Strict Lyapunov functions for dynamic consensus in linear systems interconnected over directed graphs

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***Abstract*— We study dynamic consensus for general networked (homogeneous) linear autonomous systems, that is, it is only assumed that they are stabilizable. Dynamic consensus pertains to a general form of consensus in which, as a result of the systems' interactions, they exhibit a rich collective dynamic behavior. This generalizes the classical consensus paradigm in which case all systems stabilize to a common equilibrium point. Our main statements apply to systems interconnected over generic directed connected graphs and, most significantly, the proofs are constructive. Indeed, even though our controllers are reminiscent of others previously used in the literature, to the best of our knowledge, we provide for the first time in the literature strict Lyapunov functions for fully distributed consensus over generic directed graphs.**

***Index Terms*— Multiagent system, directed graphs, Lyapunov stability, linear systems.**

I. Introduction

In the study of the collective behavior of multiagent networked systems, a common problem studied in the literature pertains to the case in which all the systems stabilize at a common equilibrium point —see [1]. The collective behavior, however, may be much richer than converging to an equilibrium. In general, it depends on the nature of the systems dynamics—they may be, e.g., linear [2]–[4] or nonlinear [5], [6]—, on whether the networked systems are homogeneous [2], [4] or heterogeneous [5], [7], [8], on the nature of the interconnections graph—whether it is undirected [6] or directed [2], [4]—, etc.

In this letter, we study the collective behavior of multiagent linear systems interconnected over directed graphs via a distributed consensus algorithm. The class of systems that we consider is fairly general since the only standing assumption regarding the systems' model is that it is stabilizable —cf. [3], [6], and [9]. This class covers stable systems, as in [2] and [7], neutrally stable systems, as in [4] and [8], but also unstable systems or oscillators. Therefore, the collective behavior of such systems is not bound to stabilizing over a common equilibrium point, but is much richer.

Following [5], we study *dynamic* consensus. This pertains to the case in which, due to the systems' interaction, an *emerging dynamics* is generated, which, roughly, corresponds to a weighted average dynamics of the interconnected systems. Then, we say that dynamic consensus is achieved if the motion of each system in the network matches asymptotically that of the emergent dynamics. For instance, a network of oscillators *may* behave collectively as a weighted averaged oscillator. Other works in which synchronization is considered with respect to a dynamical system, as opposed to an equilibrium point, include [2] and [4]. The systems studied in [3], [6], [9] and [8], too, lead to reach collective behaviors since they are assumed to be merely stabilizable in [6], [9] and [3], and neutrally stable in [8].

We analyze two controllers that are reminiscent of consensus algorithms previously reported, in which the control gain is defined upon the solution to a linear matrix inequality —cf. [3], [6], [9], and [8]. The first controller, as the one proposed in [10], relies on a constant coupling strength, so dynamic consensus is reached exponentially fast (we provide an explicit bound), but knowledge of the maximum eigenvalue of a matrix that solves a Lyapunov equation involving the Laplacian matrix is required. For the second controller this requirement is removed, so, as in [6] and [9], it is fully distributed. We show that by rendering the coupling strength time-varying and monotonically increasing, the aforementioned information is not required. To some extent, our control law recalls that proposed in [9], but even if our controller relies on the same principle, it differs from that in [9] in that in the latter the coupling gain is computed dynamically in function of the norm of the network's state, akin to high-gain adaptive control.

In regards to the network's topology, as for instance in [3], we consider networks interconnected over generic connected directed graphs, that is, containing a rooted directed spanning tree. In particular, the graph may *consist in* a spanning tree, as in [11], it may consist in a directed graph contain a root node, as in [2], [10] and [12], or in rooted graphs with a specific leader, as in [8], [9], [13], and possibly bidirectional links among the followers, as in [8], to mention a few. Similar results to those obtained here are presented in [9], but they are restricted to a leader-follower configuration, so the directed

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graph is assumed to have a directed spanning tree with a root node as a leader. See also [13]. On the other hand, in [12] the leaderless consensus problem is addressed for directed-graph networked systems, but under the assumption that the (static) graph is balanced. Thus, in none of these references generic directed graphs are considered.

