

New Convolutional Codes Derived from Algebraic Geometry Codes

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Abstract

In this paper, we construct new families of convolutional codes. Such codes are obtained by means of algebraic geometry codes. Additionally, more families of convolutional codes are constructed by means of puncturing, extending, expanding and by the direct product code construction applied to algebraic geometry codes. The parameters of the new convolutional codes are better than or comparable to the ones available in literature. In particular, a family of almost near MDS codes is presented.

Index terms— convolutional codes, algebraic geometry codes, code construction

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1 Introduction

The class of convolutional codes is a class of codes much investigated in the literature [4, 14, 18, 26–28]. Constructions of convolutional codes with good parameters or even maximum distance separable (MDS), i.e. optimal, convolutional codes (in the sense that they attain the generalized Singleton bound [27]) have also been presented in the literature [4, 11–14, 18, 26–28]. Rosenthal *et al.* introduced the generalized Singleton bound [27] (see also [28]) in 1999.

In this paper, we construct several new families of unit-memory convolutional codes derived from classical algebraic geometry (AG) codes. To do this, we apply the method introduced by Piret [26] which was generalized by Aly *et al.* [1]. Additionally, we utilize the techniques of code expansion, puncturing, extension and the product code construction in order to obtain more families of convolutional codes. An advantage of our constructions lies in the fact that the new convolutional codes are generated algebraically and not by computational search. Moreover, since there exist classical AG codes with good parameters, our new convolutional codes also have good parameters. The class of AG codes was introduced by Goppa [19] in 1981. These codes have nice properties and are asymptotically good. There exist several works dealing with investigations concerning algebraic geometry (AG) codes [6–8, 15, 23]. However, only few papers [3, 22, 25] address the construction of convolutional codes by applying AG codes as their classical counterpart.

A natural question that can arise is as follows: why it is important to obtain convolutional codes which are not MDS, since there exist MDS codes? The answer is simple: MDS codes is known to exist for specific code lengths constructed over specific alphabets. For example, in Refs. [11–13], one has convolutional MDS codes of length $n = q + 1$ or $n = \frac{(q+1)}{2}$ (in the last case, $q \equiv 3 \pmod{4}$) over \mathbb{F}_q . In Ref. [16], most of the codes are constructed over large alphabets when compared to its code length. Other example is Ref. [2], where convolutional MDS codes over \mathbb{F}_q with code length $n|(q^2 - 1)$ and $q + 1 < n \leq q^2 - 1$ were constructed.

The paper is organized as follows. In Section 2, we review basic concepts on convolutional codes. In Section 3, a review of concepts concerning algebraic geometry codes is given. In Section 4, we propose constructions of new families of convolutional codes. In particular, a family of almost near MDS (or near MDS or MDS) convolutional codes is shown. In Section 5,

we compare the new code parameters with the ones shown in the literature. Finally, in Section 6, the final remarks are drawn.

2 Review of Convolutional Codes

In this section we present a brief review of classical convolutional codes. For more details we refer the reader to [1, 2, 4, 5, 9, 12, 18, 26, 28].

Notation. Throughout this paper, p denotes a prime number, q is a prime power, \mathbb{F}_q is the finite field with q elements and F/\mathbb{F}_q is an algebraic functions field over \mathbb{F}_q of genus g .

We begin with a few usual definitions used in the theory of convolutional codes. A polynomial encoder matrix $G(D) \in \mathbb{F}_q[D]^{k \times n}$ is called *basic* if exists a polynomial right inverse for $G(D)$. A minimal-basic generator matrix is a encoder matrix which the overall constraint length $\gamma = \sum_{i=1}^k \gamma_i$ has the smallest value among all basic generator matrices (in this case, the overall constraint length γ is called the *degree* of the corresponding code).

Definition 1. [2] A rate k/n convolutional code C with parameters $(n, k, \gamma; m, d_f)_q$ is a submodule of $\mathbb{F}_q[D]^n$ generated by a reduced basic matrix $G(D) = (g_{ij}) \in \mathbb{F}_q[D]^{k \times n}$, that is, $C = \{\mathbf{u}(D)G(D) \mid \mathbf{u}(D) \in \mathbb{F}_q[D]^k\}$, where n is the length, k is the dimension, $\gamma = \sum_{i=1}^k \gamma_i$ is the degree, where $\gamma_i = \max_{1 \leq j \leq n} \{\deg g_{ij}\}$, $m = \max_{1 \leq i \leq k} \{\gamma_i\}$ is the memory and $d_f = wt(C) = \min\{wt(\mathbf{v}(D)) \mid \mathbf{v}(D) \in C, \mathbf{v}(D) \neq 0\}$ is the free distance of the code.

