Bounds on the Error Probability of Raptor Codes

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Abstract—In this paper q-ary Raptor codes under ML decoding are considered. An upper bound on the probability of decoding failure is derived using the weight enumerator of the outer code, or its expected weight enumerator if the outer code is drawn randomly from some ensemble of codes. The bound is shown to be tight by means of simulations. This bound provides a new insight into Raptor codes since it shows how Raptor codes can be analyzed similarly to a classical fixed-rate serial concatenation.

I. Introduction

Fountain codes [1] are a class of erasure codes that have the property of being rateless. Thus, they are potentially able to generate an endless amount of encoded (or output) symbols. This property makes them suitable for application in situations where the channel erasure rate is not a priori known. The first class of practical fountain codes, Luby Transform (LT) codes, was introduced in [2] together with an iterative decoding algorithm that achieves a good performance when the number of input symbols k is large. In [2] it was shown how in order to achieve a low probability of decoding error, the encoding and iterative decoding cost per output symbol is $O\left(\ln(k)\right)$.

Raptor codes were introduced in [3] and outperform LT codes in many aspects. They consist of a serial concatenation of an outer code $\mathcal{C}$ (or precode) with an inner LT code. On erasure channels, this construction allows relaxing the design of the LT code, requiring only the recovery of a fraction $1 - \gamma$ of the input symbols with γ small. This can be achieved with linear encoding complexity and also linear decoding complexity using iterative decoding. The outer code is responsible for recovering the remaining fraction of input symbols, γ. If the outer code $\mathcal{C}$ is linear-time encodable and decodable then the Raptor code has linear encoding and iterative decoding complexity over erasure channels.

Most of the existing works on LT and Raptor codes consider iterative decoding and assume large input block lengths (k at least in the order of a few tens of thousands). However, in practice, smaller values of k are more commonly used. For example, for the binary Raptor codes standardized in [4] and

[5] the recommended values of k range from 1024 to 8192. For these input block lengths, iterative decoding performance degrades considerably. In this context, a different decoding algorithm is adopted that is an efficient maximum likelihood (ML) decoder, in the form of inactivation decoding [6].

An inactivation decoder solves a system of equations in several stages. First a set of variables is declared *inactive*. Next a system of equations involving the set of inactive variables needs to be solved, for example using Gaussian elimination. Finally, once the value of the inactive variables is known, all other variables are recovered using iterative decoding.

Recently there have been several works addressing the complexity of inactivation decoding for Raptor and LT codes [7]–[10]. The probability of decoding failure of LT and Raptor codes under ML decoding has also been subject of study in several works. In [11] upper and lower bounds to the intermediate symbol erasure rate were derived for LT codes and Raptor codes with outer codes in which every element of the parity check matrix is independent and identically distributed (i.i.d.) Bernoulli random variables with parameter p. This work was extended in [12], where lower an upper bounds to the performance of LT codes under ML decoding were derived. A further extension was presented in [13], where an approximation to the performance of Raptor codes under ML decoding is derived under the assumption that the number of erasures correctable by the outer code is small. Hence, this approximation holds only if the rate of the outer code is sufficiently high. In [14] it was shown by means of simulations how the error probability of q-ary Raptor codes is very close to that of linear random fountain codes. In [15] upper and lower bounds to the probability of decoding failure of Raptor codes were derived. The outer codes considered in [15] are binary linear random codes with a systematic encoder. Recently, ensembles of Raptor codes with linear random outer codes were also studied in a fixed-rate setting in [16], [17]. Although a number of works has studied the probability of decoding failure of Raptor codes, to the best of the knowledge of the authors, up to now the results hold only for specific binary outer codes (see [11], [15]–[17]).

In this paper an upper bound on the probability of decoding failure of Raptor codes is derived, based on the weight enumerator of their outer codes. The bound is extended to ensembles of Raptor codes where the outer code is drawn randomly from an ensemble. In this case, it is necessary to know the average

weight enumerator for the outer code ensemble. By means of simulations, the derived bound is shown to be tight, specially in the error floor region, for Raptor codes with Hamming and linear random outer codes. In contrast to [11], [15]–[17] not only binary Raptor codes are considered, but also q-ary Raptor codes. The bounds presented in this paper can be seen as an extension of the upper bound in [12] to Raptor codes.