From the viewpoint of analysis, in many cases consensus among linear systems may be assessed by relying on linear algebra and graph theory. However, these methods fail for nonlinear systems —cf. [5], [6]. In this case, the consensus analysis (resp. design) problem may be broached as one of stability analysis (resp. stabilization), relying on the construction of a Lyapunov function. There are many articles in which Lyapunov functions are proposed, for multiagent systems interconnected over undirected graphs [6] as well as over directed graphs [8]–[10], [12]–[14]. Actually, our proofs follow a similar rationale as that in [6], but they certainly do not constitute a straightforward extension.

Furthermore, we stress that except for [14], which is restricted to systems of the form $\ddot{x}_i = u_i$, in none of these references a strict Lyapunov function is proposed. Furthermore, we stress that except for [14], which is restricted to systems of the form $\ddot{x}_i = u_i$, in none of these references a strict Lyapunov function (i.e., having a negative-definite derivative) is proposed. The analysis methods used most commonly in the literature relies on tools such as La Salle's invariance principle or Barbalat's lemma. The former does not apply to networks with time-varying topology and the latter does not lead to uniform asymptotic stability. Only the latter, however, guarantees robustness with respect to bounded additive disturbances. Disposing of strict Lyapunov functions is an important step to consider more general scenarrii, such as that of linear heterogeneous systems or with added nonlinearities, but they may also be useful in the study of perturbed networked systems or of robustness with respect to neglected dynamics.

Thus, to the best of our knowledge, although several aspects studied in this letter have been addressed in the literature, separately, never have they been considered simultaneously. Beyond the generality of the network topology as well as in the class of systems that we consider, our primary contribution resides in the construction of *strict* Lyapunov functions for multiagent systems in such a general scenario. Indeed, we are unaware of strict Lyapunov functions for generic linear systems interconnected over arbitrary connected directed graphs. The construction is based on that for second-order integrators, proposed in [14], and as a byproduct of our main results, we provide explicit exponential stability bounds on the synchronization error trajectories. We show explicitly how the speed of convergence depends on the maximum eigenvalue of the matrix that solves an algebraic Riccati equation with unity weighting matrix.

The remainder of this Letter is organized as follows. In the next section we present the problem statement and its solution. In Section III we present our main results, in Section IV we provide illustrative numerical examples, before wrapping up the paper with some concluding remarks in Section V.

II. Problem formulation and its solution

We consider N multiagent linear systems with identical dynamics,

$$\dot{x}_i = Ax_i + Bu_i, \quad i \in \{1, 2, \cdots, N\}, \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^p$, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, and $N \in \mathbb{N}$. For the systems (1) we address the consensus problem, i.e., to guarantee that $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0 \quad \forall i \neq j$, under the following

Standing Assumption:

1) the agents communicate over a directed connected graph,
2) the pair (A, B) is stabilizable.

That is, it is assumed that the systems communicate through reliable *unidirectional* channels of intensity $a_{ij} \geq 0$. That is, if there exists an unidirectional edge ε_{ji} interconnecting the node j to the node i, we have $a_{ij} > 0$ and if no information flows from the jth to the ith node $a_{ij} = 0$. In general, for any pair (i, j), $a_{ij} \neq a_{ji}$. As is customary, the graph topology may be modeled using the Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$, where

$$l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}, \quad l_{ij} = -a_{ij}, i \neq j. \tag{2}$$

The first part of the Standing assumption is that the directed-interconnection graph *contains* a rooted spanning tree. The latter is a necessary condition for consensus in such networks [16] and it means that there exists a subgraph containing a node called root which has no incoming edge and from which all nodes may be reached. It seems important to stress that we do *not* assume that the graph contains a root node with no incoming edges, as in a leader-follower configuration [8], [9], [13]. Under the assumption that the graph is connected we have the following.

Lemma 1: [16], [17] If the directed graph $\mathcal{G}$ has a rooted spanning tree then its associated Laplacian matrix has a simple zero eigenvalue with $1_N := \begin{bmatrix}1, \cdots, 1\end{bmatrix}^\top$ as its eigenvector of dimension N and all of the remaining eigenvalues lie in the open right half plane.

