A Novel Approach for Trajectory Tracking Control of an Under-actuated Quad-rotor UAV

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***Abstract*—A novel Udwadia-Kalaba approach for the trajectory tracking control of an under-actuated quad-rotor unmanned aerial vehicle (UAV) is presented. Compared to standard control approaches, the desired trajectories are treated as constraints called trajectory tracking constraints in this approach. Neither making any approximations or linearization of the nonlinear system nor imposing any a priori structure on the nature of the nonlinear controller, this methodology provides closed-form nonlinear control. The control inputs satisfying the desired trajectory requirements can be obtained explicitly in compact closed form by solving Udwadia-Kalaba equation. Nonlinear dynamics modeling of a quad-rotor UAV is processed and a desired trajectory is given in this paper to illustrate this approach. The theoretical analysis and MATLAB simulation results verify the validity and efficiency of this approach. The real-time servo constraint forces are obtained conveniently and the quad-rotor UAV's movement meets the designed trajectory precisely.**

***Index Terms*—Servo constraint force, trajectory tracking constraint, under-actuated, quad-rotor UAV, Udwadia-kalaba approach, .**

I. INTRODUCTION

RECENTLY various researches are carried out for developing unmanned aerial vehicles (UAVs). UAVs have broad practical application prospects in both military and public services such as in environmental research, resource exploration, national defense, material transport, logistics, search and rescue, space detection and other fields where it is dangerous or difficult for human. In order to accomplish these unmanned autonomous objects, UAVs are often required to follow desired trajectories autonomously.

As a kind of UAVs, a quad-rotor UAV has easier implementation and more excellent maneuverability compared to other UAVs. The quad-rotor is a classical under-actuated system with characteristics of nonlinear and strong coupling, such that its tracking control becomes especially difficult. Various studies have been done focusing on the trajectory tracking control problem of under-actuated quad-rotor UAVs in recent years $[1-8]$. Linear control methods such as classical PID (Proportional-Integral-Differential) and LQR (Linear Quadratic Regulator) control [9] – [11] produce unsatisfactory performance because of the nonlinearity of the system. In order to improve control stability, various nonlinear control methods are proposed, such as backstepping control [12], [13], sliding mode control [14], [15] and nonlinear controller design based on visual feedback [16], [17]. However, all these methods mentioned above either make some assumptions or a linearization of the nonlinear system or impose a priori structure on the controller's nature, which increases control complication largely, and the tracking performance is also not very optimistic.

In this paper, a novel Udwadia-Kalaba approach is considered to solve the trajectory tracking problem of the quad-rotor. The Udwadia-Kalaba approach provides a new perspective for dealing with the under-actuated quad-rotor system. Firstly, this approach provides closed-form nonlinear control, neither making any linearization or approximations of the nonlinear system nor imposing any a priori structure on the nature of the nonlinear controller as common control methods usually do especially for an under-actuated system. Furthermore, this approach treats the desired trajectory as a constraint of the mechanical system called trajectory tracking constraint. By solving Udwadia-Kalaba equation, the control inputs (lift forces in this paper) can be obtained explicitly in closed form. An excellent tracking performance of the quad-rotor is acquired in this paper by using Udwadia-Kalaba approach.

Classical theories for dynamic modeling of constrained mechanical system (e.g., Newton-Euler equation, lagrangian equation, Maggi equation, Boltzmann and Hamel equation, etc.) treat d'Alembert's principle and principle of virtual displacements as their starting point [18] – [24]. However, these assumptions do not apply well to all situations. Moreover, dynamic modeling process becomes complicated and difficult especially for many-degree-of-freedom mechanical systems by using lagrangian equation. Pars (1965) indicates the lagrangian equations of the unconstrained motion of mechanical systems yield non-singular, symmetric and positive definite mass matrix while the minimum number of coordinates are employed. This restricts the flexibility and multiplicity of one's modeling since systems with singular mass matrices are not common in classical dynamics when dealing with unconstrained motion.

Udwadia and Kalaba (1992, 1996) derived general and explicit fundamental dynamic equation of constrained systems from Gauss's principle in their theory, no matter its constraints are holonomic or not [25] – [29]. This theory can deal with ideal and also non-ideal constraints, producing explicit and closed-form formulation. The motion equation of constrained system can be obtained concisely by using this theory, without consideration of the system'sphysical structure's function as in classical methods. Thus it is especially applicable for dynamic

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modeling of many-degree-of-freedom mechanical systems. And the number of coordinates turns out less important, as long as the constraint equations are consistent and the mass matrix is positive definite.

The trajectory tracking control is in fact an inverse dynamics. In analytical dynamics, the tracking control problem can be redefined as servo constraint control problem. The Udwadia-Kalaba theory simplifies this problem in a new way. Using the research results of Udwadia-Kalaba, Chen systematically put forward the concept of servo constraint control of mechanical systems and achieved the design of constraint force by servo control [30] – [32]. Also Chen studied servo constraint problems on the basis of Maggi equation and indicated that the required constraint force can be obtained by servo control [33]. Bajodah studied some mathematical computation problems in the servo control problem on Udwadia-Kalaba equation, providing some theoretical basis for the practical application of the equation [34]. Using this equation, Chen also carried on the thorough research on the adaptive robust control of uncertain systems [35] – [39]. Schutte studied the control problem of nonlinear mechanical systems with holonomic and non-holonomic constraints on the basis of Udwadia-Kalaba equation and proposed two types of nonlinear state feedback controllers which were shown to provide exact tracking and stabilization to the constrained system under certain conditions [40]. Udwadia firstly applied the servo constrained control method in the tracking control of nonlinear structural and mechanical systems and made a preliminary research in this field [41].

