Dynamic Collective Choice: Social Optima

Rabih Salhab, Jerome Le Ny and Roland P. Malhamé

Abstract

We consider a dynamic collective choice problem where a large number of players are cooperatively choosing between multiple destinations while being influenced by the behavior of the group. For example, in a robotic swarm exploring a new environment, a robot might have to choose between multiple sites to visit, but at the same time it should remain close to the group to achieve some coordinated tasks. We show that to find a social optimum for our problem, one needs to solve a set of Linear Quadratic Regulator problems, whose number increases exponentially with the size of the population. Alternatively, we develop via the Mean Field Games methodology a set of decentralized strategies that are independent of the size of the population. When the number of agents is sufficiently large, these strategies qualify as approximately socially optimal. To compute the approximate social optimum, each player needs to know its own state and the statistical distributions of the players' initial states and problem parameters. Finally, we give a numerical example where the cooperative and non-cooperative cases have opposite behaviors. Whereas in the former the size of the majority increases with the social effect, in the latter, the existence of a majority is disadvantaged.

Index Terms

Mean Field Games, Collective Choice, Multi-Agent Systems, Social Optimum.

I. INTRODUCTION

Discrete choice models were developed in economics to understand human choice behavior. A concern of these models is predicting the decision of an individual in face of a set of alternatives, for example, anticipating a traveler's choice between different modes of transportation [1]. These choices depend on some personal characteristics, such as the traveler's financial situation, on

some attributes of the alternatives, such as their prices, and on some unobservable attributes, e.g., the traveler's taste. The first static discrete choice model was proposed by McFadden in [2].

In some situations, the individuals' choices are socially influenced, that is, an individual's choice is affected by the others' choices, for example entry or withdrawal from the labor market in cooperative families [3]. The main goal of this paper is to model within the framework of dynamic *cooperative* game theory situations where a large number of players/agents are making socially influenced choices among a finite set of alternatives. The players involved in this game are weakly coupled, that is, the individual choices are considerably influenced by a functional of the others' choice distribution (in this paper the mean), but for a sufficiently large population, an isolated individual's choice has a negligible influence on the others' choices. Moreover, the players' states contributing to the social effect are assumed indistinguishable. In navigation applications for example, a planner might want to deploy a swarm of robots cooperating to explore an unknown terrain. A robot faces a situation where it should choose between multiple sites to visit. At the same time, it should remain closed to the group to achieve some coordinated tasks [4]–[6].

In non-cooperative games, perfectly rational players act selfishly by minimizing their individual costs irrespective of making the other players better off or worse off. This "utilitarianist" aspect of non-cooperative games neglects the social context where the social norms, social values, the presence of a social planner or the social structures impose a kind of cooperation between the players. An example of the influence of the social context on the behavior of players was given in [7], where the author shows how at the Chicago Options Exchange the relations among the traders, supposed to be noncooperative, affect their trades. In the robotic swarm example, the cooperative behavior of the robots results from the intention of the planner to optimize a total cost. Whereas in the non-cooperative case the agents search for a Nash equilibrium, the players seek in the cooperative case a totally different type of solution, namely a social optimum.

The Mean Field Games (MFG) methodology, which we follow in this paper, is concerned with dynamic games involving a large number of weakly coupled agents. It was originally developed in a series of papers to study dynamic non-cooperative games [8]–[14]. The cooperative Linear Quadratic Gaussian (LQG) MFG formulation was developed later in [15], where the authors investigate the structure of the LQG costs to develop for a continuum of agents a set of

decentralized person-by-person optimal strategies (a weaker solution concept than the social optimum that coincides under some conditions with the social optimum [16], [17]). Moreover, they show that these strategies, when applied by a finite population, converge to an exact social optimum as the number of players increases to infinity.

The main contribution of this paper is as follows. We consider a cooperative collective choice model where the number of candidate optimal control laws increases exponentially with the size of the population. Then, we develop a set of decentralized strategies of dimensions independent of the size of the population and that converge to the social optimum as the size of the population increases to infinity. Although the methodology used to solve the game follows [15], the non-smoothness and non-convexity of our final costs, which involve a minimum function, require different proofs for the convergence of the mean field based decentralized strategies to the social optimum, see Lemmas 5, 8, Theorem 6 and Remark 3. In particular, our problem formulation results in decentralized strategies that are *discontinuous* with respect to the agents' initial conditions, capturing the issue of choosing between a finite set of alternatives, which cannot be modeled using the standard LQG MFG setup considered in [15].

In [18], we studied the non-cooperative version of our model and developed via the MFG methodology approximate Nash strategy profiles that converge to exact Nash equilibria as the number of players increases to infinity. Since the person-by-person solutions are Nash-like solutions, we rely in this note on some results established in [18] to establish the existence of the person-by-person solutions and compute them. A static discrete choice model with social interactions was also studied by Brock and Durlauf in [19], where the authors develop a non-cooperative and a cooperative game involving a large number of players. Each player makes a choice between two alternatives while being affected by the average of its peers' decisions. Inspired by the statistical mechanics approach, Brock and Durlauf propose a methodology to solve the game that is similar to the MFG approach.

The cooperative dynamic discrete choice model is formulated in Section II. We show in Section III that to find an exact social optimum, one can naively solve l^N Linear Quadratic Regulator (LQR) problems, each of dimensions Nn, where l is the number of choices, N the number of players, and n the dimension of the individual state spaces. Alternatively, we develop in Section IV via the MFG approach and within the so-called person to person optimization setting a set of decentralized strategies that are asymptotically socially optimal. The dimensions

of the decentralized strategies are independent of the size of the population. In Section V we give some simulation results, while Section VI presents our conclusions.

II. MATHEMATICAL MODEL

We consider a cooperative game model involving N players with linear dynamics

$$\dot{x}_i = A_i x_i + B_i u_i \qquad\qquad i = 1, \ldots, N, \tag{1}$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $x_i \in \mathbb{R}^n$ is the state of agent i, $u_i \in U = L_2([0, T], \mathbb{R}^m)$ its control input and x_i^0 its initial state. The players cooperate to minimize a common social cost

$$J_{soc}\left(u_1, \ldots, u_N, x^{(N)}\right) = \sum_{i=1}^{N} J_i\left(u_i, x^{(N)}\right), \tag{2}$$

where

$$J_i\left(u_i, x^{(N)}\right) = \int_0^T \left\{ \frac{q}{2} \left\| x_i - Z x^{(N)} \right\|^2 + \frac{r_i}{2} \|u_i\|^2 \right\} \mathrm{d}t + \min_{j=1,\ldots,l} \left\{ \frac{M_{ij}}{2} \|x_i(T) - p_j\|^2 \right\} \tag{3}$$

are the individual costs, $q, r_i, M_{ij} > 0$, $Z \in \mathbb{R}^{n \times n}$, and $p_j \in \mathbb{R}^n$, $j = 1, \ldots, l$, are the destination points. The individual cost functions penalize along the path the effort and the deviation from the mean. Moreover, each agent must be close at time T to one of the destination points. Otherwise, it is strongly penalized by the final cost. The agents are cost coupled via the average $x^{(N)} = \frac{1}{N} \sum_{i=1}^{N} x_i$. The coefficient r_i depends on the agent i. In the robotic swarm example, this reflects, for instance, the intention of the social planner to limit the mobility of some robots. We assume that the coefficient M_{ij} depends on the agent i and the destination point p_j to impose initial preferences towards the alternatives, as discussed later in Remark 1. When considering the limiting population ($N \to \infty$), it is convenient to represent the limiting sequences of $(\theta_i)_{i=1,\ldots,N} := \{(A_i, B_i, r_i, M_{i1}, \ldots, M_{il})\}_{i=1,\ldots,N}$ and $\{x_i^0\}_{i=1,\ldots,N}$ by two independent random variables θ and x^0 on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that θ is in a compact set Θ. Let us denote the empirical measures of the sequences θ_i and x_i^0, $\mathbb{P}_\theta^N(\mathcal{A}) = \frac{1}{N} \sum_{i=1}^{N} 1_{\{\theta_i \in \mathcal{A}\}}$ and $\mathbb{P}_0^N(\mathcal{A}) = \frac{1}{N} \sum_{i=1}^{N} 1_{\{x_i^0 \in \mathcal{A}\}}$ for all (Borel) measurable sets $\mathcal{A}$. We assume that $\mathbb{P}_\theta^N$ and $\mathbb{P}_0^N$ have weak limits $\mathbb{P}_\theta$ and $\mathbb{P}_0$. For further discussions about this assumption, one can refer to [15].

