Synchronization of oscillators by nonlinear measurements with application to VLC

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***Abstract*— This note studies the problem of master-slave phase synchronization between two oscillators described by Andronov-Hopf equations, where pulse-like signals, which disturbances can be heavily corrupt, are used for communication. The synchronization conditions are established using the Lyapunov function theory, and the synchronization error bounds are evaluated. The results are motivated and illustrated by designing a synchronization scheme for visible light communication (VLC), where a periodic sequence is usually sent at the beginning of the communication to synchronize the transmitter and the receiver clocks.**

I. Introduction

Synchronization is a complex and omnipresent phenomenon in different areas of science and technology [5], [20], [14], [12], [15]. Many applications require a fast and robust synchronization between two systems, which are frequently called transmitter and receiver, or master and follower, related to clock synchronization [6], [17]. The example that motivated the research of this paper comes from the Visible Light Communication (VLC) domain [3], where the dual usage of light infrastructures for illumination and communication using Light Emitting Diodes (LEDs) is considered.

LEDs emit incoherent light in VLC, and the information is sent by the intensity of the optical signal (*i.e.*, its actual value), which is unipolar and non-negative. The baseband signal modulates the carrier frequency's intensity rather than its amplitude or phase, and a photodiode conducts direct detection at the receiver stage. Pulse-position modulation (PPM) technique is usually employed in VLC, which requires that the sampling occurs at the correct time interval to ensure proper data decoding and minimize transmission errors. Thus, the receiver and transmitter clocks should be well adjusted [4]. The scheme applied in VLC to ensure this is based on a pilot-based approach [22] with a synchronization preamble, which represents a periodic sequence of pulses preceding the message during which the receiver has to adjust its clocks in a finite time and in the presence of significant noises. In [7], it was investigated how well a phase-lock loop (PLL) synchronizing optical PPM communication with a variable index performed. However, free-space optics (FSO) for a real-world scenario is still a new field, and pertinent literature appeared in the early 2000s.

So, the clock synchronization problem between a transmitter and a receiver in a VLC system is the motivating application of this paper. To this end, our primary goal is to lower the synchronization convergence time in scenarios with varying noise levels and, more importantly, to keep the convergence time within bounds (predefined by the preamble length). The synchronization approach we propose is based on dynamical system theory; the principal assumption we make is that the input signals must be pulse-like periodic (or close to it, taking into account perturbations). Every periodic signal may be connected to the output of an oscillator, which is a dynamical system whose state moves periodically. The space-state of an oscillator may include many periodic trajectories, and if such a trajectory is locally attracting/repulsing, it is said to form a limit cycle [13], [16]. In our approach, we investigate oscillators with globally attractive limit cycles and concentrate the analysis on the Andronov-Hopf oscillator, a popular benchmark system for synchronization [19].

There are plenty of methods oriented toward the analysis and design of synchronization patterns in nonlinear systems; see [11], [21], [9], [1] to mention a few pertinent solutions (in complement to previously mentioned works). Despite that, fast pulse-based robust synchronization of oscillators is still an open area of research, and in this work, we propose a solution considering Andronov-Hopf oscillators.

The rest of the paper is organized as follows. In Section II, the synchronization problem is formulated, and the interconnection of the considered oscillators is described. We explain the synchronization procedure in Section III, where the applicability conditions and the estimates on error bounds are provided. Finally, in Section IV, the protocol is applied to VLC data to illustrate our results. The Appendix contains the auxiliary lemmas used to prove our main result.

Notation

- Denote by $\mathbb{R}$ and $\mathbb{N}^*$ the sets of real numbers and nonzero positive integers.
- Denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$ for any $n \in \mathbb{N}^*$.
- Denote by I_n the identity matrix of dimension $n \times n$ and by $\lambda_{\min}(P)$ the minimum eigenvalue of a real

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symmetric matrix $P \in \mathbb{R}^{n\times n}$.

