A Few steps more towards NPT bound entanglement

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***Abstract*—We consider the problem of existence of bound entangled states with non-positive partial transpose (NPT). As one knows, existence of such states would in particular imply nonadditivity of distillable entanglement. Moreover it would rule out a simple mathematical description of the set of distillable states. Distillability is equivalent to so called n-copy distillability for some n. We consider a particular state, known to be 1-copy nondistillable, which is supposed to be bound entangled. We study the problem of its two-copy distillability, which boils down to show that maximal overlap of some projector Q with Schmidt rank two states does not exceed $1/2$. Such property we call the *half-property*. We first show that the maximum overlap can be attained on vectors that are not of the simple product form with respect to cut between two copies. We then attack the problem in twofold way: a) prove the half-property for *some classes* of Schmidt rank two states b) bound the required overlap from above for *all* Schmidt rank two states. We have succeeded to prove the half-property for wide classes of states, and to bound the overlap from above by $c < 3/4$. Moreover, we translate the problem into the following matrix analysis problem: bound the sum of the squares of the two largest singular values of matrix $A \otimes I + I \otimes B$ with A, B traceless 4×4 matrices, and $\mathrm{Tr}A^\dagger A + \mathrm{Tr}B^\dagger B = \frac{1}{4}$.**

***Index Terms*—Quantum Physics, Quantum Information Theory, Bound entanglement, Entanglement distillation**

I. Introduction

The Phenomenon of bound entanglement lies at the heart of entanglement theory [1]. A bound entangled state of a bipartite system is one which is entangled, but cannot be used for quantum communication. A possibility of transmitting qubits via bipartite states is connected with their *distillability* [2], [3] i.e. the possibility of obtaining asymptotically pure maximally entangled states by local operations and classical communication from many copies of a given state. Such maximally entangled states can be then used for transmitting qubits by means of teleportation. It is known that all entangled two qubit states are distillable [4]; however, already for $3 \otimes 3$ or $2 \otimes 4$ systems there exist bound entangled states — entangled states that cannot be distilled. Such states involve irreversibility: to create them by LOCC one needs pure entanglement [5], [6], but no pure entanglement can be obtained back from them. They constitute a sort of a "black hole" of quantum information theory [7], and have been also compared to a single heat bath in thermodynamics, since to create the latter one has to spend work (as in Joule experiment), yet no work can be obtained back from it by a cyclic process [8], [9].

Bound entangled states, although directly not useful for quantum communication, are not entirely useless. They can be helpful indirectly, via activation like process: in conjunction with some distillable state, they allow for better performance of some tasks [10], [11]. It was even recently shown that any bound entangled state can perform nonclassical task via kind of activation [12]. This is the first result showing that entanglement always allows for nonclassical tasks. Finally, it was also shown that some bound entangled states can be useful for production of secure cryptographic key [13], [14], [15]. This has lead to the possibility of obtaining unconditionally secure key via channels which cannot reliably convey quantum information [16], [17].

Since bound entangled states present qualitatively different type of entanglement from the distillable states behaving in a strange way, it is more than desired to have some characterization of the set. It has been shown [18] that any state with *positive partial transpose* (PPT) [19] is non-distillable. A long standing open problem is whether the converse is also true. Since the discovery of bound entanglement the question "Are all states which do not have positive partial transpose distillable?" has remained open.

Provided it has a positive answer, we would have computable criterion allowing to distinguish between bound and free entanglement. However the importance of the problem is not merely due to technical (in)convenience. As a matter of fact, in [20] dramatic consequences of a negative answer have been discovered. Namely, for some hypothetical bound entangled state ϱ with a non-positive partial transpose (NPT) there exists another bound entangled state σ such that the joint state $\varrho \otimes \sigma$ is no longer a bound entangled state. In [11] it was shown that an arbitrary NPT bound entangled state would exhibit such a phenomenon (it also follows from [21] via Jamiołkowski isomorphism). Such a phenomenon of "superactivation" has been indeed found in a multipartite case [22] and translated into extreme nonadditivity of multipartite quantum channel capacities [23]. (In a multipartite case, though still very strange, this can be easier to understand than in a bipartite case due to a rich state structure allowed by many possible splits between the parties.) In quantum communication language the phenomenon of "superactivation" would mean that two channels (supported by two-way classical communication) none of them separately can convey quantum information if put together, can be used for reliable transmission of qubits.

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Analogous problem for channels that are not supported by classical communication was recently solved by Smith and Yard [24] (see also [25] in this context). Another implication of the existence of NPT bound entangled states is that the basic measure of entanglement — the *distillable entanglement* — would be non-convex.

The problem of existence of NPT bound entanglement has been attacked many times since the beginning. In [26] it was shown that it is enough to concentrate on one parameter family of the Werner states [27]: if NPT bound entangled states exist at all, some of the Werner states must be NPT bound entangled too. There also exists the following characterization of distillable states [18]: A state is distillable, if some number of copies $\varrho^{\otimes n}$ can be locally projected to obtain a two-qubit NPT state. The state is then called n-copy distillable. Therefore, a state is non-distillable if it is not n-copy distillable for all n. The whole problem is to relate this rather non operational characterization to the NPT property.

Subsequently, two attempts to solve the problem have been then made independently [28], [29]. In particular the authors have singled out a set of the Werner states which is expected to contain only non-distillable states. Moreover for any n they have shown a subset of the Werner states containing solely n-copy non-distillable states (see also [30] in this context). However the subsets are decreasing when n increases. One might ask at this point, whether n-copy non-distillability implies the same for $n+1$. Then to solve the problem it would be enough to check whether a state is 1-copy non-distillable, which for the Werner states is not hard to do. However it was shown in [31] that this is not true. For any n states have been found, which are n-copy non-distillable, but are $(n+1)$-copy distillable.

Another way to attack the problem would be the following: let us take a larger but mathematically more tractable class of operations than LOCC — the ones that preserve PPT states [32], [21]. If one can show that there are some NPT states that are not distillable by this larger class of operations, then it would be also true for LOCC, and the problem would be solved. However in [21] it have been shown that all NPT states are distillable by PPT preserving operations. This shows that such an approach cannot solve our problem.

There are some sufficient conditions for distillability. E.g. if a state violates the reduction criterion, then it is distillable [26]. In [33], [34] Clarisse provided a systematic way of finding such conditions. His conditions are related to a description of the set of 1-copy distillable states by means of some maps and associated witnesses, in analogy to describing the set of separable states by means of entanglement witnesses and positive maps [35], [36]. There remains the main problem of checking such conditions on n-copies, to be able to prove also n-copy distillability. Another connection with separability problem was found in [37] where it was shown that the problem of existence of NPT bound states is equivalent to showing that some operators labeled by n are entanglement witnesses. This connection was exploited in [38] to provide exact numerical evidence for 2-copy undistillability of one-copy undistillable qutrit Werner states.

For further attempts to solve the problem see [39] where one can also find relevant literature. There have been several more recent attempts. Unfortunately the proofs given in two of them [40], [41] turned out to have some gaps. The last partial result is due to [42] where a notion of n-copy correlated distillability was introduced, and used to characterize the convex hull of the non-distillable states.

We have seen that a considerable effort has been put so far without providing the final solution, but definitely enriching "phenomenology" of the problem. In such situation we have decided to consider a modest goal. Namely we analyze two-copy distillability only, and we focus on a single state, drawn from the "suspicious" family of the Werner states. We choose a dimension $C^4 \otimes C^4$, in which case, the problem reduces to analysis of suitable properties of some *projector*. Namely, we ask whether

$$\sup_{\phi_2} \langle \phi_2 | Q | \phi_2 \rangle \leq \frac{1}{2} \tag{1}$$

where Q is our projector on bipartite system $C^{16} \otimes C^{16}$, and supremum is taken over all states with at most two Schmidt coefficients. If it is true it would mean that our state would be two-copy non-distillable. The above condition is essentially a special case of the condition obtained in [28], [29]. There exists numerical evidence that it is indeed true, however the analytical proof is still lacking.

To begin with, we have not been able to solve even this modest problem. However we have obtained numerous partial results. First of all we have shown that the maximum overlap can be attained on vectors that are not of the simple product form with respect to cut between two copies. Then we have focused research on two main approaches. One is to provide the largest class of Schmidt rank two states ϕ_2 which satisfy the above inequality (a state ϕ_2 satisfying the inequality is said to have the *half-property*). The other is to provide some nontrivial bound on the quantity $\langle \phi_2 | Q | \phi_2 \rangle$. Regarding the first approach we have provided several classes of states satisfying the half-property. In particular we have translated the problem into a concise matrix analysis problem, and have solved it for wide class of matrices — normal matrices. This translates into a wide class of states ϕ_2 possessing the half-property. We have also shown that the problem reduces to determining whether some family of symmetric mixed states has Schmidt number greater than two (i.e. cannot be written as mixture of states with Schmidt rank two). This allows to attack the problem by means of entanglement measures. We have performed suitable analysis for the negativity, which however provided smaller class of states with the half-property than the previous method.

As far as the second approach is concerned, we have first analyzed the easier problem, of supremum over *product* states (Schmidt rank one). We obtained that it gives $3/8$. By Schwarz inequality one obtains that the supremum over Schmidt rank two states can be at most twice as much, giving then $3/4$. However, as we argue, such approach, if continued for larger number of copies, can give only the trivial bound 1 for $n \to \infty$. We subsequently prove that our quantity is for sure *strictly less* than $3/4$. By continuity we are able to push it to ≈ 0.7497. We also provide a couple of other results, that may be useful for further investigation of the problem.

The paper is organized as follows. In section II we specify the main problem. In particular we introduce projector Q related to two-copy distillability (and its generalizations to more copies) and define the *half-property*. Then we show (Sec. III) that one cannot solve the problem by showing that the Schmidt rank two states ϕ_2 achieving the maximum are product with respect to cut between the copies. Subsequently (Sec. IV) the problem of the half-property is translated into matrix analysis problem, regarding maximization of the sum of the squares of the two largest singular values of matrix $A \otimes I + I \otimes B$ under some constraints. We solve the problem for normal matrices A, B and obtain a wide class of states satisfying the half-property. Next we show (Sec. V) that any two pair state for which at least one system from each pair is effectively two-level one, satisfies the half-property. Then we turn to an easier problem of optimizing the overlap of Q with product states (Sec. VI). We compute maximum for general case of n-copies, obtaining $3/8$ for two copies. This gives bound $3/4$ for the overlap of all Schmidt rank-two states with Q. We then show the half-property for superpositions of the product states attaining maximum. Then (Sec. VII) we observe a trade-off between two parts of the overlap $\langle\phi_2|Q|\phi_2\rangle$ — the "diagonal" and the "coherence" part, if the former is large, then the latter must be small. Since coherence part is bounded by diagonal one, this allows us to go slightly below $3/4$, namely we obtain ≈ 0.7497. Finally we apply entanglement measures, and two-positive maps to the problem in Sec. VIII, providing some exemplary results, which for a while are not stronger than the ones obtained in previous sections. We also point that entanglement measure that would distinguish between separable, bound entangled and distillable states must be discontinuous.