Another important characteristic of the Laplacian corresponding to a directed graph is that it is not symmetric. This is significant because it adds considerable difficulty to the task of constructing strict Lyapunov functions to study consensus. Yet, the following statement holds.

Lemma 2: [14] Let $\mathcal{G}$ be a directed graph of order N and $\mathcal{L} \in \mathbb{R}^{N\times N}$ be the associated non-symmetric Laplacian matrix. Then, the following statements are equivalent:

1) the graph $\mathcal{G}$ has a spanning tree,
2) for any matrix $Q_\mathcal{L} \in \mathbb{R}^{N\times N}, Q_\mathcal{L} = Q_\mathcal{L}^\top > 0$ and for $\alpha > 0$, there exists matrix $P = P^\top > 0$ such that

$$P\mathcal{L} + \mathcal{L}^\top P = Q_\mathcal{L} - \alpha[P1_N v_l^\top + v_l 1_N^\top P], \tag{3}$$

where v_l is the left eigenvector associated to the single zero eigenvalue of $\mathcal{L}$.

Under the standing assumptions above, we propose two solutions to the consensus problem. Both involve the use of a control input of the form

$$u_i = -c(\cdot)F\sum_{j=1}^{N} a_{ij}(x_i - x_j), \qquad (4)$$

where $c(\cdot) > 0$ is a coupling weight that may be constant or a time-varying function $c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ and $F \in \mathbb{R}^{p\times n}$ is the feedback matrix defined as $F = B^\top M$, where M is solution to the Riccati equation

$$MA + A^\top M - MBB^\top M = -Q_0, \qquad (5)$$

for any given $Q_0 = Q_0^\top > 0$ —cf. [3], [6], [8]–[10]. The existence of such matrix M is guaranteed by the assumption that the pair (A, B) is stabilizable —cf. [18], [19]. Indeed, in this case, for any symmetric positive definite matrices Q and $R \in \mathbb{R}^{n\times n}$, there exists a positive definite matrix $M = M^\top \in \mathbb{R}^{n\times n}$ that satisfies the matrix algebraic Riccati equation

$$MA + A^\top M - MBR^{-1}B^\top M = -Q.$$

Thus, the existence of the control parameters c and F is guaranteed by Lemmata 1–2 and the standing assumption.

III. Consensus analysis: main results

Our main statements, which are presented in this section farther below, establish consensus of the systems (1) under the control law (4) and, more significantly, provide a strict Lyapunov function for the closed-loop system. In the first statement it is assumed that the coupling strength c is a constant majorating the largest eigenvalue of P in (3) —cf. [3], [8], [10], [12]. Hence, the computation of the appropriate gain c relies on the solution P to Eq. (3) and, indirectly, on the knowledge of $\mathcal{L}$. The second statement relaxes this dependence by introducing a time-varying, strictly increasing, gain $t \mapsto c(t)$.

A. The networked system's equations

The rest of the paper is devoted to the analysis of the consensus manifold $\{x_i = x_j\}$ for all $i, j \leq N$ for the systems (1) under the control law (4). To that end, we write the closed-loop equations in the compact matrix form

$$\dot{x} = (I_N \otimes A)x + (I_N \otimes B)u, \qquad (6)$$

where '$\otimes$' denotes the Kronecker product, I_N denotes the identity matrix of dimension $\mathbb{R}^N \times \mathbb{R}^N$, and $u := [u_1^\top \ \cdots \ u_N^\top]^\top$ is given by

$$u = -(\mathcal{L} \otimes cF)x, \qquad (7)$$

so, replacing (7) in (6), we obtain

$$\dot{x} = [(I_N \otimes A) - (\mathcal{L} \otimes cBF)]x. \qquad (8)$$

Then, we recall that, in contrast to systems that have asymptotically stable equilibria, in general, for the multiagent system (8) consensus, if it takes place, is *dynamic*. That is, on the consensus manifold, on one hand a collective dynamic behavior arises and, on the other, the synchronization errors with respect to such behavior, converge to zero. To analyze the dynamic consensus we follow the framework laid in [5].