II. Problem statement

Our approach is based on the observation that any periodic signal can be related to the output of an oscillator, *i.e.*, a dynamical system whose state performs periodical movements. In the sequel, the following Andronov-Hopf oscillator will be investigated:

$$\dot{x}_i(t) = Ax_i(t) + \omega k(1-|x_i(t)|^2)x_i(t) + \delta_i(t),\ t \geq 0, \quad (1)$$
$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix},$$

for all $i \in \mathbb{N}^*$, where $x_i(t) = \begin{pmatrix} x_{i1}(t) & x_{i2}(t) \end{pmatrix}^\top \in \mathbb{R}^2$ is the state of the ith oscillator; $\delta_i(t) \in \mathbb{R}^2$ is an external (essentially bounded) input; $\omega > 0$ is the oscillation frequency; $k > 0$ is the attraction gain. It can be shown (see the Appendix) that for any $i \in \mathbb{N}^*$, these oscillators for $\delta_i \equiv 0$ have its limit cycles on the unit circle with $|x_i| \equiv 1$, where all solutions are harmonic signals of frequency ω. There is also an equilibrium at the origin, which is unstable. For any bounded input δ_i, the trajectories of (1) stay bounded approaching a vicinity of the limit cycle (or the origin), whose size is proportional to the amplitude of the input, *i.e.*, (1) is input-to-state stable [19] in the sense of [2].

For the VLC application, the oscillators in (1) can then model both the transmitter and the receiver with $i = 1$ and $i = 2$, respectively, with trajectories on the limit cycle having different initial conditions and $\delta_i \equiv 0$. Let

$$y_i(t) = h(x_i(t)),\ t \geq 0,$$

be the pulse generated by the oscillators for a suitable defined function h. Then the signal y_1 is sent by the transmitter, corrupted by a bounded noise $v(t) \in \mathbb{R}$, and recovered by the receiver as

$$y(t) = y_1(t) + v(t).$$

An example of such an output map is

$$h(\xi) = \begin{cases} \mathfrak{a} & \text{if } \arcsin(\xi_1) \geq \mathfrak{b} \\ 0 & \text{if } \arcsin(\xi_1) < \mathfrak{b} \end{cases},\ \xi = \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}^\top \in \mathbb{R}^2, \quad (2)$$

where $\mathfrak{a} > 0$ and $\mathfrak{b} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ are parameters regulating the amplitude and the pulse width, respectively, or

$$h(\xi) = a\left(b - \tanh\left(\kappa(1+\xi_1)\right)\right), \quad (3)$$

where a, $b > 0$ determine the amplitude and $\kappa > 0$ parameterizes the pulse width.

The function in (2), taking as the input a harmonic signal, generates a proper square pulse, while function (3) produces its smoothed version. For $\mathfrak{a} = 1$, $\mathfrak{b} = 0.3\pi$, $a = b = 1$ and $\kappa = 5$ the outputs generated by these functions for $\xi(t) = [\sin(t)\cos(t)]^\top$ are shown in Fig. 1, where (2) is presented by the solid line, and (3) by the dash one. In the presence of output perturbations, applying the continuous output map (3) is more reasonable than the discontinuous one (2).

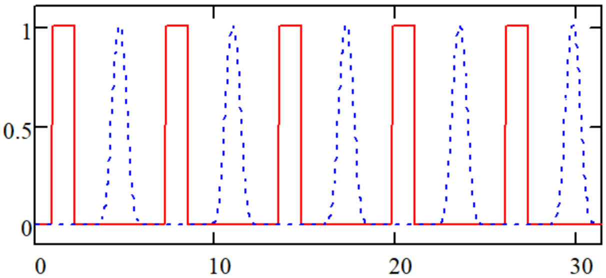

Fig. 1. Behavior of $h(\xi(t))$ for $\xi(t) = [\sin(t)\cos(t)]^\top$ versus time: (2) solid line, (3) dash line

The goal of the synchronization protocol that we wish to design is to guarantee that $y_2(t) \to y_1(t)$ using the measurements of $y(t)$. Such an issue can be classified as the *phase synchronization* between oscillators (the phase determines the position of the trajectories $x_i(t)$ on the limit cycle). If we introduce the synchronization error $e = x_1 - x_2 = \begin{pmatrix} e_1 & e_2 \end{pmatrix}^\top \in \mathbb{R}^2$, then the posed problem can be formulated as a robust (concerning the noise v) stabilization of e.