The rest of the paper is organized as follows. In Section II some preliminary definitions are presented. Section III presents the upper bounds on the probability of decoding failure for the case in which the outer code is deterministic. In Section IV these bounds are extended to the case in which the outer code is drawn from a linear parity-check based ensemble. Numerical results are presented in Section V. Section VI presents the conclusions of our work.

II. PRELIMINARIES

We consider Raptor codes constructed over $\mathbb{F}_q$ with an (h, k) outer linear block code $\mathcal{C}$. We shall denote the k input (or source) symbols of a Raptor code as $\mathbf{u} = (u_1, u_2, \ldots, u_k)$. The elements of $\mathbf{u}$ belong to $\mathbb{F}_q$. Out of the k input symbols, the outer code generates a vector of h intermediate symbols $\mathbf{v} = (v_1, v_2, \ldots, v_h) \in \mathcal{C}$. Denoting by $\mathbf{G_o}$ the employed generator matrix of the outer code, of dimension $(k \times h)$ and with elements in $\mathbb{F}_q$, the intermediate symbols can be expressed as

$$\mathbf{v} = \mathbf{u} \mathbf{G_o}.$$

These intermediate symbols serve as input to an LT encoder, which can generate an unlimited number of output symbols, $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, where n can grow unbounded. Again, the elements of $\mathbf{c}$ belong to $\mathbb{F}_q$. For any n the output symbols can be expressed as

$$\mathbf{c} = \mathbf{v} \mathbf{G_{LT}} = \mathbf{u} \mathbf{G_o} \mathbf{G_{LT}}$$

where $\mathbf{G_{LT}}$ is an $(h \times n)$ matrix whose elements belong to $\mathbb{F}_q$. Each column of $\mathbf{G_{LT}}$ is associated with c_i. More specifically, each column of $\mathbf{G_{LT}}$ is generated by first selecting an output degree d according to the degree distribution $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_{d_{\max}})$, and then selecting d different indexes uniformly at random between 1 and h. Finally, the elements of the column corresponding to these indexes are drawn independently and uniformly at random from $\mathbb{F}_q \backslash \{0\}$, while all other elements of the column are set to zero.

The output symbols $\mathbf{c}$ are transmitted over a q-ary erasure channel (q-EC) at the output of which each transmitted symbol is either correctly received or erased.[1] We denote by m the number of output symbols collected by the receiver of interest, and we express it as $m = k + \delta$. Let us denote by $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ the m received output symbols. Denoting by $\mathcal{I} = \{i_1, i_2, \ldots, i_m\}$ the set of indices corresponding to the m non-erased symbols, we have

$$y_j = c_{i_j}.$$

[1]The results developed in this paper remain valid regardless the statistic of the erasures introduced by the channel.

An ML decoder (for example, an inactivation decoder) proceeds by solving the linear system of equations

$$\mathbf{y} = \mathbf{u} \tilde{\mathbf{G}}$$

where

$$\tilde{\mathbf{G}} = \mathbf{G_o} \tilde{\mathbf{G}}_{LT} \tag{1}$$

with $\tilde{\mathbf{G}}_{LT}$ given by the m columns of $\mathbf{G_{LT}}$ with indices in $\mathcal{I}$.

Given a block code $\mathcal{C}$ of length h we shall denote its weight enumerator as $A = \{A_0, A_1 \ldots A_h\}$, where A_i denotes the multiplicity of codewords of weight i. Similarly, given an ensemble of block codes, all with the same length h, along with a probability distribution on the codes in the ensmble, we shall denote its average weight enumerator as $\mathsf{A} = \{\mathsf{A}_0, \mathsf{A}_1 \ldots \mathsf{A}_h\}$, where A_i denotes the expected multiplicity of codewords of weight i of a code drawn randomly from the ensemble.

