On converse bounds for classical communication over quantum channels

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We explore several new converse bounds for classical communication over quantum channels in both the one-shot and asymptotic regimes. First, we show that the Matthews-Wehner meta-converse bound for entanglement-assisted classical communication can be achieved by activated, no-signalling assisted codes, suitably generalizing a result for classical channels. Second, we derive a new efficiently computable meta-converse on the amount of classical information unassisted codes can transmit over a single use of a quantum channel. As applications, we provide a finite resource analysis of classical communication over quantum erasure channels, including the second-order and moderate deviation asymptotics. Third, we explore the asymptotic analogue of our new meta-converse, the Υ-information of the channel. We show that its regularization is an upper bound on the classical capacity, which is generally tighter than the entanglement-assisted capacity and other known efficiently computable strong converse bounds. For covariant channels we show that the Υ-information is a strong converse bound.

I. INTRODUCTION

One of the central problems in quantum information theory is to determine the capability of a noisy quantum channel to transmit classical messages faithfully. The classical capacity of a quantum channel is the highest rate (in bits per channel use) at which it can convey classical information such that the error probability vanishes asymptotically as the code length increases. The Holevo-Schumacher-Westmoreland (HSW) theorem [1–3] establishes that the classical capacity of a noisy quantum channel is given by its regularized Holevo information.

However, in realistic settings, there are natural restrictions imposed on the code length. One fundamental question thus asks how much classical information can be transmitted over a single use of a quantum channel when a finite decoding error is tolerated. Of particular interest is the converse bound given by Polyanskiy, Poor and Verdú (PPV) for classical channels [4]. Their bound, named as "meta-converse", was established based on hypothesis testing and it limits the performance of a coding scheme given fixed resources. They showed by numerical examples that the bound is quite tight for several channels of interest, even at small blocklengths. Since then, converse bounds with a similar structure to the PPV bound are also called meta-converse. For quantum channels, Matthews and Wehner [5] extended the hypothesis testing approach to the task of transmitting classical bits over quantum channels and formulated converse bounds for codes with or without entanglement assistance. Several other upper and lower bounds on the one-shot classical capacity were explored, e.g. in [6–9], but these in general do not match and are

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often hard to compute.

In Section III we build on an exact expression, provided in [10], for the amount of classical information that can be transmitted over a single use of a quantum channel using codes that are assisted by no-signalling correlations. Using this result we show that the hypothesis testing relative entropy converse bound by Matthews and Wehner [5] can be achieved and is optimal for activated, no-signalling assisted codes. This generalizes to the quantum setting a result by Matthews [11] for no-signalling assisted classical codes, with the additional twist that in the quantum setting the codes require a classical noiseless channel as a catalyst.

In Section IV we provide a new efficiently computable (as a semi-definite program) meta-converse that upper bounds the amount of information that can be transmitted with a single use of the channel by unassisted codes. This meta-converse, in the spirit of the classical meta-converse by Polyanskiy, Poor and Verdú [4], relates the channel coding problem to a binary composite hypothesis test between the actual channel and a class of subchannels that are generalizations of the useless channels for classical communication. As a simple application, in Section VI, we apply our meta-converse to establish second-order asymptotics [12] and moderate deviation asymptotics [13, 14] for the classical capacity of the quantum erasure channel.

In Section V we give a new upper bound for the classical capacity of quantum channels inspired by our meta-converse, which we call Υ-information of the channel. We again interpret this bound as a relative entropy distance between the quantum channel and a class of useless completely positive trace non-increasing maps. We show that the regularized Υ-information is a weak converse bound that is always smaller than the entanglement-assisted classical capacity and the semi-definite program strong converse bound in [10]. Furthermore, for covariant channels, we show that the Υ-information is in fact a strong converse bound.

II. UNASSISTED, ENTANGLEMENT-ASSISTED AND NO-SIGNALLING ASSISTED CODES

For our purposes, a quantum channel $\mathcal{N}_{A'\to B}$ is a completely positive (CP) and trace-preserving (TP) linear map from operators on a finite-dimensional Hilbert space A' to operators on a finite-dimensional Hilbert space B. We are interested in sending classical messages from Alice to Bob via a given quantum channel $\mathcal{N}$. The usual coding scheme is as follows. Alice encodes her message via an operation $\mathcal{E}_{A\to A'}$ and sends the encoded message to Bob through the channel $\mathcal{N}_{A'\to B}$. After receiving the message, Bob performs an operation $\mathcal{D}_{B\to B'}$ to decode it. More generally, instead of considering the encoding and decoding operations separately, one could imagine the coding protocol as a single super-operator $\Pi_{AB\to A'B'}$. The authors of Ref. [15] showed that a two-input and two-output CPTP map $\Pi_{AB\to A'B'}$ sends any CPTP map $\mathcal{N}_{A'\to B}$ to another CPTP map $\mathcal{M}_{A\to B'}$ if and only if $\Pi_{AB\to A'B'}$ is B to A no-signalling (see also [16]). We denote by $\mathcal{M}_{A\to B'} = \Pi_{AB\to A'B'} \circ \mathcal{N}_{A'\to B}$ the resulting composite channel of the super-operator $\Pi_{AB\to A'B'}$ and the channel $\mathcal{N}_{A'\to B}$. Then the classical communication task is equivalent to Alice sending the classical messages to Bob using the effective channel $\mathcal{M}_{A\to B'}$. We say Π is an Ω-assisted code if it can be implemented by local operations with Ω-assistance. In the following, we eliminate Ω for the case of unassisted codes and write $\Omega = \mathrm{E}$ and $\Omega = \mathrm{NS}$ for entanglement-assisted and no-signalling-assisted (NS-assisted) codes, respectively. In particular,

- an unassisted code reduces to the product of encoder and decoder, i.e., $\Pi = \mathcal{D}_{B\to B'}\mathcal{E}_{A\to A'}$;
- an entanglement-assisted code corresponds to a superchannel of the form $\Pi = \mathcal{D}_{B\widehat{B}\to B'}\mathcal{E}_{A\widehat{A}\to A'}\Psi_{\widehat{A}\widehat{B}}$, where $\Psi_{\widehat{A}\widehat{B}}$ can be any entangled state shared between Alice and Bob;
- a NS-assisted code corresponds to a superchannel which is no-signalling from Alice to Bob and vice-versa.

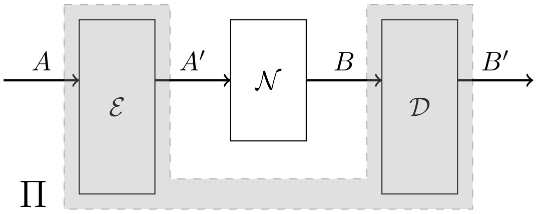

FIG. 1: General code scheme

Given a quantum channel $\mathcal{N}_{A\to B}$ and any Ω-assisted code Π with size m, the optimal average success probability of $\mathcal{N}$ to transmit m messages is given by

$$p_{\text{succ},\Omega}(\mathcal{N}, m) := \frac{1}{m} \sup \sum_{k=1}^{m} \operatorname{Tr} \mathcal{M}(|k\rangle\langle k|)|k\rangle\langle k|, \\ \text{s.t. } \mathcal{M} = \Pi \circ \mathcal{N} \text{ is the effective channel.} \tag{1}$$

With this in hand, we now say that a triplet (r, n, ε) is achievable on the channel $\mathcal{N}$ with Ω-assisted codes if

$$\frac{1}{n} \log m \geq r, \text{ and } p_{\text{succ},\Omega}(\mathcal{N}^{\otimes n}, m) \geq 1 - \varepsilon. \tag{2}$$

Throughout the paper we take the logarithm to be base two unless stated otherwise. We are interested in the following boundary of the non-asymptotic achievable region:

$$C_{\Omega}^{(1)}(\mathcal{N}, \varepsilon) := \sup \left\{ \log m \,\middle|\, p_{\text{succ},\Omega}(\mathcal{N}, m) \geq 1 - \varepsilon \right\}. \tag{3}$$

We also define $p_{\text{succ},\Omega}(\mathcal{N}, \rho_A, m)$ and $C_{\Omega}^{(1)}(\mathcal{N}, \rho_A, \varepsilon)$ as the same optimization but only using codes with a fixed average input ρ_A. The Ω-assisted classical capacity of a quantum channel is

$$C_{\Omega}(\mathcal{N}) = \lim_{\varepsilon \to 0} \lim_{n\to\infty} \frac{1}{n} C_{\Omega}^{(1)}(\mathcal{N}^{\otimes n}, \varepsilon). \tag{4}$$

III. MATTHEWS-WEHNER CONVERSE VIA ACTIVATED, NO-SIGNALLING ASSISTED CODES

For classical communication over quantum channels assisted by entanglement, Matthews and Wehner [5] proved a meta-converse bound $R(\mathcal{N}, \varepsilon)$ in terms of the hypothesis testing relative entropy which generalizes Polyanskiy, Poor and Verdú's approach [4] to quantum channels assisted by entanglement. Given a quantum channel $\mathcal{N}$, they proved that [5] $C_{E}^{(1)}(\mathcal{N}, \varepsilon) \leq R(\mathcal{N}, \varepsilon)$ where

$$R(\mathcal{N}, \varepsilon) := \max_{\rho_{A'}} \min_{\sigma_B} D_H^{\varepsilon}(\mathcal{N}_{A\to B}(\phi_{A'A}) \| \rho_{A'} \otimes \sigma_B), \tag{5}$$