This paper proposes a dynamic modeling of an underactuated four-rotor UAV and a desired flying path is given. A novel control design is worked out based on Udwadia-Kalaba approach. By solving Udwadia-Kalaba equation, the control inputs satisfying the desired trajectory requirements are obtained explicitly and in compact closed form. To verify the tracking performance, MATLAB simulation by ode45 integrator is processed. The simulation results indicate that the servo forces are solved conveniently and the quad-rotor shows an excellent tracking performance.

II. The Fundamental Equation of Udwadia-Kalaba Approach

We first consider the unconstrained discrete mechanical system. The n generalized coordinates of the system are assumed as $q := (q_1\, q_2\, \cdots\, q_n)^T$. By using Newtonian or lagrangian equation [25], the equation of motion of the unconstrained system can be written as

$$M(q,t)\ddot{q} = Q(q,\dot{q},t), \quad q(0) = q_0;\ \dot{q}(0) = \dot{q}_0 \tag{1}$$

where $M(q,t)$ is a positive definite $n \times n$ inertia matrix. $Q(q,\dot{q},t)$ is an $n \times 1$ vector denoting the known force imposed on the system whose constraints are released, and may include centrifugal force, gravitational force and control input. $\dot{q}$ is the $n \times 1$ vector of velocity and $\ddot{q}$ is the vector of acceleration respectively. The initial conditions at time t_0 are defined by q_0 and $\dot{q}_0$ respectively. The generalized acceleration at time t of the unconstrained system, which is defined by $a(q,\dot{q},t)$, is thus given by

$$\ddot{q} = M^{-1}(q,t)Q(q,\dot{q},t) = a(q,\dot{q},t). \tag{2}$$

Then, constraints presented in the system are considered. We assume that the system is subjected to h holonomic constraints in the form of

$$\varphi_i(q,t) = 0, \quad i = 1, 2, \cdots, h. \tag{3}$$

And also there are $m - h$ non-holonomic constraints of the form

$$\varphi_i(q,\dot{q},t) = 0, \quad i = h+1, h+2, \cdots, m. \tag{4}$$

Here the equations must be consistent in description of any given set of constraints, and we do not care whether the constraints are linearly independent or not. Under the assumption of sufficient smoothness, by differentiating holonomic constraints (3) twice and non-holonomic constraints (4) once with respect to time t, 2-order constraint equations in the form of matrix equation can be acquired, which is written as

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t) \tag{5}$$

where $A(q,\dot{q},t)$ is an $m \times n$ matrix denoting the constraint matrix and $b(q,\dot{q},t)$ is an $m \times 1$ vector.

Remark 1: The constraint used in classical mechanics, such as lagrangian equation, Maggi equation, Boltzmann and Hamel equation, Gibbs and Appell equation, etc., is either in the 0-order or 1-order form. The Udwadia-Kalaba equation first converted all the constraints (holonomic constraints as well as non-holonomic constraints) into 2-order forms (Chen 1998), which is significant for the flexible modeling of the equation of motion. When modeling of a constrained mechanical system by using Udwadia-Kalaba equation, one needs to do is just considering the constraints of the unconstrained system and then transforming them into 2-order matrix equations in the form of (5), by differentiating the constraints with respect to time t once or twice.

Therefore the constraints of the system are conveniently modeled. By combining the unconstrained equation and the constraints, the explicit equation of motion with constraints can be acquired. Additional generalized forces of constraints resulting from the constraints should be imposed on the system. We assumed the actual explicit equation of motion of the constrained system in the form of

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q^c(q,\dot{q},t) \tag{6}$$

where $Q^c(q,\dot{q},t) \in \mathbb{R}^n$ is the additional generalized forces imposed on the system, arising due to the holonomic and non-holonomic constraints. $Q^c(q,\dot{q},t)$ is considered to be ideal in lagrangian mechanics, which is derived based on d'Alembert's principle indicating that the constraint forces do zero work under virtual displacements. The ideal constraints generate non-ideal constraint forces on the basis of d'Alembert's principle. In a practical mechanical system, non-ideal constraints also exist and generate non-ideal constraint forces such as friction force, electro-magnetic force, etc., [27]. Considering a constraints mechanical system with ideal as well as non-ideal constraints, $Q^c(q,\dot{q},t)$ can be given by

$$Q^c(q,\dot{q},t) = Q^c_{id}(q,\dot{q},t) + Q^c_{nid}(q,\dot{q},t) \tag{7}$$

where $Q^c_{id}(q,\dot{q},t)$ represents the ideal constraint force and $Q^c_{nid}(q,\dot{q},t)$ the non-ideal one respectively.

Udwadia extends the lagrangian form of d'Alembert's principle to include non-ideal constraints as described in (7). He generalizes d'Alembert's principle to include forces of constraint that may do positive, negative, or zero work under virtual displacement at any instant of time during the motion of the constrained system. We denote constraint force as $c(q,\dot{q},t) \in \mathbb{R}^n$ and its work $W = v^T c$ in any displacement v subjecting to $A(q,\dot{q},t)v = 0$. Since the work done by $c(q,\dot{q},t)$ equals to that done by $Q^c(q,\dot{q},t)$, we have

$$W = v^T Q^c = v^T c \tag{8}$$

which is the extended lagrangian form of d'Alembert's principle. The work done by the ideal constraint force Q^c_{id} under virtual displacements is

$$v^T Q^c_{id} = 0 \tag{9}$$

and the work done by non-ideal constraint force Q^c_{nid} is

$$v^T Q^c_{nid} \neq 0. \tag{10}$$

Udwadia and Kalaba have proved that the ideal constraint force takes the form

$$Q^c_{id} = M^{1/2}B^+(b - AM^{-1}Q) \tag{11}$$

and non-ideal constraint force takes the form

$$Q^c_{nid} = M^{1/2}(I - B^+B)M^{-1/2}c \tag{12}$$

where $B = AM^{-1/2}$, and the superscript " $+$ " denotes the Moore-Penrose generalized inverse.