To that end, we start by recalling a suitable convertible transformation that maps the space of the states x into two orthogonal spaces, one containing the states of the "averaged" states x_m and one containing the synchronization errors e. More precisely, we have

$$\begin{bmatrix} x_m \\ e \end{bmatrix} = Tx, \qquad T := \begin{bmatrix} v_l^\top \otimes I_n \\ (I_N - 1_N v_l^\top) \otimes I_n \end{bmatrix}. \qquad (9)$$

It is important to remark that the matrix T is invertible and it exists under the mild assumption that the Laplacian has a simple eigenvalue equal to zero and all others have positive real parts. That is, after Lemma 2 under the assumption that the graph is directed and connected.

We remark that the state $x_m = (v_l^\top \otimes I_n)x$ may be regarded as a weighted average of the individual systems' states and the synchronization errors are defined relative to it, i.e.,

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix} = \begin{bmatrix} x_1 - \sum_{j=1}^N v_{lj}x_j \\ x_2 - \sum_{j=1}^N v_{lj}x_j \\ \vdots \\ x_N - \sum_{j=1}^N v_{lj}x_j \end{bmatrix} = \begin{bmatrix} x_1 - x_m \\ x_2 - x_m \\ \vdots \\ x_N - x_m \end{bmatrix}. \qquad (10)$$

Therefore the collective behavior of the networked system is completely described by the dynamics of x_m and of e. These are computed by differentiating on both sides of the first equality in (9) and using (8). For the mean-field dynamics we obtain

$$\begin{aligned} \dot{x}_m &= (v_l^\top \otimes I_n)[(I_N \otimes A) - (\mathcal{L} \otimes cBF)]x \\ &= (v_l^\top \otimes A)x = \ Ax_m, \end{aligned} \qquad (11)$$

—cf. [2], [4]. This dynamics, which is inherent to the network, is clearly influenced by the systems dynamics.

On the other hand, the dynamics of the synchronization errors e, yield

$$\dot{e} = [(I_N - 1_N v_l^\top) \otimes A - (\mathcal{L} - 1_N v_l^\top \mathcal{L}) \otimes cBF)]x. \qquad (12)$$

Then, since v_l is a left eigenvector of $\mathcal{L}$, we have $\mathcal{L}^\top v_l = 0_N$ and $\mathcal{L}1_N = 0_N$. Using the latter in (12) we obtain

$$\dot{e} = [(I_N \otimes A) - (\mathcal{L} \otimes cBF)][(I_N - 1_N v_l^\top) \otimes I_n]x. \qquad (13)$$

Hence, replacing $e = [(I_N - 1_N v_l^\top) \otimes I_n]x$ —cf. Eq. (9), in (13), we obtain

$$\dot{e} = [(I_N \otimes A) - (\mathcal{L} \otimes cBF)]e. \qquad (14)$$

In the sequel, we establish exponential stability of $\{e = 0\}$ for (14) and provide a strict Lyapunov function for this system under the condition that c is positive and either constant or a strictly increasing function of time. Exponential stability of $\{e = 0\}$ implies that all the systems synchronize and their motions tend to that of the emergent dynamics (11); note that the behavior of this system is purely determined by that of the original systems and the network topology (through the eigenvector v_ℓ), independently of the coupling strength.

B. Consensus with prescribed convergence rate

Remark 1 (notation): We use p_m and p_M to denote, respectively, the smallest and largest eigenvalues of P; *mutatis mutandis* for Q, M and any other square matrices.

Proposition 1: Let $P = P^\top > 0$ be a solution of (3) with $Q_\mathcal{L} = I_N$ and an arbitrary $\alpha > 0$ and let M be the solution of (5) for any *given* $Q_0 = Q_0^\top > 0$. Consider N identical linear systems (1), with (A, B) stabilizable, in closed loop with (4) with $c \geq p_M$, $F := B^\top M$, and let the coefficients $a_{ij} \geq 0$ be such that they generate, through (2), a non-symmetric Laplacian matrix $\mathcal{L}$ that has a simple zero eigenvalue and all others have real positive parts.