$\phi_{AA'} = (\mathbb{1}_A \otimes \rho_{A'}^{1/2}) \widetilde{\Phi}_{AA'} (\mathbb{1}_A \otimes \rho_{A'}^{1/2})$ is a purification of $\rho_{A'}$ and $\widetilde{\Phi}_{AA'} = \sum_{ij} |i_A i_{A'}\rangle\langle j_A j_{A'}|$ denotes the unnormalized maximally entangled state. In the above expression the quantum hypothesis testing relative entropy is defined as [7] $D_H^{\varepsilon}(\rho_0 \| \rho_1) := -\log \beta_{\varepsilon}(\rho_0 \| \rho_1)$ with $\beta_{\varepsilon}(\rho_0 \| \rho_1) = \min \left\{ \operatorname{Tr} Q\rho_1 \,\middle|\, 1 - \operatorname{Tr} Q\rho_0 \leq \varepsilon, 0 \leq Q \leq \mathbb{1} \right\}$, which is the minimum type-II error for the test while the type-I error is no greater than ε. Note that β_{ε} is a fundamental quantity in quantum theory

[17–19] with many applications (e.g., [7, 20–26]) and can be solved by a semi-definite program (SDP). The Matthews-Wehner bound in Eq. (5) thus constitutes an SDP itself, i.e.

$$\begin{aligned} R(\mathcal{N},\varepsilon) = -\log \quad \underset{F_{AB},\,\rho_A,\,\lambda}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}\rho_A = 1, \\ & \operatorname{Tr}_A F_{AB} \le \lambda \mathbb{1}_B \\ & \operatorname{Tr} J_{\mathcal{N}} F_{AB} \ge 1-\varepsilon. \end{aligned} \tag{6}$$

Here the Choi-Jamiołkowski matrix [27, 28] of $\mathcal{N}$ is given by $J_{\mathcal{N}} = \sum_{ij} |i_A\rangle\langle j_A| \otimes \mathcal{N}(|i_{A'}\rangle\langle j_{A'}|)$, where $\{|i_A\rangle\}$ and $\{|i_{A'}\rangle\}$ are orthonormal bases on isomorphic Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_{A'}$, respectively.

For classical channels, the Matthews-Wehner bound is exactly equal to the one-shot classical capacity assisted by NS codes [11]. For quantum channels the one-shot ε-error capacity assisted by NS codes is given by [10]

$$\begin{aligned} C^{(1)}_{\text{NS}}(\mathcal{N},\varepsilon) = -\log \quad \underset{F_{AB},\,\rho_A,\,\eta}{\text{minimize}} \quad & \eta \\ \text{subject to} \quad & 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}\rho_A = 1, \\ & \operatorname{Tr}_A F_{AB} = \eta \mathbb{1}_B, \\ & \operatorname{Tr} J_{\mathcal{N}} F_{AB} \ge 1-\varepsilon. \end{aligned} \tag{7}$$

Note that the only difference between the SDPs (6) and (7) is the partial trace constraint of F_{AB}. However, unlike in the classical special case, the SDPs in (6) and (7) are not equal in general [10].

In this section we show that this gap can be closed by considering activated, NS-assisted codes. The concept of activated capacity follows the idea of potential capacities of quantum channels [29–31]. The model is described as follows. For a quantum channel $\mathcal{N}$ assisted by NS codes, we can first borrow a noiseless classical channel $\mathcal{I}_m$ whose capacity is $\log m$, then we can use $\mathcal{N} \otimes \mathcal{I}_m$ to transmit classical messages. After the communication finishes, we just pay back the capacity of $\mathcal{I}_m$. The code scheme in this scenario is what we call activated code. Note that this kind of communication method was also studied in zero-error information theory [32, 33].

Definition 1 *For any quantum channel $\mathcal{N}$, we define*

$$C^{(1)}_{\text{NS,a}}(\mathcal{N},\varepsilon) := \sup_{m\ge 1} \left[C^{(1)}_{\text{NS}}(\mathcal{N}\otimes\mathcal{I}_m,\varepsilon) - \log m \right], \tag{8}$$

where $\mathcal{I}_m(\rho) := \sum_{i=1}^m \operatorname{Tr}(\rho|i\rangle\langle i|)|i\rangle\langle i|$ the classical noiseless channel with capacity $\log m$.

The following is the main result of this section.

Theorem 2 *For any quantum channel $\mathcal{N}_{A\to B}$ and error tolerance $\varepsilon \in (0,1)$, we have*

$$C^{(1)}_{\text{NS,a}}(\mathcal{N},\varepsilon) = R(\mathcal{N},\varepsilon). \tag{9}$$

The proof outline is as follows. We first show that $\mathcal{I}_2$ is enough to activate the channel to achieve the bound $R(\mathcal{N},\varepsilon)$ in the following Lemma 3, i.e.,

$$C^{(1)}_{\text{NS,a}}(\mathcal{N},\varepsilon) \ge C^{(1)}_{\text{NS}}(\mathcal{N}\otimes\mathcal{I}_2,\varepsilon) - 1 \ge R(\mathcal{N},\varepsilon). \tag{10}$$

We then show that $R(\mathcal{N},\varepsilon)$ is additive for noiseless channel in the following Lemma 4, i.e.,

$$R(\mathcal{N}\otimes\mathcal{I}_m,\varepsilon)=R(\mathcal{N},\varepsilon)+\log m. \tag{11}$$

This implies that $R(\mathcal{N},\varepsilon)$ is also a converse bound for the activated capacity, i.e.,

$$\begin{aligned} C^{(1)}_{\mathrm{NS,a}}(\mathcal{N},\varepsilon) &= \sup_{m\geq 1}\left[C^{(1)}_{\mathrm{NS}}(\mathcal{N}\otimes\mathcal{I}_m,\varepsilon)-\log m\right] && (12)\\ &\leq \sup_{m\geq 1}\left[R(\mathcal{N}\otimes\mathcal{I}_m,\varepsilon)-\log m\right] && (13)\\ &= R(\mathcal{N},\varepsilon). && (14)\end{aligned}$$

Then Theorem 2 directly follows from Lemmas 3 and 4.

Lemma 3 *We have* $C^{(1)}_{\mathrm{NS}}(\mathcal{N}\otimes\mathcal{I}_2,\varepsilon)-1\geq R(\mathcal{N},\varepsilon)$.

Proof This proof is based on a key observation that the additional one-bit noiseless channel can provide a larger solution space to help the activated capacity achieve the quantum hypothesis testing converse. The dual SDP of $R(\mathcal{N},\varepsilon)$ is given in the following Eq. (23). By Slater's theorem [34], the strong duality holds. Suppose that the optimal solution to SDP (6) of $R(\mathcal{N},\varepsilon)$ is $\{\lambda,\rho_{A_1},F_{A_1B_1}\}$. We are going to use this optimal solution to construct a feasible solution of the SDP (7) of $C^{(1)}_{\mathrm{NS}}(\mathcal{N}\otimes\mathcal{I}_2,\varepsilon)$.

Let us choose

$$\begin{aligned} \rho_{A_1A_2} &= \rho_{A_1}\otimes\frac{1}{2}(|0\rangle\langle 0|+|1\rangle\langle 1|)_{A_2}, \quad \text{and} && (15)\\ F_{A_1A_2B_1B_2} &= \frac{1}{2}F_{A_1B_1}\otimes G_{A_2B_2}+\frac{1}{2}\widetilde{F}_{A_1B_1}\otimes\widetilde{G}_{A_2B_2}, && (16)\end{aligned}$$

$$\begin{aligned} \text{with} \qquad G_{A_2B_2} &= (|00\rangle\langle 00|+|11\rangle\langle 11|)_{A_2B_2}, && (17)\\ \widetilde{G}_{A_2B_2} &= (|01\rangle\langle 01|+|10\rangle\langle 10|)_{A_2B_2}, && (18)\\ \widetilde{F}_{A_1B_1} &= \rho_{A_1}\otimes(\lambda\mathbb{1}_{B_1}-\mathrm{Tr}_{A_1}F_{A_1B_1}). && (19)\end{aligned}$$

We see that $F_{A_1A_2B_1B_2}\geq 0$, $\rho_{A_1A_2}\geq 0$ and $\mathrm{Tr}\,\rho_{A_1A_2}=1$. Moreover, this construction ensures that

$$\begin{aligned} &\mathrm{Tr}_{A_1A_2}F_{A_1A_2B_1B_2}\\ &\quad=\frac{1}{2}\mathrm{Tr}_{A_1}\left[(F_{A_1B_2}+\widetilde{F}_{A_1B_1})\otimes\mathbb{1}_{B_2}\right]=\frac{\lambda}{2}\mathbb{1}_{B_1B_2}, && (20)\end{aligned}$$

and

$$\begin{aligned} &\mathrm{Tr}(J_{\mathcal{N}}\otimes D_{A_2B_2})F_{A_1A_2B_1B_2}\\ &\quad=\frac{1}{2}\mathrm{Tr}\,J_{\mathcal{N}}F_{A_1B_1}\otimes\mathrm{Tr}\,D_{A_2B_2}G_{A_2B_2} && (21)\\ &\quad=\mathrm{Tr}\,J_{\mathcal{N}}F_{A_1B_1}\geq 1-\varepsilon, && (22)\end{aligned}$$

where $D_{A_2B_2}=\sum_{i=0}^{1}|ii\rangle\langle ii|$ is the Choi-Jamiołkowski matrix of $\mathcal{I}_2$. Furthermore, $\rho_{A_1}\otimes\mathbb{1}_{B_1}-\widetilde{F}_{A_1B_1}\geq 0$ and consequently we find that $\rho_{A_1A_2}\otimes\mathbb{1}_{B_1B_2}-F_{A_1A_2B_1B_2}\geq 0$. Hence, $\left\{\frac{1}{2}\lambda,\rho_{A_1A_2},F_{A_1A_2B_1B_2}\right\}$ is a feasible solution, ensuring that $C^{(1)}_{\mathrm{NS}}(\mathcal{N}\otimes\mathcal{I}_2,\varepsilon)-1\geq R(\mathcal{N},\varepsilon)$. ∎

Lemma 4 *We have* $R(\mathcal{N}\otimes\mathcal{I}_m,\varepsilon)=R(\mathcal{N},\varepsilon)+\log m$.