Remark 2: When describing the constrained motion by using Udwadia-Kalaba equation, the Moore-Penrose generalized inverse of constraint matrix A as shown in (11) and (12) is a substantial tool in the calculation of the constraint force. To obtain the explicit equation of motion of the constrained mechanical system, the rank of the matrix A is not essential. The Moore-Penrose generalized inverse gives a deeper insight into the nature of constrained motion of mechanical system. The Moore-Penrose generalized inverse A^+ which is unique possesses the following characteristics

$$\begin{aligned} AA^+ &= (AA^+)^T, \quad & A^+A &= (A^+A)^T, \\ AA^+A &= A, & A^+AA^+ &= A^+ \end{aligned} \tag{13}$$

From (6), (7), (11) and (12), the explicit equation of motion that governs the evolution of the constrained system including both ideal and non-ideal constraints is given by

$$M\ddot{q} = Q + M^{1/2}B^+(b - AM^{-1}Q) + M^{1/2}(I - B^+B)M^{-1/2}c \tag{14}$$

where the vector c is determined by the engineer, which can be obtained by experiment or observation.

Equation (14) is called the Udwadia-Kalaba fundamental equation of motion. When c equals zero, say, the constraints are ideal and the total work done under virtual displacement is zero according to the d'Alembert's principle, equation (14) becomes

$$Q^c = Q^c_{id} = M^{1/2}B^+(b - AM^{-1}Q) \tag{15}$$

and the explicit equation of motion of the constrained system including only ideal constraints can be written as

$$M\ddot{q} = Q + M^{1/2}B^+(b - AM^{-1}Q). \tag{16}$$

Thus, at any instant of time t, the constrained system is subjected to an additional constraint force $F^c(t)$, given by

$$F^c(t) = M^{1/2}B^+(b - AM^{-1}Q). \tag{17}$$

When the matrix M is a constant diagonal matrix, so we have $M = mI$. Equation (17) simplifies to

$$F^c(t) = mA^+(b - AM^{-1}Q). \tag{18}$$

Furthermore, when the unconstrained acceleration $M^{-1}Q$ is zero, equation (18) becomes

$$F^c(t) = mA^+b. \tag{19}$$

If the mass matrix M in (14) is singular, the Udwadia-Phohomsiri equation is utilized (Udwadia & Phohomsiri 2006) instead of Udwadia-Kalaba equation (Udwadia & Kalaba 2000) to acquire the equation of motion of the constrained system.The equation of motion is given by [29]

$$\ddot{q} = \begin{bmatrix} [I - A^+A]M \\ A \end{bmatrix}^+ \begin{bmatrix} Q \\ b \end{bmatrix} \tag{20}$$

where the superscript " $+$ " denotes the Moore-Penrose generalized inverse (Moore 1920; Penrose 1955). Equation (20) is valid when the matrix $[M|A]T$ is in full rank (Udwadia & Phohomsiri 2006). The full rank condition is essential for the equation of motion of the constrained system to be unique, which can be used to check whether the proposed model is correct.

III. Trajectory Tracking Control of a Quad-rotor UAV

A. Dynamics of the Quad-rotor

As is shown in Fig. 1, every rotor driven by a DC servo motor produces lift force as well as moment [3], [11]. It is assumed that the body fixed frame $B\{x_b, y_b, z_b\}$ is created at the mass center of the rigid quad-rotor body where the z-axis is pointing upwards. The frame B has six degrees of freedom with respect to the earth fixed frame $I\{x, y, z\}$ which is assumed as an inertial frame. Therefore the position and orientation of the quad-rotor can be described as a position vector $p = (x\ y\ z)^T$ and an orientation vector $r = (\theta\ \psi\ \phi)^T$. θ, ψ and ϕ are Euler angles corresponding to x_b-axis, y_b-axis and z_b-axis respectively. Let $q = (x\ y\ z\ \theta\ \psi\ \phi)^T \in \mathbb{R}^6$ denote the generalized coordinates of the system.

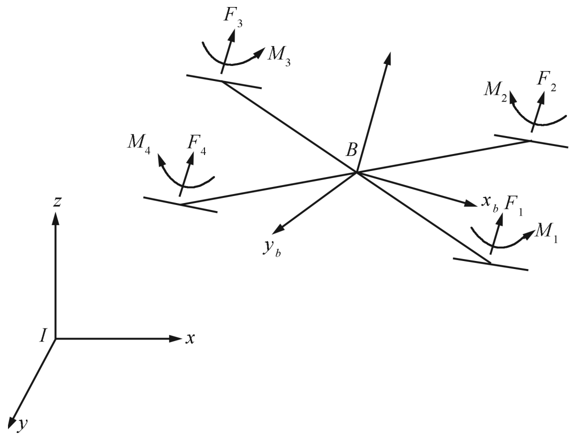

Fig. 1. The quad-rotor modeling

The total lift force $\sum_{i=1}^{4} F_i$ generated by the four rotors points at z_b-axis, thus the total lift force vector can be described in the body fixed frame B as

$$F_B = \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^{4} F_i \end{pmatrix}, \quad i = 1, 2, 3, 4 \tag{21}$$

where, F_B is the total lift force vector, the subscript B indicates this vector is described in frame B. $F_i(i = 1, 2, 3, 4)$ is the lift force produced by the $i-$th rotor respectively.

In order to move the quad-rotor model from earth to the fixed mass center, the direction cosine matrix from frame I to frame B is donated by R [15], which is given by

$$R = \begin{bmatrix} \cos\psi\cos\phi & \sin\theta\sin\psi\cos\phi - \cos\theta\sin\phi & \cos\theta\sin\psi\cos\phi + \sin\theta\sin\phi \\ \cos\psi\sin\phi & \sin\theta\sin\psi\sin\phi + \cos\theta\cos\phi & \cos\theta\sin\psi\sin\phi - \sin\theta\cos\phi \\ -\sin\psi & \sin\theta\cos\psi & \cos\theta\cos\psi \end{bmatrix} \tag{22}$$

where, the transformation matrix R is the direction cosine matrix. θ, ψ and ϕ are the roll angle, the pitch angle and yaw angle, respectively.

