Lyapunov-based Transient Stability Analysis

Jianli Gao, Balarko Chaudhuri, and Alessandro Astolfi

Abstract— **The paper presents an analytical control solution to the problem of transient stabilization of lossy multi-machine power systems. Firstly, a new form of control Lyapunov function candidates with a flexible potential-energy-like term is proposed. This is achieved mainly by introducing an auxiliary state that contributes to the derivation of a cross-term. Based on the Lyapunov function candidates, a new control law ensuring asymptotic stability of the desired closed-loop operating equilibrium is proposed. Finally, a case study on the model of a two-machine system to illustrate the effectiveness of the proposed control solution is presented.**

I. INTRODUCTION

Transient stability analysis of power systems is a long-standing challenge because of several intrinsic complex characteristics, such as the nonlinear dynamic behaviours of the synchronous generators (SGs), the network discontinuity caused by faults or switching operations and the saturation effects due to limited feasible operating ranges, see *e.g.* [1]–[4]. Transient stability is mainly concerned with the ability of the power system to maintain synchronism when subject to large disturbances such as a fault on the transmission facilities [1]. The fault modifies the network topology, therefore driving the state away from the pre-fault operating equilibrium; after the fault is cleared, the question becomes whether the state trajectory converges back to a desired post-fault operating equilibrium [1], [5]. Typically, if the post-fault rotor angular separations between all SGs remain within certain bounds, the system maintains synchronism; otherwise, the system is deemed to be transient unstable and corrective actions have to be undertaken [1], [3], [4]. To assess the transient stability of a post-fault power system, time domain simulations (TDS) offer one of the most widely accepted methods [1], [3]. This is usually achieved through the numerical integration of the model of the considered post-fault power system. However, since the scale of power systems has been ever increasing, TDS methods have become computationally expensive and therefore not well-suited for real-time implementations [3]. Compared to TDS methods, direct methods have a distinct advantage in that they allow assessing transient stability of a post-fault trajectory without time-consuming numerical integration [1], [3]. This is achieved mainly by using a suitable Lyapunov or energy function to analyze the stability properties of the desired post-fault operating equilibrium, and to check whether the initial post-fault state is inside the region of attraction of the desired post-fault operating equilibrium [3]. Therefore, direct methods are expected to offer a promising solution to the problem of real-time transient stability analysis of power systems [1], [3].

The key issue to the realization of direct methods lies in the construction of a well-defined Lyapunov or energy function for the model of the considered post-fault power system [3]. Historically, considerable efforts have been concentrated on the construction of Lyapunov or energy functions especially for the models of multi-machine systems with lossy transmission lines (see the detailed illustration in [3, Chapter 6] and *e.g.* [6]–[9]). However, these efforts have been partially in vain, since the proposed Lyapunov or energy functions are either not well-defined or such that their time-derivatives along the state trajectories are not strictly negative definite [3].

The main obstacle to the task of constructing an explicit control Lyapunov function (CLF) stems from the deleterious effects of resistive elements that hamper the assignment of the potential-energy-like term [5]. To address this problem, [5] has proposed a nonstandard cross-term that modifies the energy transfer and therefore improves the general form of traditional CLFs. By applying the *Implicit Function Theorem*, [5] has proved the existence of a CLF for the transient stability analysis of lossy multi-machine models. However, neither an explicit form of CLFs nor a feasible control law has been suggested. As step forward, a well-defined asymptotically stabilizing controller has been firstly proposed in [10]. However, the negative definiteness of the time-derivative of the proposed CLF has not been verified. In [11], a control law with particular emphasis on limited feasible operating ranges has been proposed. The corresponding CLF is such that its time-derivative along the closed-loop state trajectories is strictly negative definite; moreover, it has been shown to be applicable to the transient stability analysis of a single-machine model. However, the feasibility of extending such a control law and CLF to the transient stability analysis of multi-machine models has not been demonstrated.

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As a natural extension of [11], the main contribution of the paper is twofold.