A social optimum is defined as the optimal control law $(u_1^*, \ldots, u_N^*)$ of (2). We start in the following section by solving for such a social optimum in a centralized manner.

III. Centralized Social Optimum

In this section, we assume that each player can observe the states and the parameters of the other players. We define $x = (x_1, \ldots, x_N)^T$ the state of the population and $u = (u_1, \ldots, u_N)^T$ its strategy profile. The population's dynamics is then

$$\dot{x} = \tilde{A}x + \tilde{B}u, \tag{4}$$

where $\tilde{A} = \text{diag}(A_1, \ldots, A_N)$ and $\tilde{B} = \text{diag}(B_1, \ldots, B_N)$. The individual costs can be written

$$J_i\left(u_i, x^{(N)}\right) = \min_{p_j \in \Delta} J_i^{p_j}\left(u_i, x^{(N)}\right), \tag{5}$$

where $\Delta = \{p_1, \ldots, p_l\}$ and

$$J_i^{p_j}\left(u_i, x^{(N)}\right) = \int_0^T \left\{ \frac{q}{2} \left\| x_i - Z x^{(N)} \right\|^2 + \frac{r_i}{2} \|u_i\|^2 \right\} \mathrm{dt} + \frac{M_{ij}}{2} \|x_i(T) - p_j\|^2.$$

Using the equality $a + \min(b, c) = \min(a+b, a+c)$, one can prove by induction that the social cost (2) can be written

$$J_{soc}\left(u, x^{(N)}\right) = \min_{d=(d_1,\ldots,d_N)\in\Delta^N} \sum_{i=1}^N J_i^{d_i}\left(u_i, x^{(N)}\right).$$

Noting that

$$\inf_{u\in U^N} J_{soc}\left(u, x^{(N)}\right) = \min_{d\in\Delta^N} \inf_{u\in U^N} \sum_{i=1}^N J_i^{d_i}\left(u_i, x^{(N)}\right),$$

one can optimize the l^N costs $J^d(u) = \sum_{i=1}^N J_i^{d_i}(u_i, x^{(N)})$ and choose the less costly combination of destination points $d^* \in \Delta^N$ which corresponds to the minimum of the optima of J^d. The costs J^d, for $d \in \Delta^N$, can be written

$$J^d(u) = \int_0^T \left\{ \frac{1}{2} x^T \tilde{Q} x + \frac{1}{2} u^T \tilde{R} u \right\} \mathrm{dt} + \frac{1}{2}(x(T) - d)^T \tilde{M}^d(x(T) - d), \tag{6}$$

where $\tilde{Q} = I_n \otimes I_N + \frac{1}{N}(11^T) \otimes L$, $\tilde{R} = \text{diag}(r_1 I_m, \ldots, r_N I_m)$, $\tilde{M}^d = \text{diag}(M_{1d_1} I_n, \ldots, M_{Nd_N} I_n)$, and

$$L = Z^T Z - Z - Z^T, \tag{7}$$

with $\otimes$ denoting the Kronecker product, $1 = [1, \ldots, 1]^T$, diag(.) denoting a block diagonal matrix.

The LQR problem defined by (6) and (4) has a unique optimal control law [20]

$$u_*^d(t) = -\tilde{R}^{-1}\tilde{B}^T\left(\tilde{\Gamma}^d(t)x + \tilde{\beta}^d(t)\right) \tag{8}$$

with the corresponding optimal cost

$$J_*^d(x(0)) = \frac{1}{2}x(0)^T\tilde{\Gamma}^d(0)x(0) + \tilde{\beta}^d(0)^Tx(0) + \tilde{\delta}^d(0), \tag{9}$$

where $\tilde{\Gamma}^d$, $\tilde{\beta}^d$ and $\tilde{\delta}^d$ are respectively matrix-, vector-, and real-valued functions satisfying the following backward propagating differential equations:

$$\dot{\tilde{\Gamma}}^d - \tilde{\Gamma}^d\tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{\Gamma}^d + \tilde{\Gamma}^d\tilde{A} + \tilde{A}^T\tilde{\Gamma}^d + \tilde{Q} = 0 \tag{10a}$$

$$\dot{\tilde{\beta}}^d = \left(\tilde{\Gamma}^d\tilde{B}\tilde{R}^{-1}\tilde{B}^T - \tilde{A}^T\right)\tilde{\beta}^d \tag{10b}$$

$$\dot{\tilde{\delta}}^d = \frac{1}{2}(\tilde{\beta}^d)^T\tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{\beta}^d \tag{10c}$$

with the final conditions $\tilde{\Gamma}^d(T) = \tilde{M}^d$, $\tilde{\beta}^d(T) = -\tilde{M}^d d$ and $\tilde{\delta}^d(T) = \frac{1}{2}d^T\tilde{M}^d d$.

We summarize the above analysis in the following theorem.

Theorem 1: The social planner problem (2) has an optimal control law u_*^v defined in (8) and corresponding to

$$J_*^v = \min_{d \in \Delta^N} J_*^d.$$

As discussed in Section II, to capture the discrete choice phenomenon, the final cost forces the agents to be at time T in the vicinity of one of the destination points. Indeed, the following theorem establishes that for sufficiently large M_{ij}, each player reaches an arbitrarily small neighborhood of a destination point. Moreover, it asserts that there is only one set of destination points $p^* \in \mathbb{R}^{Nn}$ that the agents can reach exactly under an optimal control law, namely, the final state $x_0(T)$ under the control law u_0 optimizing

$$J_0(u) = \int_0^T \left\{\frac{1}{2}x^T\tilde{Q}x + \frac{1}{2}u^T\tilde{R}u\right\} \mathrm{dt}, \tag{11}$$

i.e., (6) without the final cost.

Theorem 2: Suppose that (A_i, B_i), $i = 1, \ldots, N$, are controllable and the agents are minimizing (2). Then,

   i. for any $\epsilon > 0$, there exists $M_0 > 0$ such that for all $M_{ij} > M_0$, each agent is at time T inside a ball of radius ϵ and centered at one of the destination points.

  ii. the agents $1, \ldots, N$ reach at time T the destination points $d = (d_1, \ldots, d_N) \in \Delta^N$ if and only if $d = p^*$.

Proof: Let $\epsilon > 0$ and $d \in \Delta^N$. The pairs (A_i, B_i), for $i = 1, \ldots, N$, are controllable. Therefore, there exist N continuous control laws $\tilde{u}_i$, $i = 1, \ldots, N$, such that the corresponding final states satisfy $\tilde{x}_i(T) = d_i$, $i = 1, \ldots, N$. Let $\tilde{u} = (\tilde{u}_i, \tilde{u}_{-i})$. We have

$$J^d(\tilde{u}) = \int_0^T \left\{ \frac{1}{2} \tilde{x}^T \tilde{Q} \tilde{x} + \frac{1}{2} \tilde{u}^T \tilde{R} \tilde{u} \right\} \mathrm{dt}.$$

By optimality, we have

$$\sum_{i=1}^N \frac{M_{id_i}}{2} \left\| x_i(u_*^d)(T) - d_i \right\|^2 \leq J_*^d \leq J^d(\tilde{u}).$$

The cost $J^d(\tilde{u})$ is independent of M_{ij}. Therefore, there exists $M_d^0 > 0$ such that for all $M_{id_i} > M_d^0$, $\|x_i(u_*^d)(T) - d_i\|^2 < \epsilon$, for $i = 1, \ldots, N$. By choosing $M_0 = \max\limits_{d \in \Delta^N} M_d^0$, we get i).