III. Synchronization protocols

The simplest synchronization scheme includes a direct synchronization error injection in the receiver:

$$\delta_2(t) = L(y(t) - y_2(t)) = L(y_1(t) - y_2(t) + v(t)), \quad (4)$$

where $L \in \mathbb{R}^2$ is the coupling gain. Note that (4) is a nonlinear bounded function of the state vectors $x_1(t)$ and $x_2(t)$.

If the measurement noise is small, while the time of convergence is not constrained, then the theory of pulse synchronization can be applied, as it can be found in many papers (see, for instance, [10]). It is mainly based on a phase dynamics analysis with output as in (2). The phase dynamics of an oscillator (independently of the dimension of the state-space vector) is one-dimensional, so its pulse response can be easily studied using the phase response curve (PRC) approach [10].

The synchronization problem becomes complicated if the measurement noise is significant and if the measured signal y has a shape severely deviated from a pulse. In such a case, the continuous output function (3) can be used.

Hence, we have the following result for (1):

Theorem 1: Consider two oscillators (1) with $\delta_1 \equiv 0$ and $\delta_2(t)$ given in (4), where the output (3) is used. Assume $|x_i(0)| = 1$ for $i = 1, 2$. Suppose there exist

$$0 < P = P^\top \in \mathbb{R}^{2\times 2},$$

$$L \in \mathbb{R}^2,\ \alpha > 0,\ \rho > 0 \text{ and } \gamma > 0$$

such that

$$(A-\beta LC)^\top P + P(A-\beta LC) + \varrho C^\top C \le -2\alpha P,$$
$$-PL = C^\top = [1\ 0]^\top,\ P \le \gamma I_2,$$
$$\varrho = 2\left(\frac{2\gamma}{\alpha} + \frac{\rho}{\omega k}\right)|L|^2 a^2 \kappa^2,$$

where

$$e_1\left(\tanh\left(\kappa(1+x_{11}-e_1)-\tanh\left(\kappa(1+x_{11})\right)-\frac{\beta}{a}e_1\right) \le 0 \quad (5)$$

for any $x_{11}\in[-1,1]$, any $|e_1|\le 2.15$ and some $\beta\in\mathbb{R}$. If the measurement noise admits a known upper bound $\overline{v}>0$ (so $|v(t)|\le\overline{v}$ for almost all $t\ge 0$) and

$$\frac{|L|^2}{\omega^2 k^2}(4a^2+\overline{v}^2) \le \frac{1}{16},$$

then the synchronization error e stays bounded and

$$\limsup_{t\to+\infty}|e(t)| \le \sqrt{\frac{\frac{2}{\lambda_{\min}(P)}\left(3\frac{\gamma}{\alpha}+\frac{\rho}{\omega k}\right)}{\alpha\min\left\{1,\frac{\rho\omega k}{2\gamma\omega k+\alpha\rho}\right\}}}\,|L|\overline{v}.$$

Proof: Since $|x_1(0)|=1$ and $\delta_1\equiv 0$, then $|x_1|\equiv 1$, due to Lemma 1 (Appendix). Next, the proof is divided into three steps: first, the behavior of the driven system (1) with $i=2$ is evaluated, and the boundedness of x_2 is shown. Second, the stability of the synchronization error e is investigated. Third, the composed dynamics is analyzed, showing that the systems are synchronized on the unit cycle with the error proportional to the amplitude of the measurement noise.

1) Following the proof of Lemma 1, consider the Lyapunov function

$$W(x_2) = \frac{1}{2}\left(1-|x_2|^2\right)^2$$

for the driven system, whose derivative for the dynamics of (1) with $i=2$ admits an estimate:

$$\dot{W} \le -\omega k(1-|x_2|^2)^2|x_2|^2 + \frac{1}{\omega k}\delta_2^\top\delta_2.$$

Note that

$$\begin{aligned}\delta_2^\top\delta_2 &= (y_1+v-y_2)^\top L^\top L(y_1+v-y_2)\\ &\le 2(y_1-y_2)^\top L^\top L(y_1-y_2)+2v^\top L^\top Lv\\ &= 2a^2\left(\tanh(\kappa(1+x_{21}))-\tanh(\kappa(1+x_{11}))\right)^\top L^\top\\ &\quad\times L(\tanh(\kappa(1+x_{21}))-\tanh(\kappa(1+x_{11})))\\ &\quad+2v^\top L^\top Lv.\end{aligned}$$

