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Input-to-State Stabilization in Probability for Nonlinear Stochastic Systems Under Quantization Effects and Communication Protocols

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***Abstract*—In this paper, the observer-based stabilization problem is investigated for a class of discrete-time nonlinear stochastic networked control systems (NCSs) with exogenous disturbances. The signal transmission from the sensors to the observer is implemented via a shared digital network, in which both uniform quantization effect and stochastic communication protocol (SCP) are taken into account to reflect several network-induced constraints. The notion of input-to-state stability in probability is introduced to describe the dynamical behaviors of the closed-loop stochastic NCS that is effectively characterized by a general nonlinear stochastic difference equation with Markovian jumping parameters. A theoretical framework is first established to felicitate the dynamics analysis of the closed-loop system in virtue of the switched Lyapunov function method and the stochastic analysis techniques. By making full use of the quantized measurement output under the scheduling of the SCP, the existence conditions for an observer-based controller are established under which the closed-loop system is input-to-state stable in probability. Then, the explicit expression of the gain matrices of the desired controller is given by resorting to a set of feasible solutions of certain matrix inequalities. The effectiveness of the theoretical results is demonstrated by a numerical simulation example.**

***Index Terms*—Communication protocol, input-to-state stability (ISS), networked control system (NCS), observer-based control, quantization effects.**

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I. Introduction

The past decade has witnessed a rapidly growing research interest in networked control systems (NCSs) that have found extensive applications in a variety of industrial processes including manufacturing, monitoring, communication, and so on [8], [43]. Compared with the conventional point-to-point control structures, NCSs are more flexible in the system design with lower cost in installation/maintenance due to the participation of digital communication networks that may be shared by sensors, controllers and actuators [11], [37], [39]. Owing to their attractive advantages in engineering applications, NCSs have stirred particular research attention from control community. In recent years, a rich body of literature has been available concerning the issues of modeling, analysis and control synthesis for NCSs (see [2]–[4], [9], [21], [22], [38], [40] and references therein).

In real-world control systems, exogenous disturbances (e.g., unmoderated dynamics and measurement errors) may influence the dynamical behaviors and control performance seriously [13], [14]. As such, it is of practical significance to investigate the stability robustness against the exogenous disturbances for control systems. Initially proposed in [27], the notion of input-to-state stability (ISS) is capable of effectively characterizing the response of asymptotically stable systems to the bounded exogenous disturbances. Over the past years, a large number of research results have been reported to cope with the issues of ISS for dynamical systems with exogenous disturbances [10], [16], [24], [25], [29], [34], [35], [41]. Nonetheless, comparing to the fruitful results for the conventional *point-to-point control systems*, the corresponding advances on the ISS behaviors for *nonlinear stochastic NCSs with exogenous disturbances* have been relatively slow and the related results have been scattered despite their great significance in practical applications, and this constitutes the first motivation for us to carry out this paper.

It is well known that, in many computer-based control loops, the quantizer is usually adopted to convert the analog signal (such as states, measured outputs, or control inputs) to the corresponding discrete-time digital one because of the limitation on bit rate. In the conversion process, the operation of rounding or truncation will inevitably result in the quantization error which can be described as an unknown but bounded disturbance [19]. Up to now, there have been plenty of research papers published concerning the stability

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and stabilization problems of deterministic NCSs with quantization effects [7], [26], [42]. In recent years, some progress has also been made on the control synthesis for stochastic NCSs with quantization effects [5], [6], [12], [33].

In a shared communication network, it is quite common that the number of sensors getting access to the shared channel is limited due primarily to the bandwidth constraint [28], [47]. In order to avoid the data congestion/collision, it is necessary to propose an appropriate communication (or scheduling) protocol to orchestrate the order of sensors being given the license at each transmission instant [36], [45]. So far, a substantial amount of research work has been reported to address the issues of performance analysis and control synthesis for NCSs subjected to the protocol scheduling [1], [17], [30], [32], [44], [46]. For instance, in [46], the problems of set-membership filtering have been investigated for a class of time-varying NCSs under the round-Robin and the weighted try-once-discard (TOD) protocols, respectively. A new concept of stochastic protocol has been proposed in [30] and sufficient criteria have been obtained to render the NCS with exogenous disturbances L_p stable. In [17], the time-delay approach has been utilized to investigated the network-based control for systems in which two classes of stochastic protocols have been introduced to schedule the activation of sensor nodes.

Recently, some initial effort has been devoted to the study of *deterministic NCSs* with both quantization effects and communication protocols [18], [20], [23], [31]. In [23], by employing an emulation-like approach, a unified framework has been established for the controller design of NCSs with the dynamic quantization and the time scheduling. Within the dynamic quantization and the Round-Robin protocol, the exponential stability of a discrete-time linear plant has been derived by utilizing feedback control in [18]. The synchronization problem for chaotic neural networks has been studied in [31], in which the quantization effect and the TOD scheduling protocol have been taken into account. Nevertheless, the problems of dynamics analysis and control synthesis for *nonlinear stochastic NCSs* with both quantization effects and communication protocols have not been adequately investigated due primarily to the substantial complexities/difficulties in dealing with the simultaneous presence of quantization errors and stochastic hybrid dynamics [48]. To the best of our knowledge, under the uniform quantization effect and the stochastic communication protocol (SCP), the problems of ISS in probability and design of stabilizing controller for nonlinear stochastic NCSs with exogenous disturbances are still open and remain challenging, and the second aim of this paper is therefore to shorten such a gap.

In this paper, we endeavor to address the problems of input-to-state stabilization and controller design for a class of nonlinear stochastic NCSs with bounded exogenous disturbances. The measurement output of the plant is first quantized by the uniform quantizer with finite levels and then transmitted to the observer under the scheduling of the SCP. By fully utilizing the output signals featured with the quantization and the SCP scheduling, an observer-based controller is developed with hope to guarantee the desired performance of the closed-loop system. The main contributions of this paper are highlighted as follows.

1) The observer-based stabilization problem is, for the first time, investigated for the discrete-time nonlinear stochastic NCSs with both uniform quantization effect and SCP.
2) The notion of ISS in probability is extended to the case of discrete-time stochastic systems with Markovian jumping parameters with hope to reflect the stochastic characteristics and bounded disturbances more accurately.
3) A theoretical framework is established to investigate the property of ISS in probability for the addressed controlled system by employing the switched Lyapunov function and stochastic analysis techniques.
4) The synthesis problem of the observer-based controller depending on the SCP is addressed for the closed-loop system and the gain matrices for the desired controller are given by resorting to the feasibility of a set of matrix inequalities.

The rest of this paper is organized as follows. The problem of the input-to-state stabilization in probability for a class of nonlinear stochastic NCSs with exogenous disturbances is formulated in Section II, in which the uniform quantization effect and the SCP are also presented. In Section III, a Lyapunov-like theorem is established, by which several sufficient conditions are derived to guarantee the performance of the ISS in probability and the observer-based controller design algorithm is also established for the controlled system. In Section IV, a numerical example is given to illustrate the usefulness and flexibility of the theoretical result developed in this paper. The conclusions are outlined in Section V.

Notations: Let $\mathbb{R}^+$ and $\mathbb{Z}^+$ be the set of nonnegative real numbers and nonnegative integers, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The symbol $|x|$ stands for the Euclidean norm of a real vector x, and $\|A\|$ denotes the induced matrix norm. I represents the identity matrix of compatible dimensions. The symbol $\star$ will be used in some matrix expressions to indicate a symmetric structure. For symmetric matrices A and B, $A > B$ means that $A - B$ is positive definite. A^{T} and A^{-1} represent the transpose and inverse matrix of A, respectively. Let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ be the smallest and the largest eigenvalue of a symmetric matrix, respectively. $\mathcal{L}_\infty^n$ denotes the class of measurable and essentially bounded functions $v : \mathbb{Z}^+ \rightarrow \mathbb{R}^n$ with the infinity norm $|v|_\infty = \operatorname{ess\,sup}_{k\in\mathbb{Z}^+}\{|v(k)|\} < \infty$. A function $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and also $\gamma(s) \rightarrow \infty$ as $s \rightarrow \infty$; it is a $\mathcal{VK}_\infty$-function if it is a $\mathcal{K}_\infty$-function and also convex. A function $\beta : \mathbb{R}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ is a $\mathcal{KL}$-function if, for each fixed $k \in \mathbb{Z}^+$, the function $\beta(\cdot, k)$ is a $\mathcal{K}$-function, and for each fixed $s \in \mathbb{R}^+$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, k) \rightarrow 0$ as $k \rightarrow \infty$. Let Id be the identity function and $\psi \circ \phi$ represent the composition of two functions ϕ and ψ. $(\Omega, \mathscr{F}, \{\mathscr{F}_k\}_{k\geq 0}, \mathbf{P})$ represents the complete probability space with Ω being a sample space, $\mathscr{F}$

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being a σ-field, $\{\mathscr{F}_k\}_{k\geq 0}$ being a filtration and $\mathbf{P}$ being a probability measure.

II. Problem Formulation and Preliminaries

Consider the following general nonlinear stochastic discrete-time system:

$$\begin{cases} x(k+1) = Ax(k) + f(x(k)) + Bu(k) + Dv_1(k) \\ \qquad\qquad +g(x(k),\ v_1(k))w(k) \\ y(k) = Cx(k) + Ev_2(k) \\ x(0) = x_0 \in \mathbb{R}^{n_x} \end{cases} \tag{1}$$

where for $k \in \mathbb{Z}^+$, $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$, and $y(k) \in \mathbb{R}^{n_y}$ denote the state, control input, and measured output vectors, respectively. $v_1(k)$ and $v_2(k)$ represent the exogenous disturbances belonging to $\mathcal{L}_\infty^{n_x}$ and $\mathcal{L}_\infty^{n_y}$, respectively. $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ and $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ are smoothly nonlinear vector-value functions with $f(0) = 0$ and $g(0, 0) = 0$. $w(k) \in \mathbb{R}$ is a zero-mean random sequence on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_k\}_{k\geq 0}, \mathbf{P})$ and satisfies $\mathbb{E}\{w^2(k)\} = 1$. A, B, C, D, and E are known constant matrices with compatible dimensions.