Next, suppose that $d \neq p^*$ for all $d \in \Delta^N$. The optimal social cost is J_*^d, for some d and some M_{id_i}, $i = 1, \ldots, N$. We suppose that the players reach under their optimal strategies the destination points $d_1, \ldots, d_N$. Let $M'_{id_i} > M_{id_i}$ for $i = 1, \ldots, N$. We have, for all $u \in U^N$, $J'(u) \geq J^d(u)$ where

$$J'(u) = \int_0^T \left\{ \frac{1}{2} x^T \tilde{Q} x + \frac{1}{2} u^T \tilde{R} u \right\} \mathrm{dt} + \sum_{i=1}^N \frac{M'_{id_i}}{2} \|x_i(T) - d_i\|^2.$$

Under u_*^d, the players $1, \ldots, N$ reach $d_1, \ldots, d_N$. Therefore,

$$J'\left(u_*^d\right) = J^d\left(u_*^d\right) = \min_u J^d(u) = J_*^d.$$

Therefore, $\min\limits_u J'(u) = \min\limits_u J^d(u)$. This equality holds for all $M'_{id_i} > M_{id_i}$, $i = 1, \ldots, N$. The solutions of (10a)-(10c) are analytic functions of $\tilde{M}^d$ (for a proof of the analyticity one can refer to [21]). Therefore, the optimal cost $\min_u J'(u)$ defined in (9) is an analytic function of M'_{id_i}. But $\min_u J'(u)$ is constant for all $M'_{id_i} > M_{id_i}$. Therefore, by analyticity, it is constant for all $M'_{id_i} \geq 0$, and more precisely for $M'_{id_i} = 0$. This implies that u_*^d is the optimal control law of $J_0(u)$ defined in (11). The definition of p^* implies that $x\left(u_*^d\right)(T) = p^* \neq d$. This is a contradiction, so in fact some of the agents cannot reach their destination point.

Now suppose that there exists $v \in \Delta^N$ such that $v = p^*$. We have $J^v(u) \geq J_0(u)$ for all u. Following the definition of p^*, we have

$$\min_u J_0(u) = J_0(u_0) = J^v(u_0).$$

Therefore, the optimal control of J^v is $u_*^v = u_0$. Hence, the agents reach $p^* = v$. ∎

Remark 1: We show in this remark that in the absence of a social effect ($q = 0$), the number of agents that go towards a destination point p_j decreases as M_{ij} increases. To simplify things, we consider the binary choice case $l = 2$. In the absence of a social effect, each agent i minimizes its individual cost (5). In the following, we write $J_i^{p_j}\left(u_i, x^{(N)}\right)$ as $J_i^{p_j}(u_i, M)$ to emphasize that the coefficient M_{ij} in $J_i^{p_j}\left(u_i, x^{(N)}\right)$ is equal to M, and that the cost does not depend on $x^{(N)}$ ($q = 0$). Following Theorem 1 and the absence of a social effect, for $M_{i1} = M_1 > 0$ and $M_{i2} = M_2 > 0$, an agent i goes towards p_1 if and only if $\min J_i^{p_1}(u_i, M_1) < \min J_i^{p_2}(u_i, M_2)$. For an $M_2' > M_2$, $\min J_i^{p_1}(u_i, M_1) < \min J_i^{p_2}(u_i, M_2) \leq \min J_i^{p_2}(u_i, M_2')$. Therefore, by increasing M_2, the number of agents that go towards p_2 decreases.

A naive approach to find an exact social optimum would be to solve the l^N LQR problems corresponding to the different combinations of destinations. This is obviously computationally expensive, and moreover, with this approach each player needs to observe the states and parameters of all the other players. Instead, we develop in the following sections a set of decentralized strategies that are asymptotically optimal. These strategies are decentralized in the sense that an agent i's strategy depends only on its state x_i and on the distributions $\mathbb{P}_0$ and $\mathbb{P}_\theta$ of the initial conditions and parameters respectively.

IV. DECENTRALIZED SOCIAL OPTIMUM

A weaker solution concept than the social optimum is the person-by-person optimal solution [16], [17].

Definition 1: A strategy profile (u_i^*, u_{-i}^*) is said to be person-by-person optimal with respect to the social cost $J_{soc}(u_i, u_{-i})$ if for all $i \in \{1, \ldots, N\}$, for all $u_i \in U$, $J_{soc}(u_i, u_{-i}^*) \geq J_{soc}(u_i^*, u_{-i}^*)$. A social optimum is necessarily a person-by-person optimal solution. Following the methodology proposed in [15], we compute in the following section a set of decentralized approximate person-by-person solutions. Moreover, we show under some technical assumptions that these solutions become socially optimal as $N \to \infty$.

A. Person-by-Person Optimality

Assuming that the other players fixed their person-by-person optimal strategies u_{-i}^*, an agent i computes its person-by-person optimal strategy u_i^* by minimizing the cost $J_{soc}(u_i, u_{-i}^*)$ over

the strategies $u_i \in U$. Similarly to [15], one can show that the social cost can be written

$$J_{soc}(u_i, u^*_{-i}) = J_{1,i}\left(u_i, x^{*(N)}_{-i}\right) + J_{2,i}(u^*_{-i}),$$

where $x^{*(N)}_{-i} = 1/N \sum^N_{j=1, j\neq i} x^*_j$,

$$J_{1,i}\left(u_i, x^{*(N)}_{-i}\right) = \int_0^T \left\{ x_i^T \hat{Q} x_i + \left(x^{*(N)}_{-i}\right)^T \hat{L} x_i + \frac{r_i}{2}\|u_i\|^2 \right\} dt + \min_{j=1,\ldots,l} \frac{M_{ij}}{2}\|x_i(T) - p_j\|^2$$

$$\hat{Q} = \frac{q}{2}\left(I_n - \frac{Z}{N}\right)^T \left(I_n - \frac{Z}{N}\right) + \frac{q(N-1)}{2N^2} Z^T Z$$

$$\hat{L} = -qZ^T\left(I_n - \frac{Z}{N}\right) - qZ + \frac{q(N-1)}{N} Z^T Z.$$

The term $J_{2,i}(u^*_{-i})$ does not depend on the strategy u_i of player i. Therefore, minimizing $J_{soc}(u_i, u^*_{-i})$ reduces to minimizing $J_{1,i}\left(u_i, x^{*(N)}_{-i}\right)$.

The person-by-person optimal solutions (u_i^*, u^*_{-i}) are fixed points of the following system of equations:

$$u_i^* = \operatorname*{argmin}_{u_i \in U} J_{1,i}\left(u_i, x^*_{-i}{}^{(N)}\right) \qquad\qquad i = 1, \ldots, N.$$

Equivalently, these solutions are the Nash equilibria of a noncooperative game involving the N players defined in (1) but associated with the individual costs

$$J_{1,i}\left(u_i, x_{-i}{}^{(N)}\right) \qquad\qquad i = 1, \ldots, N. \qquad (12)$$

The players are cost coupled through the average of the population. In the following we develop via the MFG approach a decentralized approximate Nash strategy profile with respect to (12), or equivalently a set of decentralized approximately person-by-person optimal strategies with respect to (2).

B. Mean Field Equation System

According to the MFG approach, each agent assumes a continuum of agents and computes its best response to an assumed given continuous path $\bar{x}$. This path represents the mean path of the infinite size population under a Nash strategy profile. Since the players must collectively reproduce this assumed mean path when applying their best responses to it, this path can be computed by a fixed point argument. Under the infinite size population assumption, the costs (12) reduce to the cost of a generic agent with state x, control input u and parameters θ:

$$J(u, \bar{x}, x^0, \theta) = \int_0^T \left\{ \frac{q}{2}\|x\|^2 + q\bar{x}^T Lx + \frac{r_\theta}{2}\|u\|^2 \right\} dt + \min_{j=1,\ldots,l} \left\{ \frac{M_{\theta j}}{2}\|x(T) - p_j\|^2 \right\}, \quad (13)$$

where $\bar{x} = \mathbb{E}x$ is the mean trajectory of the infinite size population. The generic agent's state x satisfies (1) where (A_i, B_i, u_i) is replaced by (A_θ, B_θ, u), with an initial state $x^0(\omega)$ drawn from $\mathbb{P}_0$ and parameters $\theta(\omega) = (A_\theta, B_\theta, r_\theta, M_{\theta 1}, \ldots, M_{\theta l})(\omega)$ drawn from $\mathbb{P}_\theta$. In the following, we omit ω from the notation.