II. Specifying the problem

It is known that if NPT bound entangled states exist then such state must exist among the Werner states. The latter states are of the form

$$\varrho_W = p\varrho_s + (1-p)\varrho_a \tag{2}$$

where

$$\varrho_s = \frac{P_s}{d_s}, \quad \varrho_a = \frac{P_a}{d_a} \tag{3}$$

with P_s and P_a being the projectors onto the symmetric and the antisymmetric subspaces of the Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$ and $d_s = d(d+1)/2$ and $d_a = d(d-1)/2$ their dimensions. Alternatively the Werner states may be written as

$$\varrho_W = \frac{I + \alpha V}{d^2 + \alpha d} \tag{4}$$

where $\alpha \in [-1, 1]$ ($V = P_s - P_a$ is a swap operator). It is known that they are separable and PPT for $p \geq \frac{1}{2}$ while for $p < p_0 = \frac{d+1}{4d-2}$ they are distillable and for $p \in [p_0, \frac{1}{2})$ they are NPT and it is not known whether they are distillable. Actually it is conjectured that for the whole region $p \in [p_0, \frac{1}{2})$ the states are NPT bound entangled [28], [29] (We will call them the *suspicious* Werner states).

In [18] the characterization of the distillable states was obtained in terms of so called n-copy distillability. Namely we say that a state is n-copy distillable, if $\varrho^{\otimes n}$ can be locally projected to a obtain two-qubit NPT state. Equivalently a state ϱ is n-copy distillable if it satisfies

$$\inf_{\phi_2} \langle\phi_2|\varrho^{\Gamma\otimes n}|\phi_2\rangle < 0 \tag{5}$$

where the infimum is taken over all pure states with Schmidt rank two, and the superscript Γ denotes the partial transposition. Now a state is distillable iff it is n-copy distillable for some n. Hence to prove that a state is non-distillable one has to show that for all n

$$\inf_{\phi_2} \langle\phi_2|\varrho^{\Gamma\otimes n}|\phi_2\rangle \geq 0. \tag{6}$$

For the suspicious Werner states it is known that they are one copy undistillable more over it was numerically checked that they are also two and three copy undistillable [28], [29]. As a matter of fact for all n an n-copy undistillable subset of the suspicious Werner states is known, but the subsets are shrinking with n giving an empty set in the limit of $n \to \infty$.

Anyway, it is likely that even the most entangled state from the suspicious region is undistillable. In this paper we will focus just on this boundary state (i.e. with $p = p_0$) and moreover we consider only the $\mathbb{C}^4 \otimes \mathbb{C}^4$ case (this gives $p = \frac{5}{14}$ or $\alpha = -\frac{1}{2}$). The reason is that the problem of n-copy distillability for the boundary state in this dimension reduces to analyzing the overlap of rank two states with some *projector*.

Since we will be mostly concerned with two copy undistillability let us begin with $n = 2$. The normalization of ${\varrho_W^\Gamma}^{\otimes 2}$ has no impact on the existence of ϕ_2 satisfying (6), thus for $d = 4$ we can simplify the expression of ${\varrho_W^\Gamma}^{\otimes 2}$ to

$$\begin{aligned} {\varrho_W^\Gamma}^{\otimes 2} &\sim (I - \tfrac{1}{2}V)^{\Gamma\otimes 2} = \left(I - \tfrac{d}{2}P_+\right)^{\otimes 2} &(7)\\ &= (P_+^\perp \otimes P_+^\perp + P_+ \otimes P_+) - (P_+^\perp \otimes P_+ + P_+ \otimes P_+^\perp) &(8) \end{aligned}$$

where

$$P_+^\perp = I - P_+, \quad P_+ = |\psi_+\rangle\langle\psi_+|, \quad |\psi_+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle. \tag{9}$$

If we replace the minus sign with the plus sign in formula (7) we get the identity. Thus it is evident that two-copy undistillability, i.e. (6) with $n = 2$, is equivalent to

$$\langle\phi_2|Q|\phi_2\rangle \leq \frac{1}{2} \tag{10}$$

for all Schmidt rank two states ϕ_2 in the cut $AA' : BB'$, or, using a shorthand notation, for all $\phi_2 \in \mathrm{SR}_2(AA' : BB')$, with

$$Q = P_+^\perp \otimes P_+ + P_+ \otimes P_+^\perp. \tag{11}$$

We will call equation (10) the *half-property*. Thus our Werner state is two copy undistillable iff all rank two states ϕ_2 satisfy the half-property. In particular, equality in the half-property (10) for some ϕ_2 is equivalent to equality in (6) with $n = 2$.

Thus to prove two copy undistillability we would have to show that all two pair rank two states ϕ_2 satisfy the half-property. We will show that this is the case for a wide range of ϕ_2 states.

We will use the notion of ϕ_k to denote the state of Schmidt rank k in Alice versus Bob cut. If not explicitly specified it should be clear from the context whether we mean a state on a single pair, i.e. $\phi_k \in \mathrm{SR}_k(A:B)$ or on both pairs, i.e $\phi_k \in \mathrm{SR}_k(AA':BB')$.

In some cases we will consider the projector Q for any dimension d, though only for $d=4$ it is connected with two copy distillability of the boundary state.

Analogously to the two copy case one can relate n-copy distillability of the boundary Werner state with the overlap of rank two states with some projectors Q_n. Namely for $d=4$ we have

$$\varrho_W^{\Gamma\otimes n} \sim (I - \tfrac{d}{2}P_+)^{\otimes n} = \mathcal{P}_+ - \mathcal{P}_- = I^{\otimes n} - 2\mathcal{P}_- \qquad (12)$$

where $\mathcal{P}_+$ and $\mathcal{P}_-$ are projectors satisfying $\mathcal{P}_+ + \mathcal{P}_- = I^{\otimes n}$. We define Q_n as

$$Q_n \equiv \mathcal{P}_- = \frac{1}{2}\left(I^{\otimes n} - \left(I - \tfrac{d}{2}P_+\right)^{\otimes n}\right) \qquad (13)$$

so that $\langle\phi_2|Q_n|\phi_2\rangle \le \frac{1}{2}$ iff $\langle\phi_2|\varrho_W^{\otimes n}|\phi_2\rangle \ge 0$.

Lemma 1. *For $d=4$ projectors Q_n satisfy the following recursive formula*

$$Q_1 = P_+, \qquad (14)$$
$$Q_{n+1} = Q_n \otimes Q_1^{\perp} + Q_n^{\perp} \otimes Q_1. \qquad (15)$$

Proof: For $n=1$ it is evident, for $n>1$ by substituting Q_n transformed to

$$\left(I - \tfrac{d}{2}P_+\right)^{\otimes n} = I^{\otimes n} - 2Q_n \qquad (16)$$

into Q_{n+1} we obtain the recursive formula. ■

We have $Q_2 = Q$ and Q_3 has the form

$$Q_3 = P_+ \otimes P_+^{\perp} \otimes P_+^{\perp} + P_+^{\perp} \otimes P_+ \otimes P_+^{\perp} + P_+^{\perp} \otimes P_+^{\perp} \otimes P_+ + P_+ \otimes P_+ \otimes P_+. \qquad (17)$$

III. Existence of nontrivial maxima of $\langle\phi_2|Q|\phi_2\rangle$

In [43] a class of states of the form $\phi_1 \otimes \phi_2$ was shown to provide local minimum for (6) with $d=3$, $\alpha = -\frac{1}{2}$, $n=2$. This suggests the following question: is it that all local minima are of the form $\phi_1 \otimes \phi_2$? In our specific case it translates into the same question about the maximum. It is easy to see that states of the form $\phi_2 \otimes \phi_1$ may attain equality in the half-property and nothing more. We will now examine a question whether there are other rank two states which attain equality in the half-property and are not of this form. The answer is unfortunately positive.

A. Example of equality in superpositions

We show that there are nontrivial superpositions of $\phi_2 \otimes \phi_1$ and $\phi_1' \otimes \phi_2'$ which are rank two states and attain equality in the half-property.

For any state of the form $\phi_2 \otimes \phi_1$ its projection on Q is given by

$$\langle\phi_2 \otimes \phi_1|\, Q\, |\phi_2 \otimes \phi_1\rangle = p + q - 2pq \le \frac{1}{2} \qquad (18)$$

where

$$p = \langle\phi_2|P_+|\phi_2\rangle \le \frac{2}{d}, \quad q = \langle\phi_1|P_+|\phi_1\rangle \le \frac{1}{d} \qquad (19)$$

and the maximal value is attainable for $p = \frac{2}{d}$ and, for $d=4$, any q.

If we take superpositions of two states of that form with one of them swapped

$$|\psi\rangle = \sqrt{r}|\phi_2\rangle \otimes |\phi_1\rangle + \sqrt{1-r}|\phi_1'\rangle \otimes |\phi_2'\rangle \qquad (20)$$

satisfying

$$\langle\phi_2|P_+|\phi_2\rangle = \langle\phi_2'|P_+|\phi_2'\rangle = \frac{2}{d}, \qquad (21)$$
$$\langle\phi_1|P_+|\phi_1\rangle = \langle\phi_1'|P_+|\phi_1'\rangle = 0 \qquad (22)$$

then

$$\langle\psi|Q|\psi\rangle = \frac{1}{2}. \qquad (23)$$

States of the form ψ have in general Schmidt rank higher than two but there are also rank two states among them such as the following class of states

$$|\phi\rangle = \sqrt{r}\,|01\rangle \otimes |\psi_+^2\rangle + \sqrt{1-r}\,|\psi_+^2\rangle \otimes |01\rangle \qquad (24)$$

where

$$|\psi_+^2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \qquad (25)$$

The class ϕ can be rewritten in Alice versus Bob cut as

$$|\phi^{AA':BB'}\rangle = \frac{1}{\sqrt{2}}|00\rangle \otimes \left(\sqrt{r}|01\rangle + \sqrt{1-r}|10\rangle\right) + \frac{1}{\sqrt{2}}\left(\sqrt{r}|10\rangle + \sqrt{1-r}|01\rangle\right) \otimes |11\rangle \qquad (26)$$

which shows that ϕ are rank two states in this cut.

B. Form of ϕ_2 states maximizing overlap with $I \otimes P_+$

In contrast to the previous section we shall show here that two pair ${\phi_2}^{AA':BB'}$ which maximizes overlap with $I \otimes P_+$ must be of the form ${\phi_1}^{A:B} \otimes {\phi_2}^{A':B'}$. (This result is inspired by [40]) Of course, the maximum attainable value of the projection on P_+ for one pair Schmidt rank two state is $2/d$. Let ϕ_2 be a two pair state which attains this value. Then we have

$$|\langle\phi_2|\phi\rangle|^2 = 2/d, \qquad (27)$$

where ϕ is some normalized state from subspace $I \otimes P_+$, i.e. it is of the form

$$|\phi\rangle = \sum_j a_j |e_j f_j\rangle_{AB} \otimes \frac{1}{\sqrt{d}}\sum_i |ii\rangle_{A'B'}. \qquad (28)$$

Moreover also

$$\sup_{\phi_2 \in SR2} |\langle\phi_2|\phi\rangle|^2 = \frac{2}{d}. \qquad (29)$$

On the other hand we know that for any ψ

$$\sup_{\phi_2\in SR2} |\langle\phi_2|\psi\rangle|^2 = \mu_1^2+\mu_2^2, \tag{30}$$

where μ_1, μ_2 are the two largest Schmidt coefficients of ψ in the same cut that ϕ_2 has rank two, i.e. $AA' : BB'$. Thus, as the Schmidt coefficients of ϕ has the form $a_j/\sqrt{d}$ and each of them occurs d times in the composition, we have

$$|\langle\phi_2|\phi\rangle|^2 = \frac{2a_{\max}^2}{d}, \tag{31}$$

where $a_{\max} = \max_j a_j$. Therefore $a_{\max} = 1$, i.e.