A generator matrix $G(D)$ is called *catastrophic* if there exists a information sequence $\mathbf{u}(D)^k \in \mathbb{F}_q((D))^k$ of infinite Hamming weight such that results in a codeword $\mathbf{v}(D)^k = \mathbf{u}(D)^k G(D)$ with finite Hamming weight. Since a basic generator matrix is non-catastrophic, the convolutional codes constructed in this paper have non catastrophic generator matrices.

The Euclidean inner product of two vectors $\mathbf{u}(D) = \sum_i \mathbf{u}_i D^i$ and $\mathbf{v}(D) = \sum_j \mathbf{v}_j D^j$ in $\mathbb{F}_q[D]^n$ is defined as $\langle \mathbf{u}(D) \mid \mathbf{v}(D) \rangle = \sum_i \mathbf{u}_i \cdot \mathbf{v}_i$. For a convolutional code C , the Euclidean dual of C is defined by $C^\perp = \{\mathbf{u}(D) \in \mathbb{F}_q[D]^n \mid \langle \mathbf{u}(D) \mid \mathbf{v}(D) \rangle = 0 \text{ for all } \mathbf{v}(D) \in C\}$.

Let $[n, k, d]_q$ be a linear code with parity check matrix H . One first splits H into $m + 1$ disjoint submatrices H_i such that

$$H = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_m \end{bmatrix}. \quad (1)$$

After this, we consider the polynomial generator matrix given by

$$G(D) = \tilde{H}_0 + \tilde{H}_1 D + \tilde{H}_2 D^2 + \dots + \tilde{H}_m D^m, \quad (2)$$

where the matrices \tilde{H}_i , for all $1 \leq i \leq m$, are derived from the respective matrices H_i by adding zero-rows at the bottom in such a way that the matrix \tilde{H}_i has κ rows in total, where κ is the maximal number of rows among all the matrices H_i . The matrix $G(D)$ generates a convolutional code with memory m .

Theorem 1. [1, Theorem 3] *Let $C \subseteq \mathbb{F}_q^n$ be a linear code with parameters $[n, k, d]_q$. Assume also that $H \in \mathbb{F}_q^{(n-k) \times n}$ is a parity check matrix for C partitioned into submatrices H_0, H_1, \dots, H_m as in Eq. (1) such that $\kappa = rkH_0$ and $rkH_i \leq \kappa$ for $1 \leq i \leq m$ and consider the polynomial matrix $G(D)$ as in Eq. (2). Then the matrix $G(D)$ is a reduced basic generator matrix. Additionally, if d_f denotes the free distances of the convolutional code V generated by $G(D)$, and d^\perp is the minimum distance of C^\perp , then one has $d_f \geq d^\perp$.*

To finish this section, we recall the generalized Singleton bound [28] of an $(n, k, \gamma; m, d_f)_q$ convolutional code, which says that the free distance is upper bounded by $d_f \leq (n - k)[\lceil \gamma/k \rceil + 1] + \gamma + 1$.

3 Review of Algebraic Geometry Codes

In this section, we introduce some basic notation and results of algebraic geometry codes. For more details, the reader can see [29, 31].

Let F/\mathbb{F}_q be an algebraic functions field of genus g . A place P of F/\mathbb{F}_q is the maximal ideal of some valuation ring \mathcal{O} of F/\mathbb{F}_q . We also define $\mathbb{P}_F := \{P \mid P \text{ is a place of } F/\mathbb{F}_q\}$. A divisor of F/\mathbb{F}_q is a formal sum of places

given by $D := \sum_{P \in \mathbb{P}_F} n_P P$, with $n_P \in \mathbb{Z}$, almost all $n_P = 0$. The support of D is defined as $\text{supp}D := \{P \in \mathbb{P}_F | n_P \neq 0\}$. The discrete valuation corresponding to a place P is written as ν_P . For every element x of F/\mathbb{F}_q , we can define a principal divisor of x by $(x) := \sum_P \nu_P(x)P$. For a given divisor G , we denote the Riemann-Roch space associated to G by $\mathcal{L}(G) = \{x \in F/K \setminus \{0\} | (x) \geq -G\}$.