Proof On the one hand, it is easy to prove that $R(\mathcal{N}\otimes\mathcal{I}_m,\varepsilon)\geq R(\mathcal{N},\varepsilon)+\log m$. To see the other direction, we are going to use the dual SDP of $R(\mathcal{N},\varepsilon)$:

$$\begin{aligned} R(\mathcal{N},\varepsilon) = -\log \underset{X_{AB},\,Y_B,\,s,\,t}{\text{maximize}}\ & [s(1-\varepsilon)-t] \\ \text{subject to}\ & X_{AB}+\mathbb{1}_A\otimes Y_B \geq sJ_{\mathcal{N}}, \\ & \operatorname{Tr}_B X_{AB}\leq t\mathbb{1}_A, \\ & \operatorname{Tr} Y_B \leq 1, \\ & X_{AB},\, Y_B,\, s\geq 0. \end{aligned} \tag{23}$$

We note that the strong duality holds here by Slater's theorem [34]. Suppose that the optimal solution to the dual SDP (23) of $R(\mathcal{N},\varepsilon)$ is $\{\widehat{X}_{AB},\widehat{Y}_B,\widehat{s},\widehat{t}\,\}$. Let us choose $X_{AA'BB'}=\frac{1}{m}\widehat{X}_{AB}\otimes D_m$, $Y_{BB'}=\frac{1}{m}\widehat{Y}_B\otimes\mathbb{1}_m$, $s=\frac{1}{m}\widehat{s}$, $t=\frac{1}{m}\widehat{t}$, with $D_m=\sum_{i=0}^{m-1}|ii\rangle\langle ii|$. Then it can be easily checked that

$$\begin{aligned} & X_{AA'BB'}+\mathbb{1}_{AA'}\otimes Y_{BB'} \\ & \quad \geq (\widehat{X}_{AB}+\mathbb{1}_A\otimes\widehat{Y}_B)\otimes\frac{D_m}{m}\geq sJ_{\mathcal{N}}\otimes D_m. \end{aligned} \tag{24}$$

The other constraints can be verified similarly. Thus, $\{X_{AA'BB'},Y_{BB'},s,t\}$ is a feasible solution to the SDP (23) of $R(\mathcal{N}\otimes\mathcal{I}_m,\varepsilon)$, which implies that

$$R(\mathcal{N}\otimes\mathcal{I}_m,\varepsilon)\leq-\log[s(1-\varepsilon)-t]=R(\mathcal{N},\varepsilon)+\log m,$$

and completes the proof. ■

IV. NEW META-CONVERSE FOR UNASSISTED CLASSICAL COMMUNICATION

In the following we will use the concept of *subchannels*. Denote $\mathcal{S}(A):=\{\rho_A\geq 0\mid \operatorname{Tr}\rho_A=1\}$ as the set of quantum states on A. A subchannel $\mathcal{N}_{A\to B}$ is a CP linear map that is trace non-increasing, i.e., $\operatorname{Tr}\mathcal{N}(\rho)\leq 1$ for all quantum states $\rho\in\mathcal{S}(A)$.

Recall that the only useless quantum channel for classical communication is the *constant channel* $\mathcal{N}(\cdot)=\sigma$ [1–3, 35, 36] , which maps all states ρ on A to a constant state σ on B. As a natural extension, we say a subchannel $\mathcal{N}$ is *constant-bounded* if it maps all states ρ to positive definite operators that are smaller than or equal to a constant state σ, i.e.,

$$\mathcal{N}(\rho)\leq\sigma,\forall\rho\in\mathcal{S}(A). \tag{25}$$

We also define the set of constant-bounded subchannels as $\mathcal{V}:=\big\{\mathcal{M}\in\text{CP}(A:B)\,\big|\,\exists\,\sigma\in\mathcal{S}(B)\text{ s.t. }\mathcal{M}(\rho)\leq\sigma,\forall\rho\in\mathcal{S}(A)\big\}$, where $\text{CP}(A:B)$ denotes the set of all CP linear maps from A to B. Clearly, the set $\mathcal{V}$ is convex and closed. This inspires the following new one-shot converse bound.

Theorem 5 *For any quantum channel $\mathcal{N}_{A'\to B}$ and error tolerance $\varepsilon\in(0,1)$, we have*

$$\begin{aligned} & C^{(1)}(\mathcal{N},\varepsilon) \\ & \leq \max_{\rho_{A'}}\min_{\mathcal{M}\in\mathcal{V}} D_H^{\varepsilon}(\mathcal{N}_{A'\to B}(\phi_{A'A})\|\mathcal{M}_{A'\to B}(\phi_{A'A})) \qquad (26) \\ & = \min_{\mathcal{M}\in\mathcal{V}}\max_{\rho_{A'}} D_H^{\varepsilon}(\mathcal{N}_{A'\to B}(\phi_{A'A})\|\mathcal{M}_{A'\to B}(\phi_{A'A})), \qquad (27) \end{aligned}$$

where $\phi_{A'A}$ is a purification of $\rho_{A'}$.

Proof Consider an unassisted code with inputs $\{\rho_k\}_{k=1}^m$ and POVM $\{M_k\}_{k=1}^m$ whose average input state is $\rho_{A'} = \sum_{k=1}^m \frac{1}{m}\rho_k$, the success probability to transmit m messages is given by

$$\begin{aligned} p_{\text{succ}} &= \frac{1}{m}\sum_{k=1}^m \operatorname{Tr} \mathcal{N}(\rho_k) M_k && (28)\\ &= \operatorname{Tr} J_{\mathcal{N}}\Big(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes M_k\Big) && (29)\\ &= \operatorname{Tr} \mathcal{N}_{A'\to B}(\phi_{AA'})E, && (30)\end{aligned}$$

where

$$E := (\rho_A^T)^{-1/2}\Big(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes M_k\Big)(\rho_A^T)^{-1/2}. \tag{31}$$

Then we have

$$0 \le E \le (\rho_A^T)^{-1/2}\left(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes \mathbb{1}_B\right)(\rho_A^T)^{-1/2} = \mathbb{1}_{AB}. \tag{32}$$

Let us fix $\mathcal{M} \in \mathcal{V}$ and assume that the output states of $\mathcal{M}$ are bounded by the state σ_B, then

$$\begin{aligned} \operatorname{Tr} \mathcal{M}_{A'\to B}(\phi_{AA'})E &= \operatorname{Tr} J_{\mathcal{M}}\Big(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes M_k\Big) && (33)\\ &= \frac{1}{m}\sum_{k=1}^m \operatorname{Tr} \mathcal{M}(\rho_k) M_k && (34)\\ &\le \frac{1}{m}\sum_{k=1}^m \operatorname{Tr} \sigma_B M_k = \frac{1}{m}. && (35)\end{aligned}$$

The second line follows from the fact that $J_{\mathcal{M}} = (\rho_A^T)^{-1/2}\mathcal{M}_{A'\to B}(\phi_{AA'})(\rho_A^T)^{-1/2}$. In the third line, we use the inverse Choi-Jamiołkowski transformation $\mathcal{M}_{A'\to B}(\rho_{A'}) = \operatorname{Tr}_A J_{\mathcal{M}}(\rho_A^T \otimes \mathbb{1}_B)$. The forth line follows since any output state of $\mathcal{M}$ is bounded by the state σ_B. Therefore, combining Eqs. (30) and (35), we know that $\operatorname{Tr} \mathcal{N}_{A'\to B}(\phi_{AA'})E \ge 1-\varepsilon$ and $\operatorname{Tr} \mathcal{M}_{A'\to B}(\phi_{AA'})E \le \frac{1}{m}$. Thus $C^{(1)}(\mathcal{N}, \rho_{A'}, \varepsilon) \le \min_{\mathcal{M}\in\mathcal{V}} D_H^\varepsilon(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \mathcal{M}_{A'\to B}(\phi_{AA'}))$. Maximizing over all average input $\rho_{A'}$, we can obtain the desired result of (26).

Since $\beta_\varepsilon(\mathcal{N}_{A'\to B}(\phi_{A'A}) \| \mathcal{M}_{A'\to B}(\phi_{A'A}))$ is convex in $\rho_{A'}$ and concave in $\mathcal{M}$ [5], we can exchange the maximization and minimization by applying Sion's minimax theorem [37] and obtain the result of (27). ■

Remark Noting that E above also satisfies $0 \le E^{T_B} \le \mathbb{1}$, we can further obtain an upper bound of $C^{(1)}(\mathcal{N}, \varepsilon)$ as

$$\max_{\rho_{A'}} \min_{\mathcal{M}\in\mathcal{V}} D_{H,PPT}^\varepsilon(\mathcal{N}_{A'\to B}(\phi_{A'A}) \| \mathcal{M}_{A'\to B}(\phi_{A'A})), \tag{36}$$

where $D_{H,PPT}^\varepsilon(\rho_0\|\rho_1)$ is defined as the optimal value of

$$-\log \min\{\operatorname{Tr} E\rho_1 \,|\, 1 - \operatorname{Tr} E\rho_0 \le \varepsilon, 0 \le E, E^{T_B} \le \mathbb{1}\}. \tag{37}$$

If we consider $\max_{\rho_{A'}} D_H^\varepsilon(\mathcal{N}_{A'\to B}(\phi_{A'A}) \| \mathcal{M}_{A'\to B}(\phi_{A'A}))$ as the "distance" between the channel $\mathcal{N}$ and CP map $\mathcal{M}$, then our new meta-converse can be treated as the "distance" between the given channel $\mathcal{N}$ with the set of all constant-bounded subchannels.