1) The Generic Agent's Best Response to $\bar{x}$: We define $\Gamma_k^\theta \in C([0, T], \mathbb{R}^{n \times n})$, $\beta_k^\theta \in C([0, T], \mathbb{R}^n)$ and $\delta_k^\theta \in C([0, T], \mathbb{R})$ to be the unique solutions of the following backward propagating differential equations:

$$\dot{\Gamma}_k^\theta - \frac{1}{r_\theta} \Gamma_k^\theta B_\theta B_\theta^T \Gamma_k^\theta + \Gamma_k^\theta A_\theta + A_\theta^T \Gamma_k^\theta + qI_n = 0 \tag{14a}$$

$$\dot{\beta}_k^\theta = \left(\frac{1}{r_\theta} \Gamma_k^\theta B_\theta B_\theta^T - A_\theta^T \right) \beta_k^\theta - qL\bar{x} \tag{14b}$$

$$\dot{\delta}_k^\theta = \frac{1}{2r_\theta} (\beta_k^\theta)^T B_\theta B_\theta^T \beta_k^\theta, \tag{14c}$$

with the final conditions

$$\Gamma_k^\theta(T) = M_{\theta k} I_n, \quad \beta_k^\theta(T) = -M_{\theta k} p_k, \quad \delta_k^\theta(T) = \frac{1}{2} M_{\theta k} p_k^T p_k.$$

Lemma 3: Given the initial condition and the parameters, an agent's best response to $\bar{x}$ and the corresponding optimal cost are

$$\hat{u}(t, x^0, \theta) = \sum_{j=1}^l -\frac{1}{r_\theta} B_\theta^T \left(\Gamma_j^\theta(t) \hat{x}(t, x^0, \theta) + \beta_j^\theta(t) \right) \mathbb{1}_{D_j^\theta(\bar{x})}(x^0) \tag{15}$$

$$J^*(\bar{x}, x^0, \theta) = \sum_{j=1}^l \left(\frac{1}{2} (x^0)^T \Gamma_j^\theta(0) x^0 + (\beta_j^\theta)(0)^T x^0 + \delta_j^\theta(0) \right) \mathbb{1}_{D_j^\theta(\bar{x})}(x^0), \tag{16}$$

where $\hat{x}(t, x^0, \theta)$ is the generic agent's state under the feedback law (15), Γ_k^θ, β_k^θ, δ_k^θ are the unique solutions of (14a)-(14c), and

$$D_j^\theta(\bar{x}) = \left\{ x \in \mathbb{R}^n \middle| \forall k = 1, \ldots, l, \; \frac{1}{2} x^T \left(\Gamma_j^\theta(0) - \Gamma_k^\theta(0) \right) x + \right.$$
$$\left. \left(\beta_j^\theta(0) - \beta_k^\theta(0) \right)^T x + \delta_j^\theta(0) - \delta_k^\theta(0) \le 0 \right\}. \tag{17}$$

Proof: See [18, Lemma 1]. ∎

The cost function (13) can be written as the minimum of l LQR cost functions each corresponding to a distinct possible destination point. When minimizing one of these LQR costs, an agent

goes towards the corresponding destination point. The region $D_j^\theta(\bar{x})$ defined in (17) includes the initial conditions for which the LQR problem corresponding to p_j is the less costly LQR problem. Therefore, there exist l basins of attraction $D_j^\theta(\bar{x})$, $j = 1, \ldots, l$, where the players initially present in $D_j^\theta(\bar{x})$ go towards p_j.

We define $\Psi_j^\theta(\eta_1, \eta_2, \eta_3, \eta_4) = \Phi_j^\theta(\eta_1, \eta_2)^T B_\theta B_\theta^T \Phi_j^\theta(\eta_3, \eta_4)$, where $\Pi_j^\theta(t) = \frac{1}{r_\theta} \Gamma_j^\theta(t) B_\theta B_\theta^T - A_\theta^T$ and Φ_j^θ is the unique solution of

$$\frac{d\Phi_j^\theta(t, \eta)}{dt} = \Pi_j^\theta(t)\Phi_j^\theta(t, \eta), \quad \Phi_j^\theta(\eta, \eta) = I_n. \tag{18}$$

The state trajectory of the generic agent is then [18]

$$\hat{x}\left(t, x^0, \theta\right) = \sum_{j=1}^{l} \mathbb{1}_{D_j^\theta(\bar{x})}(x^0)\Big\{ \Phi_j^\theta(0, t)^T x^0 + \frac{M_{\theta j}}{r_\theta} \int_0^t \Psi_j^\theta(\sigma, t, \sigma, T)p_j \, \mathrm{d}\sigma$$

$$+ \frac{q}{r_\theta} \int_0^t \int_T^\sigma \Psi_j^\theta(\sigma, t, \sigma, \tau)L\bar{x}(\tau) \, \mathrm{d}\tau\mathrm{d}\sigma \Big\}. \tag{19}$$

2) Existence of a Solution for the Mean Field Fixed Point Equation System: The mean field equation system is determined by (14a)-(14c) plus the infinite size population mean equation

$$\bar{x}(t) = \int \hat{x}\left(t, x^0, \theta\right) \mathrm{d}\mathbb{P}_0 \times \mathbb{P}_\theta. \tag{20}$$

This equation system defines an operator $G(.)$ from the Banach space $(C([0, T], \mathbb{R}^n), \|\|_\infty)$ into itself. In fact, given a continuous path $\bar{x}$, one can solve (14a)-(14c) and compute by (20) the mean trajectory $G(\bar{x})$ of the generic agent when it optimally tracks $\bar{x}$. We define

$$k_1 = \mathbb{E}\|x^0\| \times \left(\sum_{j=1}^{l} \max_{(\theta, t) \in \Theta \times [0, T]} \|\Phi_j^\theta(0, t)\| \right)$$

$$k_2 = \sum_{j=1}^{l} \max_{(\theta, t) \in \Theta \times [0, T]} \left\| \frac{M_{\theta j}}{r_\theta} \int_0^t \Psi_j^\theta(\sigma, t, \sigma, T)p_j \, \mathrm{d}\sigma \right\| \tag{21}$$

$$k_3 = \sum_{j=1}^{l} \max_{(\theta, t, \sigma, \tau) \in \Theta \times [0, T]^3} \frac{q}{r_\theta} \|\Psi_j^\theta(\sigma, t, \sigma, \tau)L\|.$$

Since Θ and $[0, T]$ are compact and Φ_j^θ is continuous with respect to time and parameter θ, then k_1, k_2 and k_3 are well defined.

Assumption 1: We assume that $\sqrt{\max(k_1 + k_2, k_3)}T < \pi/2$.

Noting that the left hand side of the inequality tends to zero as T goes to zero, Assumption 1 can be satisfied for short time horizon T for example.

Assumption 2: We assume that $L \succeq 0$, where L is defined in (7).

Assumption 2 is satisfied, for example, when $Z = -\alpha I_n$, $\alpha > 0$. In this case, the social effect pushes the agents away from the mean of the population.

Assumption 3: We assume that $\mathbb{P}_0$ is such that the $\mathbb{P}_0-$measure of quadric surfaces is zero.

Assumption 4: We assume that $\mathbb{E}\|x^0\|^2 < \infty$.

Theorem 4: Under Assumptions 1, 3 and 4, G has a fixed point. If $(A_\theta, B_\theta, M_{\theta j}, r_\theta) = (A, B, M_j, r)$, i.e., the parameters are the same for all the agents, the result holds with Assumption 1 replaced by Assumption 2.

Proof: See [18, Theorems 6 and 8]. ∎

Theorem 4 provides conditions under which a solution of the mean field equations (14a)-(14c) and (20) exists. In case of nonuniform parameters, i.e. $(A_\theta, B_\theta, M_{\theta j}, r_\theta)$ are not the same for all the agents, the existence of a fixed point is proved by Schauder's fixed point theorem [18, Theorem 8] , where Assumption 1 is used to construct a bounded set that is mapped by G into itself. When the parameters are the same for all the agents, by similar techniques than those used in [18, Theorem 6], one can show that a fixed point of G is the optimal state of an LQR problem of running cost $\frac{q}{2}x^T(L + I_n)x + \frac{r}{2}\|u\|^2$. The existence and uniqueness of an optimal solution of this LQR problem is a consequence of Assumption 2. In the following, (15) and (19) are considered for a fixed point path $\bar{x}$. We define

$$\hat{x}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N}\hat{x}_i(t) = \int \hat{x}\left(t, x_i^0, \theta_i\right) \mathrm{d}\mathbb{P}_0^N(x_i^0)\mathrm{d}\mathbb{P}_\theta^N(\theta_i), \tag{22}$$

and $\hat{u}^{(N)} = (\hat{u}_i, \hat{u}_{-i})$, where $\hat{u}_i(t) = \hat{u}(t, x_i^0, \theta_i)$ and $\hat{x}_i(t) = \hat{x}(t, x_i^0, \theta_i)$.

