing edges from every codeword node, and a total of

$$2^{nR} \binom{n}{e} \binom{n-e}{w} \quad (17)$$

edges connecting the left and right sides of the bipartite graph. Define r such that

$$\binom{n}{e} 2^{n-e} 2^r = 2^{nR} \binom{n}{e} \binom{n-e}{w} \quad (18)$$

and assume parameters have been chosen such that r is positive.

Now Alice picks a codeword c uniformly at random and sends it over the channel: one error pattern of weight w and e erasures will happen, all of them are equally likely. This means that after transmission an edge of the graph has been chosen uniformly among the total number (17) of edges. We now argue that most edges are connected to a right vertex with large degree.

Let $\alpha > 0$. The number of edges connected to binary strings on the right of the graph, whose degree is smaller than $2^{r-\alpha}$ is $N_1 + 2N_2 + 3N_3 + \dots + 2^{r-\alpha} N_{2^{r-\alpha}}$ where N_i counts the number of right nodes whose degree is i , and $\sum N_i = \binom{n}{e} 2^{n-e}$. Since $i \leq 2^{r-\alpha}$ for every i , $N_1 + 2N_2 + 3N_3 + \dots + 2^{r-\alpha} N_{2^{r-\alpha}} \leq 2^{r-\alpha} \sum N_i = 2^{r-\alpha} \binom{n}{e} 2^{n-e}$. The number of edges connected to right nodes whose degree is bigger than $2^{r-\alpha}$ is then bounded from below by

$$\binom{n}{e} (2^{n-e} 2^r - 2^{r-\alpha} 2^{n-e}) = \binom{n}{e} 2^{n-e} 2^r (1 - 2^{-\alpha}).$$

This shows that for any $0 < \alpha \leq r$, with probability at least $1 - 1/2^\alpha$, Bob receives a vector v such that, for any codeword c ,

$$P(C_X = c | Z = z) \leq \frac{1}{2^{r-\alpha}}$$

where C_X is the input to the channel, with uniform distribution on the code C , and Z is the random variable consisting of the received vector.

If the number w of errors equals the expected number of errors for $n-e$ transmitted bits over a BSC, i.e. $w = p(n-e)$, then we get from (18)

$$r = R - (1 - e/n)(1 - h(p)) + o(1).$$

This proves a version of Lemma 3 for a constant, rather than binomially distributed number of errors. For a binomially distributed number of errors, i.e. the result of an actual binary symmetric channel, we may proceed as above by making the bipartite graph weighted. We put an edge for every possible number of errors w , $0 \leq w \leq n-e$, and associate to every such edge the weight $p^w(1-p)^{n-e-w}$. Concentration of measure around the mean number $p(n-e)$ of errors ensures the same behaviour as in the constant error case: we leave out the cumbersome details.

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