To make this meta-converse bound efficiently computable, we can restrict the set of constant-bounded subchannels $\mathcal{V}$ to an SDP-tractable set of CP maps. Let us define

$$\mathcal{V}_\beta := \{\mathcal{M} \in \mathrm{CP}(A:B) \mid \beta(J_\mathcal{M}) \leq 1\}, \tag{38}$$

where $\beta(J_\mathcal{M})$ is given by the following SDP

$$\begin{aligned} \beta(J_\mathcal{M}) := \underset{S_B, R_{AB}}{\text{minimize}} \ & \operatorname{Tr} S_B \\ \text{subject to } & -R_{AB} \leq J_\mathcal{M}^{T_B} \leq R_{AB}, \\ & -\mathbb{1}_A \otimes S_B \leq R_{AB}^{T_B} \leq \mathbb{1}_A \otimes S_B. \end{aligned} \tag{39}$$

Here $J_\mathcal{M}$ is the Choi-Jamiołkowski matrix of $\mathcal{M}$ and T_B means the partial transpose on system B. We note that $\beta(\cdot)$ for a quantum channel $\mathcal{N}$ is faithful in the sense that $\beta(J_\mathcal{N}) = 1$ if and only if $C(\mathcal{N}) = 0$ [10]. Thus the set $\mathcal{V}_\beta$ contains all the constant channels, which makes it reasonable, to some extent, to introduce the set $\mathcal{V}_\beta$ here. Moreover, the set $\mathcal{V}_\beta$ also satisfies some basic properties such as convexity and invariance under composition with unitary maps. These are shown in Appendix A.

Lemma 6 *The set $\mathcal{V}_\beta$ is a subset of $\mathcal{V}$, i.e., $\mathcal{V}_\beta \subseteq \mathcal{V}$.*

Proof Note that the strong duality of SDP (39) holds due to the Slater's theorem [34]. Given a CP map $\mathcal{M}$ in $\mathcal{V}_\beta$, we suppose that the optimal solution of $\beta(J_\mathcal{M})$ is $\{R_{AB}, S_B\}$. Then, we know $\beta(J_\mathcal{M}) = \operatorname{Tr} S_B \leq 1$. Furthermore, for any input ρ_A, the output $\mathcal{M}(\rho_A)$ satisfies that

$$\begin{aligned} \mathcal{M}_{A\to B}(\rho_A) &= \operatorname{Tr}_A \sqrt{\rho_A^T} J_\mathcal{M} \sqrt{\rho_A^T} && (40) \\ &= (\operatorname{Tr}_A \sqrt{\rho_A^T} J_\mathcal{M}^{T_B} \sqrt{\rho_A^T})^T && (41) \\ &\leq (\operatorname{Tr}_A \sqrt{\rho_A^T} R_{AB} \sqrt{\rho_A^T})^T && (42) \\ &= \operatorname{Tr}_A \sqrt{\rho_A^T} R_{AB}^{T_B} \sqrt{\rho_A^T} && (43) \\ &\leq \operatorname{Tr}_A \sqrt{\rho_A^T} (\mathbb{1}_A \otimes S_B) \sqrt{\rho_A^T} && (44) \\ &= S_B. && (45) \end{aligned}$$

■

As a consequence of Theorem 5 and Lemma 6, we have the following meta-converse.

Theorem 7 *For any quantum channel $\mathcal{N}_{A'\to B}$ and error tolerance $\varepsilon \in (0,1)$, we have*

$$\begin{aligned} & C^{(1)}(\mathcal{N}, \varepsilon) \\ & \leq \max_{\rho_{A'}} \min_{\mathcal{M} \in \mathcal{V}_\beta} D_H^\varepsilon(\mathcal{N}_{A'\to B}(\phi_{A'A}) \| \mathcal{M}_{A'\to B}(\phi_{A'A})) && (46) \\ & = \min_{\mathcal{M} \in \mathcal{V}_\beta} \max_{\rho_{A'}} D_H^\varepsilon(\mathcal{N}_{A'\to B}(\phi_{A'A}) \| \mathcal{M}_{A'\to B}(\phi_{A'A})), && (47) \end{aligned}$$

where $\phi_{A'A}$ is a purification of $\rho_{A'}$. Note that this bound can be computed via SDP (see Appendix D).

There are several other converses for the one-shot ε-error capacity of a general quantum channel, e.g., the Matthews-Wehner converse [5], the Datta-Hsieh converse [38], and the recent SDP converse via no-signaling (NS) and positive-partial-transpose-preserving (PPT) codes [10]. Note that the Datta-Hsieh converse is not known to be efficiently computable. Also, our meta-converses in Theorem 5 and 7 are always tighter than the Matthews-Wehner converse in Eq. (5) since we can rewrite $R(\mathcal{N}, \varepsilon)$ as

$$\max_{\rho_{A'}} \min_{\mathcal{M} \in \mathcal{W}} D_H^{\varepsilon}(\mathcal{N}_{A' \to B}(\phi_{A'A}) \| \mathcal{M}_{A' \to B}(\phi_{A'A})), \tag{48}$$

where $\mathcal{W}$ is the set of all constant channels and $\mathcal{W} \subsetneq \mathcal{V}_\beta \subsetneq \mathcal{V}$. But our relaxed meta-converse in Theorem 7 is no tighter than the SDP converse via NS and PPT codes (cf. Theorem 4 in [10]).

As we will show later, our meta-converse will lead to new results in both the finite blocklength and asymptotic regimes. In particular, our new bounds allow us to establish finite blocklength analysis for quantum channels beyond classical-quantum channels (cf. Section VI), which haven't been done via previous converse bounds.

V. COMPARISON OF ASYMPTOTIC CONVERSE BOUNDS

By substituting the relative entropy for the hypothesis testing relative entropy in our meta-converse we define the following quantity, which we call the Υ-*information* of the channel $\mathcal{N}$,

$$\Upsilon(\mathcal{N}) := \max_{\rho_{A'}} \min_{\mathcal{M} \in \mathcal{V}} D(\mathcal{N}_{A' \to B}(\phi_{A'A}) \| \mathcal{M}_{A' \to B}(\phi_{A'A})), \tag{49}$$

where the relative entropy is defined as $D(\rho\|\sigma) := \operatorname{Tr} \rho(\log \rho - \log \sigma)$ if $\operatorname{supp} \rho \subseteq \operatorname{supp} \sigma$ and $+\infty$ otherwise. We also introduce its regularization,

$$\Upsilon^\infty(\mathcal{N}) := \limsup_{n \to \infty} \frac{1}{n} \Upsilon(\mathcal{N}^{\otimes n}). \tag{50}$$

Recently, one of us and his collaborators [10] derived an SDP strong converse bound $C_\beta(\mathcal{N})$ for the classical capacity of a general quantum channel, which means that any code with a rate exceeding this bound will have a vanishing success probability. To be specific, for any quantum channel $\mathcal{N}$, it holds that $C(\mathcal{N}) \leq C_\beta(\mathcal{N}) := \log \beta(J_\mathcal{N})$. In this section our goal is to compare Υ and Υ^∞ with other known quantities: the Holevo capacity χ, the classical capacity C (or regularized Holevo capacity), the entanglement-assisted classical capacity C_E, and the strong converse bound C_β. The graph of relations among these quantities is displayed in Fig. 2.

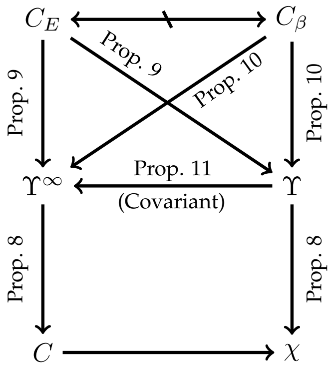

FIG. 2: Relation graph of converse bounds. An arrow $A \longrightarrow B$ indicates that $A(\mathcal{N}) \geq B(\mathcal{N})$ for any channel $\mathcal{N}$. $A \longleftrightarrow\!\!\!\!\!\!\!\!\!\!/\;\;\;\; B$ indicates that A and B are not comparable, i.e, $A(\mathcal{N}) > B(\mathcal{N})$ for some channel $\mathcal{N}$ and $A(\mathcal{M}) < B(\mathcal{M})$ for some channel $\mathcal{M}$.

Proposition 8 *For any quantum channel $\mathcal{N}$, we have*

$$\chi(\mathcal{N}) \leq \Upsilon(\mathcal{N}) \quad \textit{and} \quad C(\mathcal{N}) \leq \Upsilon^\infty(\mathcal{N}). \tag{51}$$

Proof We first need to prove that the quantity $D(\mathcal{N}_{A'\to B}(\phi_{A'A})\|\mathcal{M}_{A'\to B}(\phi_{A'A}))$ is concave in $\rho_{A'}$. For any convex combination $\rho_{A'} = \sum_i p_i \rho^i_{A'}$, suppose $\rho^i_{A'}$ has a purification $\phi^i_{A'A}$. Then $|\psi_{PAA'}\rangle = \sum_i \sqrt{p_i}|i\rangle \otimes |\phi^i_{AA'}\rangle$ is a purification of the state $\rho_{A'}$. By the data-processing inequality of the relative entropy under the channel $\sum_i |i\rangle\langle i| \cdot |i\rangle\langle i|$, we have

$$D(\mathcal{N}_{A'\to B}(\psi_{PAA'})\big\|\mathcal{M}_{A'\to B}(\psi_{PAA'})) \geq D(G_1\|G_2),$$

$$\text{with} \qquad G_1 = \sum_i p_i |i\rangle\langle i| \otimes \mathcal{N}_{A'\to B}(\phi^i_{AA'}), \tag{52}$$

$$G_2 = \sum_i p_i |i\rangle\langle i| \otimes \mathcal{M}_{A'\to B}(\phi^i_{AA'}). \tag{53}$$

Then the concavity follows from

$$D(G_1\|G_2) = \sum_i p_i D(\mathcal{N}_{A'\to B}(\phi^i_{AA'})\|\mathcal{M}_{A'\to B}(\phi^i_{AA'})).$$