Without loss of generality, we assume that the output $y(k)$ of the plant is measured by sensor nodes which are divided into n_y groups. The information exchange between the sensors and the observer takes place over a shared communication network with limited resources. In the following, we present the transmission of the measurement output under the quantization effect and the SCP in a step-by-step manner.

A. Quantization of the Measurement Output

According to [19], the quantization of signals is implemented by a quantizer $q : \mathbb{R} \to \mathbb{D}$, which is a piecewise constant function with a finite subset $\mathbb{D} \subset \mathbb{R}$ and satisfies

$$\begin{aligned} &1)\ \text{if } |z| \leq M,\ \text{then } |q(z) - z| \leq \Delta \\ &2)\ \text{if } |z| > M,\ \text{then } |q(z)| > M - \Delta \end{aligned}$$

where $z \in \mathbb{R}$. $M > 0$ and $\Delta \geq 0$ are referred to as the quantization range and quantization error of q, respectively. Condition 1) indicates that the quantization error possesses a bound if the quantizer does not saturate, while condition 2) offers a way to check the possibility of saturation.

In this paper, we consider the uniform quantization effect of the measurement output $y(k)$. Denote by

$$\hat{y}(k) \triangleq q(y(k)) = \left(q_1(y_1) \cdots q_{n_y}(y_{n_y})\right)^{\mathrm{T}} \tag{2}$$

the quantized measurement output, in which q_j $(1 \leq j \leq n_y)$ is chosen to be the finite-level uniform quantizer described by the following static nonlinear function:

$$q_j(z) = \begin{cases} \Delta_q\left[\frac{z}{\Delta_q}\right], & -(M_q + 0.5)\Delta_q \leq z \\ & \leq (M_q + 0.5)\Delta_q \\ M_q\Delta_q, & z > (M_q + 0.5)\Delta_q \\ -M_q\Delta_q, & z < -(M_q + 0.5)\Delta_q \end{cases}$$

where $[\,\cdot\,]$ is the operation of round for a real number, $\Delta_q \geq 0$ is a constant, and M_q is a positive integer.

It is easy to see that, if there is an integer $i \in [-M_q,\ M_q]$ such that $y_j(k) \in [(i - 0.5)\Delta_q,\ (i + 0.5)\Delta_q]$, then the component $q_j(y_j(k))$ of the quantized measurement output takes the value $i\Delta_q$. Letting

$$\Delta(k) \triangleq \hat{y}(k) - y(k) \tag{3}$$

be the quantization error, one concludes that $\Delta(k)$ is a norm-bounded vector satisfying

$$|\Delta(k)| \leq \frac{\sqrt{n_y}}{2}\Delta_q$$

by directly following the definition of the finite-level uniform quantizer.

B. Stochastic Communication Protocol

Since the bandwidth constraint limits the amount of sensors that can transmit signals in parallel, the SCP is adopted to schedule the transmission of the quantized measurement outputs for preventing the data from collision. By an SCP, we mean that only one node of sensors $i \in \{1, 2, \ldots, n_y\}$ is chosen to acquire the channel access at each transmission instant, that is, only one component of the latest quantized measurement output $\hat{y}_i(k)$ will be transmitted via the shared network at time k, while other components without having permission are held by the zero-order holders. For any transmission time k, we denote by $r(k)$ the label of the active sensor node getting access to the network. As pointed out by Donkers *et al.* [4], $r(k)$ is a stochastic process which is determined through a Markov chain on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_k\}_{k\geq 0}, \mathbf{P})$ taking values in a finite set $\mathcal{S} \triangleq \{1, 2, \ldots, n_y\}$. Under the condition $r(k) = i$, the conditional probability of the active node $r(k+1) = j$ is given by

$$\mathbf{P}\{r(k+1) = j | r(k) = i\} = \pi_{ij} \tag{4}$$

where $\pi_{ij} \geq 0$ $(i, j \in \mathcal{S})$ and $\sum_{j=1}^{n_y} \pi_{ij} = 1$ for any $i \in \mathcal{S}$. Clearly, $\Pi = (\pi_{ij})_{n_y \times n_y}$ is the transition probability matrix of $r(k)$.

For $i \in \mathcal{S}$, $k \in \mathbb{Z}^+$, let $\delta(i, r(k))$ represent the scheduling operator defined by the Kronecker delta function as follows:

$$\delta(i, r(k)) = \begin{cases} 1, & i = r(k) \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Let $\bar{y}_i(k)$ $(i \in \mathcal{S})$ be the final measurement output (received by the observer) of the ith sensor node at time k after the network scheduling. Thus, for $k \in \mathbb{Z}^+$, one has

$$\bar{y}_i(k) = \delta(i, r(k))\hat{y}_i(k) + (1 - \delta(i, r(k)))\bar{y}_i(k-1). \tag{6}$$

Here, we assume that $\bar{y}_i(-1) \in \mathbb{R}$ is a known constant.

Denote

$$\bar{y}(k) \triangleq \left(\bar{y}_1(k) \cdots \bar{y}_{n_y}(k)\right)^{\mathrm{T}} \tag{7}$$

$$\Phi_{r(k)} \triangleq \operatorname{diag}\left\{\delta(1, r(k)), \ldots, \delta(n_y, r(k))\right\}. \tag{8}$$

From (3) and (6), the network-based measurement output available to the observer is calculated as follows:

$$\begin{aligned} \bar{y}(k) &= \Phi_{r(k)}\hat{y}(k) + (I - \Phi_{r(k)})\bar{y}(k-1) \\ &= \Phi_{r(k)}Cx(k) + (I - \Phi_{r(k)})\bar{y}(k-1) \\ &\quad + \Phi_{r(k)}Ev_2(k) + \Phi_{r(k)}\Delta(k) \end{aligned} \tag{9}$$

where $\bar{y}_{-1} \triangleq \bar{y}(-1) = (\bar{y}_1(-1) \cdots \bar{y}_{n_y}(-1))^{\mathrm{T}} \in \mathbb{R}^{n_x}$.

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C. Observer-Based Stochastic NCSs

In reality, the system states might not be completely accessible due to the physical limitation or the implementation cost. Thus, it is more practical to employ an observer-based controller [38]. According to (9), we note that the final measurement output $\bar{y}(k)$ (available to observer) is characterized by a formula with Markovian switching parameters. As such, in this paper, a mode-dependent observer-based controller is proposed as follows:

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + f(\hat{x}(k)) + Bu(k) \\ \qquad\qquad + L_{r(k)}\big(\bar{y}(k) - C\hat{x}(k)\big) \\ u(k) = K_{r(k)}\hat{x}(k) \end{cases} \tag{10}$$

with the initial condition $\hat{x}(0) = \hat{x}_0 \in \mathbb{R}^{n_x}$, where the vector $\hat{x}_0$ could be zero or any known vector.

By letting the estimate error vector (between the plant and the observer) be

$$e(k) \triangleq x(k) - \hat{x}(k) \tag{11}$$

the dynamics of the estimate error system is obtained as follows:

$$\begin{aligned} e(k+1) = {} & \big(A - L_{r(k)}C\big)e(k) + f(x(k)) - f(\hat{x}(k)) \\ & - L_{r(k)}\big(\Phi_{r(k)}C - C\big)x(k) - L_{r(k)}(I - \Phi_{r(k)}) \\ & \times \bar{y}(k-1) + Dv_1(k) - L_{r(k)}\Phi_{r(k)}Ev_2(k) \\ & - L_{r(k)}\Phi_{r(k)}\Delta(k) + g(x(k), v_1(k))w(k) \end{aligned} \tag{12}$$

where $e_0 \triangleq e(0) = x(0) - \hat{x}(0)$.

Denoting

$$\xi(k+1) \triangleq \big(x^{\mathrm{T}}(k+1) \quad e^{\mathrm{T}}(k+1) \quad \bar{y}^{\mathrm{T}}(k)\big)^{\mathrm{T}}$$

and combining (1) and (9) with (12), the closed-loop stochastic NCS can be reformulated as the following nonlinear stochastic system with Markovian jumping parameters:

$$\xi(k+1) = \mathcal{F}_{r(k)}(\xi(k), v(k)) + \mathcal{G}_{r(k)}(\xi(k), v(k))w(k) \tag{13}$$

where

$$\begin{aligned} \mathcal{F}_{r(k)}(\xi(k), v(k)) &= \mathcal{A}_{r(k)}\xi(k) + F(\xi(k)) + \mathcal{D}_{r(k)}v(k) \\ \mathcal{G}_{r(k)}(\xi(k), v(k)) &= \big(g^{\mathrm{T}}(x(k), v_1(k)) \;\; g^{\mathrm{T}}(x(k), v_1(k)) \;\; 0\big)^{\mathrm{T}} \\ \mathcal{A}_{r(k)} &= \begin{pmatrix} A + BK_{r(k)} & -BK_{r(k)} & 0 \\ \mathcal{A}^{21}_{r(k)} & A - L_{r(k)}C & \mathcal{A}^{23}_{r(k)} \\ \Phi_{r(k)}C & 0 & I - \Phi_{r(k)} \end{pmatrix} \\ \mathcal{A}^{21}_{r(k)} &= -L_{r(k)}\big(\Phi_{r(k)}C - C\big) \\ \mathcal{A}^{23}_{r(k)} &= -L_{r(k)}(I - \Phi_{r(k)}) \\ \mathcal{D}_{r(k)} &= \begin{pmatrix} D & 0 & 0 \\ D & -L_{r(k)}\Phi_{r(k)}E & -L_{r(k)}\Phi_{r(k)} \\ 0 & \Phi_{r(k)}E & \Phi_{r(k)} \end{pmatrix} \\ F(\xi(k)) &= \begin{pmatrix} f(x(k)) \\ f(\bar{x}(k)) - f(\hat{x}(k)) \\ 0 \end{pmatrix}, \; v(k) = \begin{pmatrix} v_1(k) \\ v_2(k) \\ \Delta(k) \end{pmatrix}. \end{aligned}$$

In addition, the initial condition of system (13) is assumed to be $r(0) \triangleq r_0 \in \mathcal{S}$ and $\xi(0) \triangleq \xi_0 = (x_0^{\mathrm{T}} \;\; e_0^{\mathrm{T}} \;\; \bar{y}_{-1}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2n_x+n_y}$.