C. Asymptotic Social Optimum

In this section, we show that when the agents apply the strategy profile $\hat{u}$ defined below Theorem 4 and in (15), the corresponding per agent social cost (2) converges to the optimal per agent social cost as the size of the population increases to infinity. At the end of this section, we also give an explicit form of the asymptotic per agent optimal social cost.

Assumption 5: We assume that $\frac{1}{N}\sum_{i=1}^{N}\|x_i^0\|^2 < C$ for all $N > 0$.

Remark 2: Assumption 5 implies Assumption 4. In fact, $\mathbb{P}_0^N$ converges in distribution to $\mathbb{P}_0$. Therefore, there exists on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a sequence of random variables X_N^0 of distribution $\mathbb{P}_0^N$ and a random variable X^0 of distribution $\mathbb{P}_0$ such that X_N^0 converges with

probability one to X^0. We may consider, without loss of generality, that $(\Omega, \mathcal{F}, \mathbb{P})$ is the same as the one defined in Section II. By Fatou's Lemma [22],

$$C \geq \liminf_N \frac{1}{N} \sum_{i=1}^N \left\| x_i^0 \right\|^2 = \liminf_N \int \left\| X_N^0 \right\|^2 \mathrm{d}\mathbb{P}$$

$$\geq \int \liminf_N \left\| X_N^0 \right\|^2 \mathrm{d}\mathbb{P} = \int \left\| X^0 \right\|^2 \mathrm{d}\mathbb{P} = \mathbb{E} \left\| x^0 \right\|^2.$$

The functions defined by (14a), (14b) and (14c) are continuous with respect to θ, which belongs to a compact set Θ. The random variables θ and x^0 are assumed to be independent. Therefore, under Assumption 4 and by Fubini-Tonelli's theorem [22], the operator G defined in paragraph IV-B2 by (14a)-(14c) and (20) has the following form:

$$\mathbb{E}\hat{x}\left(t, x^0, \theta\right) = G(\bar{x})(t) = \int_\Theta \int_{\mathbb{R}^n} \hat{x}\left(t, x^0, \theta\right) \mathrm{d}\mathbb{P}_0(x^0) \mathrm{d}\mathbb{P}_\theta(\theta). \tag{23}$$

In the following lemma, we show that when applying the decentralized person-by-person control laws, the finite population average path converges to the fixed point path $\bar{x}$ that the agents are optimally tracking. In the standard LQG MFG literature, the proof of this result relies on the uniform boundedness and equicontinuity of the generic agent's state trajectory with respect to the initial conditions and parameters. In our case, this trajectory (19), considered as a function of the time t, the initial condition x^0 and the parameter θ, is discontinuous. In fact, it has on each basin of attraction D_j^θ a different structure that depends on the corresponding p_j. Hence, the proof requires some additional constructions to deal with the discontinuity.

Lemma 5: Under Assumptions 3 and 5,

$$\lim_{N \to \infty} \int_0^T \left\| \hat{x}^{(N)} - \bar{x} \right\|^2 \mathrm{d}t = 0. \tag{24}$$

Proof: In view of (22) and (23), we have

$$\hat{x}^{(N)}(t) - \bar{x}(t) = \int \hat{x}\left(t, x_i^0, \theta_i\right) \mathrm{d}\mathbb{P}_0^N(x_i^0) \mathrm{d}\mathbb{P}_\theta^N(\theta_i) - \int \hat{x}\left(t, x^0, \theta\right) \mathrm{d}\mathbb{P}_0(x^0) \mathrm{d}\mathbb{P}_\theta(\theta).$$

If $\hat{x}(t, x_i^0, \theta_i)$ and $\hat{x}(t, x^0, \theta)$ were uniformly bounded and equicontinuous with respect to the initial conditions and parameters, then one can show the convergence by [23, Corollary 1.1.5]. But $\hat{x}(t, x_i^0, \theta_i)$ and $\hat{x}(t, x^0, \theta)$ are discontinuous. Alternatively, we show that the set of discontinuity points has a measure zero under Assumption 3. We then we show that $\hat{x}^{(N)}$ converges pointwise to $\bar{x}$. Finally, We prove the uniform convergence, from which the result follows.

Pointwise convergence. $\mathbb{P}_0^N$ and $\mathbb{P}_\theta^N$ converge respectively in distribution to $\mathbb{P}_0$ and $\mathbb{P}_\theta$. Therefore, there exist on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a sequence of random variables X_N^0 of distribution $\mathbb{P}_0^N$ (resp. a sequence of random variables ξ_N^θ of distribution $\mathbb{P}_\theta^N$), and a random variable X^0 of distribution $\mathbb{P}_0$ (resp. a random variable ξ^θ of distribution $\mathbb{P}_\theta$) such that X_N^0 (resp. ξ_N^θ) converges with probability one to X^0 (resp. ξ^θ). Thus,

$$\hat{x}^{(N)}(t) - \bar{x}(t) = \int \left(\hat{x}\left(t, X_N^0, \xi_N^\theta\right) - \hat{x}\left(t, X^0, \xi^\theta\right) \right) \mathrm{d}\mathbb{P}.$$

For a fixed t, the discontinuity points of $\hat{x}\left(t, x^0, \theta\right)$ (considered now as a function of x^0 and θ) are included in the set $D = \{(x^0, \theta) \in \mathbb{R}^n \times \Theta \,|\, x^0 \in \partial D_j^\theta(\bar{x})\}$. Under Assumption 3 and the independence of x^0 and θ, one can prove that $\mathbb{P}_0 \times \mathbb{P}_\theta(D) = 0$. Hence, $\hat{x}\left(t, X_N^0, \xi_N^\theta\right)$ converges with probability one to $\hat{x}\left(t, X^0, \xi^\theta\right)$. The compactness of $[0, T]$ and Θ, and the continuity of Π_j^θ imply

$$\left\| \hat{x}\left(t, X_N^0, \xi_N^\theta\right) - \hat{x}\left(t, X^0, \xi^\theta\right) \right\| \leq K_1 \|X_N^0\| + K_2 \|X^0\| + K_3,$$

for some finite $K_1, K_2, K_3 > 0$. $\hat{x}^{(N)}(t)$ converges pointwise to $\bar{x}(t)$ for all $t \in [0, T]$ as a consequence of Assumption 5, Remark 2 and Lebesgue's dominated convergence theorem.

Uniform convergence. As in the proof of Theorem 4, see [18, Theorem 8], one can show that for all t_1, t_2, $\left\| \hat{x}^{(N)}(t_1) - \hat{x}^{(N)}(t_2) \right\| \leq K|t_1 - t_2|$ and $\|\bar{x}(t_1) - \bar{x}(t_2)\| \leq K|t_1 - t_2|$, where $K > 0$ is independent of N. We fix an $\epsilon > 0$ and consider a partition $0 = t_0 < t_1 < \cdots < t_j = T$ of $[0, T]$ such that for all $t, t' \in [t_k, t_{k+1}]$, for $N \geq 1$, $\|\hat{x}^{(N)}(t) - \hat{x}^{(N)}(t')\| < \epsilon$ and $\|\bar{x}(t) - \bar{x}(t')\| < \epsilon$. By the pointwise convergence, there exists N_0 such that for all $N > N_0$, for $k = 1, \ldots, j$, $\|\hat{x}^{(N)}(t_k) - \bar{x}(t_k)\| < \epsilon$. We fix $N > N_0$. For an arbitrary $t \in [0, T]$, there exists k such that $t \in [t_k, t_{k+1}]$. We have

$$\|\hat{x}^{(N)}(t) - \bar{x}(t)\| \leq \|\hat{x}^{(N)}(t) - \hat{x}^{(N)}(t_k)\| + \|\hat{x}^{(N)}(t_k) - \bar{x}(t_k)\| + \|\bar{x}(t_k) - \bar{x}(t)\| \leq 3\epsilon.$$

This inequality holds for an arbitrary $t \in [0, T]$, therefore, $\lim_{N \to \infty} \sup_{t \in [0, T]} \|\hat{x}^{(N)}(t) - \bar{x}(t)\|^2 = 0$. This implies (24). ∎

We now state the main result of this paper, which asserts that under appropriate conditions, when the agents apply the mean field person to person optimization based decentralized strategies (15), the per agent social cost converges to the per agent optimal social cost as the size of the population increases to infinity. To compute its control strategy (15), each agent only needs to

know its initial condition, current state, the distributions $\mathbb{P}_0$ and $\mathbb{P}_\theta$ and a fixed point path $\bar{x}$ of the operator G defined in (23).