Therefore, the corresponding x-axis, y-axis and z-axis component force vector in frame I can be written as

$$\begin{aligned} U_I = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = RF_B \\ = \begin{pmatrix} cos\theta sin\psi cos\phi + sin\theta sin\phi \\ cos\theta sin\psi sin\phi - sin\theta cos\phi \\ cos\theta cos\psi \end{pmatrix} \sum_{i=1}^{4} F_i, \quad i = 1, 2, 3, 4 \end{aligned} \tag{23}$$

where, U_I is the total lift force vector, the subscript I indicates this vector is described in frame I.

Then by using force and moment balance, the dynamic equation of motion can be written as

$$m\ddot{x} = U_1 - K_1\dot{x} \tag{24}$$

$$m\ddot{y} = U_2 - K_2\dot{y} \tag{25}$$

$$m\ddot{z} = U_3 - K_3\dot{z} \tag{26}$$

$$I_x\ddot{\theta} = l(-F_1 + F_2 + F_3 - F_4) - K_4\dot{\theta} \tag{27}$$

$$I_y\ddot{\psi} = l(F_1 + F_2 - F_3 - F_4) - K_5\dot{\psi} \tag{28}$$

$$I_z\ddot{\phi} = M_1 - M_2 + M_3 - M_4 - K_6\dot{\phi} \tag{29}$$

where, m is the mass of the quad-rotor. $F_i(i = 1, 2, 3, 4)$ is the lift force generated by the $i-$th rotor and $M_i(i = 1, 2, 3, 4)$ is the additional moment due to rotation of the corresponding rotor, imposed on the quad-rotor body. $K_i(i = 1, 2, 3, 4, 5, 6)$ is the aerodynamic drag coefficient corresponding to the UAV′s velocity and angular velocity $\dot{q} = (\dot{x}\ \dot{y}\ \dot{z}\ \dot{\theta}\ \dot{\psi}\ \dot{\phi})^T \in R^6$. l is the distance from the center of rotation of the rotor to x_b-axis or y_b-axis. I_x, I_y and I_z are the moment of inertia of the quad-rotor around x-axis, y-axis and z-axis respectively.

Remark 3: The moment M_i has been experimentally observed to be linearly dependent on the force F_i for low speeds. Since the four forces are the input parameters to be controlled, the relationship between M_i and F_i can be modeled. Equation (29) becomes

$$I_z\ddot{\phi} = lC(F_1 - F_2 + F_3 - F_4)) - K_6\dot{\phi} \tag{30}$$

where the constant C is the force to moment scaling factor, in this paper, $C = 0.05$.

Since drag is negligible at low speeds, the drag coefficients given above are assumed to be zero. By combining (23) – (28), the mathematical model of the quad-rotor dynamics becomes

$$\begin{aligned} \begin{bmatrix} \frac{m}{R_{13}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{m}{R_{23}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m}{R_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{I_x}{l} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{I_y}{l} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{I_z}{lC} \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\theta} \\ \ddot{\psi} \\ \ddot{\phi} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{mg}{R_{33}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \end{aligned} \tag{31}$$

where $R_{13} = cos\theta sin\psi cos\phi + sin\theta sin\phi$, $R_{23} = cos\theta sin\psi sin\phi - sin\theta cos\phi, R_{33} = cos\theta cos\psi$.

Matrix (31) can be rewritten in the form of equation (6)

$$M(q, t)\ddot{q} = Q(q, \dot{q}, t) + Q^c(q, \dot{q}, t) \tag{32}$$

where

$$M = \begin{bmatrix} \frac{m}{R_{13}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{m}{R_{23}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m}{R_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{I_x}{l} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{I_y}{l} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{I_z}{lC} \end{bmatrix} \tag{33}$$

$$Q = -\begin{pmatrix} 0 \\ 0 \\ \frac{mg}{R_{33}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{34}$$

$$Q^c = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \tag{35}$$

$$q = \begin{pmatrix} x \\ y \\ z \\ \theta \\ \psi \\ \phi \end{pmatrix}, \quad \dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix}, \ddot{q} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\theta} \\ \ddot{\psi} \\ \ddot{\phi} \end{pmatrix} \tag{36}$$

Remark 4: The second part of the right side of (32) is the control input (i.e., the lift forces F_i generated by servo DC motors) desired to be applied to the quad-rotor to realize the trajectory tracking control. That is, while the quad-rotor is desired to fly along a designed specific trajectory, every servo DC motor must drive its corresponding rotor to generate a certain level of lift force F_i respectively to make sure the mechanical system realize the required trajectory tracking control target.

Therefore, if there are no trajectory constraints, the unconstrained dynamic equation of the quad-rotor can be written as

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) \tag{37}$$

B. Trajectory Dracking Constraints

In Udwadia-Kalaba theory, the problem of constrained motion in analytical dynamics can also be described as a trajectory tracking control problem. Based on Udwadia-Kalaba theory, the desired trajectories are treated as constraints named trajectory tracking constraints, which can be written in the form of constant (5). Thus the servo constraint force $F^c(t)$ can be redefined as the control input required to apply to the mechanical system to realize the trajectory tracking control.

The desired trajectory tracking constraints of the constrained mechanical system can be modeled as

$$\sum_{i=1}^{n} A_{li}(q,t)\dot{q}_l + A_l(q,t) = 0, \quad l = 1,2,\ldots,m \tag{38}$$

where $n \geq m \geq 1$, $A_{li}(\cdot)$ and A_l are both C^1 in q and t. These constraints imply restrictions on the velocities as well as the displacements, and are the 1-order forms of the constraints.