$$|\phi\rangle = |x\rangle_A|y\rangle_B|\psi_+\rangle_{A'B'}, \tag{32}$$

where $|x\rangle, |y\rangle$ are some states. Writing $|\phi_2\rangle = c_1|r_1\rangle|s_1\rangle + c_2|r_2\rangle|s_2\rangle$ we get

$$|\langle\phi_2|\phi\rangle|^2 = \frac{1}{d}\left|c_1\alpha_1+c_2\alpha_2\right|^2 \le \frac{1}{d}\left(c_1|\alpha_1|+c_2|\alpha_2|\right)^2 \tag{33}$$

where

$$\alpha_1 = \sum_i ({}_{AA'}\langle r_1|x\rangle_A|i\rangle_{A'})({}_{BB'}\langle s_1|y\rangle_B|i\rangle_{B'}) \tag{34}$$

$$\alpha_2 = \sum_i ({}_{AA'}\langle r_2|x\rangle_A|i\rangle_{A'})({}_{BB'}\langle s_2|y\rangle_B|i\rangle_{B'}) \tag{35}$$

Since $|\alpha_1|, |\alpha_2| \le 1$, to get $|\langle\phi_2|\phi\rangle| = \frac{2}{d}$ we must have $|\alpha_1| = |\alpha_2| = 1$ and $c_1 = c_2 = \frac{1}{\sqrt{2}}$. It follows that $|r_1\rangle$ and $|r_2\rangle$ belong to the subspace $|x\rangle\langle x|\otimes I$. Which means that $|r_{1(2)}\rangle = |x\rangle_A|\tilde{r}_{1(2)}\rangle_{A'}$, where $|\tilde{r}_{1(2)}\rangle_{A'}$ are some orthogonal states. Similar relations hold for $|s_1\rangle$ and $|s_2\rangle$. Thus

$$\phi_2 = |x\rangle_A|y\rangle_B\left(|\tilde{r}_1\rangle_{A'}|\tilde{s}_1\rangle_{B'} + |\tilde{r}_2\rangle_{A'}|\tilde{s}_2\rangle_{B'}\right)/\sqrt{2}, \tag{36}$$

i.e. we obtain the desired form.

IV. States having "normal" projection on Q

Here we show that if a two pair Schmidt rank two state ϕ_2 has the projection on Q which is isomorphic to a normal operator through a state–operator isomorphism then it satisfies the half-property. To this end we will reformulate our optimization task in terms of the two largest Schmidt coefficients of states of the subspace defined by the projector Q. Then we will use the state–operator isomorphism to obtain optimization problem involving matrices and finally will solve the problem for normal matrices.

We have the following lemma, which is a generalization of a similar one for product states [44]

Lemma 2. *For any projector P acting on a bipartite system*

$$\sup_{\phi_2\in \mathrm{SR}_2} \langle\phi_2|P|\phi_2\rangle = \sup_{\psi\in\mathcal{H}_P} (\mu_1^2+\mu_2^2) \tag{37}$$

where μ_1 and μ_2 are the two largest Schmidt coefficients of ψ and $\mathcal{H}_P$ is the subspace defined by the projector P.

Note that this lemma immediately generalizes to rank k states for arbitrary fixed $k \ge 1$.

Proof: Let us observe that for all $\psi \in \mathcal{H}_P$

$$\langle\phi_2|P|\phi_2\rangle \ge \langle\phi_2|\psi\rangle\langle\psi|\phi_2\rangle. \tag{38}$$

Moreover there exists $\psi \in \mathcal{H}_P$ which reaches the equality

$$\langle\phi_2|P|\phi_2\rangle = \langle\phi_2|\psi\rangle\langle\psi|\phi_2\rangle, \tag{39}$$

namely $|\psi\rangle = \frac{P|\phi_2\rangle}{\|P|\phi_2\rangle\|}$ if $\|P|\phi_2\rangle\| \neq 0$ or any $\psi \in \mathcal{H}_P$ otherwise. From these two observations we get

$$\langle\phi_2|P|\phi_2\rangle = \sup_{\psi\in\mathcal{H}_P} |\langle\phi_2|\psi\rangle|^2. \tag{40}$$

From (40) and the fact stated in equation (30) we conclude

$$\sup_{\phi_2\in\mathrm{SR}_2}\langle\phi_2|P|\phi_2\rangle = \sup_{\psi\in\mathcal{H}_P}\sup_{\phi_2\in\mathrm{SR}_2}|\langle\phi_2|\psi\rangle|^2 \tag{41}$$

$$= \sup_{\psi\in\mathcal{H}_P}(\mu_1^2+\mu_2^2) \tag{42}$$

where μ_1 and μ_2 are the two largest Schmidt coefficients of ψ. ■

Let us now reformulate the problem in terms of matrices. Consider the following state–operator isomorphism

$$|\psi\rangle = \sum a_{ij}|i\rangle|j\rangle \longleftrightarrow X = \sum a_{ij}|i\rangle\langle j|. \tag{43}$$

In this isomorphism $\langle\psi|\psi\rangle = \mathrm{Tr}X^\dagger X$ and the Schmidt coefficients of a state ψ are equal to the singular values of the corresponding operator X. Therefore by lemma 2 and the equality between the Schmidt coefficients of ψ and the singular values of X we have

$$\sup_{\phi_2}\langle\phi_2|P|\phi_2\rangle = \sup_X(\sigma_1^2+\sigma_2^2) \tag{44}$$

where σ_1 and σ_2 are the two largest singular values of operator X and the supremum is taken over all operators X which correspond to states from $\mathcal{H}_P$ through the state–operator isomorphism (43).

A. Half-property in terms of matrices

Let us now apply the above consideration to our particular projector Q. All states $\psi_Q \in \mathcal{H}_Q$ where $Q = P_+^\perp\otimes P_+ + P_+\otimes P_+^\perp$ have the form

$$|\psi_Q\rangle = \sqrt{p}\,|\psi_{(1)}\rangle|\psi_+\rangle + \sqrt{1-p}\,|\psi_+\rangle|\psi_{(2)}\rangle \tag{45}$$

where $p\in[0,1]$ and

$$|\psi_{(1)}\rangle \perp |\psi_+\rangle, \quad |\psi_{(2)}\rangle \perp |\psi_+\rangle. \tag{46}$$

The image of ψ_Q states in the above state–operator isomorphism have the form

$$X = \sqrt{\frac{p}{d}}\,\tilde{A}\otimes I + \sqrt{\frac{1-p}{d}}\,I\otimes\tilde{B} \tag{47}$$

where

$$\mathrm{Tr}\tilde{A} = \mathrm{Tr}\tilde{B} = 0 \qquad \text{(orthogonality, i.e. (46))} \tag{48}$$

$$\mathrm{Tr}\tilde{A}^\dagger\tilde{A} = \mathrm{Tr}\tilde{B}^\dagger\tilde{B} = 1. \qquad \text{(normalization)} \tag{49}$$

By absorbing coefficients into operators the formulation of the image of ψ_Q states can be simplified to

$$X = A\otimes I + I\otimes B \tag{50}$$

where

$$\mathrm{Tr}A = \mathrm{Tr}B = 0, \quad \mathrm{Tr}A^\dagger A + \mathrm{Tr}B^\dagger B = \frac{1}{d}. \tag{51}$$

Thus we have reduced the problem of the half-property to the following optimization task: show that for all operators X of the form (50) satisfying constraints (51) we have

$$\sigma_1^2 + \sigma_2^2 \leq \frac{1}{2} \tag{52}$$

where σ_1 and σ_2 are the two largest singular values of operator X.

In the next section we show that this holds for normal matrices X which gives a wide class of states ϕ_2 satisfying the half-property.

B. Half-property for states having "normal" projection on Q

Let us first note that the operator X given in equation (50) is normal (i.e. $X^\dagger X = XX^\dagger$) iff operators A and B are normal. As normal matrices are diagonalizable and their singular values are equal to moduli of eigenvalues we arrive at an optimization problem over numbers rather than matrices which we will now solve. Namely we have

Theorem 1. *Let $\mathcal{X}_d$ be a subset of normal operators X of the form* (50) *satisfying constraints* (51)*. Then for $d = 4$ we have*

$$\sup_{X\in\mathcal{X}_d} (\sigma_1^2 + \sigma_2^2) \leq \frac{1}{2} \tag{53}$$

where σ_1 and σ_2 are the two largest singular values of operator X.

Proof: Since X is diagonalizable then we can replace singular values with moduli of eigenvalues. The latter are of the form

$$\lambda_{ij} = a_i + b_j \tag{54}$$

where a_i and b_j are eigenvalues of A and B respectively. We then have

$$\sup_{X\in\mathcal{X}_d} (\sigma_1^2 + \sigma_2^2) = \sup_{X\in\mathcal{X}_d} (|\lambda_1|^2 + |\lambda_2|^2) \tag{55}$$

$$= \sup_{X\in\mathcal{X}_d} \max_{\substack{i,j,k,l\in\{1,\ldots,d\},\\ (i,j)\neq(k,l)}} \left(|a_i + b_j|^2 + |a_k + b_l|^2\right) \tag{56}$$

$$= \sup_{X\in\mathcal{X}_d} \max\left\{|a_1 + b_1|^2 + |a_2 + b_2|^2, \; |a_1 + b_1|^2 + |a_1 + b_2|^2\right\} \tag{57}$$

where λ_1 and λ_2 are two eigenvalues of X with largest moduli. The constraints (51) on X imply the following constraints on a_i and b_i

$$\sum_{i=1}^{d} a_i = \mathrm{Tr}A = 0, \quad \sum_{i=1}^{d} b_i = \mathrm{Tr}B = 0, \tag{58}$$

$$\sum_{i=1}^{d} |a_i|^2 + \sum_{i=1}^{d} |b_i|^2 = \mathrm{Tr}A^\dagger A + \mathrm{Tr}B^\dagger B = \frac{1}{d}. \tag{59}$$

Equality (57) comes from the fact that there are two unique settings

1) $i \neq k \wedge j \neq l$ and
2) $i = k \wedge j \neq l \vee i \neq k \wedge j = l$.

In the second setting we consider only one term of the alternative (as under the constraints we can exchange A and B) and in both settings we take arbitrary indices (as under the constraints we can independently permute a_i and b_i).