Let $\Omega_F := \{\omega | \omega \text{ is a Weil differential of } F/K\}$ be the differential space of F/\mathbb{F}_q . Given a nonzero differential w , we denote by $(\omega) := \sum_P \nu_P(w)P$ the canonical divisor. All canonical divisor are equivalent and have degree equal to $2g - 2$. Furthermore, for a divisor A we define $\Omega_F(G) := \{\omega \in \Omega_F | \omega = 0 \text{ or } (\omega) \geq G\}$, and denote its dimension by $i(G)$.

Theorem 2. (Riemann-Roch Theorem)[29, Theorem 1.5.15, pg 30] *Let W be a canonical divisor of F/K . Then for each divisor G , the dimension of $\mathcal{L}(G)$ is given by $\ell(G) = \deg(G) + 1 - g + \ell(W - G)$, where W is a canonical divisor.*

Let P_1, \dots, P_n be pairwise distinct places of F/\mathbb{F}_q of degree 1 and $D = P_1 + \dots + P_n$. Choose a divisor G of F/\mathbb{F}_q such that $\text{supp}G \cap \text{supp}D = \emptyset$. Then one has:

Definition 2. [29, Definition 2.2.1, pg 48] *The algebraic geometry (AG) code $C_{\mathcal{L}}(D, G)$ associated with the divisors D and G is defined as $C_{\mathcal{L}}(D, G) := \{(x(P_1), \dots, x(P_n)) | x \in \mathcal{L}(G)\}$.*

Proposition 1. [29, Corollary 2.2.3, pg 49] *Let F/\mathbb{F}_q be a function field of genus g . Then the AG code $C_{\mathcal{L}}(D, G)$ is an $[n, k, d]$ -linear code over \mathbb{F}_q with parameters $k = \ell(G) - \ell(G - D)$ and $d \geq n - \deg(G)$. If $2g - 2 < \deg(G) < n$, then $k = \deg(G) - g + 1$. If $\{x_1, \dots, x_k\}$ is a basis of $\mathcal{L}(G)$, then a generator matrix of $C_{\mathcal{L}}(D, G)$ is given by*

$$G_{\mathcal{L}} = \begin{bmatrix} x_1(P_1) & x_1(P_2) & \cdots & x_1(P_n) \\ x_2(P_1) & x_2(P_2) & \cdots & x_2(P_n) \\ \vdots & \vdots & \ddots & \vdots \\ x_k(P_1) & x_k(P_2) & \cdots & x_k(P_n) \end{bmatrix}. \quad (3)$$

Definition 3. [29, Definition 2.2.6, pg 51] *Let G and $D = P_1 + \dots + P_n$ be divisors as before. Then we define the code by $C_{\Omega}(D, G) := \{(resp_{P_1}(\omega), \dots, resp_{P_n}(\omega)) | \omega \in \Omega_F(G - D)\}$, where $resp_{P_i}(\omega)$ denotes the residue of ω at P_i .*

Proposition 2. [29, Theorem 2.2.7, pg 51] Let F/\mathbb{F}_q be a function field of genus g . Let G and $D = P_1 + \dots + P_n$ be divisors as before. If $2g - 2 < \deg(G) < n$, then $C_\Omega(D, G)$ is an $[n, k', d']$ -linear code over \mathbb{F}_q , where $k' = n + g - 1 - \deg(G)$ and $d' \geq \deg(G) - (2g - 2)$.

The relationship between the codes $C_{\mathcal{L}}(D, G)$ and $C_\Omega(D, G)$ is given in the next proposition.

Proposition 3. [29, Propositions 2.2.10 and 2.2.11, pg 54] Let η be a Weil differential such that $\nu_{P_i}(\eta) = -1$ and $\eta_{P_i} = 1$ for all $i = 1, \dots, n$. Then $C_{\mathcal{L}}(D, G)^\perp = C_\Omega(D, G) = C_{\mathcal{L}}(D, D - G + (\eta))$, where $C_{\mathcal{L}}(D, G)^\perp$ is the Euclidean dual of $C_{\mathcal{L}}(D, G)$.