Remark 1: As shown in Fig. 1, the measurement output $y(k)$ of the plant is first quantized by the finite-level uniform

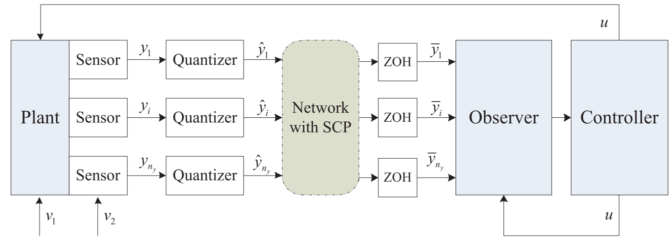

Fig. 1. Stochastic NCS with observer-based controller subject to the quantization and the SCP.

quantizer $q(\cdot)$, and it is then transmitted to the observer over the network with the SCP. It should be pointed out that, due to the presence of quantization effect and protocol scheduling, the most recent measurement output $\bar{y}(k)$ received by the observer is quite different from those in the conventional observer-based control loops [4], [6], [25], [28], [32]. On one hand, the SCP makes the closed-loop system (13) a *stochastic hybrid system with Markovian switching parameters*, which may result in substantial challenges/difficulties to the dynamics analysis/design issues. On the other hand, the finite-level uniform quantizer introduces a norm-bounded and nonvanishing quantization error to the system (13), which prevents the controlled system from becoming asymptotically stable even though the exogenous disturbances are absent.

The following definitions and lemmas will be employed in later discussions.

Definition 1: The nonlinear stochastic Markovian switching system (13) is said to be input-to-state stable in probability with respect to $v(k)$ if, for any given positive constant $\epsilon \in (0, 1)$, there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that

$$\mathbf{P}\{|\xi(k)| \le \beta(|\xi_0|, k) + \gamma(|v|_\infty)\} \ge 1 - \epsilon$$

holds for any $r_0 \in \mathcal{S}$, $\xi_0 \in \mathbb{R}^{2n_x+n_y}$ and $v \in \mathcal{L}_\infty^{2n_y+n_x}$. In particular, if $\beta(|\xi_0|, k) = \breve{\beta}(|\xi_0|)e^{-\theta k}$ with $\theta > 0$ and $\breve{\beta} \in \mathcal{K}$, then the system (13) is said to be exponentially input-to-state stable in probability.

Remark 2: This definition is capable of effectively characterizing the response of asymptotically stable systems to the bounded disturbances and the nonvanishing quantization error. If the Markov chain $r(k)$ chooses only one mode, Definition 1 introduced in this paper can be considered as a discrete-time version paralleling to the continuous-time counterparts proposed in [16] and [41], which further means that Definition 1 is quite different from the notion of γ-ISS in [29]. Furthermore, Definition 1 also extends [13, Definition 3.1] to the case of stochastic Markovian switching systems. Compared with the concept of ISS in moment sense presented in [10], Definition 1 provides a milder perspective to evaluate the dynamical behaviors for the stochastic Markovian switching systems by resorting to the ISS gain and the probability of sample trajectories entering a bounded domain.

Definition 2: System (1) is said to be input-to-state stabilizable in probability by the observer-based controller (10) if, for any $i \in \mathcal{S}$, there exist observer gain matrices L_i and control

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gain matrices K_i such that the closed-loop system (13) is input-to-state stable in probability with respect to the exogenous disturbances $v_1(k)$ and $v_2(k)$, and the quantization error $\Delta(k)$.

Lemma 1 [13]: For any $\alpha \in \mathcal{K}_\infty$, there is a $\hat{\alpha} \in \mathcal{K}_\infty$ satisfying: 1) $\hat{\alpha}(s) \leq \alpha(s)$, for any $s \in \mathbb{R}^+$ and 2) $\mathrm{Id}-\hat{\alpha} \in \mathcal{K}$.

Lemma 2 [38]: Assume that $B \in \mathbb{R}^{n\times q}$ is full column rank and $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Then, there exists a nonsingular matrix $Q \in \mathbb{R}^{q\times q}$ such that $PB = BQ$ if and only if P has the following structure:

$$P = U^{\mathrm{T}}\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}U$$

with $P_1 \in \mathbb{R}^{q\times q} > 0$, $P_2 \in \mathbb{R}^{(n-q)\times(n-q)} > 0$, and U is defined by the following singular value decomposition of B:

$$\begin{pmatrix} \Sigma \\ 0 \end{pmatrix} = UBV \triangleq \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}BV \tag{14}$$

where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{q\times q}$ are orthogonal matrices, and $U_1 \in \mathbb{R}^{q\times n}$, $U_2 \in \mathbb{R}^{(n-q)\times n}$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_q\}$, and σ_i $(i = 1, 2, \ldots, q)$ are nonzero singular values of B.

III. Main Results

In this section, a framework is, for the first time, developed to study the ISS behavior in probability for discrete-time nonlinear stochastic system with Markovian switching and bounded exogenous disturbances. Within the framework established, several sufficient criteria are given in the form of linear matrix inequalities (LMIs), by which the mode-dependent observer-based controller (10) is designed for the plant (1) with hope to provide the controlled system (13) with the exponential ISS in probability.

Theorem 1: For any $i \in \mathcal{S}$, let the observer gain matrices L_i and the controller gain matrices K_i be given. Assume that there are functions $V : \mathcal{S} \times \mathbb{R}^{2n_x+n_y} \to \mathbb{R}^+$, $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$, and $\eta \in \mathcal{K}$ such that:

1) $\alpha_1(|\xi|) \leq V(i, \xi) \leq \alpha_2(|\xi|)$, for any $(i, \xi) \in \mathcal{S} \times \mathbb{R}^{2n_x+n_y}$;
2) $\mathbb{E}\{V(r(k+1), \xi(k+1))|\mathscr{F}_k\} - V(r(k), \xi(k)) \leq -\alpha(|\xi(k)|) + \eta(|v(k)|)$, for any $k \in \mathbb{Z}^+$;
3) $\alpha \circ \alpha_2^{-1} \in \mathcal{VK}_\infty$.

Then, the nonlinear stochastic Markovian switching system (13) is input-to-state stable in probability with respect to the disturbances v.

Proof: For any initial condition

$$(r_0,\ \xi_0) \triangleq (r(0),\ \xi(0)) \in \mathcal{S} \times \mathbb{R}^{2n_x+n_y}$$

we denote by $\xi(k)$ the solution of system (13) for simplicity. It is readily seen from the condition 1) that

$$\alpha_2^{-1}(V(r(k), \xi(k))) \leq |\xi(k)|$$

which, together with $\alpha \in \mathcal{K}_\infty$, further implies that

$$\alpha \circ \alpha_2^{-1}(V(r(k), \xi(k))) \leq \alpha(|\xi(k)|) \tag{15}$$

for $k \in \mathbb{Z}^+$.

It follows directly from conditions 2), 3), and (15) that:

$$\begin{aligned}&\mathbb{E}\{V(r(k+1), \xi(k+1)) - V(r(k), \xi(k))\}\\ &\quad\leq -\alpha \circ \alpha_2^{-1}(\mathbb{E}V(r(k), \xi(k))) + \eta(|v|_\infty).\end{aligned}$$

Bearing in mind that $\mathcal{VK}_\infty \subset \mathcal{K}_\infty$ and following from Lemma 1, we conclude that there is a $\hat{\alpha} \in \mathcal{K}_\infty$ such that $\hat{\alpha} \leq \alpha \circ \alpha_2^{-1}$, which further results in:

$$\begin{aligned}\mathbb{E}V(r(k+1), \xi(k+1)) \leq \big(\mathrm{Id} - \hat{\alpha}\big)&(\mathbb{E}V(r(k), \xi(k)))\\ &+ \eta(|v|_\infty)\end{aligned} \tag{16}$$

with $\mathrm{Id}-\hat{\alpha} \in \mathcal{K}$.

For any $\mu > 1$, let

$$\mathscr{B} \triangleq \Big\{(i,\ \xi) \in \mathcal{S} \times \mathbb{R}^{2n_x+n_y} : V(i, \xi) \leq \gamma^*(\mu, |v|_\infty)\Big\} \tag{17}$$

in which $\gamma^*(\mu, |v|_\infty) \triangleq \hat{\alpha}^{-1}(\mu\eta(|v|_\infty))$. Next, we aim to prove that, for any positive scalar $\epsilon \in (0, 1)$, there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\mathbf{P}\{|\xi(k)| \leq \beta(|\xi_0|, k) + \gamma(|v|_\infty)\} \geq 1 - \epsilon. \tag{18}$$

In order to complete the proof of (18), we consider the following two mutually exclusive cases.

Case 1: $(r_0,\ \xi_0) \in \mathscr{B}$.

Case 2: $(r_0,\ \xi_0) \notin \mathscr{B}$.