- A new explicit form of CLF is proposed. Its time-derivative along the closed-loop state trajectories is proved to be negative definite. This is mainly achieved by introducing an auxiliary state that contributes to the derivation of a cross-term. Furthermore, with a suitable selection of the potential-energy-like term, the proposed CLF is applicable to the study of the transient stability of lossy multi-machine models.
- An explicit dynamic control law is derived, which ensures asymptotic stability of the desired closed-loop operating equilibrium. The resulting closed-loop transient performance is smooth, indicating that the controller injects low gains into the loop - a desired features in practical applications.

The remaining part of the paper is organized as follows. In Section II the classical flux decay model of a two-machine system is presented, followed by the problem formulation of the transient stability analysis of the considered model. In Section III some typical CLFs adopted for the transient stability analysis are briefly reviewed. A new CLF candidate is then proposed, based on which the new control law is derived. The results are summarized in **Proposition 1**. In Section IV a case study on the two-machine model to demonstrate the merits of the proposed designs is presented. Finally, conclusions are drawn and future work is discussed in Section V.

Notation: Throughout the paper, the subscripts i and j represent the index of the states or parameters of the ith and the jth SG, respectively; while double-subscripts ij represent the network connection between the ith and the jth SG. The integer n represents the number of SGs in the considered multi-machine model, termed as n-machine model. Note that $i \in \mathbb{N}$, $j \in \mathbb{N}$, $n \in \mathbb{N}$, $i \leq n$ and $j \leq n$. The superscript * attached to a variable indicates its equilibrium value.

II. Modelling

The dynamics of the ith SG in the n-machine model can be described by the classical flux decay model (see *e.g.* [5, equation (2)] in a compact form, or [12, equation (9)] in its original form):

$$\begin{aligned}
\dot{\delta}_i &= \omega_i, \\
\dot{\omega}_i &= -D_i\omega_i + P_i - G_{ii}E_i^2 - E_i \sum_{\substack{j=1,\\ j\neq i}}^{n} E_j Y_{ij} \sin\left(\delta_{ij} + \alpha_{ij}\right), \\
\dot{E}_i &= -a_i E_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} b_{ij} E_j \cos\left(\delta_{ij} + \alpha_{ij}\right) + e_{f\,i}^* + u_i,
\end{aligned} \tag{1}$$

where the dynamic states consist of the generator rotor angle $\delta_i(t) \in \mathbb{R}$, the angular speed deviation with respect to the synchronous speed $\omega_i(t) \in \mathbb{R}$, and the internal transient voltage $E_i(t) \in \mathbb{R}_{>0}$. The system parameters include the damping ratio $D_i \in \mathbb{R}_{>0}$, the mechanical power input $P_i \in \mathbb{R}_{>0}$, the self conductance $G_{ii} \in \mathbb{R}_{>0}$, the magnitude of the off-diagonal elements in the reduced admittance matrix $Y_{ij} \in \mathbb{R}_{>0}$ with the corresponding complementary angle $\alpha_{ij} \in \mathbb{R}$, two switching coefficients $a_i \in \mathbb{R}_{>0}$ and $b_{ij} \in \mathbb{R}_{>0}$, and the constant component of the field voltage $e_{f\,i}^* \in \mathbb{R}_{>0}$. Finally, $u_i(t) \in \mathbb{R}$ is the excitation control input.

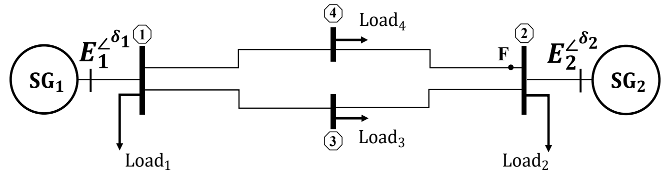

Fig. 1. A two-machine system model. Note that **F** represents the location where the fault occurs in the case study.