We have the following chain of inequalities:

$$\begin{aligned}
&\Upsilon(\mathcal{N}) \\
&\quad = \max_{\rho_{A'}} \min_{\mathcal{M}\in\mathcal{V}} D(\mathcal{N}_{A'\to B}(\phi_{A'A})\big\|\mathcal{M}_{A'\to B}(\phi_{A'A})) && (54)\\
&\quad = \min_{\mathcal{M}\in\mathcal{V}} \max_{\rho_{A'}} D(\mathcal{N}_{A'\to B}(\phi_{A'A})\big\|\mathcal{M}_{A'\to B}(\phi_{A'A})) && (55)\\
&\quad \geq \min_{\mathcal{M}\in\mathcal{V}} \max_{\rho_{A'}} D(\mathcal{N}_{A'\to B}(\rho_{A'})\big\|\mathcal{M}_{A'\to B}(\rho_{A'})) && (56)\\
&\quad \geq \min_{\mathcal{M}\in\mathcal{V}} \max_{\rho_{A'}} D(\mathcal{N}_{A'\to B}(\rho_{A'})\big\|\sigma_{\mathcal{M}}) && (57)\\
&\quad \geq \min_{\sigma_B} \max_{\rho_{A'}} D(\mathcal{N}_{A'\to B}(\rho_{A'})\big\|\sigma_B) && (58)\\
&\quad = \chi(\mathcal{N}). && (59)
\end{aligned}$$

The second line follows by Sion's minimax theorem [37] since $D(\mathcal{N}_{A'\to B}(\phi_{A'A})\big\|\mathcal{M}_{A'\to B}(\phi_{A'A}))$ is convex in $\mathcal{M}$ and concave in $\rho_{A'}$. The third line follows by tracing out the system A and the data-processing inequality of the relative entropy. The fourth line follows since for any $\mathcal{M} \in \mathcal{V}$ and $\rho_{A'}$, there exists a state $\sigma_{\mathcal{M}}$ independent of $\rho_{A'}$ such that $\mathcal{M}_{A'\to B}(\rho_{A'}) \leq \sigma_{\mathcal{M}}$. Due to the dominance property of the relative entropy, we have the inequality. The fifth line follows since we relax the feasible set of the minimization to a larger set. The last line follows from the characterization of the Holevo capacity as the divergence radius [35].

Finally, according to the HSW theorem, we have

$$\begin{aligned}
C(\mathcal{N}) &= \limsup_{n\to\infty} \frac{1}{n}\chi(\mathcal{N}^{\otimes n}) && (60)\\
&\leq \limsup_{n\to\infty} \frac{1}{n}\Upsilon(\mathcal{N}^{\otimes n}) = \Upsilon^\infty(\mathcal{N}), && (61)
\end{aligned}$$

which completes the proof. ■

Proposition 9 *For any quantum channel $\mathcal{N}$, we have*

$$\Upsilon(\mathcal{N}) \leq C_E(\mathcal{N}) \quad \textit{and} \quad \Upsilon^\infty(\mathcal{N}) \leq C_E(\mathcal{N}). \tag{62}$$

Proof For any state σ_B we introduce a trivial channel $\mathcal{M}$ that always outputs σ_B via its Choi-Jamiołkowski matrix $J_\mathcal{M} = \mathbb{1}_A \otimes \sigma_B$. Then $\mathcal{M} \in \mathcal{V}$ and we have

$$\begin{aligned} &\min_{\sigma_B} D(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \rho_A \otimes \sigma_B) \\ &\quad = \min_{\sigma_B} D\big(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \rho_A^{1/2}(\mathbb{1}_A \otimes \sigma_B)\rho_A^{1/2}\big) &(63)\\ &\quad \geq \min_{\mathcal{M}\in\mathcal{V}} D(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \mathcal{M}_{A'\to B}(\phi_{AA'})). &(64)\end{aligned}$$

Take maximization over all input state $\rho_{A'}$ on both sides, we have $C_E(\mathcal{N}) \geq \Upsilon(\mathcal{N})$. Furthermore, since $C_E(\mathcal{N})$ is additive, we have

$$\begin{aligned} C_E(\mathcal{N}) &= \limsup_{n\to\infty} \frac{1}{n} C_E(\mathcal{N}^{\otimes n}) &(65)\\ &\geq \limsup_{n\to\infty} \frac{1}{n} \Upsilon(\mathcal{N}^{\otimes n}) = \Upsilon^\infty(\mathcal{N}), &(66)\end{aligned}$$

which completes the proof. ■

Proposition 10 *For any quantum channel $\mathcal{N}$, we have*

$$\Upsilon(\mathcal{N}) \leq C_\beta(\mathcal{N}) \quad \textit{and} \quad \Upsilon^\infty(\mathcal{N}) \leq C_\beta(\mathcal{N}). \tag{67}$$

Proof Take $\widetilde{\mathcal{M}} = \frac{1}{\beta(J_\mathcal{N})}\mathcal{N}$, then $\widetilde{\mathcal{M}} \in \mathcal{V}_\beta \subseteq \mathcal{V}$ and

$$\begin{aligned} \Upsilon(\mathcal{N}) &= \max_{\rho_{A'}} \min_{\mathcal{M}\in\mathcal{V}} D(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \mathcal{M}_{A'\to B}(\phi_{AA'})) \\ &\leq \max_{\rho_{A'}} D\big(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \widetilde{\mathcal{M}}_{A'\to B}(\phi_{AA'})\big) &(68)\\ &= \max_{\rho_{A'}} D\Big(\mathcal{N}_{A'\to B}(\phi_{AA'}) \Big\| \frac{\mathcal{N}_{A'\to B}(\phi_{AA'})}{\beta(J_\mathcal{N})}\Big) &(69)\\ &= \log \beta(J_\mathcal{N}) &(70)\\ &= C_\beta(\mathcal{N}). &(71)\end{aligned}$$

Furthermore, since $C_\beta(\mathcal{N})$ is additive [10], we have

$$\begin{aligned} \Upsilon^\infty(\mathcal{N}) &= \limsup_{n\to\infty} \frac{1}{n} \Upsilon(\mathcal{N}^{\otimes n}) &(72)\\ &\leq \limsup_{n\to\infty} \frac{1}{n} C_\beta(\mathcal{N}^{\otimes n}) = C_\beta(\mathcal{N}), &(73)\end{aligned}$$

which completes the proof. ■

In the remainder we focus on covariant channels which allow us to simplify the set of input states. Let G be a finite group, and for every $g \in G$, let $g \to U_A(g)$ and $g \to V_B(g)$ be unitary representation acting on the input and output spaces of the channel, respectively. Then a quantum channel $\mathcal{N}_{A\to B}$ is *G-covariant* if $\forall \rho_A \in \mathcal{S}(A)$,

$$\mathcal{N}_{A\to B}\big(U_A(g)\rho_A U_A^\dagger(g)\big) = V_B(g)\mathcal{N}_{A\to B}(\rho_A)V_B^\dagger(g).$$

A quantum channel is *covariant* if it is covariant with respect to a finite group G for which each $g \in G$ has a unitary representation $U(g)$ such that $\{U(g)\}_{g\in G}$ is a unitary one-design. That is, the map $\frac{1}{|G|}\sum_{g\in G} U(g)(\cdot)U(g)^\dagger$ always outputs the maximally mixed state for all input states.

Proposition 11 *For any covariant channel $\mathcal{N}$, we have*

$$\Upsilon^{\infty}(\mathcal{N}) \leq \Upsilon(\mathcal{N}). \tag{74}$$

Proof Following the proof steps in Lemma 18 for the quantum relative entropy, we can fix the average input state of $\Upsilon(\mathcal{N})$ to be the maximally mixed state. Therefore, we find

$$\Upsilon(\mathcal{N}) = \min_{\mathcal{M}\in\mathcal{V}} D(\mathcal{N}_{A'\to B}(\Phi_{A'A}) \| \mathcal{M}_{A'\to B}(\Phi_{A'A})), \tag{75}$$

where $\Phi_{A'A} = \frac{1}{d}\sum_{i,j=0}^{d-1} |ii\rangle\langle jj|$. Thus it is clear that that Υ is subadditive for covariant channels, i.e., $\Upsilon(\mathcal{N}^{\otimes n}) \leq n\Upsilon(\mathcal{N})$, which implies $\Upsilon^{\infty}(\mathcal{N}) \leq \Upsilon(\mathcal{N})$. ■

Remark In an analogous spirit as in [39] we can also show that the Υ-information of a channel is a strong converse bound for covariant channels. We present this analysis in Appendix C.

We provide a summarized graph of relations among the old bounds and new bounds in Fig. 2. Since C_β and C_E are relaxations of the Υ-information, then the Υ-information is expected to be generally tighter than C_β and C_E. Similarly, since the Υ-information is a relaxation of the Holevo capacity, the inequality between them may be strict in general. However, for quantum erasure channels, our Υ-information is tight and it holds that

$$\Upsilon(\mathcal{E}_p) = \Upsilon^{\infty}(\mathcal{E}_p) = C(\mathcal{E}_p) = \chi(\mathcal{E}_p) = (1-p)\log d,$$

(see details in Section VI). Combining this property and the meta-converse in Theorem 7, we establish the finite blocklength analysis for classical communication over quantum erasure channels in Theorems 13 and 14. Another interesting case is the qubit depolarizing channel $\mathcal{N}_D(\rho) := (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$, where X, Y and Z are Pauli matrices. For this class of channels, we numerically find that the Υ-information appears to be strictly larger than the Holevo capacity but it is tighter than C_β and C_E. We expect that the Υ-information may have further applications in studying the strong converse property of other quantum channels.