III. UPPER BOUNDS ON THE ERROR PROBABILITY

The following theorem establishes an upper bound on the probability of decoding failure $\mathsf{P_F}$ under ML decoding of a Raptor code constructed over $\mathbb{F}_q$ as a function of the receiver overhead δ.

Theorem 1. *Consider a Raptor code constructed over $\mathbb{F}_q$ with an (h, k) outer code $\mathcal{C}$ characterized by a weight enumerator A, and an inner LT code with output degree distribution Ω. The probability of decoding failure under optimum erasure decoding given that $m = k + \delta$ output symbols have been collected by the receiver can be upper bounded as*

$$\mathsf{P_F} \leq \sum_{l=1}^{h} A_l \pi_l^{k+\delta}$$

where π_l is the probability that a generic output symbol is equal to 0 given that the vector $\mathbf{v}$ of intermediate symbols has Hamming weight l. The expression of π_l is [12]

$$\pi_l = \frac{1}{q} + \frac{q-1}{q} \sum_{j=1}^{d_{\max}} \Omega_j \frac{\mathcal{K}_j(l; h, q)}{\mathcal{K}_j(0; h, q)} \tag{2}$$

where $\mathcal{K}_j(l; h, q)$ is the Krawtchouk polynomial of degree j with parameters h and q.[2]

Proof. An optimum (e.g. inactivation) decoder solves the linear system of equations in (1). Decoding fails whenever the system does not admit a unique solution, that is, if and only if $\mathsf{rank}(\tilde{\mathbf{G}}) < k$, i.e. if $\exists \mathbf{u} \in \mathbb{F}_q^k \backslash \{\mathbf{0}\}$ s.t. $\mathbf{u}\tilde{\mathbf{G}} = \mathbf{0}$. Consider two vectors $\mathbf{u} \in \mathbb{F}_q^k, \mathbf{v} \in \mathbb{F}_q^h$. Let us define $E_\mathbf{u}$ as the event $\mathbf{u}\mathbf{G}_o\tilde{\mathbf{G}}_{LT} = \mathbf{0}$. Similarly, we define $E_\mathbf{v}$ as the event $\mathbf{v}\tilde{\mathbf{G}}_{LT} = \mathbf{0}$. We have

$$\mathsf{P_F} = \Pr\left\{ \bigcup_{\mathbf{u} \in \mathbb{F}_q^k \backslash \{\mathbf{0}\}} E_\mathbf{u} \right\} = \Pr\left\{ \bigcup_{\mathbf{v} \in \mathcal{C} \backslash \{\mathbf{0}\}} E_\mathbf{v} \right\} \tag{3}$$

[2]The Krawtchouk polynomial of degree j with parameters n and q is defined as [18]

$$\mathcal{K}_k(x; n, q) = \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j} (q-1)^{k-j}.$$

where we made use of the fact that due to linearity, the all zero intermediate word is only generated by the all zero input vector.

Developing (3) we have

$$\mathsf{P_F} = \Pr\left\{\bigcup_{l=1}^{h}\bigcup_{\mathbf{v}\in\mathcal{C}_l} E_{\mathbf{v}}\right\} \tag{4}$$

where, by definition

$$\mathcal{C}_l = \{\mathbf{v}\in\mathcal{C} : w_H(\mathbf{v}) = l\}$$

is the set of codewords in $\mathcal{C}$ of Hamming weight l.

Let L be a discrete random variable representing the Hamming weight of vector $\mathbf{v}\in\mathcal{C}$. Moreover, let J and I be discrete random variables representing the number of intermediate symbols which are linearly combined to generate the generic output symbol y, and the number of non-zero such intermediate symbols, respectively. Note that $I \leq L$. We can upper bound (4) as

$$\mathsf{P_F} \leq \sum_{l=1}^{h}\Pr\left\{\bigcup_{\mathbf{v}\in\mathcal{C}_l} E_{\mathbf{v}}\right\} \leq \sum_{l=1}^{h} A_l \Pr\left\{E_{\mathbf{v}}|L=l\right\}. \tag{5}$$