We now transform the constraints into 2-order forms (5). Differentiating constraint equation (38) with respect to t once or twice, yields

$$\sum_{i=1}^{n} \frac{d}{dt}A_{li}(q,t)\dot{q}_l + \sum_{i=1}^{n} A_{li}(q,t)\ddot{q}_l + \frac{d}{dt}A_l(q,t) = 0 \tag{39}$$

where

$$\frac{d}{dt}A_{li}(q,t) = \sum_{k=1}^{n} \frac{\partial A_{li}(q,t)}{\partial q_k}\dot{q}_k + \frac{\partial A_{li}(q,t)}{\partial t} \tag{40}$$

and

$$\frac{d}{dt}A_l(q,t) = \sum_{k=1}^{n} \frac{\partial A_l(q,t)}{\partial q_k}\dot{q}_k + \frac{\partial A_l(q,t)}{\partial t}. \tag{41}$$

The 2-order forms of the constraints (39) can be rewritten as

$$\sum_{i=1}^{n} A_{li}(q,t)\ddot{q}_l = -\sum_{i=1}^{n} \frac{d}{dt}A_{li}(q,t)\dot{q}_l - \frac{d}{dt}A_l(q,t)$$
$$=: b_i(q,\dot{q},t), \quad l = 1,2,\ldots,m \tag{42}$$

or, in a matrix form

$$A(q,t)\ddot{q} = b(q,\dot{q},t) \tag{43}$$

where $A = [A_{li}]_{m\times n}$ and $b = [b_1 \ b_2 \ \ldots \ b_m]^T$.

Remark 5: What one needs to do is to model the constraints first and then transform them into 2-order forms by differentiating the constraint equations with respect to time t. Specifically, if a constraint equation is given in 0-order form, we then differentiate it with respect to time t twice, and 1-order form once. Thus the 2-order form constraint equations are conveniently acquired.

Remark 6: In fact, the 2-order form constraint (43) is a very general form. It includes typical constraints as illustrated by Rosenberg (1977) and Papastavridis (2002), as well as a number of standard control problems such as stabilization, trajectory following and optimality (Chen 1998, 1999). The trajectory tracking constraint in this paper is just one of the above constraints.

C. Servo Control Input

The desired trajectory of the quad-rotor which is required to be tracked is described in the form of constraint (43), so the constraint force $Q^c(t)$ (i.e., the control force) should be applied to realize the trajectory tracking control target according to Udwadia-Kalaba equation. From (17), the control input, say, the servo constraint force F^c can be written as

$$F^c = Q^c = M^{1/2}(AM^{-1/2})^+(b - AM^{-1}Q). \tag{44}$$

The constraint force is provided by the motors′ active servo controls. Based on the available controls, the structure of the constraint force is predetermined as

$$Q^c = B\tau \tag{45}$$

where the input matrix B is determined by the structure of the available servo controls and its actual servo control input. Thus the actual control input can be given by

$$\tau = B^+Q^c \tag{46}$$

so that

$$\tau = B^+M^{1/2}(AM^{-1/2})^+(b - AM^{-1}Q) \tag{47}$$

where the superscript " + " denotes the Moore-Penrose generalized inverse.

Remark 7: One should determine the servo structure by choosing B to get Q^c in (45) for a particular mechanical system. Then, the dynamic system is formulated in the form of (6). By solving the control input F^c based on (44), the actual servo control input τ can be constructed using (46). The input matrix B is very useful in this approach. The servo control input can be obtained conventionally by determining the input matrix B, no matter the system is general, under-actuated or over-actuated.

For the under-actuated quad-rotor in this paper, from (32), we have

$$Q^c = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = B\tau \tag{48}$$

where

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad \tau = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \tag{49}$$

The real-time lift forces generated by the servo DC motors can be acquired by solving (47).

IV. Trajectory Tracking Simulation

A. The Desired Trajectories

We assume every degree of freedom of the quad-rotor is given a constraint equation respectively, to accomplish a certain task. The designed UAV movement is assumed to tracking a helical trajectory described as

$$\begin{aligned} x &= 10\sin(t) \\ y &= 10\cos(t) \\ z &= 3t \\ \theta &= \frac{\pi}{4} + \frac{\pi}{8}\sin\left(\frac{\pi}{2}t\right) \\ \psi &= \frac{\pi}{2} + \frac{\pi}{3}\cos\left(\frac{\pi}{2}t\right) \\ \phi &= \frac{\pi}{4}\sin\left(\frac{\pi}{3}t\right) \end{aligned} \tag{50}$$

Differentiate the constraint (50) with respect to t twice, we have

$$\begin{aligned} \ddot{x} &= -10\sin(t) \\ \ddot{y} &= -10\cos(t) \\ \ddot{z} &= 0 \\ \ddot{\theta} &= -\frac{\pi^3}{32}\sin\left(\frac{\pi}{2}t\right) \\ \ddot{\psi} &= -\frac{\pi^3}{12}\cos\left(\frac{\pi}{2}t\right) \\ \ddot{\phi} &= -\frac{\pi^3}{36}\sin\left(\frac{\pi}{3}t\right) \end{aligned} \tag{51}$$

The system constraints can be written in the form of (52)

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t) \tag{52}$$

where

$$A = I_{6\times6}, \quad b = \begin{pmatrix} -10\sin(t) \\ -10\cos(t) \\ 0 \\ -\frac{\pi^3}{32}\sin(\frac{\pi}{2}t) \\ -\frac{\pi^3}{12}\cos(\frac{\pi}{2}t) \\ -\frac{\pi^3}{36}\sin(\frac{\pi}{3}t) \end{pmatrix}, \quad \ddot{q} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\theta} \\ \ddot{\psi} \\ \ddot{\phi} \end{pmatrix} \tag{53}$$

where $I_{6\times6}$ represents the unit matrix.