Since

$$\begin{aligned}(\tanh(\kappa(1+x_{21}))-\tanh(\kappa(1+x_{11})))^2 &\le \kappa^2(x_{21}-x_{11})^2\\ &\le \kappa^2 e_1^\top e_1,\end{aligned}$$

for all x_{11}, $x_{21}\in\mathbb{R}$ and $e_1=x_{11}-x_{21}$, finally we obtain:

$$\dot{W} \le -\omega k(1-|x_2|^2)^2|x_2|^2 + 2|L|^2\frac{a^2\kappa^2 e_1^\top e_1 + v^\top v}{\omega k}. \quad (6)$$

From another side,

$$\begin{aligned}\delta_2^\top\delta_2 &\le 2(y_1-y_2)^\top L^\top L(y_1-y_2)+2v^\top L^\top Lv\\ &\le 2|L|^2\left((y_1-y_2)^\top(y_1-y_2)+v^\top v\right)\\ &\le 2|L|^2(4a^2+\overline{v}^2),\end{aligned}$$

hence,

$$\dot{W} \le -\omega k(1-|x_2|^2)^2|x_2|^2 + \frac{2|L|^2}{\omega k}(4a^2+\overline{v}^2), \quad (7)$$

and it is straightforward to check that if

$$\frac{2|L|^2}{\omega^2 k^2}(4a^2+\overline{v}^2) \le \frac{1}{8},$$

then the set

$$\mathcal{X} = \{x_2\in\mathbb{R}^2 : \sqrt{0.5}\le|x_2|\le 1.144\}$$

is forward invariant for the driven system. Therefore, $x_2(t)\in\mathcal{X}$ for all $t\ge 0$ since $x_2(0)\in\mathcal{X}$, which also leads to the constraint:

$$|e(t)| \le 2.144,\ \forall t\ge 0.$$

2) The dynamics of the estimation error e can be written as

$$\begin{aligned}\dot{e} &= Ae-\omega k(1-|x_2|^2)x_2 - L(y_1+v-y_2)\\ &= (A-\beta LC)e-\omega k(1-|x_2|^2)x_2\\ &\quad -L(y_1-y_2-\beta Ce)-Lv\end{aligned}$$

with $\beta\in\mathbb{R}$ given in the theorem's statement. Consider a Lyapunov function for the synchronization error dynamics:

$$V(e) = e^\top Pe,$$

where the matrix P is defined in the formulation of the theorem. Its derivative takes the form (recall that $-e^\top PL(y_1-y_2-\beta Ce)\le 0$ by the imposed condition (5)):

$$\begin{aligned}&\dot{V} \le -2\alpha e^\top Pe - \varrho e^\top C^\top Ce\\ &-2e^\top P\left(\omega k(1-|x_2|^2)x_2 + L(y_1-y_2-\beta Ce)+Lv\right)\\ &\le -2\alpha e^\top Pe-\varrho e^\top C^\top Ce - 2e^\top P\left(\omega k(1-|x_2|^2)x_2+Lv\right)\\ &\le -\alpha e^\top Pe - \varrho e^\top C^\top Ce\\ &+\frac{2}{\alpha}\left(\omega^2k^2(1-|x_2|^2)^2 x_2^\top Px_2 + v^\top L^\top PLv\right)\\ &\le -\alpha e^\top Pe - \varrho e^\top C^\top Ce\\ &+\frac{2\gamma}{\alpha}\left(\omega^2k^2(1-|x_2|^2)^2|x_2|^2+|L|^2v^\top v\right).\end{aligned} \quad (8)$$