Observing that the output symbols are independent of each other, we have

$$\Pr\left\{E_{\mathbf{v}}|L=l\right\} = \pi_l^{k+\delta}$$

where $\pi_l = \Pr\{y=0|L=l\}$. An expression for π_l may be obtained observing that

$$\pi_l = \sum_{j=1}^{d_{\max}}\Pr\{y=0|L=l, J=j\}\Pr\{J=j|L=l\}$$

$$\overset{(a)}{=} \sum_{j=1}^{d_{\max}}\Omega_j \Pr\{y=0|L=l, J=j\}$$

$$\overset{(b)}{=} \sum_{j=1}^{d_{\max}}\Omega_j \sum_{i=0}^{\min\{j,l\}}\Pr\{y=0|I=i\}\Pr\{I=i|L=l, J=j\}$$

where equality '(a)' is due to $\Pr\{J=j|L=l\} = \Pr\{J=j\} = \Omega_j$ and equality '(b)' to $\Pr\{y=0|L=l, J=j, I=i\} = \Pr\{y=0|I=i\}$. Letting $\vartheta_{i,l,j} = \Pr\{I=i|L=l, J=j\}$, since the j intermediate symbols are chosen uniformly at random by the LT encoder we have

$$\vartheta_{i,l,j} = \frac{\binom{l}{i}\binom{h-l}{j-i}}{\binom{h}{j}}. \tag{6}$$

Let us denote $\Pr\{y=0|I=i\}$ by φ_i and let us observe that, due to the elements of $\mathbf{G}$ being i.i.d. and uniformly drawn in $\mathbb{F}_q\setminus\{0\}$, on invoking Lemma 1 in the Appendix[3] we have

$$\varphi_i = \frac{1}{q}\left(1 + \frac{(-1)^i}{(q-1)^{i-1}}\right). \tag{7}$$

[3]The proof in the Appendix is only valid for fields with characteristic 2, the case of most interest for practical purposes. The proof of the general case is a trivial extension of Lemma 1 in the Appendix.

We conclude that π_l is given by

$$\pi_l = \sum_{j=1}^{d_{\max}}\Omega_j \sum_{i=0}^{\min\{j,l\}}\vartheta_{i,l,j}\,\varphi_i$$

where $\vartheta_{i,l,j}$ and φ_i are given by (6) and (7), respectively.

Expanding this expression and rewriting it using Krawtchouk polynomials and making use of the Chu-Vandermonde identity, one obtains (2). This completes the proof. $\square$

The following theorem makes the bound in Theorem 1 tighter for $q > 2$. It is equivalent to Theorem 1 for $q = 2$.

Theorem 2. *Consider a Raptor code constructed over $\mathbb{F}_q$ with an (h, k) outer code $\mathcal{C}$ characterized by a weight enumerator A, and an inner LT with output degree distribution Ω. The probability of decoding failure under optimum erasure decoding given that $m = k + \delta$ output symbols have been collected by the receiver can be upper bounded as*

$$\mathsf{P_F} \leq \sum_{l=1}^{h}\frac{A_l}{q-1}\pi_l^{k+\delta}$$

Proof. The bound (5) can be tightened by a factor $q - 1$ exploiting the fact that for a linear block code $\mathcal{C}$ constructed over $\mathbb{F}_q$, if $\mathbf{c}$ is a codeword, $\alpha\mathbf{c}$ is also a codeword, $\forall\alpha\in\mathbb{F}_q\setminus\{0\}$ [19]. $\square$

The upper bound in Theorem 2 also applies to LT codes. In that case, A_l is simply the total number of sequences of Hamming weight l and length k,

$$A_l = \binom{k}{l}(q-1)^{l-1}.$$

The upper bound obtained for LT codes coincides with the bound in [12] (Theorem 1).