For case 1, it follows from $(r_0,\ \xi_0) \in \mathscr{B}$ that:

$$\mathbb{E}V(r(0), \xi(0)) \leq \gamma^*(\mu, |v|_\infty). \tag{19}$$

Recalling $\mathrm{Id}-\hat{\alpha} \in \mathcal{K}$, we get from (16) and (19) that

$$\begin{aligned}\mathbb{E}V(r(1), \xi(1)) &\leq \big(\mathrm{Id} - \hat{\alpha}\big)(\mathbb{E}V(r(0), \xi(0))) + \eta(|v|_\infty)\\ &\leq \gamma^*(\mu, |v|_\infty) - \hat{\alpha} \circ \gamma^*(\mu, |v|_\infty) + \eta(|v|_\infty)\\ &< \gamma^*(\mu, |v|_\infty).\end{aligned} \tag{20}$$

By the mathematical induction, one has for any $k \in \mathbb{Z}^+$

$$\mathbb{E}V(r(k), \xi(k)) \leq \gamma^*(\mu, |v|_\infty). \tag{21}$$

Applying Chebyshev's inequality yields that

$$\mathbf{P}\bigg\{V(r(k), \xi(k)) \geq \frac{1}{\epsilon}\gamma^*(\mu, |v|_\infty)\bigg\} \leq \frac{\mathbb{E}V(r(k), \xi(k))\epsilon}{\gamma^*(\mu, |v|_\infty)} \leq \epsilon. \tag{22}$$

It should be kept in mind that the condition 1 implies

$$\begin{aligned}&\bigg\{\omega \in \Omega : \alpha_1(|\xi(k)|) \geq \frac{1}{\epsilon}\gamma^*(\mu, |v|_\infty)\bigg\}\\ &\quad\subset \bigg\{\omega \in \Omega : V(r(k), \xi(k)) \geq \frac{1}{\epsilon}\gamma^*(\mu, |v|_\infty)\bigg\}\end{aligned}$$

which gives rise to

$$\begin{aligned}&\mathbf{P}\bigg\{\alpha_1(|\xi(k)|) \geq \frac{1}{\epsilon}\gamma^*(\mu, |v|_\infty)\bigg\}\\ &\quad\leq \mathbf{P}\bigg\{V(r(k), \xi(k)) \geq \frac{1}{\epsilon}\gamma^*(\mu, |v|_\infty)\bigg\}.\end{aligned} \tag{23}$$

Taking (22) into account, we derive that

$$\mathbf{P}\bigg\{\alpha_1(|\xi(k)|) \geq \frac{1}{\epsilon}\gamma^*(\mu, |v|_\infty)\bigg\} \leq \epsilon$$

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which leads to

$$\mathbf{P}\left\{|\xi(k)| \le \alpha_1^{-1} \circ \frac{\gamma^*}{\epsilon}(\mu, |v|_\infty)\right\} \ge 1 - \epsilon. \tag{24}$$

By denoting

$$\gamma \triangleq \alpha_1^{-1} \circ \frac{\hat{\alpha}^{-1}}{\epsilon} \circ \eta \tag{25}$$

and letting $\mu \to 1^+$ on both sides of (24), we deduce that, for $k \in \mathbb{Z}^+$

$$\mathbf{P}\{|\xi(k)| \le \gamma(|v|_\infty)\} \ge 1 - \epsilon. \tag{26}$$

For any $\beta \in \mathcal{KL}$, it is worth noting that

$$\mathbf{P}\{|\xi(k)| \le \gamma(|v|_\infty)\} \le \mathbf{P}\{|\xi(k)| \le \beta(\xi_0, k) + \gamma(|v|_\infty)\}. \tag{27}$$

Thus, by following from (26) and (27), we arrive at (18) under case 1.

For case 2, it is immediately known from $(r_0, \ \xi_0) \notin \mathscr{B}$ that $\mathbb{E}V(r(0), \xi(0)) > \gamma^*(\mu, |v|_\infty)$. Now, if $\mathbb{E}V(r(k), \xi(k)) > \gamma^*(\mu, |v|_\infty)$ for all $k \in \mathbb{Z}^+$, then one obtains that

$$\eta(|v|_\infty) < \frac{1}{\mu}\hat{\alpha}(\mathbb{E}V(r(k), \xi(k))). \tag{28}$$

Substituting (28) into (16) gives

$$\begin{aligned} &\mathbb{E}V(r(k+1), \xi(k+1)) - \mathbb{E}V(r(k), \xi(k)) \\ &\quad \le \left(\frac{1}{\mu} - 1\right)\hat{\alpha}(\mathbb{E}V(r(k), \xi(k))). \end{aligned} \tag{29}$$

Noting that $\mu > 1$ and applying the standard comparison lemma in [14], we conclude that there exists a function $\hat{\beta} \in \mathcal{KL}$ such that

$$\mathbb{E}V(r(k), \xi(k)) \le \hat{\beta}(V(r(0), \xi(0)), k) \tag{30}$$

for $k \in \mathbb{Z}^+$.

By making use of Chebyshev's inequality again, we acquire

$$\begin{aligned} &\mathbf{P}\left\{V(r(k), \xi(k)) \ge \frac{1}{\epsilon}\hat{\beta}(V(r(0), \xi(0)), k)\right\} \\ &\quad \le \frac{\mathbb{E}V(r(k), \xi(k))\epsilon}{\hat{\beta}(V(r(0), \xi(0)), k)} \le \epsilon. \end{aligned} \tag{31}$$

Along the similar line to obtain (23), we have

$$\begin{aligned} &\mathbf{P}\left\{\alpha_1(|\xi(k)|) \ge \frac{1}{\epsilon}\hat{\beta}(V(r(0), \xi(0)), k)\right\} \\ &\quad \le \mathbf{P}\left\{V(r(k), \xi(k)) \ge \frac{1}{\epsilon}\hat{\beta}(V(r(0), \xi(0)), k)\right\}. \end{aligned} \tag{32}$$

Since $\hat{\beta}$ is a $\mathcal{KL}$-function, it is readily deduced from the condition 1 that

$$\begin{aligned} &\mathbf{P}\left\{\alpha_1(|\xi(k)|) \ge \frac{1}{\epsilon}\tilde{\beta}(\alpha_2(|\xi(0)|), k)\right\} \\ &\quad \le \mathbf{P}\left\{\alpha_1(|\xi(k)|) \ge \frac{1}{\epsilon}\hat{\beta}(V(r(0), \xi(0)), k)\right\}. \end{aligned} \tag{33}$$

From (31)–(33), it is readily observed that

$$\mathbf{P}\left\{|\xi(k)| \ge \alpha_1^{-1}\left(\frac{1}{\epsilon}\hat{\beta}(\alpha_2(|\xi(0)|), k)\right)\right\} \le \epsilon \tag{34}$$

for $k \in \mathbb{Z}^+$.

Let

$$\beta(s, k) \triangleq \alpha_1^{-1}\left(\frac{\hat{\beta}}{\epsilon}(\alpha_2(s), k)\right) \tag{35}$$

which is also a $\mathcal{KL}$-function due to [15, Lemma 4.2]. Then, it follows from (34) that:

$$\mathbf{P}\{|\xi(k)| \le \beta(|\xi(0)|, k)\} \ge 1 - \epsilon \tag{36}$$

which further results in

$$\mathbf{P}\{|\xi(k)| \le \beta(|\xi(0)|, k) + \gamma(|v|_\infty)\} \ge 1 - \epsilon \tag{37}$$

for $k \in \mathbb{Z}^+$, where $\gamma \in \mathcal{K}$ is defined in (25).

On the other hand, if there is a positive integer $k > 0$ such that $\mathbb{E}V(r(k), \xi(k)) \le \gamma^*(\mu, |v|_\infty)$, then by denoting

$$k^* \triangleq \min\{k : \mathbb{E}V(r(k), \xi(k)) \le \gamma^*(\mu, |v|_\infty)\}$$

one obtains

$$\mathbb{E}V(r(k), \xi(k)) > \gamma^*(\mu, |v|_\infty) \tag{38}$$

for $k \in [0, k^* - 1]$.

Considering (28), we have

$$\begin{aligned} &\mathbb{E}V(r(k+1), \xi(k+1)) - \mathbb{E}V(r(k), \xi(k)) \\ &\quad \le \left(\frac{1}{\mu} - 1\right)\hat{\alpha}(\mathbb{E}V(r(k), \xi(k))) \end{aligned} \tag{39}$$

which, by following the similar line to the proof of (36), leads to:

$$\mathbf{P}\{|\xi(k)| < \beta(|\xi(0)|, k)\} > 1 - \epsilon \tag{40}$$

for $k \in [0, k^* - 1]$.

When $k = k^*$, it is readily obtained from the definition of k^* that

$$\mathbb{E}V\big(r\big(k^*\big), \xi\big(k^*\big)\big) \le \gamma^*(\mu, |v|_\infty) \tag{41}$$

which is followed immediately by:

$$\mathbb{E}V(r(k), \xi(k)) \le \gamma^*(\mu, |v|_\infty) \tag{42}$$

for any $k \ge k^*$ by taking advantage of the analogy to (21).

Repeating the proof process from (22) to (26) yields that

$$\mathbf{P}\left\{\sup_{k \ge k^*} |\xi(k)| \le \gamma(|v|_\infty)\right\} > 1 - \epsilon \tag{43}$$

in which γ is defined in (25). Thus, combining (40) with (43) indicates that for $k \in \mathbb{Z}^+$

$$\mathbf{P}\{|\xi(k)| \le \beta(|\xi(0)|, k) + \gamma(|v|_\infty)\} \ge 1 - \epsilon. \tag{44}$$

Accordingly, (37) and (44) show that the assertion (18) is true under case 2. Therefore, the proof is now complete. ■

Remark 3: In Theorem 1, a Lyapunov-like framework is established, by which the dynamical behavior is investigated for a class of discrete-time Markovian switching stochastic systems with exogenous disturbances. By applying the switched Lyapunov function method and stochastic analysis techniques, the ISS property in probability for addressed system is guaranteed under several very general conditions described by $\mathcal{K}$-functions. The evolution of states is characterized in (18), in which the ISS gain $\gamma(|v|_\infty)$ depicts the final upper bound for sample trajectories while the positive scalar $1-\epsilon$ is used to quantify the possibility that sample trajectories are bounded by $\gamma(|v|_\infty)$ as $k \to \infty$.

Remark 4: It is readily seen that Theorem 1 is a generalization of [13, Lemma 3.5] in the case of stochastic Markovian switching systems. Moreover, it can also be regarded as a discrete-time counterpart of [34, Th. 1] and [16, Th. 2] [41, Th. 2]. In view of the random nature (from the noises and the SCP) and the nonvanishing disturbances (from the quantization errors) of the addressed stochastic system with Markovian switching, Theorem 1 focuses mainly on the ultimate upper-bound of state trajectories as well as the probability of state trajectories entering a bounded domain, which is different from and less conservative than those results in moment sense [10], [33].