3) Finally, for given $\rho>0$, introduce a Lyapunov function

$$U(e,x_2) = V(e) + \left(\frac{2\gamma}{\alpha}\omega k+\rho\right)W(x_2),$$

then according to the derived estimates (6) and (8) we obtain (recall that $\sqrt{0.5} \le |x_2|$):

$$\begin{aligned}\dot{U} &\le -\alpha e^\top Pe - \rho\omega k(1-|x_2|^2)^2|x_2|^2 \\ &\quad + 2|L|^2\left(3\frac{\gamma}{\alpha} + \frac{\rho}{\omega k}\right)v^\top v \\ &\le -\alpha e^\top Pe - 0.5\rho\omega k(1-|x_2|^2)^2 + 2|L|^2\left(3\frac{\gamma}{\alpha} + \frac{\rho}{\omega k}\right)v^\top v \\ &\le -\alpha\min\left\{1, \frac{\rho\omega k}{2\gamma\omega k + \alpha\rho}\right\}U + 2|L|^2\left(3\frac{\gamma}{\alpha} + \frac{\rho}{\omega k}\right)\overline{v}^2,\end{aligned}$$

which implies the result and the given asymptotic estimate on the synchronization error amplitude. ■

If (5) is verified for smaller values of e_1, then the obtained result holds locally in the synchronization error. The linear matrix inequalities introduced in this theorem can be simplified at the price of a less precision estimate for the asymptotic behavior of the synchronization error:

Corollary 1: Consider two oscillators (1) with $\delta_1 \equiv 0$ and δ_2 given in (4), where the output (3) is used. Assume $|x_i(0)| = 1$ for $i = 1, 2$. Suppose there exist

$$0 < P = P^\top \in \mathbb{R}^{2\times 2},$$

$$L \in \mathbb{R}^2,\ \alpha > 0,\ \rho > 0 \ \text{ and } \gamma > 0$$

such that

$$\begin{aligned}(A-\beta LC)^\top P + P(A-\beta LC) &\le -2\alpha P, \\ -PL = C^\top = [1\ 0]^\top,\ P &\le \gamma I_2,\end{aligned}$$

with (5) for any $x_{11} \in [-1, 1]$, any $|e_1| \le 2.15$ and some $\beta \in \mathbb{R}$. Assume that the measurement noise admits a known upper bound $\overline{v} > 0$ (so $|v(t)| \le \overline{v}$ for almost all $t \ge 0$), and

$$\frac{[L]^2}{\omega^2 k^2}(4a^2 + \overline{v}^2) \le \frac{1}{16}.$$

Then the synchronization error e stays bounded and

$$\limsup_{t\to+\infty}|e(t)| \le \sqrt{\frac{\frac{2}{\lambda_{\min}(P)}\left(3\frac{\gamma}{\alpha} + \frac{\rho}{\omega k}\right)}{\alpha\min\left\{1, \frac{\rho\omega k}{2\gamma\omega k + \alpha\rho}\right\}}}\,|L|(\overline{v} + 2a).$$

Proof: The proof follows the line of the main result, but instead of using (6) we will apply (7), and since the matrix inequalities have been modified for the derivative of Lyapunov function V the following inequality can be obtained:

$$\dot{V} \le -\alpha e^\top Pe + \frac{2\gamma}{\alpha}\left(\omega^2 k^2(1-|x_2|^2)^2|x_2|^2 + |L|^2\overline{v}^2\right). \tag{9}$$

For given $\rho > 0$, applying the derived estimates (7) and (9) to the derivative of the Lyapunov function U we have:

$$\begin{aligned}\dot{U} &\le -\alpha e^\top Pe - \rho\omega k(1-|x_2|^2)^2|x_2|^2 \\ &\quad + 2|L|^2\left(\left(\frac{3\gamma}{\alpha} + \frac{\rho}{\omega k}\right)\overline{v}^2 + 4\left(\frac{2\gamma}{\alpha} + \frac{\rho}{\omega k}\right)a^2\right) \\ &\le -\alpha\min\left\{1, \frac{\rho\omega k}{2\gamma\omega k + \alpha\rho}\right\}U \\ &\quad + 2|L|^2\left(\left(\frac{3\gamma}{\alpha} + \frac{\rho}{\omega k}\right)(\overline{v}^2 + 4a^2)\right),\end{aligned}$$

which implies the result. ■

The derived estimates of the asymptotic behavior of the synchronization error are rather conservative (which is a frequent issue in the analysis of nonlinear systems). However, the actual precision of the synchronization scheme is much better, as we are going to demonstrate through its application.