IV. Case of Random Outer Codes from Linear Parity-Check Based Ensembles

Both Theorem 1 and Theorem 2 apply to the case of a specific outer code. Next we extend these results to the case of a random outer code drawn from an ensemble of codes. Specifically, we consider a parity-check based ensemble of outer codes, denoted by $\mathscr{C}^{\mathrm{o}}$, defined by a random matrix of size $(h-k)\times h$ whose elements belong to $\mathbb{F}_q$. A linear block code of length h belongs to $\mathscr{C}^{\mathrm{o}}$ if and only if at least one of the instances of the random matrix is a valid parity-check matrix for it. Moreover, the probability measure of each code in the ensemble is the sum of the probabilities of all instances of the random matrix which are valid parity-check matrices for that code. Note that all codes in $\mathscr{C}^{\mathrm{o}}$ are linear, have length h, and have dimension $k_{\mathcal{C}} \geq k$. In the following we use the expression "Raptor code ensemble" to refer to the set of Raptor codes obtained by concatenating an outer code belonging to the ensemble $\mathscr{C}^{\mathrm{o}}$ with an LT encoder having distribution Ω. We shall denote this ensemble as $(\mathscr{C}^{\mathrm{o}}, \Omega)$.

Theorem 3. *Consider a Raptor code ensemble $(\mathscr{C}^o, \Omega)$ and let $\mathsf{A} = \{\mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_h\}$ be the expected weight enumerator*

of a code $\mathcal{C}$ that is randomly drawn from $\mathscr{C}^o$, i.e., let $\mathsf{A}_l = \mathbb{E}_{\mathscr{C}^o}[A_l(\mathcal{C})]$ for all $l \in \{0, 1, \ldots, h\}$. Let

$$\bar{\mathsf{P}}_\mathsf{F} = \mathbb{E}_{\mathscr{C}^o}[\mathsf{P}_\mathsf{F}(\mathcal{C})]$$

be the average probability of decoding failure of the Raptor code obtained by concatenating an instance of $\mathcal{C}$ with the LT encoder, under optimum erasure decoding and given that $m = k + \delta$ output symbols have been collected by the receiver. Then

$$\bar{\mathsf{P}}_\mathsf{F} \le \sum_{l=1}^{h} \frac{\mathsf{A}_l}{q-1} \pi_l^{k+\delta}.$$

Proof. Due to Theorem 2 we may write

$$\bar{\mathsf{P}}_\mathsf{F} \le \mathbb{E}_{\mathscr{C}^o}\left[\sum_{l=1}^{h} \frac{A_l(\mathcal{C})}{q-1} \pi_l^{k_\mathcal{C}+\delta}\right]. \tag{8}$$

For all outer codes $\mathcal{C} \in \mathscr{C}^o$ we have $k_\mathcal{C} \ge k$. Since $\pi_l \le 1$ we can write

$$\pi_l^{k_\mathcal{C}+\delta} \le \pi_l^{k+\delta}$$

which allows us to upper bound (8) as

$$\bar{\mathsf{P}}_\mathsf{F} \le \mathbb{E}_{\mathscr{C}^o}\left[\sum_{l=1}^{h} \frac{A_l(\mathcal{C})}{q-1} \pi_l^{k+\delta}\right] = \sum_{l=1}^{h} \frac{\mathsf{A}_l}{q-1} \pi_l^{k+\delta}$$

where the last equality follows from linearity of expectation. $\square$

V. NUMERICAL RESULTS

All results presented in this section use the LT output degree distribution employed by standard R10 Raptor codes, [4], [5],

$$\Omega(\mathsf{x}) = \sum_{j=1}^{d_{\max}} \Omega_j \mathsf{x}^j$$
$$= 0.0098\mathsf{x} + 0.4590\mathsf{x}^2 + 0.2110\mathsf{x}^3 + 0.1134\mathsf{x}^4$$
$$+ 0.1113\mathsf{x}^{10} + 0.0799\mathsf{x}^{11} + 0.0156\mathsf{x}^{40}. \tag{9}$$

A. Binary Raptor Codes with Hamming Outer Codes

In this section we consider binary Raptor codes with (deterministically known) Hamming outer codes. The weight enumerator of a binary Hamming code of length $h = 2^t - 1$ and dimension $k = h - t$ can be derived easily using the recursion

$$(i+1)A_{i+1} + A_i + (h-i+1)A_{i-1} = \binom{h}{i}$$

with $A_0 = 1$ and $A_1 = 0$ [18]. The weight distribution obtained from this recursion can then incorporated in Theorem 1 to derive the corresponding upper bound on the probability of Raptor decoding failure under optimum decoding.