Theorem 6: Under Assumptions 2, 3, and 5,

$$\lim_{N \to \infty} \left| \inf_{u \in U^N} \frac{1}{N} J_{soc} \left(u, x^{(N)} \right) - \frac{1}{N} J_{soc} \left(\hat{u}^{(N)}, \hat{x}^{(N)} \right) \right| = 0. \tag{25}$$

Proof: Let $u \in U^N$ such that $J_{soc} \left(u, x^{(N)} \right) \le J_{soc} \left(\hat{u}^{(N)}, \hat{x}^{(N)} \right)$. Noting (19), the compactness of Θ, the continuity of $\Pi_j^\theta(t)$ with respect to t and θ and Assumption 5, one can prove that $(1/N) J_{soc} \left(\hat{u}^{(N)}, \hat{x}^{(N)} \right) < c_0$, where c_0 is independent of N. Therefore, $(1/N) J_{soc} \left(u, x^{(N)} \right) < c_0$ and

$$\frac{1}{N} \sum_{i=1}^N \int_0^T \left\{ \|u_i\|^2 + \|\hat{u}_i\|^2 + \|x_i\|^2 + \|\hat{x}_i\|^2 \right\} \mathrm{dt} < c_1,$$

where $c_1 > 0$ is independent of N. Let $\tilde{x}_i = x_i - \hat{x}_i$ and $\tilde{u}_i = u_i - \hat{u}_i$. We have (26) below

$$\frac{1}{N} J_{soc} \left(u, x^{(N)} \right) = \frac{1}{N} J_{soc} \left(\hat{u}^{(N)}, \hat{x}^{(N)} \right) + \frac{1}{N} \sum_{i=1}^N \int_0^T r_i \tilde{u}_i^T \hat{u}_i \mathrm{dt}$$

$$+ \frac{1}{N} \sum_{i=1}^N \int_0^T \left\{ \frac{q}{2} \left\| \tilde{x}_i - Z \tilde{x}^{(N)} \right\|^2 + \frac{r_i}{2} \|\tilde{u}_i\|^2 + q \left(\tilde{x}_i - Z \tilde{x}^{(N)} \right)^T \left(\hat{x}_i - Z \hat{x}^{(N)} \right) \right\} \mathrm{dt} \tag{26}$$

$$+ \frac{1}{N} \sum_{i=1}^N \min_{j=1,\dots,l} \frac{M_{ij}}{2} \|x_i(T) - p_j\|^2 - \frac{1}{N} \sum_{i=1}^N \min_{j=1,\dots,l} \frac{M_{ij}}{2} \|\hat{x}_i(T) - p_j\|^2.$$

For a fixed point $\bar{x}$ of G, and recalling (13) we have

$$J(u_i, \bar{x}, x_i^0, \theta_i) = J(\hat{u}_i, \bar{x}, x_i^0, \theta_i) + \int_0^T \left\{ \frac{q}{2} \|\tilde{x}_i\|^2 + \frac{r_i}{2} \|\tilde{u}_i\|^2 + q \bar{x}^T L \tilde{x}_i + q \tilde{x}_i^T \hat{x}_i + r_i \tilde{u}_i^T \hat{u}_i \right\} \mathrm{dt}$$

$$+ \min_{j=1,\dots,l} \frac{M_{ij}}{2} \|x_i(T) - p_j\|^2 - \min_{j=1,\dots,l} \frac{M_{ij}}{2} \|\hat{x}_i(T) - p_j\|^2, \tag{27}$$

Now (26) and (27) yield

$$\frac{1}{N} J_{soc} \left(u, x^{(N)} \right) = \frac{1}{N} J_{soc} \left(\hat{u}^{(N)}, \hat{x}^{(N)} \right) + \frac{1}{N} \sum_{i=1}^N \left(J(u_i, \bar{x}, x_i^0, \theta_i) - J(\hat{u}_i, \bar{x}, x_i^0, \theta_i) \right) \tag{28}$$

$$+ q \int_0^T \left(\tilde{x}^{(N)} \right)^T L \tilde{x}^{(N)} \mathrm{dt} + q \int_0^T \left(\hat{x}^{(N)} - \bar{x} \right)^T L \tilde{x}^{(N)} \mathrm{dt}.$$

By the bounds c_0 and c_1, the Cauchy-Schwarz inequality and Lemma 5, we deduce that $\epsilon_N = q \int_0^T (\hat{x}^{(N)} - \bar{x})^T L \tilde{x}^{(N)} \mathrm{dt}$ converges to 0 as N goes to infinity. The optimization of $\hat{u}_i$ with respect to J and Assumption 2 imply $\frac{1}{N} J_{soc}(u, x^{(N)}) \ge \frac{1}{N} J_{soc}(\hat{u}^{(N)}, \hat{x}^{(N)}) + \epsilon_N$. ∎

Remark 3 (Need for Assumption 2): In static games, a sufficient condition of the person-by-person solution to be a social optimum is the convexity and smoothness of the costs [17,

Lemma 2.6.1]. Although not explicitly mentioned by the authors [15], this condition (which is automatically satisfied in the LQG setting) guarantees also the convergence of the person-by-person solution to the social optimum in case of dynamic LQG MFG problems [15, Theorem 4.2]. In fact, if we follow the techniques used in [15, Theorem 4.2], then by the convexity of the running cost, (26) implies

$$\frac{1}{N} J_{soc}\left(u, x^{(N)}\right) \geq \frac{1}{N} J_{soc}\left(\hat{u}^{(N)}, \hat{x}^{(N)}\right) + \frac{1}{N} \sum_{i=1}^{N} \int_0^T \left\{ r_i \tilde{u}_i^T \hat{u}_i + q\left(\tilde{x}_i\right)^T \left(\hat{x}_i + L\bar{x}\right) \right\} \mathrm{dt} + \epsilon_N$$

$$+ \frac{1}{N} \sum_{i=1}^{N} \min_{j=1,\ldots,l} \frac{M_{ij}}{2} \|x_i(T) - p_j\|^2 - \frac{1}{N} \sum_{i=1}^{N} \min_{j=1,\ldots,l} \frac{M_{ij}}{2} \|\hat{x}_i(T) - p_j\|^2. \tag{29}$$

We have

$$\frac{d}{dt} \tilde{x}_i^T (\Gamma_k^{\theta_i} \hat{x}_i + \beta_k^{\theta_i}) = -r_i \tilde{u}_i^T \hat{u}_i - q\left(\tilde{x}_i\right)^T \left(\hat{x}_i + L\bar{x}\right).$$

Hence,

$$\frac{1}{N} J_{soc}\left(u, x^{(N)}\right) \geq \frac{1}{N} J_{soc}\left(\hat{u}^{(N)}, \hat{x}^{(N)}\right) \tag{30}$$

$$+ \frac{1}{N} \sum_{i=1}^{N} \left(\phi_i\left(x_i(T)\right) - \phi_i\left(\hat{x}_i(T)\right) - \tilde{x}_i^T(T) \frac{d}{dx} \phi_i\left(\hat{x}_i(T)\right) \right) + \epsilon_N,$$

where ϕ_i is the final cost of agent i. If the final costs are convex (which is not the case), then (30) implies (25). To deal with the non-convexity of the final costs, steps (29) and (30) are replaced by (27), (28) and Assumption 2.

D. Asymptotic Optimal Social Cost

In this section, we give an explicit form of the asymptotic per agent optimal social cost $\lim_{N \to \infty} \inf_{u \in U^N} \frac{1}{N} J_{soc}(u, x^{(N)})$. In the following lemmas, we start by approximating this asymptotic per agent social cost.