IV. Application to VLC data

For illustration, we will use the VLC data given in [8]. In such a case, the sampling period $T_s = 5 \times 10^{-7}$ [sec], the pulse width $T_{pulse} = 25 \times 10^{-7}$ [sec] and the length of the communication message $T = 16T_{pulse}$ [sec], other details on the used experimental setup can be found in [8]. Select

$$\alpha = 0.01,\ \rho = 0.1,\ \beta = -10^{-5},\ \gamma = 10^{-3},\ \omega = \frac{2\pi}{T},$$
$$L = \begin{pmatrix}-10^5 \\ 1\end{pmatrix},\ C = \begin{pmatrix}1 & 0\end{pmatrix},\ \kappa = 5,\ k = 10\omega$$

then the conditions of the corollary can be checked. The property (5) for this value of β and $a \ge 0.25$ is verified in Fig. 2 (the left-hand side is plotted), and it is satisfied for a limited domain of values of the synchronization error. The found solution for linear matrix inequalities is

$$P = \begin{pmatrix}10^{-5} & -4.867\times 10^{-11} \\ -4.867\times 10^{-11} & 10^{-5}\end{pmatrix}.$$

The synchronization results for a frequency 450 Hz and a frequency 940 Hz, with different levels of measurement perturbations (represented by parasitic ambient light) are shown in figures 3 and 4, respectively. As we can conclude, in the former case the noise hides completely the transmitted pulse, while in the latter scenario the pulse is less damaged. Anyway, in both situations the suggested synchronization protocol ensures the phase alignment with a good precision.

Remark 1: In addition, the algorithm was equipped with a noise filter and a procedure for estimating the minimum and maximum value of the pulse following the measurement signal.

We refer to [18], where numerical modeling and experimental data have validated phase alignment and noise robustness using our synchronization method. Also, the same paper has provided a comparison with commonly used synchronization approaches based on Phase Locked

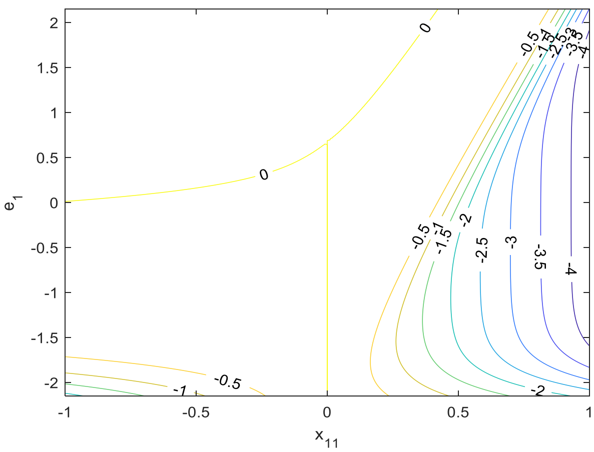

Fig. 2. Verification of (5)

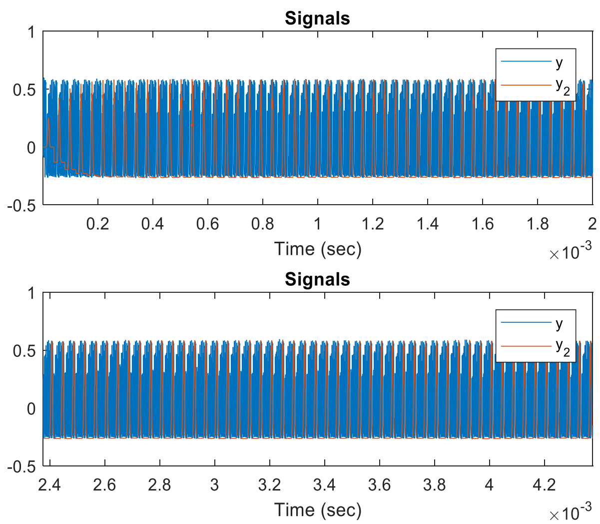

Fig. 3. Synchronization for a frequency 450 Hz

Loop (PLL). Through correct steady state phase delay and a decreased Synchronization Error Rate, even in the face of extremely noisy environmental conditions, experimental data demonstrate that our method surpasses PLL techniques in terms of noise robustness. In [18], we have carried out a set of focused tests utilizing both numerically generated signals and tested on an actual VLC architecture, using optical signals based on PPM (Pulse-position modulation) with various indexes, to compare our results with existing methodologies based on PLL. Specifically, we used synchronization frames based on 2, 4, 8, and 16 PPM to verify the efficiency of our method.