VI. FINITE BLOCKLENGTH ANALYSIS FOR QUANTUM ERASURE CHANNEL

The quantum erasure channel is denoted by

$$\mathcal{E}_p(\rho) := (1-p)\rho + p|e\rangle\langle e|, \tag{76}$$

where $|e\rangle$ is orthogonal to the input Hilbert space. The classical capacity of a quantum erasure channel is given by $C(\mathcal{E}_p) = (1-p)\log d$, where d is the dimension of input space [40]. In [41], the strong converse property for the classical capacity of $\mathcal{E}_p$ is established.

In this section, applying our new meta-converse, we derive the second-order expansion and moderate deviation analysis of quantum erasure channel in Theorem 13 and 14, respectively. To our knowledge, this is the first second-order or moderate deviation expansion of classical capacity beyond entanglement-breaking channels.

We first show that the Υ-information matches the classical capacity for erasure channels.

Lemma 12 *For any quantum erasure channel $\mathcal{E}_p$ with input dimension d, we have $\Upsilon(\mathcal{E}_p) = (1-p)\log d$.*

Proof Since quantum erasure channels are covariant, we can restrict the input state to the maximally mixed state, i.e.,

$$\Upsilon(\mathcal{E}_p) = \min_{\mathcal{M}\in\mathcal{V}} D(\mathcal{E}_p(\Phi_{A'A}) \| \mathcal{M}(\Phi_{A'A})), \tag{77}$$

where $\Phi_{A'A} = \frac{1}{d}\sum_{i,j=0}^{d-1} |ii\rangle\langle jj|$ is the maximally entangled state. Denote

$$J_{\mathcal{M}} = \frac{1-p}{d}\sum_{i,j=0}^{d-1} |ii\rangle\langle jj| + p\sum_{i=0}^{d-1} |i\rangle\langle i| \otimes |d\rangle\langle d| \tag{78}$$

as the Choi-Jamiołkowski matrix of the CP map $\mathcal{M}$. Then we have $\mathcal{M} \in \mathcal{V}_\beta \subseteq \mathcal{V}$ and

$$\Upsilon(\mathcal{E}_p) \leq D(\mathcal{E}_p(\Phi_{A'A}) \| \mathcal{M}(\Phi_{A'A})) = (1-p)\log d. \tag{79}$$

On the other hand, since Υ is an upper bound on the classical capacity for covariant channels due to Proposition 8 and 11, we have $(1-p)\log d = C(\mathcal{E}_p) \leq \Upsilon(\mathcal{E}_p)$. Together with Eq. (79), we have the desired result. ■

A. Second-order asymptotics of quantum erasure channel

Theorem 13 *For any quantum erasure channel $\mathcal{E}_p$ with parameter p and input dimension d, we have*

$$\begin{aligned} C^{(1)}(\mathcal{E}_p^{\otimes n}, \varepsilon) &= n(1-p)\log d \\ &\quad + \sqrt{np(1-p)(\log d)^2}\,\Phi^{-1}(\varepsilon) + O(\log n), \end{aligned} \tag{80}$$

where Φ is the cumulative distribution function of a standard normal random variable.

Proof For the direct part, denote

$$\mathcal{F}_1(\rho) := \sum_{i=0}^{d-1} \langle i|\rho|i\rangle |i\rangle\langle i|, \qquad \text{and} \tag{81}$$

$$\mathcal{F}_2(\rho) := \sum_{i=0}^{d} \langle i|\rho|i\rangle |i\rangle\langle i|, \tag{82}$$

which are both classical channels. Then $\mathcal{N}_p = \mathcal{F}_2 \circ \mathcal{E}_p \circ \mathcal{F}_1$ is a classical erasure channel. We have

$$\begin{aligned} C^{(1)}(\mathcal{E}_p^{\otimes n}, \varepsilon) \geq C^{(1)}(\mathcal{N}_p^{\otimes n}, \varepsilon) &= n(1-p)\log d \\ &\quad + \sqrt{np(1-p)(\log d)^2}\,\Phi^{-1}(\varepsilon) + O(\log n), \end{aligned} \tag{83}$$

where the equality comes from the result in [4].

For the converse part, we have

$$\begin{aligned} & C^{(1)}(\mathcal{E}_p^{\otimes n}, \varepsilon) \\ & \quad \leq \min_{\mathcal{M}\in\mathcal{V}} D_H^{\varepsilon}(\mathcal{E}_p^{\otimes n}(\Phi_{A'A}^{\otimes n}) \| \mathcal{M}_{A'^n \to B^n}(\Phi_{A'A}^{\otimes n})). \end{aligned} \tag{84}$$

Take $\mathcal{M}_{A'^n\to B^n} = \mathcal{M}_{A'\to B}^{\otimes n}$, where $\mathcal{M}_{A'\to B}$ is the same CP map as given by Eq. (78), we have

$$D_H^{\varepsilon}(\mathcal{E}_p^{\otimes n}(\Phi_{A'A}^{\otimes n}) \| \mathcal{M}_{A'\to B}^{\otimes n}(\Phi_{A'A}^{\otimes n})) \tag{85}$$

$$\begin{aligned} &= nD(\mathcal{E}_p(\Phi_{A'A}) \| \mathcal{M}(\Phi_{A'A})) \\ &\quad + \sqrt{nV(\mathcal{E}_p(\Phi_{A'A}) \| \mathcal{M}(\Phi_{A'A}))}\,\Phi^{-1}(\varepsilon) + O(\log n) \\ &= n(1-p)\log d + \sqrt{np(1-p)(\log d)^2}\,\Phi^{-1}(\varepsilon) + O(\log n). \end{aligned} \tag{86}$$

In the second line, we use second-order expansion of quantum hypothesis testing relative entropy and $V(\rho\|\sigma) := \operatorname{Tr}\rho(\log\rho - \log\sigma)^2 - D(\rho\|\sigma)^2$ is the quantum information variance [42, 43]. The third line follows by direct calculation. Combining this with (84) leads to the desired bound. ■

B. Moderate deviation of quantum erasure channel

Theorem 14 *For any squence* $\{a_n\}$ *such that* $a_n \to 0$ *and* $\sqrt{n}a_n \to \infty$, *let* $\varepsilon_n = e^{-na_n^2}$. *For any quantum erasure channel* $\mathcal{E}_p$ *with parameter* p *and input dimension* d, *it holds*

$$\begin{aligned} \frac{1}{n}C^{(1)}(\mathcal{E}_p^{\otimes n}, \varepsilon_n) &= (1-p)\log d \\ &\quad - \sqrt{2p(1-p)(\log d)^2}\, a_n + o(a_n), \end{aligned} \tag{87}$$

$$\begin{aligned} \frac{1}{n}C^{(1)}(\mathcal{E}_p^{\otimes n}, 1-\varepsilon_n) &= (1-p)\log d \\ &\quad + \sqrt{2p(1-p)(\log d)^2}\, a_n + o(a_n). \end{aligned} \tag{88}$$

Proof We only need to prove Eq. (87), and Eq. (88) can be proved with the same argument. For the converse part, we apply the moderate deviation of hypothesis testing in [13, 14] to our meta-converse in Eq. (84). Specifically,

$$C^{(1)}(\mathcal{E}_p^{\otimes n}, \varepsilon) \leq D_H^{\varepsilon}(\mathcal{E}_p^{\otimes n}(\Phi_{A'A}^{\otimes n}) \| \mathcal{M}_{A'\to B}^{\otimes n}(\Phi_{A'A}^{\otimes n})), \tag{89}$$

where $\mathcal{M}_{A'\to B}$ is the CP map given by Eq. (78). Thus

$$\begin{aligned} &\frac{1}{n}C^{(1)}(\mathcal{E}_p^{\otimes n}, \varepsilon_n) \\ &\leq \frac{1}{n}D_H^{\varepsilon}(\mathcal{E}_p^{\otimes n}(\Phi_{A'A}^{\otimes n}) \| \mathcal{M}_{A'\to B}^{\otimes n}(\Phi_{A'A}^{\otimes n})) && (90) \\ &= D(\mathcal{E}_p(\Phi_{A'A}) \| \mathcal{M}_{A'\to B}(\Phi_{A'A})) && (91) \\ &\qquad - \sqrt{2V(\mathcal{E}_p(\Phi_{A'A}) \| \mathcal{M}_{A'\to B}(\Phi_{A'A}))}\, a_n + o(a_n) \\ &= (1-p)\log d - \sqrt{2p(1-p)(\log d)^2}\, a_n + o(a_n). && (92) \end{aligned}$$

The direct part proceeds analogously to the direct part in Theorem 13. ■

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Appendix A SOME PROPERTIES OF $\mathcal{V}_\beta$

Lemma 15 *The set $\mathcal{V}_\beta$ is convex.*

Proof Due to the Choi-Jamiołkowski isomorphism, we only need to prove that the set $\{K \geq 0 \mid \beta(K) \leq 1\}$ is convex. That is, for any $K_1, K_1 \in \{K \geq 0 \mid \beta(K) \leq 1\}$ we prove that for any $p \in (0,1)$,

$$K = pK_1 + (1-p)K_2 \in \{K \geq 0 \mid \beta(K) \leq 1\}. \tag{93}$$

It is clear that $K \geq 0$. Suppose optimal solutions of $\beta(K_1)$ and $\beta(K_2)$ are $\{R_1, S_1\}$ and $\{R_2, S_2\}$, respectively. Then we can verify that $\{pR_1 + (1-p)R_2,\, pS_1 + (1-p)S_2\}$ is a feasible solution of $\beta(K)$. Thus $\beta(K) \leq \operatorname{Tr} pS_1 + (1-p)S_2 = p \operatorname{Tr} S_1 + (1-p) \operatorname{Tr} S_2 \leq 1$. ■

Lemma 16 *For any local unitary $U_A \otimes V_B$ and $K \geq 0$, it holds $\beta\big((U_A \otimes V_B)K\big(U_A^\dagger \otimes V_B^\dagger\big)\big) = \beta(K)$.*