Lemma 7: Under Assumptions 2, 3 and 5,

$$\lim_{N \to \infty} \left| \inf_{u \in U^N} \frac{1}{N} J_{soc}\left(u, x^{(N)}\right) - \frac{1}{N} J_{soc}\left(\hat{u}^{(N)}, \bar{x}\right) \right| = 0$$

Proof: We have

$$\frac{1}{N} J_{soc}\left(\hat{u}^{(N)}, \hat{x}^{(N)}\right) - \frac{1}{N} J_{soc}\left(\hat{u}^{(N)}, \bar{x}\right) = \int_0^T \frac{q}{2N} \sum_{i=1}^{N} \left(\left\|\hat{x}_i - Z\hat{x}^{(N)}\right\|^2 - \left\|\hat{x}_i - Z\bar{x}\right\|^2 \right) \mathrm{dt}$$

$$= \frac{q}{2} \int_0^T \left\| Z\left(\hat{x}^{(N)} - \bar{x}\right) \right\|^2 \mathrm{dt} + q \int_0^T \left(\hat{x}^{(N)} - Z\bar{x}\right)^T Z\left(\bar{x} - \hat{x}^{(N)}\right) \mathrm{dt}.$$

The Cauchy-Schwarz inequality and Lemma 5 imply

$$\lim_{N \to \infty} \left| \frac{1}{N} J_{soc} \left(\hat{u}^{(N)}, \hat{x}^{(N)} \right) - \frac{1}{N} J_{soc} \left(\hat{u}^{(N)}, \bar{x} \right) \right| = 0.$$

Therefore, we deduce by Theorem 6 the result. ∎

Lemma 8: Under Assumptions 2, 3 and 5,

$$\lim_{N \to \infty} \left| \inf_{u \in U^N} \frac{1}{N} J_{soc} \left(u, x^{(N)} \right) - J_{soc}^\infty(\bar{x}) \right| = 0,$$

where

$$J_{soc}^\infty(\bar{x}) = \int \left[\int_0^T \left\{ \frac{q}{2} \left\| \hat{x}(t, x^0, \theta) - Z\bar{x} \right\|^2 + \right. \right.$$

$$\left. \left. \frac{r_\theta}{2} \left\| \hat{u}(t, x^0, \theta) \right\|^2 \right\} \mathrm{dt} + \min_{j=1,\ldots,l} \frac{M_{\theta j}}{2} \left\| \hat{x}(T, x^0, \theta) - p_j \right\|^2 \right] \mathrm{d}\mathbb{P}_0 \mathrm{d}\mathbb{P}_\theta.$$

Proof: By Lemma 7, it suffices to prove that

$$\lim_{N \to \infty} \left| J_{soc}^\infty(\bar{x}) - \frac{1}{N} J_{soc} \left(\hat{u}^{(N)}, \bar{x} \right) \right| = 0.$$

We use the same notation as in the proof of Lemma 5. We have

$$J_{soc}^\infty(\bar{x}) - \frac{1}{N} J_{soc} \left(\hat{u}^{(N)}, \bar{x} \right) = \psi_1 + \psi_2 + \psi_3$$

where

$$\psi_1 = \frac{q}{2} \int_0^T \int \left\{ \left\| \hat{x} \left(t, X^0, \xi^\theta \right) - Z\bar{x} \right\|^2 - \left\| \hat{x} \left(t, X_N^0, \xi_N^\theta \right) - Z\bar{x} \right\|^2 \right\} \mathrm{d}\mathbb{P} \mathrm{dt}$$

$$\psi_2 = \int_0^T \int \left\{ \frac{r_{\xi^\theta}}{2} \left\| \hat{u} \left(t, X^0, \xi^\theta \right) \right\|^2 - \frac{r_{\xi_N^\theta}}{2} \left\| \hat{u} \left(t, X_N^0, \xi_N^\theta \right) \right\|^2 \right\} \mathrm{d}\mathbb{P} \mathrm{dt}$$

$$\psi_3 = \int \min_{j=1,\ldots,l} \frac{M_{\xi^\theta j}}{2} \left\| \hat{x} \left(T, X^0, \xi^\theta \right) - p_j \right\|^2 \mathrm{d}\mathbb{P} - \int \min_{j=1,\ldots,l} \frac{M_{\xi_N^\theta j}}{2} \left\| \hat{x} \left(T, X_N^0, \xi_N^\theta \right) - p_j \right\|^2 \mathrm{d}\mathbb{P}.$$

Noting that $a^T a - b^T b = (a + b)^T (a - b)$ and that the minimum of l continuous functions is continuous, one can prove by the same techniques used in the proof of Lemma 5 that ψ_1, ψ_2 and ψ_3 converge to zero as N goes to infinity. ∎

In the following theorem, we give an explicit form of the asymptotic social cost. This expression depends only on the distributions $\mathbb{P}_0$, $\mathbb{P}_\theta$ and a fixed point path $\bar{x}$.

Theorem 9: Under Assumptions 2, 3 and 5,

$$\lim_{N \to \infty} \inf_{u \in U^N} \frac{1}{N} J_{soc}\left(u, x^{(N)}\right) = -\frac{1}{2} \int_0^T q\bar{x}^T L\bar{x}\mathrm{dt}+$$

$$\sum_{j=1}^l \int \mathbb{1}_{D_j^\theta(\bar{x})}(x^0) \left\{ \frac{1}{2}(x^0)^T \Gamma_j^\theta(0)x^0 + (\beta_j^\theta(0))^T x^0 + \delta_j^\theta(0) \right\} \mathrm{d}\mathbb{P}_0 \mathrm{d}\mathbb{P}_\theta.$$

Proof: Following Lemma 8, the per agent asymptotic optimal social cost is equal to $J_{soc}^\infty(\bar{x})$. Noting (20), one can write $J_{soc}^\infty(\bar{x}) = \psi_4 - \frac{1}{2} \int_0^T q\bar{x}^T L\bar{x}\mathrm{dt}$, where

$$\psi_4 = \int \left[\int_0^T \left\{ \frac{q}{2} \left\| \hat{x}\left(t, x^0, \theta\right) \right\|^2 + q\bar{x}^T L\hat{x}\left(t, x^0, \theta\right) + \frac{r_\theta}{2} \left\| \hat{x}\left(t, x^0, \theta\right) \right\|^2 \right\} \mathrm{dt} \right.$$

$$\left. + \min_{j=1,\dots,l} \frac{M_{\theta j}}{2} \left\| \hat{x}\left(T, x^0, \theta\right) - p_j \right\|^2 \right] \mathrm{d}\mathbb{P}_0 \mathrm{d}\mathbb{P}_\theta =$$

$$\sum_{j=1}^l \int \mathbb{1}_{D_j^\theta(\bar{x})}(x^0) \left\{ (x^0)^T \Gamma_j^\theta(0)x^0 + \beta_j^\theta(0)^T x^0 + \delta_j^\theta(0) \right\} \mathrm{d}\mathbb{P}_0 \mathrm{d}\mathbb{P}_\theta.$$

∎

V. SIMULATION RESULTS

In this section, we compare numerically the cooperative and the non-cooperative behaviors of a group of agents choosing between two alternatives under the social effect. We consider a uniform group of 400 players initially drawn from the Gaussian distribution $\mathcal{N}\left(\begin{bmatrix} -5 & 10 \end{bmatrix}^T, 15I_2\right)$ and moving in $\mathbb{R}^2$ according to the dynamics

$$A_i = \begin{bmatrix} 0 & 1 \\ 0.02 & -0.3 \end{bmatrix} \qquad\qquad B_i = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}$$

towards the potential destination points $p_1 = (-10, 0)$ or $p_2 = (10, 0)$. Hence we have a binary choice problem, and in this case one can characterize the way the population splits between the alternatives, in both the cooperative and non-cooperative cases, by a number λ, which is the fraction of players that go towards p_1. This number λ is a fixed point of a well defined function F and can be computed by dichotomy. Moreover, one can compute the fixed point path $\bar{x}$ that corresponds to λ. For more details one can refer to [18, Theorem 6] and [18, Section 5.A]. We set $r_i = 10$, $M_{ij} = 1200$, $T = 2$, $Z = 3.5I_2$, and we vary the social effect coefficient q. $L = Z^T Z - Z - Z^T = 5.25I_2$ satisfies Assumption 2. For $q = 0$ (no social effect), Fig. 1