It should be pointed out that the direct application of Theorem 1 is inconvenient due mainly to the difficulties/challenges in looking for appropriate functions to satisfy conditions 1)–3) for general nonlinear functions f and g. In what follows, by introducing several common assumptions on nonlinearities, we further exploit an easy-to-use criterion to address the ISS behavior in probability for the system (13).

Assumption 1: There are real matrices U_{1f} and U_{2f} such that the nonlinear function f satisfies

$$\big(f(x) - f(y) - U_{1f}(x-y)\big)^{\mathrm{T}} \times \big(f(x) - f(y) - U_{2f}(x-y)\big) \le 0$$

for any $x, y \in \mathbb{R}^{n_x}$. Here, U_{1f} and U_{2f} are known constant matrices with appropriate dimensions.

Assumption 2: There are real matrices U_g, V_g such that the nonlinear function g satisfies

$$g^{\mathrm{T}}(x,y)g(x,y) \le x^T U_g^{\mathrm{T}} U_g x + y^T V_g^{\mathrm{T}} V_g y$$

for any $x, y \in \mathbb{R}^{n_x}$.

Theorem 2: Let the observer gain matrices L_i and the feedback gain matrices K_i be given for $i \in \mathcal{S}$. Under Assumptions 1 and 2, the nonlinear stochastic Markovian switching system (13) is input-to-state stable in probability with respect to v if there exist positive definite matrices $P_i \in \mathbb{R}^{(2n_x+n_y)\times(2n_x+n_y)}$, $Q_i \in \mathbb{R}^{(n_x+2n_y)\times(n_x+2n_y)}$ and two positive constants σ_1 and σ_2 such that

$$\begin{pmatrix} \Theta_i & S_i^{\mathrm{T}}\bar{P}_i \\ \star & -\bar{P}_i \end{pmatrix} < 0 \tag{45}$$

for any $i \in \mathcal{S}$, where

$$\bar{P}_i = \sum_{j=1}^{n_y} \pi_{ij} P_j,\ S_i = \begin{pmatrix} \mathcal{A}_i & I & 0 & \mathcal{D}_i \end{pmatrix}$$

$$\Theta_i = \begin{pmatrix} \Theta_{11}^i & \sigma\tilde{\mathcal{U}}_f^{\mathrm{T}} & 0 & 0 \\ \star & -2\sigma_1 I & 0 & 0 \\ \star & \star & \bar{P}_i - \sigma_2 I & 0 \\ \star & \star & \star & -Q_i \end{pmatrix}$$

$$\Theta_{11}^i = -P_i - \sigma_1\bar{\mathcal{U}}_f + 2\sigma_2\mathcal{U}_g,\ \mathcal{U}_g = \mathrm{diag}\Big\{U_g^T U_g,\ 0,\ 0\Big\}$$
$$\tilde{\mathcal{U}}_f = \mathrm{diag}\big\{\tilde{U}_f,\ \tilde{U}_f,\ 0\big\},\ \bar{\mathcal{U}}_f = \mathrm{diag}\big\{\bar{U}_f,\ \bar{U}_f,\ 0\big\}$$
$$\tilde{U}_f = U_{1f} + U_{2f},\ \bar{U}_f = U_{1f}^{\mathrm{T}} U_{2f} + U_{2f}^{\mathrm{T}} U_{1f}.$$

Moreover, for any positive constant $\epsilon \in (0,\ 1)$, there exist a function $\check{\beta} \in \mathcal{K}$, two positive constants θ and γ_0 such that

$$\mathbf{P}\Bigg\{|\xi(k)| \le \check{\beta}(|\xi(0)|)e^{-\theta k} + \gamma_0\Bigg(|v_1|_\infty + |v_2|_\infty + \frac{\sqrt{n_y}\Delta_q}{2}\Bigg)\Bigg\} \ge 1-\epsilon \tag{46}$$

holds.

Proof: Consider a mode-dependent Lyapunov-like function as follows:

$$V(k) \triangleq V(r(k), \xi(k)) = \xi^{\mathrm{T}}(k) P_{r(k)} \xi(k). \tag{47}$$

For the brevity of notations, we denote $r(k) = i \in \mathcal{S}$ without loss of generality. By calculating the difference of $V(i, \xi(k))$ along the solution of switched closed-loop system (13), one obtains

$$\begin{aligned} \Delta V(k) &= \mathbb{E}\{V(r(k+1), \xi(k+1))|\mathscr{F}_k\} - V(i, \xi(k)) \\ &= \mathcal{F}_i^{\mathrm{T}}(\xi(k), v(k))\bar{P}_i\mathcal{F}_i(\xi(k), v(k)) \\ &\quad + \mathcal{G}_i^{\mathrm{T}}(\xi(k), v(k))\bar{P}_i\mathcal{G}_i(\xi(k), v(k)) - \xi^{\mathrm{T}}(k)P_i\xi(k). \end{aligned} \tag{48}$$

For any positive constant σ_1, it is readily derived from Assumption 1 that

$$\begin{aligned} &2\sigma_1 f^{\mathrm{T}}(x(k))f(x(k)) + \sigma x^{\mathrm{T}}(k)\bar{U}_f x(k) \\ &-\sigma_1(f^{\mathrm{T}}(x(k))\tilde{U}_f x(k) + x^{\mathrm{T}}(k)\tilde{U}_f^T f(x(k))) \le 0. \end{aligned} \tag{49}$$

It follows from the similar line to the proof of (49) that:

$$\begin{aligned} &2\sigma_1(f(x(k)) - f(\hat{x}(k)))^{\mathrm{T}}(f(x(k)) - f(\hat{x}(k))) \\ &- \sigma_1\Big(\big(f(x(k)) - f(\hat{x}(k))\big)^{\mathrm{T}}\tilde{U}_f e(k) \\ &\qquad + e^{\mathrm{T}}(k)\tilde{U}_f^{\mathrm{T}}\big(f(x(k)) - f(\hat{x}(k))\big)\Big) \\ &+ \sigma_1 e^{\mathrm{T}}(k)\bar{U}_f e(k) \le 0. \end{aligned} \tag{50}$$

Combining (49) with (50) results in

$$\begin{aligned} \tilde{\mathcal{F}}(k) &\triangleq 2\sigma_1 F^{\mathrm{T}}(\xi(k))F(\xi(k)) + \sigma_1\xi^{\mathrm{T}}(k)\bar{\mathcal{U}}_f\xi(k) \\ &\quad - \sigma_1\Big(F^{\mathrm{T}}(\xi(k))\tilde{\mathcal{U}}_f\xi(k)\Big) - \sigma_1\Big(\xi^{\mathrm{T}}(k)\tilde{\mathcal{U}}_f^{\mathrm{T}}F(\xi(k))\Big) \\ &\le 0. \end{aligned} \tag{51}$$

For any positive constant σ_2, it follows from Assumption 2 that:

$$\begin{aligned} &\sigma_2\mathcal{G}_i^{\mathrm{T}}(\xi(k), v(k))\mathcal{G}_i(\xi(k), v(k)) \\ &\quad \le 2\sigma_2\big(\xi^{\mathrm{T}}(k)\mathcal{U}_g\xi(k) + v^{\mathrm{T}}(k)\mathcal{V}_g v(k)\big) \end{aligned} \tag{52}$$

where $\mathcal{V}_g = \mathrm{diag}\{V_g^T V_g,\ 0,\ 0\}$.

By taking into account (48), (51), and (52), one deduces that

$$\begin{aligned} \Delta V(k) &\le (\mathcal{A}_i\xi(k) + F(\xi(k)) + \mathcal{D}_i v(k))^{\mathrm{T}}\bar{P}_i \\ &\quad \times (\mathcal{A}_i\xi(k) + F(\xi(k)) + \mathcal{D}_i v(k)) \\ &\quad + \mathcal{G}_i^{\mathrm{T}}(\xi(k), v(k))\bar{P}_i\mathcal{G}_i(\xi(k), v(k)) \\ &\quad - \xi^{\mathrm{T}}(k)P_i\xi(k) - v^{\mathrm{T}}(k)Q_i v(k) + v^{\mathrm{T}}(k)Q_i v(k) \\ &\quad - \tilde{\mathcal{F}}(k) + 2\sigma_2\big(\xi^{\mathrm{T}}(k)\mathcal{U}_g\xi(k) + v^{\mathrm{T}}(k)\mathcal{V}_g v(k)\big) \\ &\quad - \sigma_2\mathcal{G}_i^{\mathrm{T}}(\xi(k), v(k))\mathcal{G}_i(\xi(k), v(k)) \\ &\triangleq \eta^{\mathrm{T}}(k)\Pi_i\eta(k) + v^{\mathrm{T}}(k)\tilde{Q}_i v(k) \end{aligned} \tag{53}$$

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in which $\Pi_i \triangleq S_i^{\mathrm{T}} \bar{P}_i S_i + \Theta_i$, $\tilde{Q}_i \triangleq Q_i + 2\sigma_2 \mathcal{V}_g$ and

$$\eta(k) = \left(\xi^{\mathrm{T}}(k) \quad F^{\mathrm{T}}(\xi(k)) \quad \mathcal{G}_i^{\mathrm{T}}(\xi(k), v(k)) \quad v^{\mathrm{T}}(k)\right)^{\mathrm{T}}.$$

Clearly, we have $\tilde{Q}_i > 0$. By following the condition (45) and the well-known Schur complement lemma, we obtain $\Pi_i < 0$ which, together with (53), gives rise to:

$$\begin{aligned}\Delta V(k) &\le \lambda_{\max}(\Pi_i)\eta^{\mathrm{T}}(k)\eta(k) + \lambda_{\max}\left(\tilde{Q}_i\right)v^{\mathrm{T}}(k)v(k) \\ &\le \lambda_{\max}(\Pi_i)\xi^{\mathrm{T}}(k)\xi(k) + \lambda_{\max}\left(\tilde{Q}_i\right)v^{\mathrm{T}}(k)v(k). \quad (54)\end{aligned}$$

Let

$$\alpha_1(s) \triangleq \min_{i\in\mathcal{S}}\{\lambda_{\min}(P_i)\}s^2 \quad (55)$$

$$\alpha_2(s) \triangleq \max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}s^2. \quad (56)$$

It is readily seen that $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfy the condition 1 of Theorem 1.