Then, dynamic consensus is achieved for the multiagent closed-loop system (8) and the synchronization errors satisfy

$$|e(t)| \leq \kappa |e(0)| e^{-\gamma t} \qquad t \geq 0, \tag{15}$$

where

$$\kappa := \sqrt{\frac{m_M p_M}{m_m p_m}}, \quad \gamma := \frac{q_{0m}}{2 m_M}, \tag{16}$$

and q_{0m} is the smallest eigenvalue of Q_0 —cf. Remark 1.

Proof: Consider the Lyapunov function candidate

$$V(e) = e^\top [P \otimes M] e, \tag{17}$$

which is positive definite in the synchronization errors e, as defined in (10), but it is *not* for *all* $e \in \mathbb{R}^{nN}$. Indeed,

$$m_m p_m |e|^2 \leq V(e) \leq p_M m_M |e|^2 \tag{18}$$

(only) for all $e = [(I_N - 1_N v_l^\top) \otimes I_n] x$ —cf. Eq. (9).

Then, we use the identity $(\mathrm{A} \otimes \mathrm{B})(\mathrm{C} \otimes \mathrm{D}) = \mathrm{AC} \otimes \mathrm{BD}$, where A, B, C, and D are of suitable dimensions so that one can get the matrix products AC and BD, as well as $F = B^\top M$, to compute the total derivative of V along the trajectories of (14). We obtain

$$\dot{V}(e) = e^\top \big[P \otimes [MA + A^\top M] - c[P\mathcal{L} + \mathcal{L}^\top P] \otimes MBB^\top M \big] e,$$

so, in turn, using (3) with $Q_\mathcal{L} = I_N$, it follows that

$$\begin{aligned} \dot{V}(e) = e^\top \Big[& P \otimes [MA + A^\top M] \\ & - c\big[I_N - \alpha[P 1_N v_l^\top + v_l 1_N^\top P]\big] \otimes MBB^\top M \Big] e. \end{aligned} \tag{19}$$

However, we remark that

$$[\alpha P 1_N v_l^\top \otimes I_n] e = [\alpha P (1_N v_l^\top - 1_N v_l^\top) \otimes I_n] x = 0_{Nn},$$

so using the latter in (19) we obtain

$$\dot{V}(e) = e^\top \big[P \otimes [MA + A^\top M] - c I_N \otimes MBB^\top M \big] e.$$

Now, since P is positive definite it admits the decomposition $P = CDC^T$ where $C \in \mathbb{R}^{N \times N}$ is orthonormal and D is the diagonal matrix having the eigenvalues of P in its diagonal. Then,

$$\begin{aligned} \dot{V}(e) = \ & e^\top [C \otimes I_n] \Big[D \otimes [MA + A^\top M] \\ & - c I_N \otimes MBB^\top M \Big] [C^\top \otimes I_n] e. \end{aligned}$$

Next, we introduce $\tilde{e} := [C^\top \otimes I_n] e = [\tilde{e}_1^\top \cdots \tilde{e}_N^\top]^\top$, so

$$\dot{V}(e) = \sum_{i=1}^{N} \tilde{e}_i^\top \Big[\lambda_i(D) [MA + A^\top M] - c MBB^\top M \Big] \tilde{e}_i$$

$$\dot{V}(e) \leq p_M \sum_{i=1}^{N} \tilde{e}_i^\top \Big[MA + A^\top M - \frac{c}{p_M} MBB^\top M \Big] \tilde{e}_i. \tag{20}$$

On the other hand, by assumption, $c \geq p_M$. Therefore, after (5), we obtain

$$\dot{V}(e) \leq \ -p_M \tilde{e}^\top [I_N \otimes Q_0] \tilde{e} = \ -p_M e^\top [CC^\top \otimes Q_0] e.$$

Hence, since C is orthonormal, $CC^\top = I_N$ and, consequently, $\dot{V}(e) \leq \ -p_M e^\top [I_N \otimes Q_0] e$. Global exponential stability of the manifold $\{e = 0\}$ and the bound (15)–(16) follow from integrating the latter inequality and using (18). ■

C. Fully distributed consensus

We relax the requirement in Proposition 1 to know the maximum eigenvalue of P, which is implicit in the condition that $c \geq p_M$ and, consequently, relies on knowledge of the Laplacian $\mathcal{L}$. To that end, we redefine the coupling strength c in (7) as a "slowly" strictly increasing time-varying function.