Remark 8: Constraints (50) is defined as trajectory constraints. The four servo DC motors must drive their rotors to generate corresponding lift forces defined as the control inputs, to fulfill the trajectory tracking control task.

Initial conditions containing initial coordinate and velocity of the mass center of the quad-rotor are given in Table I.

TABLE I
Initial Conditions for Simulation

x_0	y_0	z_0
0	10	0
θ_0	ψ_0	ϕ_0
$\pi/4$	$5\pi/6$	0
$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$
10	0	3
$\dot{\theta}_0$	$\dot{\psi}_0$	$\dot{\phi}_0$
$\pi^2/16$	0	$\pi^2/12$

Remark 9: The initial conditions $q(0) = q_0$, $\dot{q}(0) = \dot{q}_0$ are required to satisfy the desired trajectory tracking constraint equations. However, it is usually difficult to achieve in practical engineering application due to various factors.

According to the Lyapunov stability theory, the following differential equation can be constructed

$$g(q,\dot{q},t) = -f(g,t;r) \tag{54}$$

where $f(g,t;r)$ is an $m\times1$ vector, r is a parameter vector related to the system's dynamic characteristics. $f(g,t;r)$ is chosen so that the system has the following two conditions:
1) $g = 0$ is an equilibrium point of the system;
2) This equilibrium point is globally asymptotically stable.

Usually numerous systems satisfying the above two conditions can be constructed, such as $\dot{g}_i = -r_i g_i$, where the constant $r_i > 0, i = 1,2,\ldots,m$.

If the m desired trajectory constraints are holonomic in the form of

$$g_i(q,t) = 0, \quad i = 1,2,\ldots,m \tag{55}$$

these constraints can be modified as

$$\ddot{g}_i = \lambda_i\dot{g}_i + \mu_i g_i = 0, \quad i = 1,2,\ldots,m \tag{56}$$

with $\lambda_i, \mu_i > 0$, so that the equilibrium solution of the system $g = \dot{g} = 0$ is asymptotically stable.

Here for the simplification of simulation, the initial conditions given above satisfy the desired trajectory tracking constraint equations.

Table II shows the parameters used in the simulation.

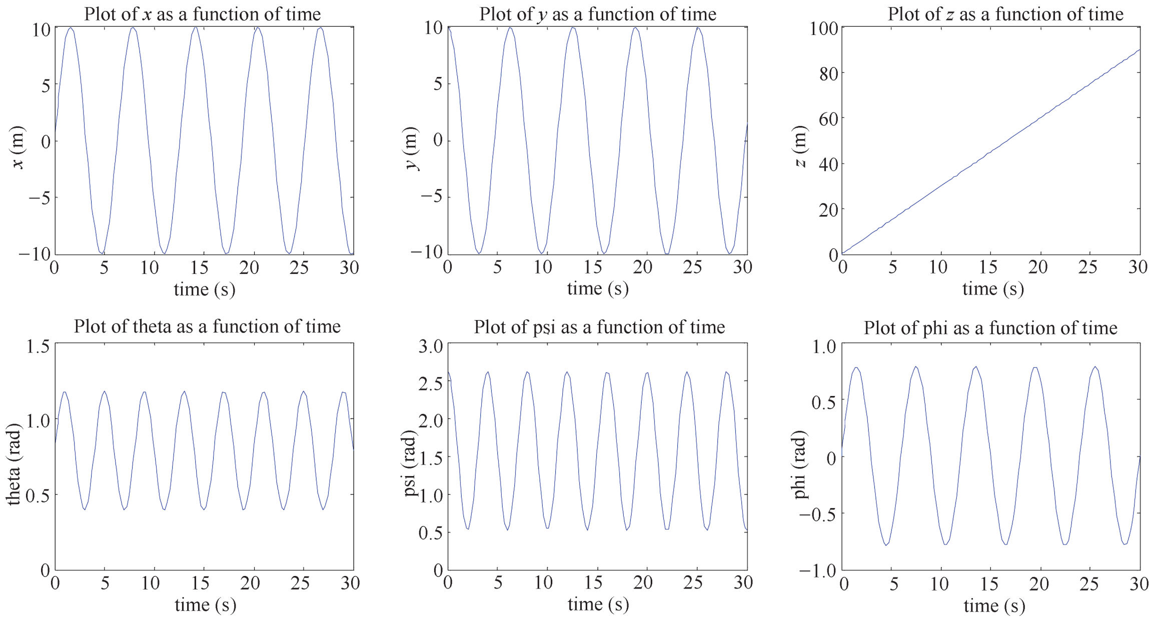

Fig. 2. The simulated position and orientation of the quad-rotor as functions of time