Note that $\delta_{ij} := \delta_i - \delta_j \in \mathbb{R}$ represents the rotor angular separation between the ith and the jth SG. Denote by $x^* = \left[x_1^{*\top}, x_2^{*\top}, \cdots, x_n^{*\top}\right]^\top \in \mathbb{R}^{3n}$ the desired stable operating equilibrium for the whole power system model, where $x_i^* = [\delta_i^*, 0, E_i^*]^\top \in \mathbb{R}^3$ is for the ith SG.

Assumption 1: The equilibrium of the rotor angular separations and of the internal transient voltages for all SGs, *i.e.* δ_{ij}^* and E_i^* for all i and j, are known.

Note that **Assumption 1** is necessary and standard in the transient stability analysis. By considering the rotor angular separation δ_{ij}, we remove a strong assumption on the requirement for the equilibrium of the rotor angle δ_i^*.

The paper focuses on a model of a power system consisting of two SGs connected to four resistive loads through lossy transmission lines, as illustrated in Fig. 1. Such a consideration is merely for ease of illustration; all results can be extended to the transient stability analysis of n-machine models for any $n \geq 3$.

Setting $n = 2$, the equations (1) describing the dynamics of the two-machine system model become

$$\begin{aligned}
\dot{\delta}_1 &= \omega_1, \\
\dot{\omega}_1 &= -D_1\omega_1 + P_1 - \left[G_{11}E_1^2 + E_1E_2Y_{12}\sin\left(\delta_{12} + \alpha_{12}\right)\right], \\
\dot{E}_1 &= -a_1E_1 + b_{12}E_2\cos\left(\delta_{12} + \alpha_{12}\right) + e_{f\,1}^* + u_1, \\
\dot{\delta}_2 &= \omega_2, \\
\dot{\omega}_2 &= -D_2\omega_2 + P_2 - \left[G_{22}E_2^2 + E_2E_1Y_{21}\sin\left(\delta_{21} + \alpha_{21}\right)\right], \\
\dot{E}_2 &= -a_2E_2 + b_{21}E_1\cos\left(\delta_{21} + \alpha_{21}\right) + e_{f\,2}^* + u_2.
\end{aligned} \tag{2}$$

With the aforementioned specifications, the problem of transient stability analysis of the two-machine system model is formulated as follows.

Problem Formulation: Consider the model (2) and a desired post-fault operating equilibrium x^*. The objective is to construct a CLF such that the following specifications are satisfied.

- The CLF is well-defined and positive definite; and its time-derivative along the post-fault closed-loop state trajectories is negative definite.
- The desired post-fault closed-loop operating equilibrium x^* is locally asymptotically stable.

III. Control Lyapunov Function Design

A. A Brief Review

Based on the theory of Hamiltonian systems, the traditional "separable" CLFs for the transient stability analysis are of the form, see *e.g.* [5, equation (10)],

$$H(\delta,\omega,E)=\psi(\delta)+\frac{1}{2}\omega^2+\frac{1}{2}(E-E^*)^2, \tag{3}$$

where the function $\psi:\mathbb{R}\to\mathbb{R}$ is the so-called potential-energy-like term, while the quadratic term in $\omega\in\mathbb{R}$ is related to the mechanical kinetic energy. Since the quadratic term in $E\in\mathbb{R}_{>0}$ has no connection to ψ, such a form is usually termed as "separable".

Remark 1: The assignment of the candidate ψ is usually accomplished through the integration of the "gradient vector" $\frac{\partial\psi}{\partial\delta}$ with respect to δ, see *e.g.* [5, Proposition 2] or [11, equation (33)]. However, for lossy n-machine models with $n\geq 2$, the "gradient vector" $\frac{\partial\psi}{\partial\delta}$ becomes non-integrable due to the asymmetry properties caused by resistive elements, *i.e.* $\alpha_{ij}\neq 0$. Therefore, the "separable" form in (3) is applicable only for single-machine models or lossless two-machine models.