Proposition 2: Consider the linear multiagent system (8), with (A, B) stabilizable, in closed loop with the consensus control law (4) with $c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ strictly increasing and F and a_{ij} as in Proposition 1. Then, the multiagent system reaches dynamic consensus exponentially.

Proof: Consider again the Lyapunov function candidate V in (17). Proceeding as in the proof of Proposition 1 we obtain that the total time derivative along the trajectories of (14) satisfies (20). Now, defining,

$$Q(c) := \frac{c}{p_M} MBB^\top M - MA - A^\top M$$

we see that, after (5) and (20),

$$\dot{V}(e) \leq -p_M \tilde{e}^\top \big[I_N \otimes [(Q(c(t)) - Q_0) + Q_0] \big] \tilde{e},$$

so, proceeding as in the proof of Proposition 1 we obtain

$$\dot{V}(e) \leq -p_M e^\top [I_N \otimes Q_0] e - p_M e^\top \big[I_N \otimes [Q(c(t)) - Q_0] \big] e^\top.$$

Now, since $t \mapsto c(t)$ is strictly increasing,

$$T' := \min\{t \geq 0 \ : \ c(t) \geq p_M\} \in [0, \infty)$$

exists. Therefore,

$$|Q(c(t)) - Q_0| \leq (m_M b_M)^2 := \beta \qquad \forall t \in [0, T'],$$

while $Q(c(t)) - Q_0 \geq 0$ for all $t \geq T'$. It follows that

$$\dot{V}(e(t)) \leq \begin{cases} p_M \beta |e(t)|^2 & \forall\, t \in [0, T'] \\ -q_{0m} p_M |e(t)|^2 & \forall\, t \geq T' \end{cases} \tag{21}$$

Then, using the inequalities in (18) and integrating both sides of (21) we obtain

$$V(e(t)) \leq \begin{cases} e^{\gamma_0 t} V(e(0)) & \forall\, t \in [0, T'] \\ e^{-\gamma_1 (t - T')} V(e(T')), & \forall\, t \geq T' \end{cases} \tag{22}$$

where $\gamma_0 := \frac{p_M \beta}{p_m m_m}$ and $\gamma_1 := \frac{q_{0m}}{m_M}$. In turn, this implies that

$$\begin{aligned} V(e(t)) &\le e^{(\gamma_0+\gamma_1)t} e^{-\gamma_1 t} V(e(0)) \qquad \forall t \in [0, T'] \\ &\le e^{(\gamma_0+\gamma_1)T'} V(e(0)) e^{-\gamma_1 t}, \end{aligned}$$

while, using $V(e(T')) \le e^{\gamma_0 T'} V(e(0))$ and the second inequality in (22), we obtain

$$V(e(t)) \le e^{(\gamma_0+\gamma_1)T'} V(e(0)) e^{-\gamma_1 t} \quad \forall t \ge T'$$

Putting the last two inequalities together and using (18) with $e = e(t)$ we obtain that (15) holds with

$$\kappa := \left[e^{(\gamma+\gamma_0)T'} \frac{m_M p_M}{m_m p_m} \right]^{1/2}, \gamma := \frac{q_{0m}}{2 m_M}. \tag{23}$$