For any positive constant ρ satisfying

$$0 < \rho < \min\left\{\min_{i\in\mathcal{S}}\{-\lambda_{\max}(\Pi_i)\}, \max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}\right\} \quad (57)$$

we choose

$$\alpha(s) \triangleq \rho s^2, \eta(s) \triangleq \max_{i\in\mathcal{S}}\left\{\lambda_{\max}\left(\tilde{Q}_i\right)\right\}s^2 \quad (58)$$

such that $\alpha, \eta \in \mathcal{K}_\infty$ and

$$\Delta V(k) \le -\alpha(|\xi(k)|) + \eta(|v(k)|) \quad (59)$$

which means that the condition 2) holds. Furthermore, by following from (56) and (58), one obtains that:

$$\alpha \circ \alpha_2^{-1}(s) = \frac{\rho}{\max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}}s \quad (60)$$

is a $\mathcal{VK}_\infty$-function, which leads to the condition 3). Consequently, according to Theorem 1, the discrete-time nonlinear stochastic Markovian switching system (13) is input-to-state stable in probability with respect to v.

Next, for any given $\epsilon \in (0,\ 1)$, we focus on finding an appropriate function $\check{\beta} \in \mathcal{K}$, two positive constants θ and γ_0 to satisfy the estimation (46), so as to eventually complete the proof. To this end, it is readily deduced from (60) that

$$\left(\mathrm{Id} - \alpha \circ \alpha_2^{-1}\right)(s) = \left(1 - \frac{\rho}{\max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}}\right)s \quad (61)$$

which is a $\mathcal{K}$-function.

By choosing $\hat{\alpha} = \alpha \circ \alpha_2^{-1}$, it follows from (29) that:

$$\begin{aligned}&\mathbb{E}V(r(k+1), \xi(k+1)) - \mathbb{E}V(r(k), \xi(k)) \\ &\le \left(\frac{1}{\mu} - 1\right)\hat{\alpha}(\mathbb{E}V(r(k), \xi(k))) \\ &= \left(\frac{1}{\mu} - 1\right)\frac{\rho\mathbb{E}V(r(k), \xi(k))}{\max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}}\end{aligned}$$

which further indicates that

$$\mathbb{E}V(r(k+1), \xi(k+1)) \le \varsigma\mathbb{E}V(r(k)\xi(k))) \quad (62)$$

for $k \in \mathbb{Z}^+$, where $\mu > 1$ and

$$\varsigma \triangleq \frac{\mu(\max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\} - \rho) + \rho}{\mu \max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}} \in (0,\ 1).$$

Applying the mathematical induction to (62) yields that

$$\begin{aligned}\mathbb{E}V(r(k), \xi(k)) &\le \varsigma^k V(r(0), \xi(0)) \\ &\triangleq \hat{\beta}(V(r(0), \xi(0)), k). \quad (63)\end{aligned}$$

Bearing in mind that (25) and (35), it is readily seen that

$$\beta(|\xi_0|, k) = \sqrt{\frac{\max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}}{\min_{i\in\mathcal{S}}\{\lambda_{\min}(P_i)\}\epsilon}}\sqrt{\varsigma}^k|\xi_0| \triangleq \check{\beta}(|\xi_0|)e^{-\theta k}$$

$$\begin{aligned}\gamma(|v|_\infty) &= \sqrt{\frac{\max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}\max_{i\in\mathcal{S}}\left\{\lambda_{\max}\left(\tilde{Q}_i\right)\right\}}{\min_{i\in\mathcal{S}}\{\lambda_{\min}(P_i)\}\rho\epsilon}}|v|_\infty \\ &\triangleq \gamma_0|v|_\infty\end{aligned}$$

in which

$$\check{\beta}(|\xi_0|) \triangleq \sqrt{\frac{\max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}}{\min_{i\in\mathcal{S}}\{\lambda_{\min}(P_i)\}\epsilon}}|\xi_0| \quad (64)$$

$$\theta \triangleq \ln\sqrt{\frac{\mu \max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}}{\mu(\max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\} - \rho) + \rho}} \quad (65)$$

$$\gamma_0 \triangleq \sqrt{\frac{\max_{i\in\mathcal{S}}\{\lambda_{\max}(P_i)\}\max_{i\in\mathcal{S}}\left\{\lambda_{\max}\left(\tilde{Q}_i\right)\right\}}{\min_{i\in\mathcal{S}}\{\lambda_{\min}(P_i)\}\rho\epsilon}}. \quad (66)$$

By following from Definition 1 and (18), we conclude that the estimation (46) is true and the Markovian switching stochastic system (13) is exponentially input-to-state stable in probability. The proof is complete. ■

Remark 5: Strictly speaking, Theorem 2 is more conservative than Theorem 1 due to Assumptions 1 and 2 on nonlinearities and the application of a quadratic switched Lyapunov function. Nevertheless, the sufficient criterion proposed in Theorem 2 is presented in the form of LMIs, which is easy to check by using the MATLAB toolbox. Moreover, it follows from (46) that Theorem 2 addresses the exponential ISS behavior in probability for the closed-loop system (13), which has not been discussed in [18], [20], [23], and [31]. The exponential decay rate θ determined in (65) quantifies the speed of the convergence of sample trajectories, while the linear ISS gain $\gamma_0|v|_\infty$ given in (66) will increase along with the increasing of $|v|_\infty$, which demonstrates the effects (from both bounded exogenous disturbances and quantization errors) on the state of the controlled system.

Having coped with the dynamics analysis of the closed-loop system in terms of the feasibility of LMIs in Theorem 2, we are now in the position to design the gain matrices K_i and L_i ($i \in \mathcal{S}$) for the observer-based controller (10) so as to stabilize the plant (1) in the sense of ISS in probability. For this purpose, we present the following assumption without any loss of generality.

Assumption 3: The control input matrix B is of full-column rank, i.e., $\mathrm{rank}(B) = n_u$, and has the singular value decomposition of the form (14).

Theorem 3: Under Assumptions 1–3, the plant (1) is input-to-state stabilizable in probability by the observer-based controller (10) if there exist a set of positive definite matrices $\hat{P}_1^{(i)} \in \mathbb{R}^{n_u\times n_u}$, $\hat{P}_2^{(i)} \in \mathbb{R}^{(n_x-n_u)\times(n_x-n_u)}$, $P_2^{(i)} \in \mathbb{R}^{n_x\times n_x}$, $P_3^{(i)} \in \mathbb{R}^{n_y\times n_y}$, $Q_1^{(i)} \in \mathbb{R}^{n_x\times n_x}$, $Q_2^{(i)} \in \mathbb{R}^{n_y\times n_y}$, and $Q_3^{(i)} \in$

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$\mathbb{R}^{n_y\times n_y}$, a set of constant matrices $Y^{(i)} \in \mathbb{R}^{n_u\times n_x}$ and $Z^{(i)} \in \mathbb{R}^{n_x\times n_y}$, and two positive constants σ_1 and σ_2 such that for any $i \in \mathcal{S}$

$$\Xi^{(i)} \triangleq \begin{pmatrix} \Xi_{11}^{(i)} & \Xi_{12}^{(i)} \\ \star & \Xi_{22}^{(i)} \end{pmatrix} < 0 \tag{67}$$

where

$$\Xi_{11}^{(i)} = \begin{pmatrix} \hat{\Theta}^{(i)} & \sigma_1\tilde{\mathcal{U}}_f^{\mathrm{T}} & 0 & 0 \\ \star & -2\sigma_1 I & 0 & 0 \\ \star & \star & \bar{P}^{(i)} - \sigma_2 I & 0 \\ \star & \star & \star & -Q^{(i)} \end{pmatrix}$$

$$\Xi_{22}^{(i)} = \mathrm{diag}\left\{P^{(1)} - 2P^{(i)}, \ldots, P^{(n_y)} - 2P^{(i)}\right\}$$

$$\Xi_{12}^{(i)} = \left(\sqrt{\pi_{i1}}\left(\hat{S}^{(i)}\right)^{\mathrm{T}} \cdots \sqrt{\pi_{in_y}}\left(\hat{S}^{(i)}\right)^{\mathrm{T}}\right)$$

$$\hat{\Theta}^{(i)} = -P^{(i)} - \sigma_1\bar{\mathcal{U}}_f + 2\sigma_2\mathcal{U}_g^{\mathrm{T}}$$

$$P^{(i)} = \mathrm{diag}\left\{P_1^{(i)},\ P_2^{(i)},\ P_3^{(i)}\right\}$$

$$Q^{(i)} = \mathrm{diag}\left\{Q_1^{(i)},\ Q_2^{(i)},\ Q_3^{(i)}\right\}$$

$$P_1^{(i)} = U^{\mathrm{T}}\begin{pmatrix} \hat{P_1}^{(i)} & 0 \\ 0 & \hat{P_2}^{(i)} \end{pmatrix}U$$

$$\bar{P}^{(i)} = \sum_{j=1}^{n_y}\pi_{ij}P^{(j)}$$

$$\hat{S}^{(i)} = \left(\hat{S}_1^{(i)} \quad \hat{S}_2^{(i)} \quad 0 \quad \hat{S}_4^{(i)}\right)$$

$$\hat{S}_1^{(i)} = \begin{pmatrix} \hat{S}_{1,11}^{(i)} & -BY^{(i)} & 0 \\ \hat{S}_{1,21}^{(i)} & \hat{S}_{1,22}^{(i)} & -Z^{(i)}(I-\Phi_i) \\ P_3^{(i)}\Phi_i C & 0 & \hat{S}_{1,33}^{(i)} \end{pmatrix}$$

$$\hat{S}_{1,11}^{(i)} = P_1^{(i)}A + BY^{(i)}, \hat{S}_{1,21}^{(i)} = -Z^{(i)}\Phi_i C + Z^{(i)}C$$

$$\hat{S}_{1,22}^{(i)} = P_2^{(i)}A - Z^{(i)}C, \hat{S}_{1,33}^{(i)} = P_3^{(i)}(I-\Phi_i)$$

$$\hat{S}_2^{(i)} = \mathrm{diag}\left\{P_1^{(i)},\ P_2^{(i)},\ P_3^{(i)}\right\}$$

$$\hat{S}_4^{(i)} = \begin{pmatrix} P_1^{(i)}D & 0 & 0 \\ P_2^{(i)}D & -Z^{(i)}\Phi_i E & -Z^{(i)}\Phi_i \\ 0 & P_3^{(i)}\Phi_i E & P_3^{(i)}\Phi_i \end{pmatrix}.$$

$\tilde{\mathcal{U}}_f$, $\bar{\mathcal{U}}_f$, and $\mathcal{U}_g$ are the same definition in Theorem 2, and U is defined by the singular value decomposition of B in (14). Furthermore, the control gain matrices K_i and the observer gain matrices L_i are designed as

$$K_i = \left(B^T P_1^{(i)} B\right)^{-1} B^T BY^{(i)} \tag{68}$$

$$L_i = \left(P_2^{(i)}\right)^{-1} Z^{(i)} \tag{69}$$

for $i \in \mathcal{S}$.