Thus to prove the theorem we have to show that the following inequalities hold

$$|a_1 + b_1|^2 + |a_2 + b_2|^2 \leq \frac{1}{2} \tag{60}$$

$$|a_1 + b_1|^2 + |a_1 + b_2|^2 \leq \frac{1}{2} \tag{61}$$

under the constraints (58) and (59) with $d = 4$. The first inequality comes directly from the parallelogram identity

$$|x + y|^2 = 2(|x|^2 + |y|^2) - |x - y|^2 \leq 2(|x|^2 + |y|^2) \tag{62}$$

which implies

$$|a_1 + b_1|^2 + |a_2 + b_2|^2 \leq 2(|a_1|^2 + |b_1|^2 + |a_2|^2 + |b_2|^2) \leq 2\frac{1}{d} = \frac{1}{2}. \tag{63}$$

The second inequality is much more involved and we have moved it to the appendix (proposition 6) where we prove that

$$|a_1 + b_1|^2 + |a_1 + b_2|^2 \leq \frac{3d - 4}{d^2} \tag{64}$$

which for $d = 4$ gives (61). ■

We are now prepared to state the main result of this section

Theorem 2. *For $d = 4$ any rank two state $\phi_2 \in SR_2(AA' : BB')$ with the projection on Q ($Q|\phi_2\rangle$) isomorphic through the state–operator isomorphism to a normal operator satisfies the half-property.*

Proof: Let us assume $\langle\phi_2|Q|\phi_2\rangle \neq 0$ (otherwise the conclusion is obvious). By hypothesis ϕ_2 reaches its projection on Q on a state $|\psi_Q\rangle = \frac{Q|\phi_2\rangle}{\|Q|\phi_2\rangle\|} \in \mathcal{H}_Q$ and ψ_Q is isomorphic through the state–operator isomorphism given by (43) to a normal operator X. Then using the fact stated in equation (30), equality of the Schmidt coefficients of ψ_Q and the singular values of operator X in the state–operator isomorphism, and theorem 1 we obtain

$$\langle\phi_2|Q|\phi_2\rangle = |\langle\phi_2|\psi_Q\rangle|^2 \leq \sup_{\phi_2\in \mathrm{SR}_2(AA':BB')} |\langle\phi_2|\psi_Q\rangle|^2 \tag{65}$$

$$= \mu_1^2 + \mu_2^2 = \sigma_1^2 + \sigma_2^2 \leq \sup_{X\in\mathcal{X}_d} (\sigma_1^2 + \sigma_2^2) \leq \frac{1}{2} \tag{66}$$

where μ_1 and μ_2 are the two largest Schmidt coefficients of ψ_Q in the same cut in which ϕ_2 has rank two (i.e $AA' : BB'$) while σ_1 and σ_2 are the two largest singular values of operator X, and $\mathcal{X}_d$ is a subset of normal operators X of the form (50) satisfying constraints (51). ■

C. Characterization of states with normal projection onto Q

A more operational characterization of the states for which the above theorem proves the half-property is the following. Suppose we project ϕ_2 state onto ψ_+ on subsystem AB. Then the subsystem $A'B'$ should collapse to a $*$-symmetric state, i.e. a state of the form

$$\sum a_i |e_i\rangle_{A'} |e_i^*\rangle_{B'}. \quad (67)$$

The same should hold for the projection on $A'B'$.

To see it let us use the state–operator isomorphism (43). In our particular case it will read as follows

$$|\phi_2\rangle = (C_{AA'} \otimes I_{BB'}) |\hat{\psi}^+\rangle_{AB} \otimes |\hat{\psi}^+\rangle_{A'B'} \quad (68)$$

with $\hat{\psi}_+ = \sum_i |ii\rangle$, or simply

$$|\phi_2\rangle = \sum_{i,i',j,j'} C_{ii'\,jj'} |ii'\rangle_{AA'} |jj'\rangle_{BB'}. \quad (69)$$

We will further write $\phi_2 \propto C$. If for an example the matrix C is normal the corresponding state is of the form

$$|\phi_2\rangle = a|e\rangle_{AA'}|e^*\rangle_{BB'} + b|f\rangle_{AA'}|f^*\rangle_{BB'} \quad (70)$$

where $e \perp f$. Here a and b are eigenvalues of C, hence Hermitian C means that they are real, while positive C matrix means that a and b are nonnegative. (We have only two terms because ϕ_2 is of Schmidt rank two).

Let us now examine the projection of ϕ_2 onto $\mathcal{H}_Q$. We have

$$Q|\phi_2\rangle = |\psi^+\rangle_{AB} \otimes \left(|\tilde{\phi}^{(2)}\rangle_{A'B'} - \frac{1}{d}\mathrm{Tr}C\, |\psi^+\rangle_{A'B'} \right) + \left(|\tilde{\phi}^{(1)}\rangle_{AB} - \frac{1}{d}\mathrm{Tr}C\, |\psi^+\rangle_{AB} \right) \otimes |\psi^+\rangle_{A'B'} \quad (71)$$

where

$$|\tilde{\phi}^{(2)}\rangle_{A'B'} = {}_{AB}\langle\psi_+|\phi_2\rangle \propto \frac{1}{d} C_{A'} \quad (72)$$

$$|\tilde{\phi}^{(1)}\rangle_{AB} = {}_{A'B'}\langle\psi_+|\phi_2\rangle \propto \frac{1}{d} C_{A} \quad (73)$$

are unnormalized states that are obtained on one pair after projecting second pair onto maximally entangled state P_+; here $C_A = \mathrm{Tr}_{A'} C_{AA'}$, $C_{A'} = \mathrm{Tr}_A C_{AA'}$. Let us now relate C_A and $C_{A'}$ with the matrices A and B from (50). Thus partial traces of matrix $C_{AA'}$ correspond to unnormalized states that emerge after projecting one pair onto P_+.

The projection of ϕ_2 onto $\mathcal{H}_Q$ can be also written as follows

$$Q|\phi_2\rangle = |\psi^+\rangle_{AB} \otimes |\phi^{(2)}\rangle_{A'B'} + |\phi^{(1)}\rangle_{AB} \otimes |\psi^+\rangle_{A'B'} \quad (74)$$

where

$$|\phi^{(1)}\rangle_{AB} = (Y_A \otimes I)|\hat{\psi}^+\rangle_{AB} \quad (75)$$

$$|\phi^{(2)}\rangle_{A'B'} = (Y'_{A'} \otimes I)|\hat{\psi}^+\rangle_{A'B'} \quad (76)$$

with

$$Y = \frac{1}{d} C_A - \frac{\mathrm{Tr}C}{d^2} I_A; \quad Y' = \frac{1}{d} C_{A'} - \frac{\mathrm{Tr}C}{d^2} I_{A'}. \quad (77)$$

(Note that Y and Y' are traceless, which means that corresponding vectors are orthogonal to ψ_+). We see that—up to a factor—A is equal to Y and B is equal to Y'. Now since we assume that A and B are normal then C_A and $C_{A'}$ must also be normal. This means that e.g. C_A is of the form

$$C_A = \sum_i c_i |e_i\rangle\langle e_i| \quad (78)$$

where c_i are complex numbers and e_i form an orthonormal basis. Thus the state (73) coming from projecting subsystem $A'B'$ onto P_+ will have the desired form

$$\sum_i a_i |e_i\rangle_A |e_i^*\rangle_B, \quad (79)$$

and similarly for projecting AB part onto P_+.

V. Half-property for low Schmidt rank states

In this section we show that any state which on each pair has at least one subsystem with one-qubit support satisfies the half-property. To this end we will use the notion of so called *common degrees of freedom* introduced in the following subsection.

A. Half-property via "common degrees of freedom"

We begin with the following definition

Definition 1. *For a given state ϕ we define a set called* common degrees of freedom *of subsystems A and B as*

$$\mathrm{cdf}(\phi, A, B) = \{i \in \mathcal{I} : \langle\phi|P_i|\phi\rangle \neq 0\} \quad (80)$$

where $\mathcal{I} = \{0, \ldots, d-1\}$ and

$$P_i = |ii\rangle\langle ii|_{AB} \otimes I_{A'B'}. \quad (81)$$

We say that subsystem A has at most k common degrees of freedom with subsystem B if $|\,\mathrm{cdf}(\phi, A, B)| \leq k$.

Proposition 1. *If for a given state ϕ subsystems A with B and A' with B' have at most $\frac{d}{2}$ common degrees of freedom then ϕ satisfies the half-property.*

Proof: We will show that if for a given state ϕ subsystems A with B and A' with B' have at most $\frac{d}{2}$ common degrees of freedom then

$$\langle\phi|Q|\phi\rangle = \frac{1}{2}\langle\phi|\tilde{Q}|\phi\rangle \leq \frac{1}{2} \quad (82)$$

where $\tilde{Q}$ is some other projector.

Let us define

$$P_d = \frac{1}{d} \sum_{i,j \in \mathcal{I}} |ii\rangle\langle jj|, \quad (83)$$

$$P_{AB} = \frac{2}{d} \sum_{i,j \in \mathcal{I}_{AB}} |ii\rangle\langle jj| \quad \text{with } |\mathcal{I}_{AB}| = \frac{d}{2} \quad (84)$$

$$\text{and } \mathrm{cdf}(\phi, A, B) \subset \mathcal{I}_{AB} \subset \mathcal{I}$$

$$P_{A'B'} = \frac{2}{d} \sum_{i,j \in \mathcal{I}_{A'B'}} |ii\rangle\langle jj| \quad \text{with } |\mathcal{I}_{A'B'}| = \frac{d}{2} \quad (85)$$

$$\text{and } \mathrm{cdf}(\phi, A', B') \subset \mathcal{I}_{AB} \subset \mathcal{I}$$

where P_d is a maximally entangled state in $d \otimes d$. P_{AB} and $P_{A'B'}$ are maximally entangled states on $\frac{d}{2} \otimes \frac{d}{2}$ subspaces chosen in such a way to contain common degrees of freedom of A with B and A' with B' respectively. $\mathcal{I}_{AB}$ and $\mathcal{I}_{A'B'}$ are extensions of the sets of common degrees of freedom (with whatever elements) to get sets of exactly $\frac{d}{2}$ elements.

One can observe that in the expression

$$\langle\phi|P_d^{AB} \otimes I^{A'B'}|\phi\rangle \quad (86)$$

ϕ projects only onto those $|ii\rangle\langle jj|$ of P_d for which $i, j \in \mathrm{cdf}(\phi, A, B)$ by the very definition of common degrees of freedom, thus we can remove any of $|ii\rangle\langle jj|$ having $i \notin \mathrm{cdf}(\phi, A, B)$ or $j \notin \mathrm{cdf}(\phi, A, B)$ in particular we can remove all those for which $i \notin \mathcal{I}_{AB}$ or $j \notin \mathcal{I}_{AB}$ which gives us

$$\langle\phi|P_d^{AB} \otimes I^{A'B'}|\phi\rangle = \langle\phi|\frac{1}{2}P_{AB} \otimes I^{A'B'}|\phi\rangle \tag{87}$$

similar consideration for other elements of Q gives us

$$\langle\phi|Q|\phi\rangle = \langle\phi|I \otimes P_d^{A'B'} + P_d^{AB} \otimes I - 2P_d^{AB} \otimes P_d^{A'B'}|\phi\rangle \tag{88}$$

$$= \langle\phi|I \otimes \frac{1}{2}P_{A'B'} + \frac{1}{2}P_{AB} \otimes I - 2\frac{1}{2}P_{AB} \otimes \frac{1}{2}P_{A'B'}|\phi\rangle \tag{89}$$

$$= \frac{1}{2}\langle\phi|I \otimes P_{A'B'} + P_{AB} \otimes I - P_{AB} \otimes P_{A'B'}|\phi\rangle \tag{90}$$

$$= \frac{1}{2}\langle\phi|\tilde{Q}|\phi\rangle \leq \frac{1}{2} \tag{91}$$

where $\tilde{Q}$ is also a projector thus the inequality holds. ∎

B. Example: states with positive matrix C

We begin by rephrasing number of cdfs in terms of the matrix C of a state (see sec. IV-C) written in block form:

$$C_{AA'} = \sum_{ij} |i\rangle_A\langle j| \otimes C_{A'}^{ij}. \tag{92}$$

The number of cdfs is the number of blocks $C^{(ii)}$, i.e. *diagonal* blocks which do not vanish (i.e. which have at least one nonzero element). The proposition 1 says that for any given state (not necessarily of Schmidt rank two) the number of cdfs is less than or equal to 2, then the state has the half-property.