To conduct the transient stability analysis of lossy n-machine models, a "non-separable" CLF has been proposed, in [5, equation (27)], *i.e.*

$$H_d(\delta,\omega,E)=\psi(\delta)+\frac{1}{2}\sum_{i=1}^{n}\omega_i^2+\frac{1}{2}\sum_{i=1}^{n}(E_i-\lambda_i(\delta)E_i^*)^2, \tag{4}$$

where $\psi:\mathbb{R}^n\to\mathbb{R}$, and $\lambda:\mathbb{R}^n\to\mathbb{R}^n$ is a vector function to be defined.

Remark 2: The vector function λ is the so-called cross-term that allows bypassing the obstacle in the integration of the "gradient vector" $\frac{\partial\psi}{\partial\delta}$, while providing the possibility to select a suitable ψ. Note that the cross-term is expected to be such that $\lambda(\delta^*):=\mathbf{1}\in\mathbb{R}^n$.

By applying the *Implicit Function Theorem*, [5] has proved the existence of a cross-term λ such that any $\mathcal{C}^1$ function is assignable for ψ. Moreover, the CLF in (4) guarantees asymptotic stability of the desired operating equilibrium x^*. However, neither an explicit form of ψ nor an explicit control law has been suggested for n-machine models in [5], which is mainly due to the difficulty in the explicit computation of the cross-term.

Hereinafter, we focus on CLF design for the two-machine model (2) for ease of illustration. Note that all the following results can be extended to the transient stability analysis of n-machine models for any $n\geq 3$.

B. Stability Analysis

To solve the problem associated with the computation of the cross-term, we introduce an auxiliary state, denoted by $\xi(t)\in\mathbb{R}^2_{>0}$. Consider now the CLF candidate

$$\begin{aligned}V(\delta,\omega,E,\xi)=\psi(\delta)&+\frac{1}{2}\sum_{i=1}^{2}\eta_i\omega_i^2\\&+\frac{1}{2}\sum_{i=1}^{2}\mu_i(E_i-\lambda_i(\delta,\xi)E_i^*)^2\\&+\frac{1}{2}\sum_{i=1}^{2}\rho_i(\lambda_i(\delta,\xi)-\xi_i)^2,\end{aligned} \tag{5}$$

where the potential-energy-like term is expected to be non-negative, *i.e.* $\psi:\mathbb{R}^2\to\mathbb{R}_{\geq 0}$; the cross-term contains the auxiliary state, *i.e.* $\lambda(\delta,\xi):\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$; while $\eta\in\mathbb{R}^2_{>0}$, $\mu\in\mathbb{R}^2_{>0}$ and $\rho\in\mathbb{R}^2_{>0}$ are tunable weighting coefficients.

Remark 3: Compared to (4), a new quadratic term is added in (5), which is motivated by the objective to drive the auxiliary state to the cross-term, *i.e.* $|\lambda(t)-\xi(t)|\to\mathbf{0}\in\mathbb{R}^2$, as $t\to\infty$. Moreover, since $\lambda(\delta^*,\xi^*):=\mathbf{1}\in\mathbb{R}^2$, we set $\xi^*:=\mathbf{1}\in\mathbb{R}^2$.

Taking the time-derivative of V along the trajectories of the two-machine system model (2) yields

$$\begin{aligned}\dot V=-\sum_{i=1}^{2}\eta_iD_i\omega_i^2&+\omega_1\Big(\frac{\partial\psi}{\partial\delta_1}+\eta_1P_1\\&-\eta_1\big[G_{11}E_1^2+E_1E_2Y_{12}\sin(\delta_{12}+\alpha_{12})\big]\Big) \qquad (6)\\&+\omega_2\Big(\frac{\partial\psi}{\partial\delta_2}+\eta_2P_2\\&-\eta_2\big[G_{22}E_2^2+E_2E_1Y_{21}\sin(\delta_{21}+\alpha_{21})\big]\Big) \qquad (7)\\&+\sum_{i=1}^{2}(E_i-\lambda_iE_i^*)\mu_i(\dot E_i-\dot\lambda_iE_i^*) \qquad (8)\\&+\sum_{i=1}^{2}(\lambda_i-\xi_i)\rho_i(\dot\lambda_i-\dot\xi_i). \qquad (9)\end{aligned}$$

Note now the identities

$$\begin{aligned}E_iE_j-\lambda_i\lambda_jE_i^*E_j^*&=(E_i-\lambda_iE_i^*)E_j+(E_j-\lambda_jE_j^*)\lambda_iE_i^*,\\E_i^2-\lambda_i^2{E_i^*}^2&=(E_i-\lambda_iE_i^*)(E_i+\lambda_iE_i^*),\end{aligned} \tag{10}$$

which are useful for rewriting the terms containing $(E_i-\lambda_iE_i^*)$.