Proof: It suffices to prove that (45) holds under the conditions in this Theorem. Following the singular value decomposition of B in (14), it is readily obtained that:

$$B = U^{\mathrm{T}}\begin{pmatrix} \Sigma \\ 0 \end{pmatrix}V^{\mathrm{T}}.$$

Thus, one has

$$B^T B = V(\Sigma, 0)UU^{\mathrm{T}}\begin{pmatrix} \Sigma \\ 0 \end{pmatrix}V^{\mathrm{T}} = V\Sigma^2 V^{\mathrm{T}}$$

$$B^T P_1^{(i)} B = V(\Sigma, 0)UU^{\mathrm{T}}\begin{pmatrix} \hat{P}_1^{(i)} & 0 \\ 0 & \hat{P}_2^{(i)} \end{pmatrix}U \times U^{\mathrm{T}}\begin{pmatrix} \Sigma \\ 0 \end{pmatrix}V^{\mathrm{T}} = V\Sigma P_1^{(i)}\Sigma V^{\mathrm{T}}$$

which are nonsingular matrices.

Considering (68), we obtain $Y^{(i)} = V\Sigma^{-1}P_1^{(i)}\Sigma V^T K_i$, that is,

$$BY^{(i)} = BV\Sigma^{-1}P_1^{(i)}\Sigma V^T K_i = P_1^{(i)}BK_i. \tag{70}$$

Note that (69) implies $P_2^{(i)}L_i = Z^{(i)}$ which, together with (70), leads to

$$\hat{S}^{(i)} = \left(P^{(i)}\mathcal{A}_i \quad P^{(i)} \quad 0 \quad P^{(i)}\mathcal{D}_i\right) = P^{(i)}S_i.$$

Thus, we have

$$\Xi_{12}^{(i)} = \left(\sqrt{\pi_{i1}}S_i^T P^{(i)} \cdots \sqrt{\pi_{in_y}}S_i^T P^{(i)}\right) \tag{71}$$

where S_i is defined in Theorem 2.

Bearing in mind that for any $i, j \in \mathcal{S}$

$$\left(P^{(i)} - P^{(j)}\right)\left(P^{(j)}\right)^{-1}\left(P^{(i)} - P^{(j)}\right) \geq 0$$

that is,

$$-P^{(i)}\left(P^{(j)}\right)^{-1}P^{(i)} \leq P^{(j)} - 2P^{(i)} \tag{72}$$

we deduce that

$$\tilde{\Xi}_{22}^{(i)} \leq \Xi_{22}^{(i)} \tag{73}$$

where

$$\tilde{\Xi}_{22}^{(i)} \triangleq \mathrm{diag}\left\{-P^{(i)}\left(P^{(1)}\right)^{-1}P^{(i)}, \ldots, -P^{(i)}\left(P^{(n_y)}\right)^{-1}P^{(i)}\right\}.$$

Substituting (71) and (73) into (67) gives

$$\begin{pmatrix} \Xi_{11}^{(i)} & \Xi_{12}^{(i)} \\ \star & \tilde{\Xi}_{22}^{(i)} \end{pmatrix} \leq \Xi^{(i)} < 0. \tag{74}$$

Pre- and post-multiplying (74) by

$$\mathrm{diag}\left\{I, \underbrace{\left(P^{(i)}\right)^{-1}, \ldots, \left(P^{(i)}\right)^{-1}}_{n_y}\right\}$$

we obtain that

$$\begin{pmatrix} \Xi_{11}^{(i)} & \sqrt{\pi_{i1}}S_i^{\mathrm{T}} & \cdots & \sqrt{\pi_{in_y}}S_i^{\mathrm{T}} \\ \star & -\left(P^{(1)}\right)^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & 0 & \cdots & -\left(P^{(n_y)}\right)^{-1} \end{pmatrix} < 0.$$

It follows from the Schur complement lemma that:

$$\Xi_{11}^{(i)} + S_i^{\mathrm{T}}\left(\sum_{j=1}^{n_y}\pi_{ij}P^{(j)}\right)S_i = \Xi_{11}^{(i)} + S_i^{\mathrm{T}}\bar{P}^{(i)}S_i < 0$$

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which can be rewritten as

$$\begin{pmatrix} \Xi_{11}^{(i)} & S_i^{\mathrm{T}} \bar{P}^{(i)} \\ \star & -\bar{P}^{(i)} \end{pmatrix} < 0.$$

Therefore, the proof is complete. ■

Remark 6: Within the theoretical framework proposed in Theorem 2, an easy-to-operate method is developed for facilitating the design of the mode-dependent observer-based controller with hope to render the plant (1) input-to-state stable in probability. The gain matrices of the desired controller are dependent on the Markovian switching signal which results from the SCP in the communication network. It should be mentioned that, when we deal with the issue of the controller design, all gain matrices K_i, L_i $(i \in \mathcal{S})$ are variables to be determined. Hence, (67) is no longer an LMI that can be solved directly by utilizing the MATLAB LMI toolbox. In order to overcome this difficulty, we first take advantage of Lemma 2 to convert the condition (67) into an LMI form with matrix equation constraint, and then form a solvable convex problem for the MATLAB LMI toolbox.

IV. Numerical Example

In this section, a simulation example is presented to illustrate the usefulness and flexibility of the theoretical result developed in this paper.

The parameters of plant (1) are given as follows:

$$A = \begin{pmatrix} 0.85 & -0.15 & 0.3 \\ 0.15 & 0.35 & 0.1 \\ -0.25 & -0.15 & 0.45 \end{pmatrix}, B = \begin{pmatrix} 2.5 & 2 \\ 2 & 3.5 \\ 1.5 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} 0.5 & 0.8 & 0.9 \\ 0.6 & -0.7 & 0.5 \end{pmatrix}, E = \begin{pmatrix} 0.5 & 0.1 \\ 0.3 & 0.1 \end{pmatrix}$$
$$D = \begin{pmatrix} 0.05 & 0.1 & 0.1 \\ -0.02 & 0.05 & 0.15 \\ 0.1 & 0.15 & 0.1 \end{pmatrix}.$$

The nonlinear functions are assumed to be

$$f(x) = (f_1(x) \;\; f_2(x) \;\; f_3(x))^{\mathrm{T}}$$

with

$$\begin{aligned} f_1(x) &= -0.5x_1 + 0.15x_2 + \tanh(0.15x_1 - 0.1x_3) \\ &\quad + \tanh(0.2x_2 + 0.1x_3) \\ f_2(x) &= 0.85x_2 - 0.05x_3 + \tanh(0.05x_1 - 0.15x_2) \\ &\quad - \tanh(0.05x_1 + 0.1x_3) \\ f_3(x) &= -0.1x_1 + 0.15x_3 - \tanh(0.05x_1 - 0.1x_2) \\ &\quad - \tanh(0.05x_2 + 0.05x_3) \end{aligned}$$

and

$$g(x, y) = \begin{pmatrix} 0.15|x_1| + y_1 \\ 0.1|x_2| + y_2 \\ 0.08|x_3| + y_3 \end{pmatrix}.$$

It can be easily calculated that there are real matrices

$$U_{1f} = \begin{pmatrix} -0.5 & 0.15 & -0.1 \\ -0.05 & 0.7 & -0.15 \\ -0.15 & -0.05 & 0.1 \end{pmatrix}$$

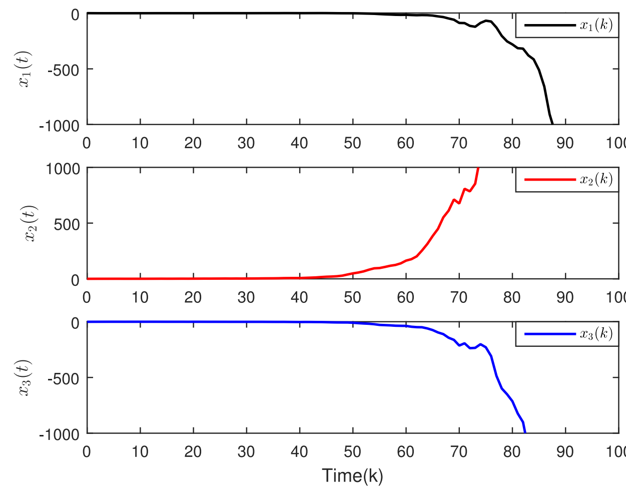

Fig. 2. State response of the open-loop system without disturbances.

$$\begin{aligned} U_{2f} &= \begin{pmatrix} -0.35 & 0.35 & 0.1 \\ 0.05 & 0.85 & -0.05 \\ -0.1 & 0.1 & 0.15 \end{pmatrix} \\ U_g &= \sqrt{2}\mathrm{diag}\{0.15, \;\; 0.1, \;\; 0.08\} \\ V_g &= \sqrt{2}\mathrm{diag}\{1, \;\; 1, \;\; 1\} \end{aligned}$$

such that Assumptions 1 and 2 hold.