Now suppose that C is positive. Then the diagonal blocks are positive matrices, and they do not vanish iff their trace is nonzero. Thus the full information about the number of cdfs is contained in the partial trace of the matrix C:

$$C_A = \mathrm{Tr}_{A'} C_{AA'} = \sum_{ij} \mathrm{Tr}(C_{A'}^{ij})|i\rangle_A\langle j| \tag{93}$$

Thus number of cdfs is equal to the number of nonzero elements on the diagonal of C_A.

Now, since Q is invariant over pairwise $U \otimes U^*$ transformations, we can rotate a state to diminish the number of cdfs as much as possible. If we can get 2 or less, then we obtain the half-property. Consider e.g. such transformation for the pair AB. The matrix C_A then transforms as $UC_AU^\dagger$. We are interested in the minimal number of nonzero diagonal elements under such transformations, which equals to the rank of the matrix C_A. We have then obtained, that any state with positive matrix C such that its partial trace has rank ≤ 2, has the half-property.

Let us note however that our result of section IV-C implies that all Schmidt rank two states with positive matrix C satisfy the half-property.

C. Application of cdf to low Schmidt rank

Here by use of proposition 1 we show that any state which on each pair has at least one subsystem with one-qubit support satisfies the half-property.

Theorem 3. *Any state ϕ that satisfies*

$$\left(\mathrm{Sch}(A : A'BB') \leq \frac{d}{2} \vee \mathrm{Sch}(B : AA'B') \leq \frac{d}{2}\right) \wedge \left(\mathrm{Sch}(A' : ABB') \leq \frac{d}{2} \vee \mathrm{Sch}(B' : AA'B) \leq \frac{d}{2}\right) \tag{94}$$

also satisfies the half-property. Here $\mathrm{Sch}(X : Y)$ denotes the Schmidt rank of the state ϕ in the X versus Y cut.

Observation 1. *The operator Q is $U_A \otimes V_{A'} \otimes U_B^* \otimes V_{B'}^*$ invariant. (Where U and V are unitaries).*

Proof of theorem 3: The hypothesis may be expanded into a four-term alternative. We prove the conclusion for one of the terms (for the others the proof is analogous). Now suppose

$$\mathrm{Sch}(A : A'BB') \leq \frac{d}{2} \wedge \mathrm{Sch}(A' : ABB') \leq \frac{d}{2} \tag{95}$$

which means that there are Schmidt decompositions of ϕ of the form

$$|\phi\rangle = \sum_{i=0}^{d/2-1} a_i|\psi_i^A\rangle|\psi_i^{A'BB'}\rangle = \sum_{i=0}^{d/2-1} a_i'|\psi_i^{A'}\rangle|\psi_i^{ABB'}\rangle \tag{96}$$

We can choose such U and V which transform ϕ to

$$|\phi'\rangle = U_A \otimes V_{A'} \otimes U_B^* \otimes V_{B'}^*|\phi\rangle \tag{97}$$

$$= \sum_{i=0}^{d/2-1} a_i|i^A\rangle|\tilde{\psi}_i^{A'BB'}\rangle = \sum_{i=0}^{d/2-1} a_i'|i^{A'}\rangle|\tilde{\psi}_i^{ABB'}\rangle \tag{98}$$

Now we can observe that A with B and A' with B' have at most $\frac{d}{2}$ degrees of freedom in common in ϕ' (as there are clearly at most $\frac{d}{2}$ degrees of freedom on A and A' subsystems) thus by applying proposition 1 we have

$$\langle\phi'|Q|\phi'\rangle \leq \frac{1}{2} \tag{99}$$

and by applying observation 1 we finally get

$$\langle\phi|Q|\phi\rangle = \langle\phi'|Q|\phi'\rangle \leq \frac{1}{2}. \tag{100}$$

∎

VI. Optimizing over product states and implications

In this section we will first consider a simpler question from the original one. Namely we will optimize the overlap of Q with product states rather than with Schmidt rank two ones. This is equivalent to optimization of the overlap of Q^Γ with product states, where Q^Γ is the partial transpose of Q. We find the maximal overlap with product states for the general case of n copies i.e. we will work with Q_n given by (13). Knowing the maximum over product states, we can bound the maximum over Schmidt rank two states. For $n = 2$ we will obtain in this way

$$\langle\phi_2|Q|\phi_2\rangle \leq \frac{3}{4}. \tag{101}$$

However the analysis of n copy case shows that in the limit of $n \to \infty$ one obtains a trivial result that the overlap does not exceed one. Nevertheless this approach will be used in subsequent section to go beyond $\frac{3}{4}$. Analysis of Q^Γ also allows for direct proof of the half-property for states with positive matrix C.

A. Maximum overlap of product states with Q_n

To find the maximum overlap of product states with Q_n given by (13) we will first analyze spectral decomposition of Q_n^Γ. We have

$$Q_n^\Gamma = \frac{1}{2}\left(I^{\otimes n} - \left(I - \tfrac{1}{2}V\right)^{\otimes n}\right) \tag{102}$$
$$= \frac{1}{2}\left(I^{\otimes n} - \left(\tfrac{1}{2}P_s + \tfrac{3}{2}P_a\right)^{\otimes n}\right) \tag{103}$$
$$= \sum_{i=0}^{n} \lambda_i A_i \tag{104}$$

where P_s and P_a are the projectors onto the symmetric and the antisymmetric subspaces and

$$\lambda_i = \frac{1}{2}\left(1 - \frac{3^i}{2^n}\right) \tag{105}$$
$$A_i = \sum_{l_j\in\{0,1\},\, \sum l_j = i} a_{l_1} \otimes \cdots \otimes a_{l_n} \tag{106}$$

with $a_0 = P_s$ and $a_1 = P_a$. (Note that $\sum_{i=0}^n A_i = I^{\otimes n}$). Thus eigenvalues of Q_n^Γ are in decreasing order and the largest eigenvalue λ_0 is associated with the eigenspace $A_0 = P_s^{\otimes n}$. In particular for $n = 2$ we have

$$\lambda_0 = \frac{3}{8},\, \lambda_1 = \frac{1}{8},\, \lambda_2 = -\frac{5}{8}, \tag{107}$$

so that

$$Q_2^\Gamma = \frac{3}{8}P_s \otimes P_s - \frac{5}{8}P_a \otimes P_a + \frac{1}{8}(P_a \otimes P_s + P_s \otimes P_a). \tag{108}$$

Let us now compute the maximum overlap of product states with Q_n. Since $(\mathrm{Tr}Q_n|\phi_1\rangle\langle\phi_1|)^\Gamma = \mathrm{Tr}Q_n^\Gamma|\tilde{\phi}_1\rangle\langle\tilde{\phi}_1|$, where $\tilde{\phi}_1$ is also a product state (with a one-to-one correspondence between ϕ_1 and $\tilde{\phi}_1$), we can replace the optimization on Q_n with an optimization on Q_n^Γ. The overlap of product states with Q_n^Γ is bounded by its largest eigenvalue λ_0 and this bound is attainable as in the eigenspace $P_s^{\otimes n}$ corresponding to λ_0 there are product states. We thus have

$$\sup_{\phi_1}\langle\phi_1|Q_n|\phi_1\rangle = \sup_{\phi_1}\langle\phi_1|Q_n^\Gamma|\phi_1\rangle = \lambda_0 = \frac{1}{2}\left(1 - \frac{1}{2^n}\right). \tag{109}$$

In particular for two copies this gives $\frac{3}{8}$.

B. Bound for $\langle\phi_2|Q|\phi_2\rangle$ in terms of $\langle\phi_1|Q|\phi_1\rangle$

As Schmidt rank two state may be decomposed to

$$|\phi_2\rangle = \sqrt{p}|\phi_1\rangle + \sqrt{1-p}|\phi_1^\perp\rangle, \tag{110}$$

we observe that

$$\sup_{\phi_2}\langle\phi_2|Q|\phi_2\rangle$$
$$= \sup_{\phi_1,\phi_1^\perp,p} (\sqrt{p}\langle\phi_1| + \sqrt{1-p}\langle\phi_1^\perp|)Q(\sqrt{p}|\phi_1\rangle + \sqrt{1-p}|\phi_1^\perp\rangle) \tag{111}$$
$$= \sup_{\phi_1,\phi_1^\perp,p} p\langle\phi_1|Q|\phi_1\rangle + (1-p)\langle\phi_1^\perp|Q|\phi_1^\perp\rangle$$
$$+ 2\sqrt{p(1-p)}\,\mathrm{Re}\langle\phi_1|Q|\phi_1^\perp\rangle \tag{112}$$
$$\leq \sup_{\phi_1,\phi_1^\perp} (\langle\phi_1|Q|\phi_1\rangle + |\langle\phi_1|Q|\phi_1^\perp\rangle|) \tag{113}$$

and thus from Schwarz inequality

$$\sup_{\phi_2}\langle\phi_2|Q|\phi_2\rangle \leq 2\sup_{\phi_1}\langle\phi_1|Q|\phi_1\rangle. \tag{114}$$

In this way we have obtained the bound for the overlap of the Schmidt rank two states with Q in terms of optimal overlap with product states. This is also true for any other projector, in particular, for Q_n.

Thus for two copies we obtain the following bound

$$\sup_{\phi_2}\langle\phi_2|Q|\phi_2\rangle \leq \frac{3}{4}. \tag{115}$$

Unfortunately this method does not lead to any bound that would hold for all n apart from the trivial bound $\langle\phi_2|Q_n|\phi_2\rangle \leq 1$.

C. The form of the rank-one states attaining maximum on Q_n

It is interesting that the product states attaining the maximum on Q_n must be of a very specific form. For $n = 2$ the partial transpose of such state (which is again a legitimate state) must belong to a subspace $P_s^{AB} \otimes P_s^{A'B'}$. One can then find that the states that are product with respect to $AA' : BB'$ cut and the same time belong to the above subspace must be of the form

$$|xx\rangle_{AB} \otimes |yy\rangle_{A'B'}. \tag{116}$$

It then follows that a product state maximizing overlap with Q_n must be of the form

$$|xx^*\rangle_{AB} \otimes |yy^*\rangle_{A'B'}. \tag{117}$$

This observation in general case of n copies is contained in the following.

Proposition 2. *For any n all rank-one states ϕ_1 reaching maximum on Q_n has the form*

$$|\phi_1\rangle = \bigotimes_{i=1}^{n} |\psi_i\rangle_{A_i}|\psi_i^*\rangle_{B_i}. \tag{118}$$

Proof: The thesis of the proposition is equivalent to the following statement: for any n all rank-one states ϕ_1 reaching maximum on Q_n^Γ have the form

$$|\phi_1\rangle = \bigotimes_{i=1}^{n} |\psi_i\rangle_{A_i} |\psi_i\rangle_{B_i}. \quad (119)$$

We prove it by induction.