4 New Convolutional AG Codes

In this section we present a general method to construct convolutional codes from AG codes. More precisely, we obtain convolutional codes whose generator matrix is derived from the AG code $C_\Omega(D, G)$. We adopt the notation given in the last section.

Our first result is given in the following:

Theorem 3. Let F/\mathbb{F}_q be a function field of genus g . Consider the AG code $C_\Omega(D, G)$ with $2g - 2 < \deg(G) < n$, where $\deg(G)$ is the degree of the divisor G . Then there exists a unit-memory convolutional code with parameters $(n, k-l, l; 1, d_f \geq d)_q$, where $l \leq k/2$, $k = \deg(G) + 1 - g$ and $d \geq n - \deg(G)$.

Proof. Let us consider the AG code $C_\Omega(D, G)$ defined over F/\mathbb{F}_q with parity check matrix

$$H_\Omega = \begin{bmatrix} x_1(P_1) & x_1(P_2) & \cdots & x_1(P_n) \\ x_2(P_1) & x_2(P_2) & \cdots & x_2(P_n) \\ \vdots & \vdots & \ddots & \vdots \\ x_k(P_1) & x_k(P_2) & \cdots & x_k(P_n) \end{bmatrix}, \quad (4)$$

where $\{x_1, \dots, x_k\}$ is a basis of $\mathcal{L}(G)$. Let $C_{\mathcal{L}}(D, G)$ be the (Euclidean) dual of the code $C_\Omega(D, G)$. A generator matrix of $C_{\mathcal{L}}(D, G)$ is equal to H_Ω . We know that $C_{\mathcal{L}}(D, G)$ is an AG code with parameters $[n, k = \deg(G) + 1 - g, d \geq n - \deg(G)]_q$, where $n = \deg(D)$. We will construct a convolutional code derived from $C_\Omega(D, G)$ as follows.

Define a convolutional code with generator matrix

$$\mathbb{G}(D) = H_0 + \tilde{H}_1 D,$$

where H_0 is the submatrix of H_Ω consisting of the $k-l$ first rows and \tilde{H}_1 is the matrix consisting of the last l rows of H_Ω by adding zero-rows at the bottom such that the matrix \tilde{H}_1 has $k-l$ rows in total. From hypothesis, it follows that $\text{rk } H_0 \geq \text{rk } \tilde{H}_1$. From Theorem 1, the matrix $\mathbb{G}(D)$ is a reduced basic matrix. The convolutional code generated by $\mathbb{G}(D)$ is a unit-memory code with dimension $k-l$, degree l and free distance d_f . From Theorem 1, it follows that $d_f \geq d$. Therefore, there exist convolutional codes with parameters $(n, k-l, l; 1, d_f)$, with $d_f \geq d$. \square

Remark 1. *It is interesting to note that Theorem 3 can be easily generalized by considering multi-memory convolutional codes. However, since unit-memory convolutional codes always achieve the largest free distance among all codes of the same rate (see [14]) we restrict ourselves to the construction of unit-memory codes.*

Corollary 1. *Assume that all the hypotheses of Theorem 3 hold. Then there exists a convolutional code with parameters $(n, k-1, 1; 1, d_f \geq d)_q$, where $k = \deg(G) + 1 - g$ and $d \geq n - \deg(G)$.*

Proof. It suffices to consider $l = 1$ in Theorem 3. \square

Remark 2. *Note that in Corollary 1, it follows from the generalized Singleton bound, that the free distance of the convolutional codes constructed here are bounded by $d_f \leq n - k + 3$ (where n and k are the parameters of $C_{\mathcal{L}}(D, G)$). Furthermore, $d_f \geq n - \deg(G) = n - (k + g - 1) = n - k + 1 - g$; so the free distance d_f is bounded by $n - k + 1 - g \leq d_f \leq n - k + 3$. In particular, for function fields F/\mathbb{F}_q with $g = 0$ the new convolutional codes have free distance bound by $n - k + 1 \leq d_f \leq n - k + 3$. In this case, observe that these codes are almost near MDS or near MDS or MDS. In other words, the Singleton defect is at most two. Therefore, we have constructed good families of convolutional codes.*

Corollary 2. *Let $F = \mathbb{F}_q(z)$ be a rational function field. For $\beta \in \mathbb{F}_q$, let P_β be the zero of $z - \beta$ and denote by P_∞ the pole of z in $\mathbb{F}_q(z)$. Then there exists a convolutional code with parameters $(q, r, 1; 1, d_f \geq q - r)_q$, where $1 < r \leq q - 1$.*