V. Conclusion

This work investigated a synchronization challenge useful in VLC, where typically, a periodic sequence is delivered at the start of the communication to synchronize the transmitter and the receiver. In this note, we proposed a synchronization tool based on the interconnection of two Andronov-Hoff oscillators. Our main result provided the conditions of synchronization error's robust stability concerning the measurement perturbations. An upper bound of this error was established. An application to some VLC data illustrated the performance of our protocol.

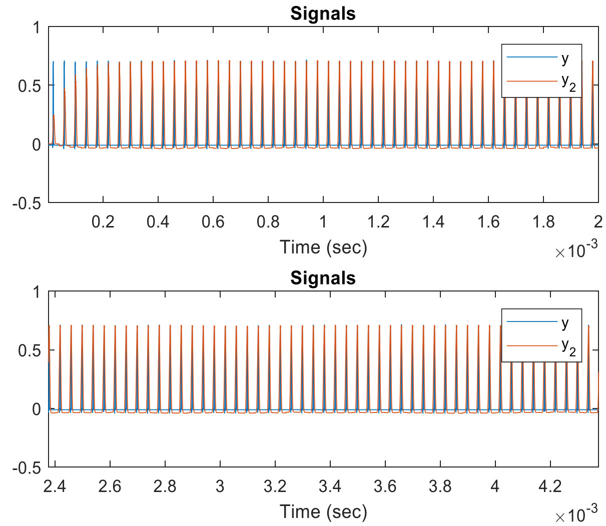

Fig. 4. Synchronization for a frequency 940 Hz

Appendix

For $\delta_i \equiv 0$, the system (1) has two compact invariant sets: the origin $x_i = 0$ and the unit circle $|x_i| = 1$. It is easy to check that the origin is locally exponentially unstable, while on the unit circle the system equations are reduced to linear ones:

$$\dot{x}_i(t) = A x_i(t),$$

whose solutions are harmonic with the frequency ω. Define the total invariant set by

$$\mathcal{W} = \{x_i \in \mathbb{R}^2 : \{0\} \cup |x_i| = 1\},$$

which is decomposable [2] in this case (for more details about definition and notions, such as input-to-state stability and invariant sets, please refer to [2]).

Lemma 1: The system (1) is input-to-state stable with respect to the set $\mathcal{W}$ and the input δ_i.

Proof: Consider a Lyapunov function candidate:

$$W(x_i) = \frac{1}{2}(1 - |x_i|^2)^2,$$

whose derivative for the system takes the form:

$$\begin{aligned} \dot{W} &= -2(1 - |x_i|^2) x_i^\top \dot{x}_i \\ &= -2\omega k (1 - |x_i|^2)^2 |x_i|^2 - 2(1 - |x_i|^2) x_i^\top \delta_i \\ &\leq -\omega k (1 - |x_i|^2)^2 |x_i|^2 + \frac{1}{\omega k} \delta_i^\top \delta_i. \end{aligned}$$

Clearly, there is a function $\psi \in \mathcal{K}_\infty$ (a continuous, zero at zero, strictly increasing and unbounded function $\mathbb{R}_+ \to \mathbb{R}_+$) such that for all $x_i \in \mathbb{R}^2$:

$$(1 - |x_i|^2)^2 |x_i|^2 \geq \psi(|x_i|_{\mathcal{W}}),$$

where $|x_i|_{\mathcal{W}} = \inf_{z\in\mathcal{W}} |z - x_i|$ is the distance to the set $\mathcal{W}$, then

$$\dot{W} \leq -\omega k\psi(|x_i|_{\mathcal{W}}) + \frac{1}{\omega k}\delta_i^\top \delta_i,$$

which is equivalent to input-to-state stability property with respect to the set $\mathcal{W}$ and the input δ_i [2]. ■

The proven stability property also implies that if $\delta_i = 0$, then almost all trajectories of (1) (excluding one initiated at the origin) are attracted by the unit circle.

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