1) For $n = 1$ only rank one states of the form $|\psi\psi\rangle$ reach maximum on $Q_1^\Gamma = \frac{1}{4}V$.
2) Suppose for some n maximal projection of rank one state on Q_n^Γ requires the from (119). From previous section a rank one state ϕ_1 defined on $n+1$ pairs to attain maximum on Q_{n+1}^Γ must be an eigenstate of $P_s^{\otimes n+1}$ which is a subspace of the symmetric space on $n+1$ pairs. Thus the Schmidt decomposition of ϕ_1 in n pairs versus single pair cut ($AB : ab$) has the form

$$|\phi_1\rangle = |\psi\rangle_{Aa}|\psi\rangle_{Bb} = \sum a_i a_j |\psi_i\psi_j\rangle_{AB}|\phi_i\phi_j\rangle_{ab} \quad (120)$$

and we have

$$\langle\phi_1|P_s^{\otimes n+1}|\phi_1\rangle = \sum a_i a_j a_k a_l \langle\psi_i\psi_j|P_s^{\otimes n}|\psi_k\psi_l\rangle\langle\phi_i\phi_j|P_s|\phi_k\phi_l\rangle \quad (121)$$

$$= \sum a_i a_j a_k a_l \langle\psi_i\psi_j|P_s^{\otimes n}|\psi_k\psi_l\rangle \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (122)$$

to obtain one above all the projections must be equal to 1. For projection on P_s given in delta-form requires $i = j = k = l$ and it is always one only if ϕ_1 is product in $AB : ab$ cut. To obtain one on $P_s^{\otimes n}$ the $\psi_i \otimes \psi_i$ state must be of the form (119) and thus ϕ_1 is of the form (119). ■

D. Superpositions of rank-one states with maximum on Q_n

One could expect that superpositions of rank-one states with maximum on Q_n has the the half-property as such rank-one states are product between the copies. Indeed this is the case, their overlap with Q_n is analyzed in the following

Proposition 3. *Let $d = 4$ and ϕ_1, $\phi_1^\perp$ be n-copy orthogonal product states with maximum overlap with Q_n, i.e. of the form*

$$|\phi_1\rangle = \bigotimes_{i=1}^{n} |\psi_i\rangle_{A_i}|\psi_i^*\rangle_{B_i}, \quad |\phi_1^\perp\rangle = \bigotimes_{i=1}^{n} |\tilde{\psi}_i\rangle_{A_i}|\tilde{\psi}_i^*\rangle_{B_i} \quad (123)$$

then their superposition

$$|\phi_2\rangle = \sqrt{p}|\phi_1\rangle + \sqrt{1-p}|\phi_1^\perp\rangle \quad (124)$$

has the following overlap with Q_n

$$\langle\phi_2|Q_n|\phi_2\rangle = \frac{1}{2}\left(1 - \frac{1}{2^n}\right) - \sqrt{p(1-p)}\prod_{i=1}^{n}\left(|\langle\psi_i|\tilde{\psi}_i\rangle|^2 - \frac{1}{2}\right). \quad (125)$$

In particular it is equal to $\frac{1}{2}$ only if $p = \frac{1}{2}$ and ϕ_1, $\phi_1^\perp$ are orthogonal on an odd number of copies and equal on the rest. Otherwise it is less than $\frac{1}{2}$.

Proof: The form of ϕ_1 and $\phi_1^\perp$ comes from proposition 2 and their overlap with Q_n from (109) thus we have

$$\langle\phi_2|Q_n|\phi_2\rangle = \frac{1}{2}\left(1 - \frac{1}{2^n}\right) + 2\sqrt{p(1-p)}\,\mathrm{Re}\langle\phi_1|Q_n|\phi_1^\perp\rangle \quad (126)$$

Thus to finish the proof we will show by induction that

$$\langle\phi_1|Q_n|\phi_1^\perp\rangle = -\frac{1}{2}\prod_{i=1}^{n}\left(|\langle\psi_i|\tilde{\psi}_i\rangle|^2 - \frac{1}{2}\right) \quad (127)$$

It is true for $n = 1$

$$\langle\phi_1|Q_1|\phi_1^\perp\rangle = \frac{1}{d}\langle\psi_1\psi_1^\perp|V|\psi_1^\perp\psi_1\rangle = \frac{1}{d} = -\frac{1}{2}(0 - \frac{1}{2}). \quad (128)$$

Suppose it is true for some n, let us show it also holds for $n+1$. Without loss of generality we can assume ϕ_1 and $\phi_1^\perp$ are orthogonal on one of the first n copies thus we can write

$$|\phi_1\rangle = |\phi\rangle|\psi\psi^*\rangle, \quad |\phi_1^\perp\rangle = |\phi^\perp\rangle|\tilde{\psi}\tilde{\psi}^*\rangle. \quad (129)$$

Then by using recursive formula (15) we have

$$\langle\phi_1|Q_{n+1}|\phi_1^\perp\rangle = \langle\phi|Q_n|\phi^\perp\rangle\left(\langle\psi\psi^*|\tilde{\psi}\tilde{\psi}^*\rangle - 2\langle\psi\psi^*|Q_1|\tilde{\psi}\tilde{\psi}^*\rangle\right) \quad (130)$$

$$= -\frac{1}{2}\prod_{i=1}^{n}\left(|\langle\psi_i|\tilde{\psi}_i\rangle|^2 - \frac{1}{2}\right)\left(|\langle\psi|\tilde{\psi}\rangle|^2 - \frac{2}{d}\langle\psi\tilde{\psi}|V|\tilde{\psi}\psi\rangle\right) \quad (131)$$

$$= -\frac{1}{2}\prod_{i=1}^{n+1}\left(|\langle\psi_i|\tilde{\psi}_i\rangle|^2 - \frac{1}{2}\right). \quad (132)$$

It is evident that to maximize (125), i.e. obtain $\frac{1}{2}$, one needs $p = \frac{1}{2}$ and (127) equal to $2^{-(n+1)}$. This requires $\left||\langle\psi_i|\tilde{\psi}_i\rangle|^2 - \frac{1}{2}\right| = \frac{1}{2}$ for all i, that is ψ_i and $\tilde{\psi}_i$ must be equal or orthogonal and further for (127) to be positive they must be orthogonal on odd number of copies and equal on the rest. ■

E. Digression: half-property for a class of states ϕ_2 via Q^Γ

We consider the following class of states

$$|\phi_2\rangle = a|e_1\rangle|e_1^*\rangle + b|e_2\rangle|e_2^*\rangle \quad (133)$$

with $a, b \geq 0$, $|e_1\rangle \perp |e_2\rangle$. In the state–operator isomorphism they correspond to positive matrices $C_{AA'}$ (see sect IV). Then C_A and $C_{A'}$ are also positive, hence normal, so that it is a subclass of states for which we have proved the half-property in section IV. Here we present another proof for this class of states (133). (In section VIII we present a third proof, which uses principle of noincreasing entanglement by LOCC).

. We can write

$$\langle\phi_2|Q|\phi_2\rangle = \mathrm{Tr}(Q^\Gamma P_{\phi_2}^\Gamma) \quad (134)$$

with $P_{\phi_2} = |\phi_2\rangle\langle\phi_2|$. We have

$$P_{\phi_2}^\Gamma = a^2 P_{|e_1\rangle|e_1\rangle} + b^2 P_{|e_2\rangle|e_2\rangle} + ab(P_{\psi_+} - P_{\psi_-}) \quad (135)$$

with

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle|e_2\rangle \pm |e_2\rangle|e_1\rangle). \tag{136}$$

Now recall that

$$Q^\Gamma = \frac{3}{8}P_s\otimes P_s - \frac{5}{8}P_a\otimes P_a + \frac{1}{8}(P_a\otimes P_s + P_s\otimes P_a). \tag{137}$$

Note that vectors $|e_1\rangle|e_1\rangle$, $|e_2\rangle|e_2\rangle$ as well as ψ_+ lie in the symmetric subspace i.e. $P_s\otimes P_s + P_a\otimes P_a$, while ψ_- lies in the antisymmetric subspace $P_s\otimes P_a + P_a\otimes P_s$. Therefore, one can estimate the expression (134) from above, by assuming, that triplet states lie solely within $P_s\otimes P_s$, obtaining

$$\langle\phi_2|Q|\phi_2\rangle = \mathrm{Tr}(Q^\Gamma P^\Gamma_{\phi_2}) \le \frac{3}{8}(a^2+b^2+ab) - \frac{1}{8}ab \le \frac{1}{2}. \tag{138}$$

VII. Bounds for maximal overlap with Q for all states ϕ_2.

In this section we show that we can improve the bound obtained by means of product states in the previous section.

A. Strictly less than 3/4

In the previous section we have provided the following bound

$$\sup_{\phi_2}\langle\phi_2|Q|\phi_2\rangle \le \frac{3}{4}. \tag{139}$$

Let us now show that the bound cannot be tight. To this end assume that we have equality. Let us recall the bound of (113) on the overlap of rank two states with Q

$$\sup_{\phi_2}\langle\phi_2|Q|\phi_2\rangle \le \sup_{\phi_1,\phi_1^\perp}(\langle\phi_1|Q|\phi_1\rangle + |\langle\phi_1|Q|\phi_1^\perp\rangle|). \tag{140}$$

Our assumption thus implies that RHS $\ge \frac{3}{4}$. As $\langle\phi_1|Q|\phi_1\rangle \le \frac{3}{8}$ this requires

$$|\,\mathrm{Re}\langle\phi_1|Q|\phi_1^\perp\rangle| \ge \frac{3}{8} \tag{141}$$

and by Schwarz inequality both ϕ_1 and $\phi_1^\perp$ must have maximal projection on Q which through proposition 2 implies they must be of the form $|xx^*\rangle_{AB}|yy^*\rangle_{A'B'}$. However for two such orthogonal states by direct calculations we obtain

$$|\,\mathrm{Re}\langle\phi_1|Q|\phi_1^\perp\rangle| \le \frac{1}{8} \tag{142}$$

which is in contradiction with (141) and hence with our assumption of equality in (139). Thus we obtain

$$\sup_{\phi_2}\langle\phi_2|Q|\phi_2\rangle < \frac{3}{4}. \tag{143}$$

Numerical optimization suggests the bound (140) is actually equal to $\frac{17}{32}$. If we want to optimize independently both terms of the bound (140) we get

$$\sup_{\phi_2}\langle\phi_2|Q|\phi_2\rangle \le \frac{3}{8} + \sup_{\phi_1,\phi_1^\perp}|\langle\phi_1|Q|\phi_1^\perp\rangle| \tag{144}$$

which numerically gives $\frac{5}{8}$. At the moment we do not have analytical proofs of these estimates.

B. Beyond 3/4

We have seen that product states attaining maximum overlap with Q have to be of the form $|\phi\rangle = |x\rangle_A|x^*\rangle_B|y\rangle_{A'}|y^*\rangle_{B'}$, i.e. the partial transpose of ϕ belongs to the product of symmetric subspaces. From continuity, if the overlap of ϕ with Q is close to maximal, the state ϕ should have big overlap with states of the above form. Here we provide quantitative estimate. First we will show that in such case ϕ has big overlap with $P_s\otimes P_s$:

Lemma 3. *For states ϕ product with respect to $AA' : BB'$ cut we have*

$$\langle\phi|P_s^{AB}\otimes P_s^{A'B'}|\phi\rangle \ge 4\langle\phi^\Gamma|Q|\phi^\Gamma\rangle - \frac{1}{2}, \tag{145}$$

where action Γ is well defined because ϕ is product.

Proof: It follows from the formula (137) and a bit of algebra.