Proof. Consider the AG code $C_{\mathcal{L}}(D, G)$ with $D = \sum_{\beta \in \mathbb{F}_q} P_{\beta}$ and $G = rP_{\infty}$, where $1 < r \leq q-1$. We know that $C_{\mathcal{L}}(D, G)$ has parameters $n = q$, $k = r+1$ and $d \geq n - r$. Applying Corollary 1 to the AG code $C_{\mathcal{L}}(D, G)^{\perp}$, one can get convolutional codes with the desired parameters. \square

Theorem 4. *Let $q = 2^t$, where $t \geq 1$ is an integer. Then there exists an $(2q^2, m - q/2, 1; 1, d_f \geq 2q^2 - m)_q$ convolutional code, where $q - 2 < m < 2q^2$.*

Proof. It follows from the fact that in the function field $F = \mathbb{F}_q(x, y)$, defined by the equation $y^2 + y = x^{q+1}$, it is possible to construct an AG code with parameters $[2q^2, m - q/2 + 1, d \geq 2q^2 - m]_q$, with $q - 2 < m < 2q^2$ (see [8, 30]). \square

Example 1. *Applying Theorem 4 we can construct an $(32, 15, 1; 1, d_f \geq 15)_4$ new convolutional code whose parameters are better than the $(32, 15, 10; \mu, d_f \geq 9)_3$ code, shown in [10], and better than the $(32, 16, \gamma; 1, d_f \geq 5)_3$ code, shown in [1]. Additionally, our new $(128, 64, 1; 1, d_f \geq 60)_8$ code is better than the $(128, 64, 35; \mu, d_f \geq 17)_7$ code, shown in [10], and better than the $(128, 64, \gamma; 1, d_f \geq 8)_7$ code, shown in [1].*

Theorem 5. *Let $q = 2^t$, where $t \geq 1$ is an odd integer. Then there exists an $(3q^2 - 2q, m - q + 1, 1; 1, d_f \geq 3q^2 - 2q - m)_q$ convolutional code, where $2q - 4 < m < 3q^2 - 2q$.*

Proof. Let F be the function field over \mathbb{F}_{q^2} defined by the equation

$$y^q + y = x^3.$$

The genus of F equals $g = q - 1$ and the number of rational places (place of degree one) is equal to $3q^2 - 2q + 1$ (see [8]). Let $D = P_1 + \dots + P_n$ be a divisor, where $n = 3q^2 - 2q$, and $G = mP_{3q^2 - 2q + 1}$, with $2g - 2 < m < n$, where $\{P_1, \dots, P_{3q^2 - 2q + 1}\}$ are all pairwise distinct rational places. Consider the AG code $C_{\mathcal{L}}(D, G)$; the parameters of $C_{\mathcal{L}}(D, G)$ are $[n = 3q^2 - 2q, k = m + 1 - g, d \geq n - m]_q$, where $2q - 4 < m < 3q^2 - 2q$.

Applying Corollary 2, we can get an $(3q^2 - 2q, m - q + 1, 1; 1, d_f \geq 3q^2 - 2q - m)_q$ convolutional code, where $2q - 4 < m < 3q^2 - 2q$. \square

The next results are obtained from Theorem 3 when considering puncturing, extending, expanding and the product code construction to AG codes.

Theorem 6. *Assume the same notation of Theorem 3, and suppose that $C_{\mathcal{L}}(D, G)$ has no minimum weight codeword with a nonzero j -th coordinate. Then there exists an $(n-1, k-l, l; 1, d_f)_q$ convolutional code, where $d_f \geq d$, $k = \deg(G) + 1 - g$, $l \leq k/2$ and $d \geq n - \deg(G)$.*