■

Remark 2 (Convergence of the control input): For the sake of generality, in the statement of Proposition 2 there is no particular choice for the function $t \to c(t)$ which is a strictly increasing control gain. It is important to stress that the monotonicity of c does not necessarily imply that the control input grows unboundedly along the trajectories. For example, in [9] a controller of the form (4) is also used and c is defined dynamically and grows monotonically as a function of the consensus errors. A simpler choice, state-independent, is the slowly increasing function $c(t) := \ln(\varepsilon + t)$, which also qualifies as a suitable function for purpose of fully decentralized consensus and the control input remains bounded. Indeed, $c(t) \ge p_M$ for all $t \ge T' := e^{p_M} - \varepsilon$. Hence, $T' \in [0, e^{p_M})$ exists for any $\varepsilon > 0$. On the other hand, the control input in (7) is bounded along trajectories and, as a matter of fact,

$$\lim_{t\to\infty} |u(e(t))| = 0.$$

This follows from the previous proof and using the explicitly exponential bounds on the synchronization errors. More precisely, note that $|u| = \left| c(t)[-\mathcal{L} \otimes B^\top M]x \right| = |c(t)[-\mathcal{L} \otimes B^\top M]e|$, so, along the synchronization-error trajectories,

$$|u(e(t))| \le c(t)| - \mathcal{L} \otimes B^\top M||e(t)|.$$

Then, using (15) with (23), we see that

$$|u| \le \kappa \left| -\mathcal{L} \otimes B^\top M \right| |e(0)| c(t) e^{-\gamma t}, \quad \forall t \ge 0.$$

Therefore, any strictly increasing function that does not grow faster than exponentially at the rate that the error trajectories converge γ is suitable. In particular,

$$\lim_{t\to\infty} \ln(t+\varepsilon) e^{-\gamma t} = 0$$

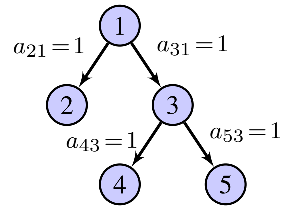
for any $\varepsilon > 0$, so (2) holds.

IV. Simulation results

To illustrate our theoretical findings, we provide some numerical simulation results done using Matlab™ R2021a. The simulation tests are done using five harmonic oscillators modeled by Eq. (1), with

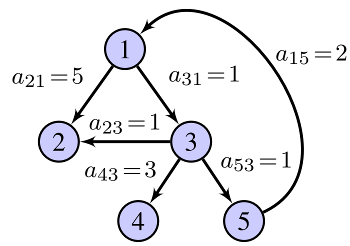
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We consider two graphs with different topology. First, as in [8], [9], and [13], we consider a leader-follower scheme in which the graph consists in a simple rooted spanning tree. In the second case the directed graph is connected. The respective graphs and the corresponding Laplacians are provided in Figures 1 and 2, below.



$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Fig. 1. Example 1: spanning-tree graph and corresponding Laplacian



$$L = \begin{bmatrix} 2 & 0 & 0 & 0 & -2 \\ -5 & 6 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Fig. 2. Example 2: connected graph and corresponding Laplacian

For both cases, we use the control law in (4) with the coupling strength c defined as $c(t) := \ln(\varepsilon + t)$, with $\varepsilon = 1.5$. To compute the control gain F, we start by setting $Q_0 = I_2$ and we solve the algebraic Riccati equation (5) for M. We obtain

$$M = \begin{bmatrix} 1.912 & 0.4142 \\ 0.4142 & 1.352 \end{bmatrix},$$

which is positive definite. Then, we compute the controller gain, $F = B^T M$, which yields $F = [0.4142 \quad 1.352]$.

Now, for the sake of fair comparison, we use the same initial conditions for both cases:

$$\begin{aligned} x_1(0) &= [\ 1 \quad -1 \quad 1.5 \quad 2 \quad 3\ ]^\top, \\ x_2(0) &= [\ 2 \quad -2 \quad 0.5 \quad 3 \quad 3.5\ ]^\top. \end{aligned}$$

For the case of leader-follower network as shown in Figure 1, the left eigenvector v_l associated with zero eigenvalue of the graph's Laplacian is given by $v_l = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$. This corroborates that the first agent is the singular root node in the network. So, the emergent dynamics corresponds to that of the first agent—see Figure 3. From the evolution of individual state variables (represented by dashed lines), it can be concluded that all the trajectories converge to that of the mean-field system (represented by a solid blue line). The control input subplot in Figure 3 shows that the control input dies down to zero as the multi-agent system reaches consensus.