We then have that large overlap of a product state ϕ with $P_s\otimes P_s$ implies large overlap with vectors of the form $|xxyy\rangle$. ■

Lemma 4. *For all states ϕ product with respect to $AA' : BB'$ cut we have*

$$\sup_{x,y}|\langle\phi|xx\rangle_{AB}|yy\rangle_{A'B'}|^2 \ge 4\langle\phi|P^s_{AB}\otimes P^s_{A'B'}|\phi\rangle - 3. \tag{146}$$

Proof: Write $|\phi\rangle = |e\rangle_{AA'}|f\rangle_{BB'}$. We then find

$$\langle\phi|P_s^{AB}\otimes P_s^{A'B'}|\phi\rangle = \frac{1}{4}(1+\mathrm{Tr}\varrho^e_A\varrho^f_B + \mathrm{Tr}\varrho^e_{A'}\varrho^f_{B'} + |\langle e|f\rangle|^2) \tag{147}$$

where ϱ^e_A is reduced density matrix of $|e\rangle$ etc. Schwarz inequality then implies

$$\langle\phi|P_s^{AB}\otimes P_s^{A'B'}|\phi\rangle \le \frac{1}{4}(1+2\max(\mathrm{Tr}\varrho_e^2,\mathrm{Tr}\varrho_f^2) + |\langle e|f\rangle|^2) \tag{148}$$

where ϱ_e is either of reduced density matrices of $|e\rangle$, similarly for ϱ_f.

On the other hand one finds

$$|\langle\phi|xxyy\rangle| = |\langle e|xy\rangle\langle f|xy\rangle| \tag{149}$$

$$\ge |\langle e|xy\rangle\langle f|e\rangle\langle e|xy\rangle| = |\langle e|xy\rangle|^2|\langle e|f\rangle| \tag{150}$$

which implies

$$\sup_{x,y}|\langle\phi|xxyy\rangle|^2 \ge \max(p_e,p_f)|\langle e|f\rangle| \tag{151}$$

where p_e, p_f are the largest eigenvalues of ϱ_e, ϱ_f respectively. Combining the two equations, and noticing that without loss of generality one can assume that $\mathrm{Tr}\varrho_e^2 = p_e^2 + (1-p_e)^2$ and the same for $\mathrm{Tr}\varrho_f^2$, one obtains

$$\sup_{x,y}|\langle\phi|xxyy\rangle|^2 \ge \frac{1}{4}(1+\alpha^2)\beta \tag{152}$$

and

$$\langle\phi|P_s\otimes P_s|\phi\rangle \le \frac{1}{4}(2+\alpha^2+\beta) \tag{153}$$

where

$$\alpha = \sqrt{2\max(\mathrm{Tr}\varrho_e^2, \mathrm{Tr}\varrho_f^2 - 1)}; \quad \beta = |\langle e|f\rangle|^2; \quad 0 \le \alpha, \beta \le 1. \tag{154}$$

Treating α and β as independent variables, after some elementary, but lengthy algebra, one gets the desired result. ■

The above lemmas lead to the following

Proposition 4. *For any product state ϕ we have*

$$\sup_\chi |\langle\phi|\chi\rangle|^2 \ge 16\langle\phi|Q|\phi\rangle - 5 \tag{155}$$

where supremum is taken over vectors $\chi = |x\rangle_A|x^*\rangle_B|y\rangle_{A'}|y^*\rangle_{B'}$.

Subsequently, writing

$$\phi = a\chi + b\psi; \quad \phi^\perp = \tilde{a}\tilde{\chi} + \tilde{b}\tilde{\psi} \tag{156}$$

where $\phi^\perp$ is a product state orthogonal to ϕ, and $\chi \perp \psi$, $\tilde{\chi} \perp \tilde{\psi}$, with $\chi, \tilde{\chi}$ being of the form $|xx^*yy^*\rangle$ and ψ, $\tilde{\psi}$ normalized, we obtain

$$|\langle\phi|Q|\phi^\perp\rangle| \le |a\tilde{a}|\,|\langle\chi|Q|\tilde{\chi}\rangle| + \sqrt{\frac{3}{8}}(|a\tilde{b}| + |b\tilde{a}|) + |b\tilde{b}| \tag{157}$$

where we have used the fact that maximal overlap of Q with a product state does not exceed $3/8$. By direct computation we also obtain

$$\langle\chi|Q|\tilde{\chi}\rangle = -\frac{1}{8} + \frac{1}{4}(\langle\chi_1|\tilde{\chi}_1\rangle + \langle\chi_2|\tilde{\chi}_2\rangle) \tag{158}$$

where $|\chi_1\rangle = |xx^*\rangle_{AB}, |\chi_2\rangle = |yy^*\rangle_{A'B'}$ and $|\tilde{\chi}_1\rangle = |\tilde{x}\tilde{x}^*\rangle_{AA'}, |\tilde{\chi}_2\rangle = |\tilde{y}\tilde{y}^*\rangle_{BB'}$. Using the fact that $\langle\phi|\phi^\perp\rangle = 0$ we get

$$|\langle\chi_1|\tilde{\chi}_1\rangle|\,|\langle\chi_2|\tilde{\chi}_2\rangle| \le |b\tilde{a}| + |a\tilde{b}|. \tag{159}$$

Since for any numbers a, b satisfying $0 \le a, b \le 1$ we have $a + b \le ab + 1$ and combining (157), (158) and (159) we get

Proposition 5. *For any product orthogonal states ϕ and $\phi^\perp$ we have*

$$|\langle\phi|Q|\phi^\perp\rangle| \le a_1a_2(-\frac{1}{8} + \frac{1}{4}(1 + a_1b_2 + a_2b_1)) + \sqrt{\frac{3}{8}}(a_1b_2 + a_2b_1) + b_1b_2 \equiv g(a_1, a_2) \tag{160}$$

where $a_1 = |a| = |\langle\phi|\chi\rangle|$, $a_2 = |\tilde{a}| = |\langle\phi|\chi\rangle|$, $b_1 = \sqrt{1 - a_1^2}$, $b_2 = \sqrt{1 - a_2^2}$, and $\chi, \tilde{\chi}$ are of the form $|xx^*yy^*\rangle$.

Let us observe that

$$\sup_{\phi_2}\langle\phi_2|Q|\phi_2\rangle \tag{161}$$

$$= \sup_{\phi_1,\phi_1^\perp,p} (\sqrt{p}\langle\phi_1| + \sqrt{1-p}\langle\phi_1^\perp|)Q(\sqrt{p}|\phi_1\rangle + \sqrt{1-p}|\phi_1^\perp\rangle) \tag{162}$$

$$= \sup_{\phi_1,\phi_1^\perp}\sup_p \begin{bmatrix}\sqrt{p}\\ \sqrt{1-p}\end{bmatrix}^T \begin{bmatrix}\langle\phi_1|Q|\phi_1\rangle & \mathrm{Re}\langle\phi_1|Q|\phi_1^\perp\rangle\\ \mathrm{Re}\langle\phi_1^\perp|Q|\phi_1\rangle & \langle\phi_1^\perp|Q|\phi_1^\perp\rangle\end{bmatrix} \begin{bmatrix}\sqrt{p}\\ \sqrt{1-p}\end{bmatrix} \tag{163}$$

$$= \sup_{\phi_1,\phi_1^\perp} \frac{1}{2}\Big(\langle\phi_1|Q|\phi_1\rangle + \langle\phi_1^\perp|Q|\phi_1^\perp\rangle \tag{164}$$

$$+ \sqrt{(\langle\phi_1|Q|\phi_1\rangle - \langle\phi_1^\perp|Q|\phi_1^\perp\rangle)^2 + 4(\mathrm{Re}\langle\phi_1|Q|\phi_1^\perp\rangle)^2}\Big) \tag{165}$$

the last expression is simply larger eigenvalue of the matrix in (163).

Now denoting $\gamma_1 = \langle\phi|Q|\phi\rangle$, $\gamma_2 = \langle\phi^\perp|Q|\phi^\perp\rangle$, we get

$$\langle\phi_2|Q|\phi_2\rangle \le \gamma_1 + \gamma_2 \tag{166}$$

from Schwarz inequality. On the other hand using (144) and proposition 4 we get

$$\langle\phi_2|Q|\phi_2\rangle \le \frac{3}{8} + \sup_{a_1,a_2} g(a_1, a_2) \tag{167}$$

where supremum is taken over a_1, a_2 satisfying

$$16\gamma_i - 5 \le a_i^2 \le 1, \quad i = 1, 2. \tag{168}$$

Finally we obtain the following estimate

$$\langle\phi_2|Q|\phi_2\rangle \le \frac{3}{8} + \min(\gamma, f(\gamma)) \tag{169}$$

where $\gamma = \min(\gamma_1, \gamma_2)$ and

$$f(\gamma) = \sup_{a_1,a_2} g(a_1, a_2) \tag{170}$$

where supremum is taken over $16\gamma - 5 \le a_i^2 \le 1$. Looking on the plot of $g(a_1, a_2)$ one can find that the maximum is obtained for $a_1 = a_2$. This leads to the bound

$$\langle\phi_2|Q|\phi_2\rangle \le 0.74971 < 3/4. \tag{171}$$

VIII. Application of Entanglement measures

Then we will show how entanglement measures can be applied to the problem of the half-property.

The formula $\langle\phi_2|Q|\phi_2\rangle$ can be written as follows:

$$\langle\phi_2|Q|\phi_2\rangle = \mathrm{Tr}(\mathcal{T}(|\phi_2\rangle\langle\phi_2|)Q) \tag{172}$$

where $\mathcal{T}$ is pairwise UU^* twirling, followed by random permutation of pairs. Since $\mathcal{T}$ is LOCC operation, the state $\sigma = \mathcal{T}(|\phi_2\rangle\langle\phi_2|)$ cannot have greater entanglement than the state ϕ_2. Then, one can hope, that if entanglement of σ is not too large, then also $\mathrm{Tr}\sigma Q$ will be bounded. Write

$$\sigma = \frac{p}{2}(\tilde{P}_+^\perp \otimes P_+ + P_+ \otimes \tilde{P}_+^\perp) + sP_+ \otimes P_+ + (1 - p - s)\tilde{P}_+^\perp \otimes \tilde{P}_+^\perp \tag{173}$$

with $\tilde{P}_+^\perp = (I - P_+)/(d^2-1)$ and probabilities p, s satisfying $p + s \leq 1$. Then we have

$$\mathrm{Tr}\sigma Q = p. \tag{174}$$

A. Negativity

We will use the negativity [45], or more precisely a closely related quantity $\|\varrho^\Gamma\|$, which is monotonous under LOCC [46]. In our case, one finds that

$$\|\sigma^\Gamma\| = \frac{1}{4}(2|1-16s| + |1+8s-4p| + 1 + 24s + 4p). \tag{175}$$

Now monotonicity requires that

$$\|\sigma^\Gamma\| \leq \|{\phi_2}^\Gamma\| = |a+b|^2 \tag{176}$$

where a, b are Schmidt coefficients of ϕ_2. This inequality together with (175) implies in particular that

$$p \leq \frac{1}{4} - 6\langle\phi_2|P_+ \otimes P_+|\phi_2\rangle + 2|a+b|^2. \tag{177}$$

Note that for fixed Schmidt coefficients a, b maximal overlap with $P_+ \otimes P_+$ cannot exceed $|a+b|^2/16$. We then obtain, that for those states which achieve this maximal overlap there holds the half-property. However such states are simply states of the form

$$\phi_2 = a|e_1\rangle_{AA'}|e_1^*\rangle_{BB'} + b|e_2\rangle_{AA'}|e_2^*\rangle_{BB'} \tag{178}$$

with $a, b, \geq 0$. Since such states have positive matrix C we end up with yet another proof of the half-property for this class of states.

For states that are orthogonal to $P_+ \otimes P_+$ negativity gives bound $3/4$. We have also tried the relative entropy of entanglement and the realignment but worse results have been obtained.