Proof. Let $C_{\mathcal{L}}(D, G)$ be the $[n, k, d]_q$ AG code considered in Theorem 3, where $D = P_1 + \dots + P_n$. Now, let $D' = D - P_j$, where $j \in \{1, 2, \dots, n\}$. We define the puncture code $C_{\mathcal{L}}(D', G)$ derived from $C_{\mathcal{L}}(D, G)$, which is also an AG code (see [24]). Note that the supports of D' and G are disjoint, i.e. the definition of $C_{\mathcal{L}}(D', G)$ makes sense. From hypothesis (see [5, Theorem 1.5.1, pg 13]), $C_{\mathcal{L}}(D', G)$ has parameters $[n-1, k, d]_q$. Applying the same construction shown in Theorem 3 we can construct a convolutional code V with parameters $(n-1, k-l, l; 1, d_f)_q$, where $d_f \geq d$. A generator matrix $\mathbb{G}^*(D)$ for V is given by

$$\mathbb{G}^*(D) = \begin{bmatrix} x_1(P_1) + x_{k-l+1}(P_1)D & x_1(P_2) + x_{k-l+1}(P_2)D & \cdots & x_1(P_{j-1}) + x_{k-l+1}(P_{j-1})D & x_1(P_{j+1}) + x_{k-l+1}(P_{j+1})D & \cdots & x_1(P_n) + x_{k-l+1}(P_n)D \\ x_2(P_1) + x_{k-l+2}(P_1)D & x_2(P_2) + x_{k-l+2}(P_2)D & \cdots & x_2(P_{j-1}) + x_{k-l+2}(P_{j-1})D & x_2(P_{j+1}) + x_{k-l+2}(P_{j+1})D & \cdots & x_2(P_n) + x_{k-l+2}(P_n)D \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_l(P_1) + x_k(P_1)D & x_l(P_2) + x_k(P_2)D & \cdots & x_l(P_{j-1}) + x_k(P_{j-1})D & x_l(P_{j+1}) + x_k(P_{j+1})D & \cdots & x_l(P_n) + x_k(P_n)D \\ x_{l+1}(P_1) & x_{l+1}(P_2) & \cdots & x_{l+1}(P_{j-1}) & x_{l+1}(P_{j+1}) & \cdots & x_{l+1}(P_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{k-l}(P_1) & x_{k-l}(P_2) & \cdots & x_{k-l}(P_{j-1}) & x_{k-l}(P_{j+1}) & \cdots & x_{k-l}(P_n) \end{bmatrix}.$$

□

Theorem 7. *Assume the same notation of Theorem 3. Then there exists an $(n+1, k-l, l; 1, d_f \geq d^e)_q$ convolutional code, where $d^e = d$ or $d^e = d+1$, where $k = \deg(G) + 1 - g$, $l \leq k/2$ and $d \geq n - \deg(G)$.*

Proof. Let us consider $C_{\mathcal{L}}(D, G)$ be the $[n, k, d]_q$ AG code considered in Theorem 3. We construct a new code $C_{\mathcal{L}}^e(D, G)$ by extending the code $C_{\mathcal{L}}(D, G)$. This new code have parameters $[n+1, k, d^e]_q$, with $d^e = d$ or $d^e = d+1$. Applying the method utilized in the proof of Theorem 3, one can get an $(n+1, k-l, l; 1, d_f \geq d^e)_q$ convolutional code, and the result follows. □

Theorem 8. *Assume the same notation of Theorem 3. Then there exists an $(mn, mk-l, l; 1, d_f \geq d)_q$ convolutional code, where $k = \deg(G) + 1 - g$, $l \leq k/2$ and $d \geq n - \deg(G)$.*

Proof. Consider that $C_{\mathcal{L}}(D, G)$ is the AG code, over \mathbb{F}_{q^m} , with parameters $[n, k, d]_{q^m}$. Let $\beta = \{b_1, \dots, b_m\}$ be a basis of \mathbb{F}_{q^m} over \mathbb{F}_q . We expand the code $C_{\mathcal{L}}(D, G)$ with respect of basis β generating the code $\beta(C_{\mathcal{L}}(D, G))$,

over \mathbb{F}_q , with parameters $[mn, mk, d^* \geq d]_q$. A parity check matrix H of $[\beta(C_{\mathcal{L}}(D, G))]^\perp$ is a generator matrix of $\beta(C_{\mathcal{L}}(D, G))$.