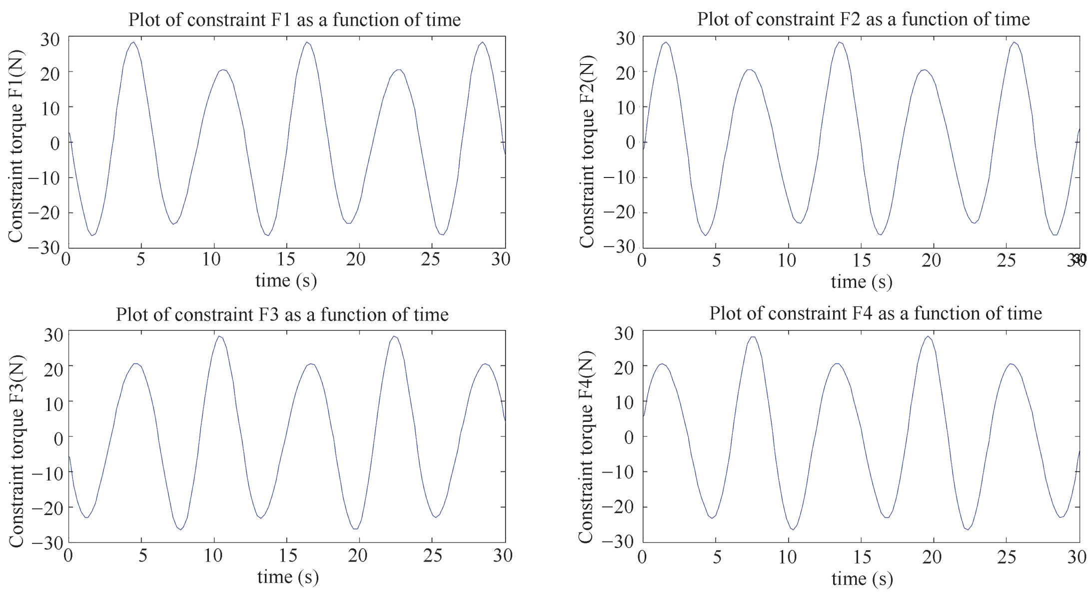

Fig. 3. The simulated lift forces generated by DC servo motors as functions of time

TABLE II
PARAMETERS FOR SIMULATION

Parameter	Name	Vale
m	Mass of the quad-rotor	2 kg
l	Arm distance of the quad-rotor	0.2 m
I_x	Moment of inertia around x-axis	1.25 kg·m^2
I_y	Moment of inertia around y-axis	1.25 kg·m^2
I_z	Moment of inertia around z-axis	1.25 kg·m^2
C	Force to moment scaling factor	0.05

B. Simulation Results

The simulation is processed in MATLAB by ode45 solver and the simulation time is 30 seconds. The real-time lift forces generated by the four DC servo motors can be obtained by solving Udwadia-Kalaba (46), as described in Fig. 3. Fig. 2 shows the simulated position and orientation as functions of time t of the quad-rotor UAV, where x, y, z represent the x-axis, y-axis, z-axis displacement and θ, ψ, ϕ the roll, pitch, yaw angle in the inertial frame respectively.

Fig. 4 shows the numerical errors of position and orientation of the mass center of the quad-rotor as functions of time. It is clear that the error between the simulated position and orientation and the desired, say, the trajectory tracking constraint (50), is small enough seen from Fig. 4. More specifically, here numerical errors $e_1 = x - x_s$, $e_2 = y - y_s$, $e_3 = z - z_s$, $e_4 = \theta - \theta_s$, $e_5 = \psi - \psi_s$ and $e_6 = \phi - \phi_s$, where x_s, y_s, z_s, θ_s, ψ_s and ϕ_s denote the simulated position and orientation of the quad-rotor. The position and orientation error of the mass center of the quad-rotor $e_i, (i = 1, 2, 3, 4, 5, 6)$ is of the order of 10^{-3}, 10^{-3}, 10^{-12}, 10^{-4}, 10^{-3}, and $10^{-4}m$ respectively,

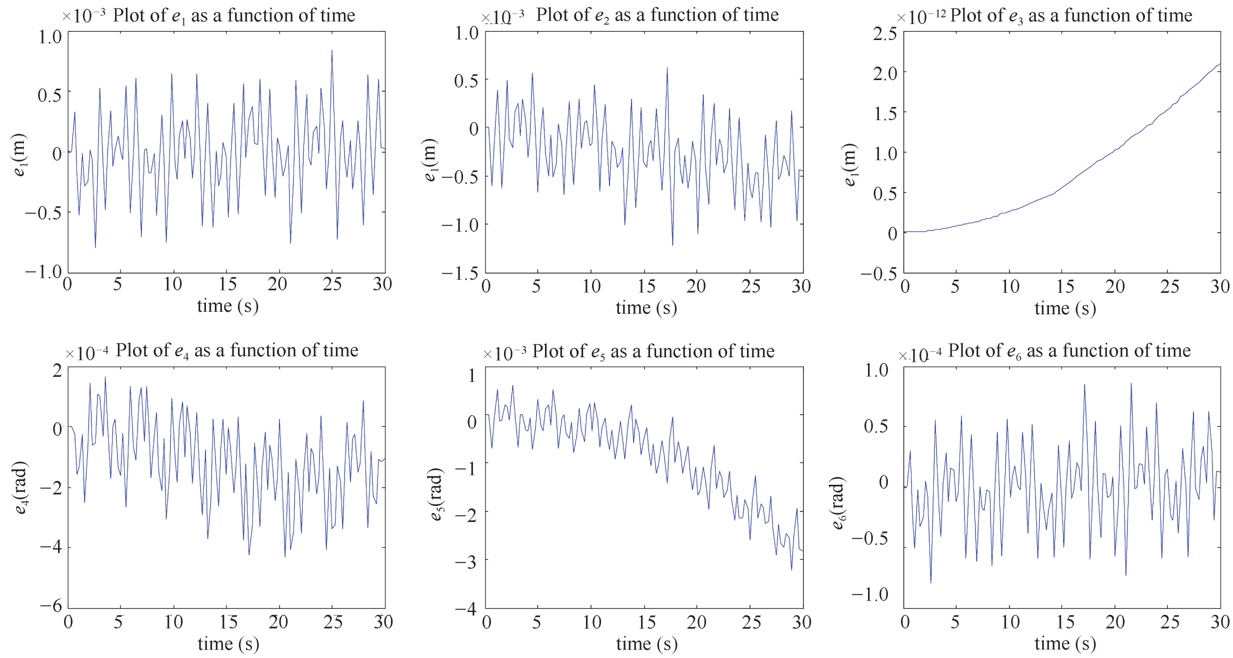

Fig. 4. The position and orientation error of the mass center of the quad-rotor as functions of time

which indicates that the computed servo constraint forces reach the required trajectory tracking constraints we designed, implying an excellent tracking performance.