B. Half-property and Schmidt rank of some symmetric states

The possibility of application of entanglement measures to the problem of the half-property can be also seen from the following different perspective. Namely, one can classify states with respect to Schmidt rank. We say that a mixed state has Schmidt rank k, if it can be written as a mixture of pure states of Schmidt rank k, but cannot be written as a mixture of pure states of Schmidt rank $k-1$ (cf [47]). We then have the following

Fact 1. *The projector Q has the half-property if and only if for all states σ of the form (173) which have Schmidt rank ≤ 2 we have $p \leq 1/2$.*

One direction is trivial, the other follows from twirling. Thus if we are able to prove that all states σ of the form (173) with $p > 1/2$ have Schmidt rank > 2, we would solve the problem of the half-property. To this end we should find a map Λ such that $I \otimes \Lambda$ is nonnegative on Schmidt rank two pure states (such maps are called two positive), and at the same time negative on all states σ with $p \geq 1/2$. Indeed, this would mean that all states σ with $p \geq 1/2$ have Schmidt rank > 2.

Using this approach one can also get bounds for our quantity $\langle\phi_2|Q|\phi_2\rangle$. For example we have checked that the following two-positive map $\Lambda(A) = I\,\mathrm{Tr}A - 1/2A$ is negative for $p > 3/4$ which reproduces the bound obtained by means of product states.

In this context we see why entanglement measures can be applied to our problem. Namely, if an entanglement measure of a given state is greater than maximum of this measure over Schmidt rank two pure states, then the state must have Schmidt rank two greater than 2.

C. Continuity of entanglement and bound entanglement

One could ask the question whether there exist a continuous entanglement measure which would detect between three kinds of states: 1) separable, 2) bound entangled, and 3) distillable ones. There are measures such as the entanglement of formation which distinguish between 1 (for which it is zero) versus 2 and 3 (for which it is nonzero), and there is a measures, the distillable entanglement, which distinguishes between 1 and 2 (for which it is zero) versus 3 (for which it is non zero). But any measure that would distinguish between the three classes of states by its value in a way that entanglement of all bound entangled states is non zero but smaller than entanglement of any distillable state must be non continuous. Indeed for such a measure there must be a range of values reserved for bound entangled states, creating a gap between separable states and distillable ones. On the other hand we can take a sequence of distillable states with a limit being a separable state (and so with zero value of entanglement), but the limit of the entanglement for this sequence must be at most supremum of its value on bound entangled states. Note that provided that NPT bound entangled states exist such a measure would also increase under tensoring because then there would exist bound entangled states whose tensor product is distillable [20], as a matter of fact the same would then hold for the distillable entanglement.

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APPENDIX

Lemma 5. *The minimum value of $\sum_{i=1}^d |\tilde{a}_i|^2$ subject to $\sum_{i=1}^d \tilde{a}_i = z$ where $\tilde{a}_i, z \in \mathbb{C}$ is obtained by settings $\tilde{a}_i = \frac{z}{d}$.*

Proof: From the parallelogram identity we have

$$\frac{1}{2}|\tilde{a}_i + \tilde{a}_j|^2 = |\tilde{a}_i|^2 + |\tilde{a}_j|^2 - \frac{1}{2}|\tilde{a}_i - \tilde{a}_j| \leq |\tilde{a}_i|^2 + |\tilde{a}_j|^2 \tag{179}$$

with equality iff $\tilde{a}_i = \tilde{a}_j$. Thus whenever for some $\tilde{a}_i$, $\tilde{a}_j$ we have $\tilde{a}_i \neq \tilde{a}_j$ we can replace them with two instances of $\frac{\tilde{a}_i + \tilde{a}_j}{2}$ decreasing the value of $\sum_{i=1}^d |\tilde{a}_i|^2$ and leaving the constrain satisfied. This implies that the optimal solution is to take all $\tilde{a}_i$ equal, i.e. $\tilde{a}_i = \frac{z}{d}$. ■

Proposition 6. *For all $d \geq 3$ dimensional vectors $\vec{a}$ and $\vec{b}$ with complex elements $\tilde{a}_i$ and $\tilde{b}_i$ and satisfying the constraints*

$$\sum_{i=1}^d \tilde{a}_i = \sum_{i=1}^d \tilde{b}_i = 0, \qquad \sum_{i=1}^d |\tilde{a}_i|^2 + \sum_{i=1}^d |\tilde{b}_i|^2 = \frac{1}{d} \tag{180}$$

the following equality holds

$$\max_{\vec{a},\vec{b}}\left(|\tilde{a}_1+\tilde{b}_1|^2+|\tilde{a}_1+\tilde{b}_2|^2\right)=\frac{3d-4}{d^2}. \tag{181}$$

Corollary 1. *For $d=4$ under this constraints we have*

$$\max_{\vec{a},\vec{b}}\left(|\tilde{a}_1+\tilde{b}_1|^2+|\tilde{a}_1+\tilde{b}_2|^2\right)=\frac{1}{2}. \tag{182}$$

Proof of proposition 6: We denote function (181) as f, the vector of all $\tilde{a}_i$ as $\vec{a}$, the vector of all $\tilde{b}_i$ as $\vec{b}$, and we use their polar decompositions

$$\tilde{a}_i=a_ie^{i\alpha_i},\quad \tilde{b}_i=b_ie^{i\beta_i},\quad a_i,b_i\in\mathbb{R}. \tag{183}$$

In optimizing function f under the constraints (180) we shrink the set of possible $\vec{a}$ and $\vec{b}$ in such a way to simplify the form of f and the constraints but keeping at least one of the global maxima within the shrinking set.

1) Without loss of generality we can take $\tilde{a}_1=a_1\geq 0$. Thus we optimize

$$f(\vec{a},\vec{b})=|a_1+\tilde{b}_1|^2+|a_1+\tilde{b}_2|^2 \tag{184}$$

$$=2a_1^2+b_1^2+b_2^2+2a_1(b_1\cos\beta_1+b_2\cos\beta_2). \tag{185}$$

2) We can consider only $\vec{b}$ for which

$$b_1\cos\beta_1+b_2\cos\beta_2\geq 0. \tag{186}$$

(If it is negative we can change its sign by multiplying $\vec{b}$ by $e^{i\pi}$ and thus increase f).

3) In maximizing f under the constraints it is always best to set

$$\tilde{a}_i=-\frac{a_1}{d-1}\qquad (i>1) \tag{187}$$

$$\tilde{b}_i=-\frac{1}{d-2}(\tilde{b}_1+\tilde{b}_2)\qquad (i>2) \tag{188}$$

Indeed whenever this setting is not used we can by lemma 5 obtain some freedom in the second constraint which we can use to increase a_1 and one of b_1 or b_2 without decreasing f. Thus it is enough to consider $\vec{a}$ and $\vec{b}$ satisfying this setting, i.e. we optimize function $f(a_1,\tilde{b}_1,\tilde{b}_2)$ subject to the following constraints

$$\frac{d}{d-1}a_1^2+b_1^2+b_2^2+\frac{1}{d-2}\left|\tilde{b}_1+\tilde{b}_2\right|^2=\frac{1}{d},$$
$$a_1\geq 0,\quad b_1\cos\beta_1+b_2\cos\beta_2\geq 0. \tag{189}$$

4) Further we show that it is enough to consider $\tilde{b}_1,\tilde{b}_2\in\mathbb{R}$ as replacing $\tilde{b}_1$ with $\tilde{b}_1'=b_1\cos\beta_1$ and $\tilde{b}_2$ with $\tilde{b}_2'=b_2\cos\beta_2$ and changing a_1 to a_1' to fit the constraint does not decrease f, i.e. $f(a_1',\tilde{b}_1',\tilde{b}_2')\geq f(a_1,\tilde{b}_1,\tilde{b}_2)$. Namely we have

$$f(a_1',\tilde{b}_1',\tilde{b}_2')=2a_1'^2+b_1^2\cos^2\beta_1+b_2^2\cos^2\beta_2 + 2a_1'(b_1\cos\beta_1+b_2\cos\beta_2) \tag{190}$$

and the main constraint is

$$\frac{d}{d-1}a_1'^2+b_1^2\cos^2\beta_1+b_2^2\cos^2\beta_2 + \frac{1}{d-2}\left|b_1\cos\beta_1+b_2\cos\beta_2\right|^2=\frac{1}{d}. \tag{191}$$

First we show that $a_1'\geq a_1$ which is evident from the difference of main constraints

$$\frac{d}{d-1}(a_1'^2-a_1^2) = b_1^2\sin^2\beta_1+b_2^2\sin^2\beta_2 + \frac{1}{d-2}\left(\left|b_1e^{i\beta_1}+b_2e^{i\beta_2}\right|^2-|b_1\cos\beta_1+b_2\cos\beta_2|^2\right) \geq 0. \tag{192}$$

Next we use this difference to show that f does not decrease after the replacement

$$\begin{aligned}f(a_1',\tilde{b}_1',\tilde{b}_2')-f(a_1,\tilde{b}_1,\tilde{b}_2) &=2(a_1'^2-a_1^2)-b_1^2\sin^2\beta_1-b_2^2\sin^2\beta_2\\ &\quad+2(a_1'-a_1)(b_1\cos\beta_1+b_2\cos\beta_2)\\ &\geq\frac{d-2}{d}(b_1^2\sin^2\beta_1+b_2^2\sin^2\beta_2)\geq 0.\end{aligned} \tag{193}$$

So we can focus on a problem with $\tilde{b}_1,\tilde{b}_2\in\mathbb{R}$

$$f(a_1,b_1,b_2)=2a_1^2+b_1^2+b_2^2+2a_1(b_1+b_2) \tag{194}$$

$$\frac{d}{d-1}a_1^2+b_1^2+b_2^2+\frac{1}{d-2}(b_1+b_2)^2=\frac{1}{d},$$
$$a_1\geq 0,\quad b_1+b_2\geq 0. \tag{195}$$

5) In analogous way we show that it is enough to consider $b_1=b_2\geq 0$ as taking $b_1'=b_2'=\frac{|b_1+b_2|}{2}$ and changing a_1 to a_1' to fit the constraint does not decrease f. Then the optimization simplifies to

$$f(a_1,b_1)=2(a_1+b_1)^2 \tag{196}$$

$$\frac{d}{d-1}a_1^2+\frac{2d}{d-2}b_1^2=\frac{1}{d},\quad a_1,b_1\geq 0. \tag{197}$$

6) We compute b_1 from the constraint and substitute to f which gives

$$f(a_1)=2\left(a_1+\sqrt{x-ya_1^2}\right)^2 \tag{198}$$

$$a_1\in\left[0,\sqrt{x/y}\right] \tag{199}$$

where

$$x=\frac{d-2}{2d^2},\qquad y=\frac{d-2}{2(d-1)}. \tag{200}$$

Function f has its maximum when the expression in the parenthesis has the maximum (as it is nonnegative). We consider its derivative

$$\frac{\partial}{\partial a_1}\left(a_1+\sqrt{x-ya_1^2}\right)=1-\frac{ya_1}{\sqrt{x-ya_1^2}} \tag{201}$$

which is zero for

$$a_1^\star=\sqrt{\frac{x}{y^2+y}} \tag{202}$$

and the second derivative is negative in $a_1^\star$ so the maximum is equal to

$$f(a_1^\star)=2\left(\sqrt{\frac{x}{y^2+y}}+\sqrt{\frac{xy}{y+1}}\right)^2 \tag{203}$$

$$=2x(y^{-1}+1)=\frac{3d-4}{d^2} \tag{204}$$

The global maximum could also be on one of the boundaries but for $d \geq 3$ $f(a_1^\star)$ is always greater than the values on the boundaries.

■

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