Let V be the convolutional code generated by the minimal-basic matrix

$$\mathbb{G}(D) = H_0 + \tilde{H}_1 D, \quad (5)$$

where H_0 is a submatrix of H consisting of the $mk - l$ first rows of H and \tilde{H}_1 is the matrix consisting of the last row of H and more $mk - 2l$ zero rows. Then, we construct a convolutional code V that has parameters $(mn, mk - l, l; 1, d_f \geq d)_q$, as desired. \square

Theorem 9. *Assume the same notation of Theorem 3. Then there exists an $(n^2, k^2 - l, l; 1, d_f \geq d^2)_q$ convolutional code, where $k = \deg(G) + 1 - g$, $l \leq k/2$ and $d \geq n - \deg(G)$.*

Proof. Let $C_{\mathcal{L}}(D, G)$ be the AG code of Theorem 3 over \mathbb{F}_q , with parameters $[n, k, d]_q$. We construct a product code $(C_{\mathcal{L}}(D, G) \otimes C_{\mathcal{L}}(D, G))$. This is an $[n^2, k^2, d^2]_q$ code. Similarly to the proof of Theorem 3, one has an $(n^2, k^2 - l, l; 1, d_f \geq d^2)$ convolutional code, as required. \square

5 Code Comparisons

In this section, we compare the parameters of the new convolutional codes with the ones available in the literature. Table 1, shows a family of almost near MDS (or near MDS or MDS) codes constructed from Corollary 2.

The codes displayed in Table 2 are obtained from Theorems 4 and 5. Note that these new $(32, 15, 1; 1, d_f \geq 15)_4$ convolutional code is better than the $(32, 15, 10; \mu, d_f \geq 9)_3$ and $(32, 16, \gamma; 1, d_f \geq 5)_3$ shown in Refs. [10] and [1], respectively. The new $(128, 64, 1; 1, d_f \geq 60)_8$ code is better than the $(128, 64, 35; \mu, d_f \geq 17)_7$ and the $(128, 64, \gamma; 1, d_f \geq 8)_7$ from Refs. [10] and [1], respectively.

The other new codes shown in Table 2 have different parameters when compared to the ones available in literature. Because of this fact, it is not possible to compare such codes with the ones available in the literature.

Table 1: New almost near MDS or near MDS or MDS codes

The new codes from Corollary 2
$(n, k, \gamma; m, d_f)$
$(8, 2, 1; 1, d_f \geq 6)_8$
$(8, 5, 1; 1, d_f \geq 3)_8$
$(37, 17, 1; 1, d_f \geq 20)_{37}$
$(37, 33, 1; 1, d_f \geq 4)_{37}$
$(71, 35, 1; 1, d_f \geq 36)_{71}$
$(71, 68, 1; 1, d_f \geq 3)_{71}$
$(128, 64, 1; 1, d_f \geq 64)_{128}$
$(128, 125, 1; 1, d_f \geq 3)_{128}$
$(256, 128, 1; 1, d_f \geq 128)_{256}$
$(256, 253, 1; 1, d_f \geq 3)_{256}$

Table 2: Code Comparison

New codes	Codes in [10]	Codes in [1]
$(32, 15, 1; 1, d_f \geq 15)_4$	$(32, 15, 10; \mu, d_f \geq 9)_3$	$(32, 16, \gamma; 1, d_f \geq 5)_3$
$(32, 1, 1; 1, d_f \geq 30)_4$	–	–
$(128, 64, 1; 1, d_f \geq 60)_8$	$(128, 64, 35; \mu, d_f \geq 17)_7$	$(128, 64, \gamma; 1, d_f \geq 8)_7$
$(176, 64, 1; 1, d_f \geq 105)_8$	–	–
$(128, 3, 1; 1, d_f \geq 122)_8$	–	–
$(176, 6, 1; 1, d_f \geq 163)_8$	–	–
$(512, 128, 1; 1, d_f \geq 376)_{16}$	–	–
$(512, 256, 1; 1, d_f \geq 248)_{16}$	–	–
$(2048, 1024, 1; 1, d_f \geq 1008)_{32}$	–	–
$(3008, 1024, 1; 1, d_f \geq 1953)_{32}$	–	–
$(2048, 15, 1; 1, d_f \geq 2017)_{32}$	–	–
$(3008, 30, 1; 1, d_f \geq 2947)_{32}$	–	–

6 Final Remarks

In this paper we have constructed new families of convolutional codes derived from algebraic geometry codes. These new codes have good param-

eters. More precisely, a family of almost near MDS codes was presented. Additionally, our codes are better than or comparable to the ones shown in [1, 10]. Furthermore, more families of convolutional codes were constructed by means of puncturing, extending, expanding and by the direct product code construction applied to algebraic geometry codes.

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