For the case of connected directed graph, the left eigenvector associated with the zero eigenvalue of the graph's Laplacian is given by $v_l = \begin{bmatrix} \frac{2}{7} & 0 & \frac{1}{7} & 0 & \frac{4}{7} \end{bmatrix}^\top$, so the mean-field state $x_m := (v_l^\top \otimes I_n)x$ corresponds to a weighted linear combination of state variables of the agents that can transfer information to all other agents, which in this case, are the first, third, and fifth agents. This is reflected in the fact that the first,

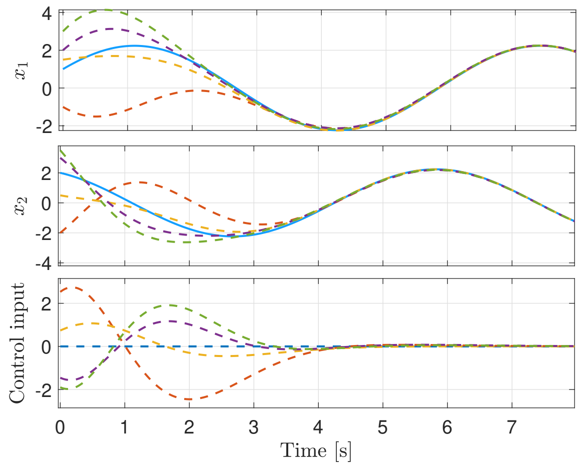


Fig. 3. Distributed consensus of x_1 and x_2 and the control input when the communication graph has a spanning tree.

third and fifth elements of v_l are non-zero. The simulation plots for this case are portrayed in Figure 4.

The states x_1 and x_2 converge to their mean-field values, showed as solid blue lines in Figure 4, so the corresponding synchronization errors converge to zero. As expected from the emergent dynamics (11), the steady state oscillator is a weighted averaged oscillator influenced by the nodes in the network that can transfer information to all the remaining nodes. The control input sub-plot is shown in Figure 4 which dies down to zero as synchronization dies down to zero.

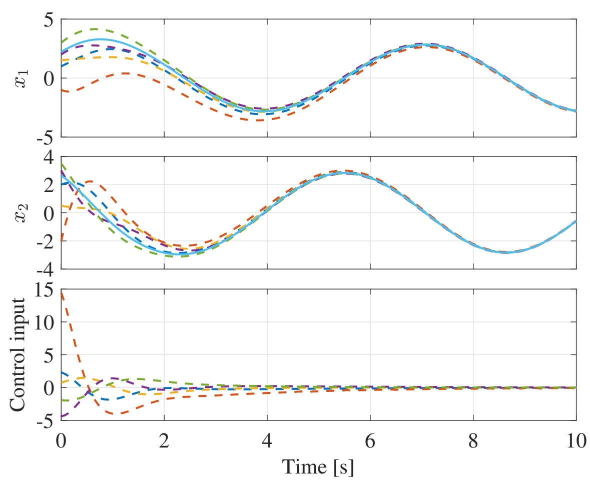


Fig. 4. Distributed consensus of x_1 and x_2 and the control input when the communication graph is connected.

V. Conclusion

The collective behaviour of "general" linear systems interconnected over a directed graph may be complex. Via a change of coordinates that depends on the network topology (the eigenvectors of the Laplacian matrix) it is possible to exhibit the dichotomous character of the resulting motion. Two variables belonging to orthogonal spaces appear, one corresponding to the synchronization errors and another to a "weighted-average" dynamics. The synchronization errors converging to zero is known as dynamic consensus.

The results presented in this letter are fairly general, as they establish dynamic consensus for merely stabilizable linear systems interconnected over arbitrary connected directed graphs. The most important contribution, however, is to provide strict Lyapunov functions (in the synchronization errors space). This is significant because it may serve as basis to extend our results to other interesting scenarios. For instance, to consider linear systems with certain degree of heterogeneity or with added nonlinearities. Disposing of strict Lyapunov functions may also be useful in the study of perturbed networked systems, of robustness with respect to neglected dynamics, and networks with time-varying topology. Such topics are currently under investigation.

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