Proof Suppose the optimal solution of $\beta(K)$ is taken at $\{R_{AB}, S_B\}$. Then it is easy to verify that $\{U_A \otimes \overline{V}_B R_{AB} U_A^\dagger \otimes V_B^T, V_B S_B V_B^\dagger\}$ is a feasible solution of $\beta\big(U_A \otimes V_B K U_A^\dagger \otimes V_B^\dagger\big)$. Thus we have

$$\beta\big(U_A \otimes V_B K U_A^\dagger \otimes V_B^\dagger\big) \leq \operatorname{Tr} V_B S_B V_B^\dagger = \operatorname{Tr} S_B = \beta(K).$$

Furthermore, we have $\beta(K) = \beta\big(\big(U_A^\dagger \otimes V_B^\dagger\big)\big(U_A \otimes V_B K U_A^\dagger \otimes V_B^\dagger\big)(U_A \otimes V_B)\big) \leq \beta\big(U_A \otimes V_B K U_A^\dagger \otimes V_B^\dagger\big)$, which completes the proof. ■

Corollary 17 *For any unitary channel $\mathcal{U}_{A'\to A'}$ and $\mathcal{V}_{B\to B}$, if $\mathcal{M}_{A'\to B}\in\mathcal{V}_\beta$, then*

$$\mathcal{V}_{B\to B}\circ\mathcal{M}_{A'\to B}\circ\mathcal{U}_{A'\to A'}\in\mathcal{V}_\beta. \tag{94}$$

Proof Denote $J_\mathcal{M}=\mathcal{M}_{A'\to B}(\widetilde{\Phi}_{A'A})$, where $\widetilde{\Phi}_{A'A}$ denotes the unnormalized maximally entangled state. Let $\mathcal{U}_{A'\to A'}(\cdot)=U_{A'}\cdot U_{A'}^\dagger$ and $\mathcal{V}_{B\to B}(\cdot)=V_B\cdot V_B^\dagger$. Since $\mathcal{M}_{A'\to B}\in\mathcal{V}_\beta$, we have $J_\mathcal{M}\geq 0$ and $\beta(J_\mathcal{M})\leq 1$. Then,

$$\begin{aligned}
K_{AB}&=\mathcal{V}_{B\to B}\circ\mathcal{M}_{A'\to B}\circ\mathcal{U}_{A'\to A'}(\widetilde{\Phi}_{A'A}) && (95)\\
&=\mathcal{V}_{B\to B}\circ\mathcal{M}_{A'\to B}(U_{A'}\widetilde{\Phi}_{A'A}U_{A'}^\dagger) && (96)\\
&=\mathcal{V}_{B\to B}\circ\mathcal{M}_{A'\to B}(U_A^T\widetilde{\Phi}_{A'A}\overline{U}_A) && (97)\\
&=\mathcal{V}_{B\to B}\Big(U_A^T\mathcal{M}_{A'\to B}(\widetilde{\Phi}_{A'A})\overline{U}_A\Big) && (98)\\
&=U_A^T\otimes V_BJ_\mathcal{M}\overline{U}_A\otimes V_B^\dagger. && (99)
\end{aligned}$$

So $K_{AB}\geq 0$ and $\beta(K_{AB})=\beta(J_\mathcal{M})\leq 1$. Thus $\mathcal{V}_{B\to B}\circ\mathcal{M}_{A'\to B}\circ\mathcal{U}_{A'\to A'}\in\mathcal{V}_\beta$. ■

Appendix B PROOF OF LEMMA 18

Let G be a finite group, and for every $g\in G$, let $g\to U_A(g)$ and $g\to V_B(g)$ be unitary representation acting on the input and output spaces of the channel, respectively. Then a quantum channel $\mathcal{N}_{A\to B}$ is G-covariant if $\mathcal{N}_{A\to B}(U_A(g)\rho_AU_A^\dagger(g))=V_B(g)\mathcal{N}_{A\to B}(\rho_A)V_B^\dagger(g)$ for all $\rho_A\in\mathcal{S}(A)$. We also introduce the average state $\widehat{\rho}_A=\frac{1}{|G|}\sum_g U_A(g)\rho_AU_A^\dagger(g)$.

For the convenience of presenting the strong converse results in Appendix C, we need to introduce the sandwiched Rényi relative entropy. For any $\rho\in\mathcal{S}$, $\sigma\geq 0$ and $\alpha\in(1,\infty)$, the sandwiched Rényi relative entropy is defined as [44, 45],

$$\widetilde{D}_\alpha(\rho\|\sigma):=\frac{1}{\alpha-1}\log\operatorname{Tr}((\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha), \tag{100}$$

if $\operatorname{supp}\rho\subseteq\operatorname{supp}\sigma$ and it is equal to $+\infty$ otherwise. We further introduce the Rényi version of Υ-information:

$$\widetilde{\Upsilon}_\alpha(\mathcal{N},\rho_{A'}):=\min_{\mathcal{M}\in\mathcal{V}}\widetilde{D}_\alpha(\mathcal{N}_{A'\to B}(\phi_{A'A})\|\mathcal{M}_{A'\to B}(\phi_{A'A})),$$

where $\phi_{AA'}$ is a purification of $\rho_{A'}$ as usual. The following is a direct adaptation of Proposition 2 in [39].

Lemma 18 *Let $\mathcal{N}_{A'\to B}$ be G-covariant with the average state $\widehat{\rho}_{A'}$. Then, $\widetilde{\Upsilon}_\alpha(\mathcal{N},\rho_{A'})\leq\widetilde{\Upsilon}_\alpha(\mathcal{N},\widehat{\rho}_{A'})$.*

Proof Consider the pure quantum state

$$|\psi\rangle_{PAA'}=\sum_g\frac{1}{\sqrt{|G|}}|g\rangle\otimes(\mathbb{1}_A\otimes U_{A'}(g))|\phi^\rho_{AA'}\rangle \tag{101}$$

which purifies $\widehat{\rho}_{A'}$. Then for any fixed CP map $\mathcal{M}_{A'\to B}\in\mathcal{V}$, we have the following chain of inequalities in (102)-(106). The second line follows from monotonicity of the sandwiched Rényi relative entropy under the channel $\sum_g|g\rangle\langle g|\cdot|g\rangle\langle g|$. The third line follows from the G-invariance of the channel $\mathcal{N}_{A'\to B}$. The fourth line follows from unitary invariance of the sandwiched Rényi

$$
\begin{aligned}
&\widetilde{D}_\alpha(\mathcal{N}_{A'\to B}(\psi_{PAA'})\|\mathcal{M}_{A'\to B}(\psi_{PAA'}))\\
&\quad\geq \widetilde{D}_\alpha\Big(\sum_g \frac{1}{|G|}|g\rangle\langle g|_P \otimes \mathcal{N}_{A'\to B}\circ\mathcal{U}_{A'}(g)(\phi_{A'A})\Big\|\sum_g \frac{1}{|G|}|g\rangle\langle g|_P \otimes \mathcal{M}_{A'\to B}\circ\mathcal{U}_{A'}(g)(\phi_{A'A})\Big) \qquad (102)\\
&\quad= \widetilde{D}_\alpha\Big(\sum_g \frac{1}{|G|}|g\rangle\langle g|_P \otimes \mathcal{V}_B(g)\circ\mathcal{N}_{A'\to B}(\phi_{A'A})\Big\|\sum_g \frac{1}{|G|}|g\rangle\langle g|_P \otimes \mathcal{M}_{A'\to B}\circ\mathcal{U}_{A'}(g)(\phi_{A'A})\Big) \qquad (103)\\
&\quad= \widetilde{D}_\alpha\Big(\sum_g \frac{1}{|G|}|g\rangle\langle g|_P \otimes \mathcal{N}_{A'\to B}(\phi_{A'A})\Big\|\sum_g \frac{1}{|G|}|g\rangle\langle g|_P \otimes \mathcal{V}_B^\dagger(g)\circ\mathcal{M}_{A'\to B}\circ\mathcal{U}_{A'}(g)(\phi_{A'A})\Big) \qquad (104)\\
&\quad\geq \widetilde{D}_\alpha\Big(\mathcal{N}_{A'\to B}(\phi_{A'A})\Big\|\sum_g \frac{1}{|G|}\mathcal{V}_B^\dagger(g)\circ\mathcal{M}_{A'\to B}\circ\mathcal{U}_{A'}(g)(\phi_{A'A})\Big) \qquad (105)\\
&\quad\geq \min_{\mathcal{M}\in\mathcal{V}} \widetilde{D}_\alpha(\mathcal{N}_{A'\to B}(\phi_{A'A})\|\mathcal{M}_{A'\to B}(\phi_{A'A})). \qquad (106)
\end{aligned}
$$

relative entropy under $\sum_g |g\rangle\langle g| \otimes V_B^\dagger(g)$. The fifth line follows from monotonicity of the sandwiched Rényi relative entropy under the partial trace over P. The last line follows from the fact that $\sum_g \frac{1}{|G|}\mathcal{V}_B^\dagger(g)\circ\mathcal{M}_{A'\to B}\circ\mathcal{U}_{A'}(g)$ is still an element in $\mathcal{V}$.

Finally, we minimize over all maps $\mathcal{M}\in\mathcal{V}$. The conclusion then follows because all purifications are related by an isometry acting on the purifying system and the quantity $\widetilde{\Upsilon}_\alpha(\mathcal{N},\rho_{A'})$ is invariant under isometries acting on the purifying system. ■

Furthermore, we should note that in the proof we only use the monotonicity of the sandwiched Rényi relative entropy. The result can thus be trivially generalized to other divergences and distance measures, including the hypothesis testing divergence and the quantum relative entropy.