Figure 1 shows the decoding failure rate for a binary Raptor code using a $(63, 57)$ binary Hamming outer code as a function of the absolute overhead, δ. The upper bound established in Theorem 1 is also shown. In order to obtain the values of failure rate, for each δ value Monte Carlo simulations were run until 200 errors were collected using inactivation decoding. It can be observed how the upper bound is tight.
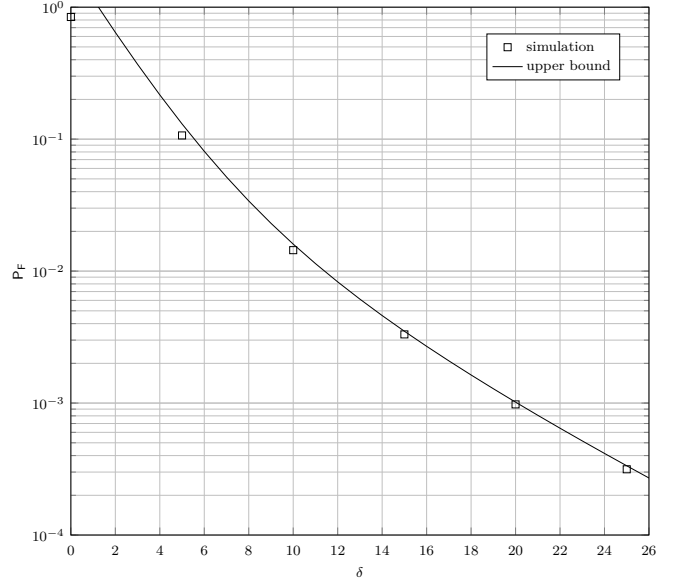

Fig. 1. Probability of decoding failure P_F versus the absolute overhead for a binary Raptor code with a $(63, 57)$ Hamming outer code. The solid line denotes the upper bound on the probability of decoding failure expressed by Theorem 1. The markers denote simulation results.

B. Linear Random Outer Code

In this subsection, we consider a $(\mathscr{C}^o, \Omega)$ Raptor code ensemble constructed over $\mathbb{F}_q$, where the LT distribution Ω is the one defined in (9) and where $\mathscr{C}^o$ is the uniform parity-check ensemble, with parity-check matrix of size $(h-k) \times h$ and characterized by i.i.d. entries with uniform distribution in $\mathbb{F}_q$. The expected multiplicity of codewords of weight l for an outer code drawn randomly in $\mathscr{C}^o$ according to the described procedure is known to be

$$\mathsf{A}_l = \binom{h}{l} q^{-(h-k)}(q-1)^l.$$

In order to obtain the experimental values of decoding failure rate, 6000 different outer codes were generated. For each outer code and for each overhead value 10^3 inactivation decoding attempts were carried out. The average failure rate was calculated by averaging the failure rates of the individual Raptor codes. In order to select the outer code an $(h-k) \times h$ parity check matrix was selected at random by generating each of its elements according to a uniform distribution in $\mathbb{F}_q$.

In Figure 2 we show simulation results for $k = 64$ and $h = 70$. Two different $(\mathscr{C}^o, \Omega)$ Raptor code ensembles were considered, one constructed over $\mathbb{F}_2$ and one constructed over $\mathbb{F}_4$. We can observe how in both cases the bounds hold and are tight except for very small values of δ.