A much more indicative view of the trajectory tracking error is provided in Fig. 5 and Fig. 6, where the blue curve represents the desired trajectory and the red curve represents the simulated tracking trajectory. Fig. 5 shows the simulated and desired helical trajectory of the mass center of the quad-rotor and Fig. 6 reflects the trajectory tracking error in $x-y$ plane, $x-z$ plane and $x-z$ plane and their corresponding partial enlarged views in above three planes respectively. It is also noted that the tracking error is small enough that these two trajectories are proved to be coincident.

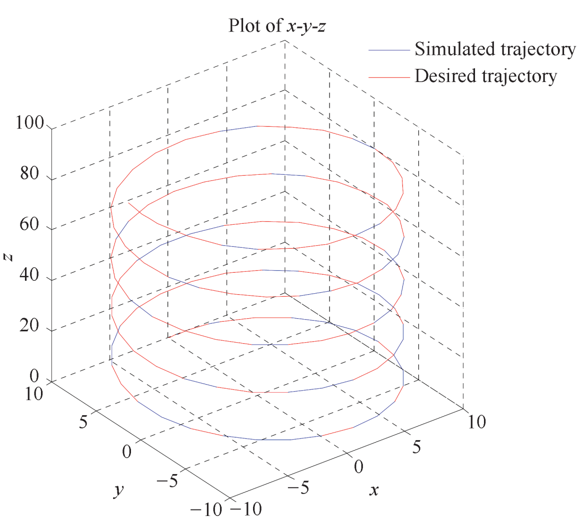

Fig. 5. The simulated/desired trajectory of the mass center of the quad-rotor

V. Conclusion

A novel approach for the trajectory tracking control of a quad-rotor UAV is processed in this paper based on Udwadia-Kalaba theory. Different from conventional approaches, this approach treats the desired trajectory as a constraint of the mechanical system called trajectory tracking constraint. The real-time forces generated by the four DC servo motors can be obtained explicitly and in compact closed form by solving Udwadia-Kalaba equation. This approach provides closed-form nonlinear control, neither making any assumptions or linearization of the nonlinear system nor imposing any a priori structure on the nature of the nonlinear controller.

Nonlinear dynamics modeling of a quad-rotor UAV is proposed and a desired trajectory is designed to illustrate this approach. The theoretical analysis and MATLAB simulation results indicate that the servo constraint control based on Udwadia-Kalaba equation fulfills the trajectory tracking task of the quad-rotor. The servo constraint forces are obtained conveniently, and the quad-rotor UAV's movement meets the designed trajectory precisely.

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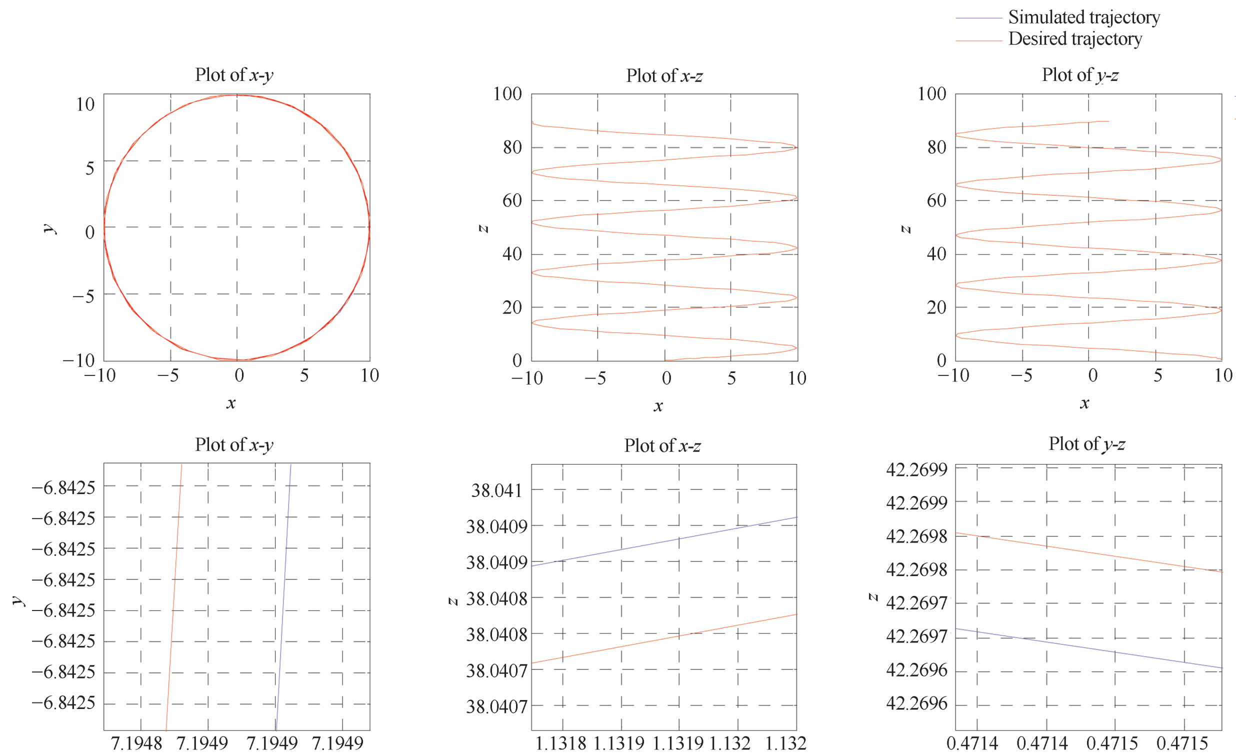

Fig. 6. The simulated/desired trajectory of the mass center of the quad-rotor from different views

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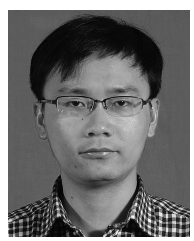

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