It is readily seen that the plant (1) with above parameters is *unstable* even though the exogenous disturbance $v_1(k) = 0$ for all $k \in \mathbb{Z}^+$. The dynamics for the plant without the control input and the exogenous disturbance is simulated in Fig. 2.

Set the transition probability matrix

$$\Pi = \begin{pmatrix} 0.55 & 0.45 \\ 0.4 & 0.6 \end{pmatrix}$$

the bounded exogenous disturbances

$$\begin{aligned} v_1(k) &= 0.05(\cos k \;\; \sin k \;\; \cos k)^{\mathrm{T}} \\ v_2(k) &= 0.05(\cos k \;\; \sin k)^{\mathrm{T}} \end{aligned}$$

and the uniform quantizer with parameters $\Delta_q = 0.02$, $M_q = 50$, as well as $\epsilon = 0.05$. By solving the LMIs (67), we derive a set of feasible solutions for the gain matrices of the mode-dependent observer-based controller as follows:

$$\begin{aligned} K_1 &= \begin{pmatrix} 0.1140 & 0.2382 & 0.0573 \\ -0.1293 & -0.2684 & -0.0798 \end{pmatrix} \\ K_2 &= \begin{pmatrix} -0.0551 & 0.4222 & -0.1447 \\ 0.0502 & -0.3910 & 0.1084 \end{pmatrix} \\ L_1 &= \begin{pmatrix} 0.2233 & 0.0022 \\ 1.2601 & -0.0098 \\ -0.2429 & 0.0011 \end{pmatrix} \\ L_2 &= \begin{pmatrix} 0.0002 & 0.0325 \\ 0.0008 & -1.2044 \\ -0.0006 & 0.3111 \end{pmatrix} \end{aligned}$$

such that the closed-loop system gains the desired control performance of ISS in probability.

For the sake of simulation, we let the time interval be $[0, \;\; 100]$ and the initial values be $r(0) = 1$, $x(0) = (0.8 \;\; 0.2 \;\; -0.9)^{\mathrm{T}}$, $\hat{x}(0) = (0 \;\; 0 \;\; 0)^{\mathrm{T}}$, and $\bar{y}(-1) = (0 \;\; 0)^{\mathrm{T}}$. Based on the MATLAB software, the simulation results are

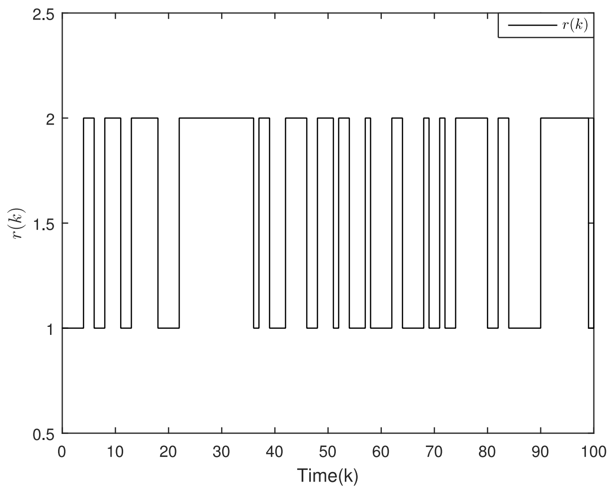

Fig. 3. Switching of sensor nodes under the scheduling of SCP.

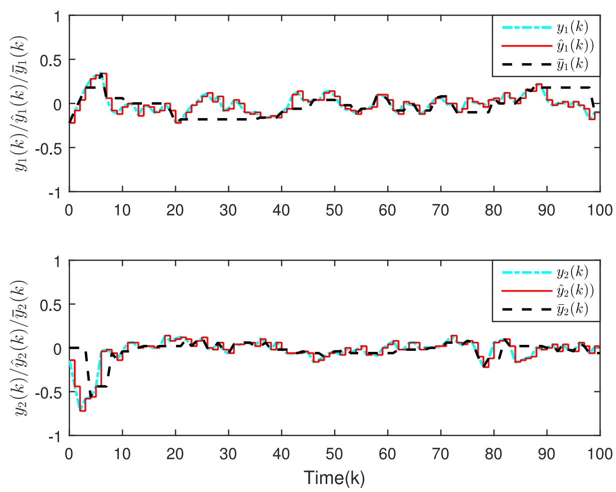

Fig. 4. Measurement outputs from sensors, quantizers, and the communication network.

shown in Figs. 3–7. In specific, under the scheduling of the SCP, the Markovian switching signal $r(k)$, which denotes the label of active sensor node getting access to the network, is depicted in Fig. 3. The measurement output $y(k)$ of the sensors, the quantized measurement output $\hat{y}(k)$ as well as the final measurement output $\bar{y}(k)$ transmitted over the network are illustrated in Fig. 4, from which we see that there are nonzero errors among these output signals. Fig. 5 illustrates the control input that is available to the plant. The state responses for the controlled plant and the observer are presented in Fig. 6, which demonstrates that the state trajectories enter a bounded domain eventually. Fig. 7 shows the tracking error between the plant and the observer, which oscillates near origin point rather than approaches zero due to the presence of bounded exogenous disturbances and the nonvanishing quantization error.

Remark 7: It should be pointed out that the results derived in [10], [16], [24], [25], and [41] cannot be applied to investigate the ISS behavior in probability because of the constraints induced by the communication network. The control strategies developed in [5], [6], [12], [19], and [42] are invalid to deal with the performance analysis and the controller design for this example due to the simultaneous presence of both uniform quantization and SCP. Furthermore, the methods proposed in [18], [20], [23], and [31] are also inapplicable to this example because of the stochastic noise and the nonvanishing disturbances (from both exogenous disturbances and quantization error).

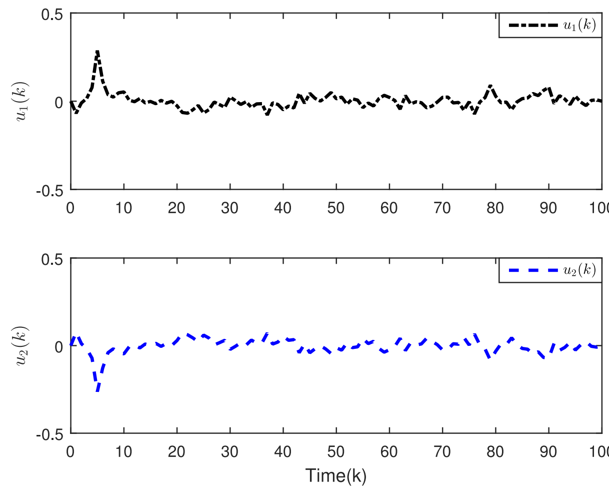

Fig. 5. Control input generated by the observer-based controller.

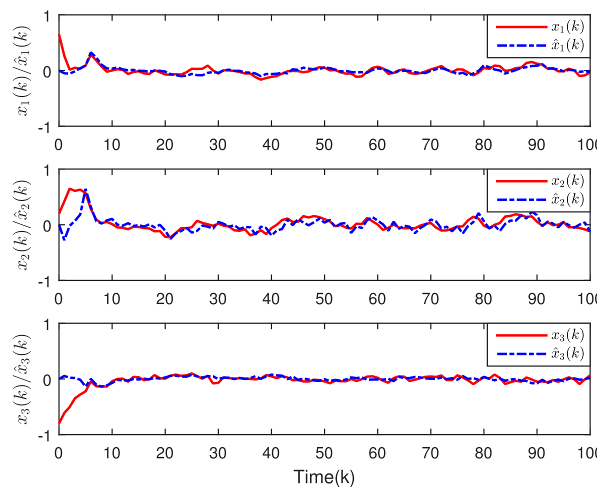

Fig. 6. State response of the closed-loop system and the observer.

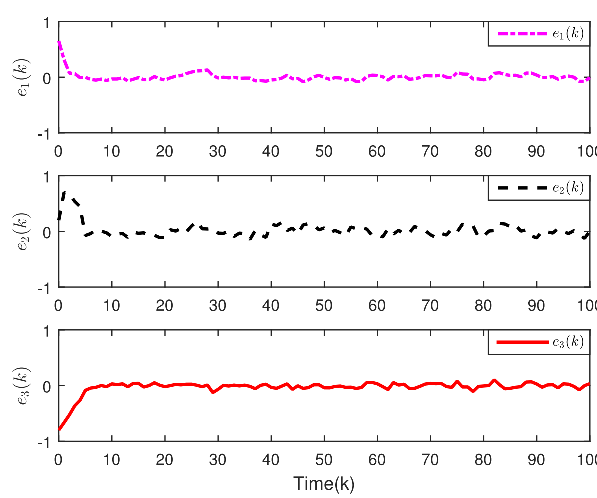

Fig. 7. Tracking error between the closed-loop system and the observer.

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V. Conclusion

In this paper, the problem of input-to-state stabilization in probability has been investigated for a class of discrete-time nonlinear stochastic NCSs with bounded exogenous disturbances. An observer-based state feedback controller has been developed to guarantee the desired dynamical performance for the controlled system. The control loop has been closed via a shared communication network, in which the measurement output of sensors has been quantized through the finite-level uniform quantizer, after which it has been transmitted to the observer under the scheduling of the SCP. The argument closed-loop system has been described as a general nonlinear Markovian switching stochastic difference equation, for which a Lyapunov-like framework has been established to address the ISS property in probability. The control gain matrices, which are dependent on the SCP, have been designed subject to the feasibility of a set of LMIs. A numerical example has been used to illustrate the effectiveness of our results. For the future research, the Round-Robin and the TOD protocols may be investigated. In addition, it is of significant importance to consider the case with both communication protocols and the dynamical quantization.

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