VI. CONCLUSIONS

In this paper we have consider Raptor codes under ML decoding. We have derived an upper bound on the probability of decoding failure of Raptor codes with generic q-ary outer codes. This bound is general and only requires the knowledge of the weight enumerator of the outer code. The bound also applies to ensembles of Raptor codes where the outer code is
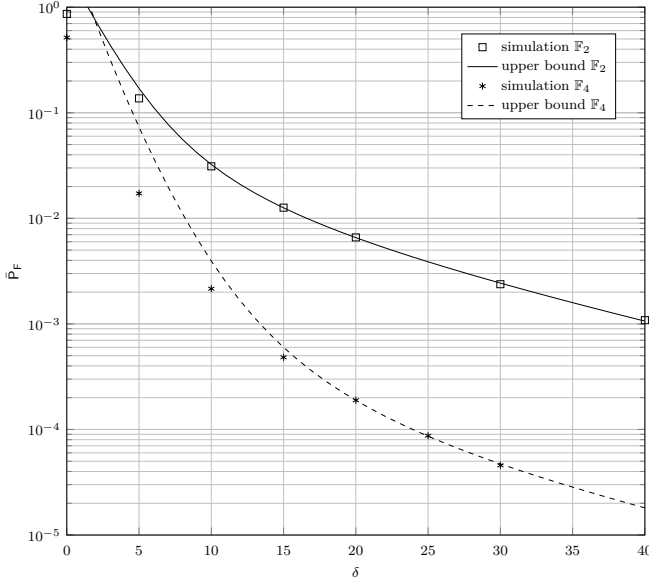
Fig. 2. Expected probability of decoding failure $\bar{\mathsf{P}}_{\mathsf{F}}$ vs absolute overhead for Raptor code ensembles where the outer code is drawn randomly from the uniform parity-check ensemble. The solid and dashed lines denote the upper bounds on the average probability of decoding failure for the ensembles constructed over $\mathbb{F}_2$ and $\mathbb{F}_4$ respectively. The markers denote simulation results.

randomly selected from an ensemble. The bound is shown to be tight, specially in the error floor, by means of simulations.

APPENDIX

The following lemma is used in the proof of Theorem 1.

Lemma 1. *Let X_1, X_2 ... X_l be discrete i.i.d random variables uniformly distributed over $\mathbb{F}_{2^m}\backslash\{0\}$. Then*

$$\Pr\{X_1 + X_2 + \ldots + X_l = 0\} = \frac{1}{q}\left(1 + \frac{(-1)^i}{(q-1)^{i-1}}\right)$$

where $q = 2^m$.

Proof. Observe that the additive group of $\mathbb{F}_{2^m}$ is isomorphic to the vector space $\mathbb{Z}_2^m$. Thus, we may let X_1, X_2 ... X_l be i.i.d random variables with uniform probability mass function over the vector space $\mathbb{Z}_2^m\backslash\{0\}$.

Let us introduce the auxiliary random variable

$$W := X_1 + X_2 + \ldots + X_l$$

and let us denote by $P_W(w)$ and by $P_X(x)$ the probability mass functions of W and X_i, respectively, where

$$P_X(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{q-1} & \text{otherwise.} \end{cases}$$

Due to independence we have

$$P_W = P_X * P_X * \ldots * P_X$$

which, taking the m-dimensional two-points discrete Fourier transform (DFT) $\mathscr{J}\{\cdot\}$ of both sides, yields

$$\mathscr{J}\{P_W(w)\} = \left(\mathscr{J}\{P_X(x)\}\right)^l.$$

Next, since

$$\hat{P}_X(t) := \mathscr{J}\{P_X(x)\} = \begin{cases} 1 & \text{if } t = 0 \\ \frac{-1}{q-1} & \text{otherwise} \end{cases}$$

we have

$$\hat{P}_W(t) := \mathscr{J}\{P_W(w)\} = \begin{cases} 1 & \text{if } t = 0 \\ \frac{(-1)^l}{(q-1)^l} & \text{otherwise.} \end{cases}$$

We are interested in $P_W(0)$ whose expression corresponds to

$$P_W(0) = \frac{1}{q}\sum_t \hat{P}_W(t) = \frac{1}{q} + \frac{1}{q}(q-1)\frac{(-1)^l}{(q-1)^l}$$

from which the statement follows. $\square$

The result in this lemma appears in [12]. However, the proof in [12] uses a different approach based on a known result on the number of closed walks of length l in a complete graph of size q from a fixed but arbitrary vertex back to itself.

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