Appendix C STRONG CONVERSE FOR Υ-INFORMATION

In this section, we are trying to establish the strong converse of Υ-information and obtain some partial results. Specifically, we show that Υ is a strong converse for covariant channels.

Proposition 19 *For any quantum channel $\mathcal{N}_{A'\to B}$ and unassisted code with achievable (r,n,ε), it holds*

$$\varepsilon \geq 1 - 2^{-n\left(\frac{\alpha-1}{\alpha}\right)\left(r-\frac{1}{n}\widetilde{\Upsilon}_\alpha(\mathcal{N}^{\otimes n})\right)}, \tag{107}$$

where $\widetilde{\Upsilon}_\alpha(\mathcal{N}) := \max_{\rho_{A'}} \widetilde{\Upsilon}_\alpha(\mathcal{N},\rho_{A'})$.

Proof Suppose (r,n,ε) is achieved by the average input state $\rho_{A'^n}$. From the proof of Theorem 7, we have the inequality that $C^{(1)}(\mathcal{N}^{\otimes n},\rho_{A'^n},\varepsilon) \leq D_H^\varepsilon\big(\mathcal{N}_{A'\to B}^{\otimes n}(\phi_{A'^nA^n})\big\|\mathcal{M}_{A'^n\to B^n}(\phi_{A'^nA^n})\big)$. Suppose $\{F_{A^nB^n}, \mathbb{1}-F_{A^nB^n}\}$ is the optimal test of $D_H^\varepsilon\big(\mathcal{N}_{A'\to B}^{\otimes n}(\phi_{A'^nA^n})\|\mathcal{M}_{A'^n\to B^n}(\phi_{A'^nA^n})\big)$. We obtain

$$nr \leq -\log f_2 \quad \text{and} \quad 1-\varepsilon \leq f_1, \tag{108}$$

$$
\begin{aligned}
\text{with}\quad f_1 &= \operatorname{Tr} F_{A^nB^n}\mathcal{N}_{A'\to B}^{\otimes n}(\phi_{A'^nA^n}), &(109)\\
f_2 &= \operatorname{Tr} F_{A^nB^n}\mathcal{M}_{A'^n\to B^n}(\phi_{A'^nA^n}). &(110)
\end{aligned}
$$

Due to the monotonicity of the sandwiched Rényi relative entropy under the test $\{F_{A^nB^n}, \mathbb{1} - F_{A^nB^n}\}$, we have

$$\widetilde{D}_\alpha\left(\mathcal{N}^{\otimes n}_{A'\to B}(\phi_{A^nA^n})\big\|\mathcal{M}_{A'^n\to B^n}(\phi_{A'^nA^n})\right) \geq \delta_\alpha(f_1\|f_2),$$

where $\delta_\alpha(p\|q) := \frac{1}{\alpha-1}\log\left(p^\alpha q^{1-\alpha} + (1-p)^\alpha(1-q)^{1-\alpha}\right)$. Using Eqs. (108), we thus find

$$\begin{aligned}\min_{\mathcal{M}\in\mathcal{V}} \widetilde{D}_\alpha\left(\mathcal{N}^{\otimes n}_{A'\to B}(\phi_{A'^nA^n})\|\mathcal{M}_{A'^n\to B^n}(\phi_{A'^nA^n})\right)& \\ \geq \delta_\alpha(\varepsilon\,\|1-2^{-nr}).& \qquad (111)\end{aligned}$$

Maximizing over all average input state $\rho_{A'^n}$, we conclude that

$$\begin{aligned}\widetilde{\Upsilon}_\alpha(\mathcal{N}^{\otimes n}) &\geq \delta_\alpha(\varepsilon\,\|1-2^{-nr}) && (112)\\ &\geq \frac{1}{\alpha-1}\log(1-\varepsilon)^\alpha(2^{-nr})^{1-\alpha} && (113)\\ &= \frac{\alpha}{\alpha-1}\log(1-\varepsilon)+nr, && (114)\end{aligned}$$

which implies that $\varepsilon \geq 1 - 2^{-n\left(\frac{\alpha-1}{\alpha}\right)\left(r-\frac{1}{n}\widetilde{\Upsilon}_\alpha(\mathcal{N}^{\otimes n})\right)}$. ■

Note that any generalization of the Rényi divergence that satisfies the data-processing inequality would suffice for this proof. But the monotonicity (in terms of α) of the sandwiched Rényi divergence is required in the following proof.

Proposition 20 *For any covariant channel $\mathcal{N}$, $\Upsilon(\mathcal{N})$ is a strong converse bound on the classical capacity.*

Proof From Lemma 18, we can fix the average input state of $\widetilde{\Upsilon}_\alpha(\mathcal{N})$ to be the maximally mixed state. Then $\widetilde{\Upsilon}_\alpha$ is subadditive, i.e., $\widetilde{\Upsilon}_\alpha(\mathcal{N}^{\otimes n}) \leq n\widetilde{\Upsilon}_\alpha(\mathcal{N})$. Thus from Eq. (107), we have

$$\varepsilon \geq 1 - 2^{-n\left(\frac{\alpha-1}{\alpha}\right)\left(r-\widetilde{\Upsilon}_\alpha(\mathcal{N})\right)}. \qquad (115)$$

The quantity $\widetilde{\Upsilon}_\alpha(\mathcal{N})$ is monotonically increasing in α. Following the proof of Lemma 3 in [39], we can also show that $\lim_{\alpha\to 1^+}\widetilde{\Upsilon}_\alpha(\mathcal{N}) = \Upsilon(\mathcal{N})$. Hence, for $r > \Upsilon(\mathcal{N})$, there always exists an $\alpha > 1$ such that $r > \widetilde{\Upsilon}_\alpha(\mathcal{N})$. Therefore the error ε will to to 1 as n goes to infinity. ■

The following two properties would be required to show that Υ is a strong converse bound for general channels.

- Weak subadditivity: $\widetilde{\Upsilon}_\alpha(\mathcal{N}^{\otimes n}) \leq n\widetilde{\Upsilon}_\alpha(\mathcal{N}) + o(n)$
- Continuity: $\lim_{\alpha\to 1^+}\widetilde{\Upsilon}_\alpha(\mathcal{N}) = \Upsilon(\mathcal{N})$.

Appendix D NEW META-CONVERSE OVER $\mathcal{V}_\beta$ IS AN SDP

In this section, we show that our new meta-converse in Theorem 7 can be written as an SDP. Let us first write

$$\begin{aligned}&\min_{\mathcal{M}\in\mathcal{V}_\beta}\max_{\rho_{A'}} D_H^\varepsilon(\mathcal{N}_{A'\to B}(\phi_{A'A})\big\|\mathcal{M}_{A'\to B}(\phi_{A'A})) && (116)\\ &= -\log\max_{\mathcal{M}\in\mathcal{V}_\beta}\min_{\rho_A}\beta_\varepsilon(\sqrt{\rho_A}J_\mathcal{N}\sqrt{\rho_A}\big\|\sqrt{\rho_A}J_\mathcal{M}\sqrt{\rho_A}). && (117)\end{aligned}$$

According to the definition of β_ε, the minimization part in (117) is equivalent to the optimization,

$$\begin{aligned} \underset{\rho_A,\, F_{AB}}{\text{minimize}} \quad & \operatorname{Tr} \sqrt{\rho_A} J_{\mathcal{M}} \sqrt{\rho_A} F_{AB} \\ \text{subject to} \quad & \operatorname{Tr} \sqrt{\rho_A} J_{\mathcal{N}} \sqrt{\rho_A} F_{AB} \geq 1-\varepsilon, \\ & 0 \leq F_{AB} \leq \mathbb{1}_{AB},\ \rho_A \geq 0,\ \operatorname{Tr} \rho_A = 1. \end{aligned} \tag{118}$$

Let $G_{AB} = \sqrt{\rho_A} F_{AB} \sqrt{\rho_A}$. We have (118) being equivalent to

$$\begin{aligned} \underset{\rho_A,\, G_{AB}}{\text{minimize}} \quad & \operatorname{Tr} J_{\mathcal{M}} G_{AB} \\ \text{subject to} \quad & \operatorname{Tr} J_{\mathcal{N}} G_{AB} \geq 1-\varepsilon, \\ & 0 \leq G_{AB} \leq \rho_A \otimes \mathbb{1}_B,\ \rho_A \geq 0,\ \operatorname{Tr} \rho_A = 1, \end{aligned} \tag{119}$$

with the dual SDP given by

$$\begin{aligned} \underset{x, y, Z_{AB}}{\text{maximize}} \quad & (1-\varepsilon)x + y \\ \text{subject to} \quad & J_{\mathcal{M}} - x J_{\mathcal{N}} + Z_{AB} \geq 0, \\ & y \mathbb{1}_A + \operatorname{Tr}_B Z_{AB} \leq 0,\ x \geq 0,\ Z_{AB} \geq 0. \end{aligned} \tag{120}$$

Finally, combining (120) with the maximization condition $\mathcal{M} \in \mathcal{V}_\beta$ in (117), we obtain the following SDP for the meta-converse (116):

$$\begin{aligned} -\log \quad \underset{\substack{x, y, J_{\mathcal{M}}, \\ Z_{AB}, S_B, R_{AB}}}{\text{maximize}} \quad & (1-\varepsilon)x + y \\ \text{subject to} \quad & J_{\mathcal{M}} - x J_{\mathcal{N}} + Z_{AB} \geq 0, \\ & y \mathbb{1}_A + \operatorname{Tr}_B Z_{AB} \leq 0, \\ & x \geq 0,\ Z_{AB} \geq 0, \operatorname{Tr} S_B \leq 1 \\ & - R_{AB} \leq J_{\mathcal{M}}^{T_B} \leq R_{AB}, \\ & - \mathbb{1}_A \otimes S_B \leq R_{AB}^{T_B} \leq \mathbb{1}_A \otimes S_B. \end{aligned} \tag{121}$$