and 2 show that the 82% of the players (green squares in Fig. 2) go towards p_2 in both the cooperative and non-cooperative cases. As the social effect increases (q increases from 0 to 45), in the non-cooperative case, the majority influences the minority whose size reduces from 18% to zero (Fig. 1 and 4). In the cooperative case however, the size of the majority decreases and the population splits more evenly between the two choices (Fig. 1 and 3). Fig. 1 also illustrates that the per agent social cost in the cooperative case is smaller than in the non-cooperative case.
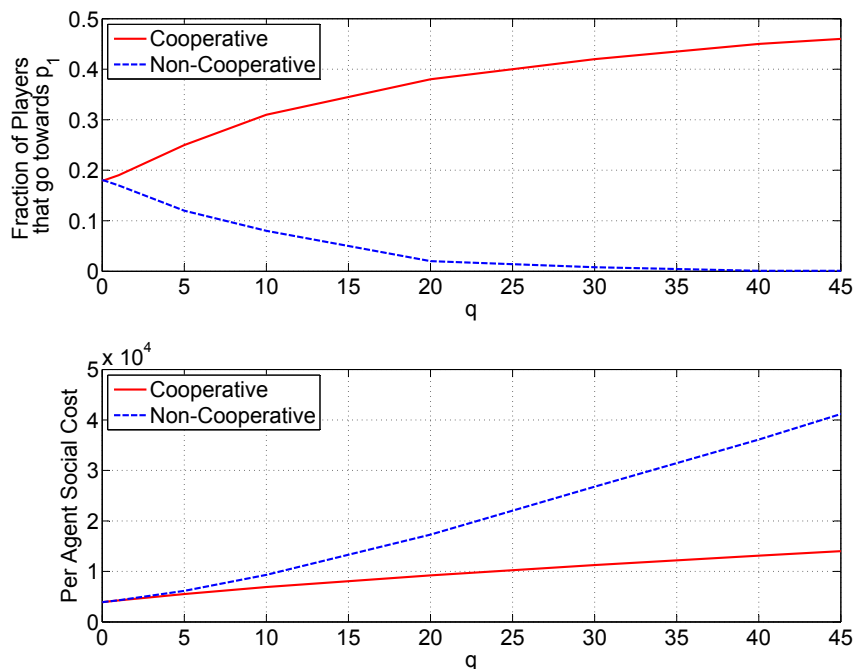
Fig. 1.   Cooperative vs. non-cooperative behavior

VI. CONCLUSION

We consider in this paper a dynamic cooperative game model where a large number of players are making a socially influenced choice between multiple alternatives. Finding an exact social optimum can be done by solving a number of LQR problems that grows exponentially with the number of players. Alternatively, we develop via the MFG methodology a set of decentralized strategies that are asymptotically socially optimal. The computation of the decentralized strategies
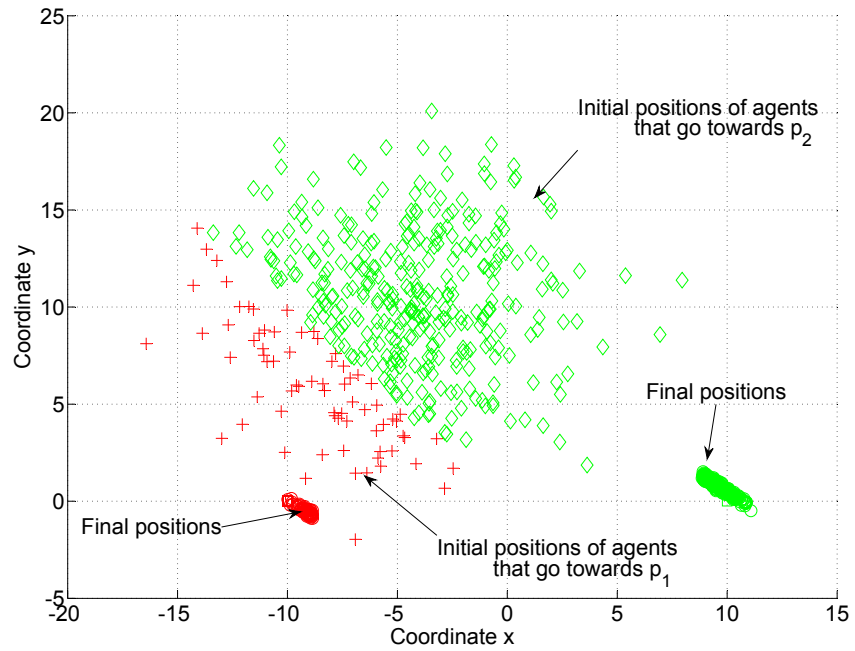
Fig. 2. Absence of social effect ($q = 0$). The majority goes towards p_2.
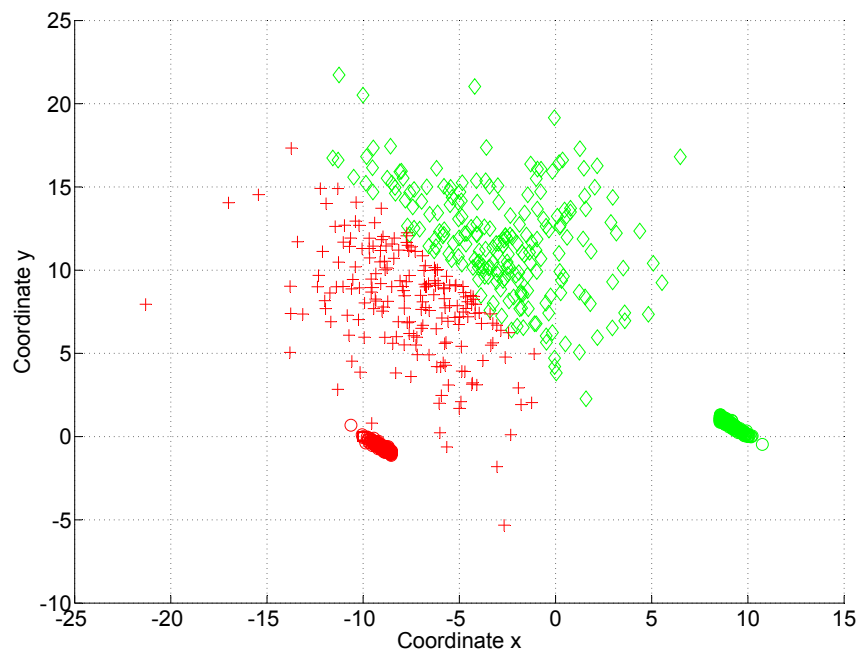

Fig. 3. Cooperative case with high social effect ($q = 40$). The population splits more evenly.
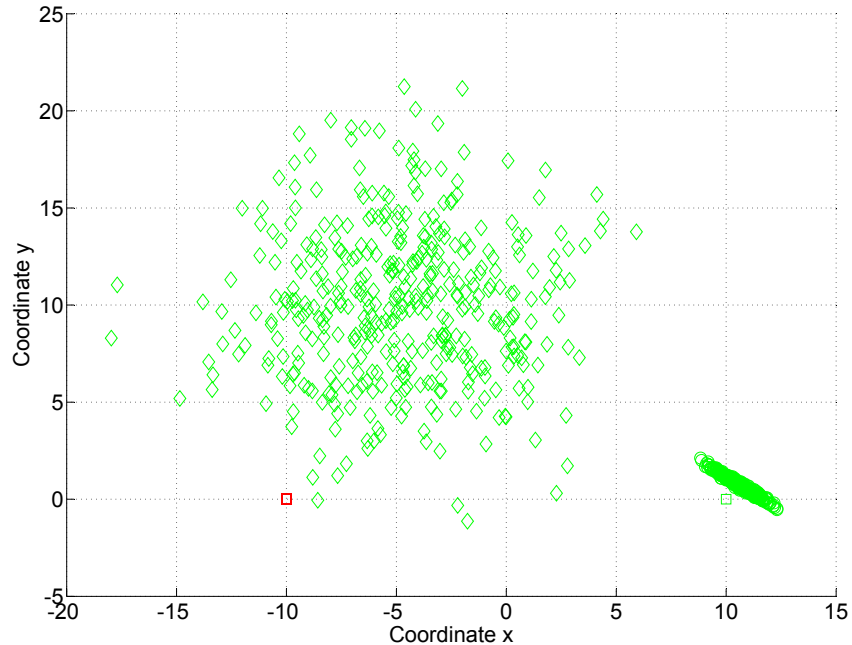
Fig. 4.   Non-cooperative case with high social effect ($q = 40$). The population reaches consensus on p_2.

assumes that each agent knows the statistical distributions of the initial states and parameters. For future work, it is of interest to consider situations where the cooperative players learn these statistical distributions while moving towards the destination points, e.g., by sharing and updating their current states and parameters through a random communication graph.

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