By (10) and other straightforward manipulations, (6) can be rewritten as

$$\begin{aligned}(6)=\omega_1\Big(\frac{\partial\psi}{\partial\delta_1}&+\eta_1P_1-\eta_1\lambda_1\big[\xi_1G_{11}{E_1^*}^2\\&+\xi_2E_1^*E_2^*Y_{12}\sin(\delta_{12}+\alpha_{12})\big]\Big) \qquad (11)\\&-\eta_1\omega_1\Big((\lambda_1-\xi_1)\lambda_1G_{11}{E_1^*}^2\\&+(\lambda_2-\xi_2)\lambda_1E_1^*E_2^*Y_{12}\sin(\delta_{12}+\alpha_{12})\Big) \qquad (12)\\&-\eta_1\omega_1\Big((E_1-\lambda_1E_1^*)G_{11}(E_1+\lambda_1E_1^*)\\&+\big((E_1-\lambda_1E_1^*)E_2+(E_2-\lambda_2E_2^*)\lambda_1E_1^*\big)\\&\times Y_{12}\sin(\delta_{12}+\alpha_{12})\Big). \qquad (13)\end{aligned}$$

Analogously, rewriting (7) yields

$$(7) = \omega_2\Big(\frac{\partial\psi}{\partial\delta_2} + \eta_2 P_2 - \eta_2\lambda_2\big[\xi_2 G_{22} E_2^{*2} + \xi_1 E_2^* E_1^* Y_{21}\sin(\delta_{21}+\alpha_{21})\big]\Big) \tag{14}$$

$$- \eta_2\omega_2\Big((\lambda_2-\xi_2)\lambda_2 G_{22}E_2^{*2} + (\lambda_1-\xi_1)\lambda_2 E_2^* E_1^* Y_{21}\sin(\delta_{21}+\alpha_{21})\Big) \tag{15}$$

$$- \eta_2\omega_2\Big((E_2-\lambda_2E_2^*)G_{22}(E_2+\lambda_2E_2^*) + \big((E_2-\lambda_2E_2^*)E_1 + (E_1-\lambda_1E_1^*)\lambda_2E_2^*\big) \times Y_{21}\sin(\delta_{21}+\alpha_{21})\Big). \tag{16}$$

We now present three design selections rendering $\dot V$ negative definite.

1) Cross-term design: By zeroing (11) and (14), we derive the explicit expressions for the cross-term, *i.e.*

$$\lambda_1(\delta,\xi) = \frac{\frac{\partial\psi}{\partial\delta_1} + \eta_1 P_1}{\eta_1\big[\xi_1 G_{11}E_1^{*2} + \xi_2 E_1^* E_2^* Y_{12}\sin(\delta_{12}+\alpha_{12})\big]},$$
$$\lambda_2(\delta,\xi) = \frac{\frac{\partial\psi}{\partial\delta_2} + \eta_2 P_2}{\eta_2\big[\xi_2 G_{22}E_2^{*2} + \xi_1 E_2^* E_1^* Y_{21}\sin(\delta_{21}+\alpha_{21})\big]}. \tag{17}$$

Assumption 2: The function ψ is at least twice differentiable and it is such that $\frac{\partial\psi}{\partial\delta_i} = 0$, for all i. In addition, the denominator of λ in (17) remains positive along the trajectories of the system (2).

Note that, at the desired operating equilibrium, we have

$$\eta_i\big[\xi_i^* G_{ii}E_i^{*2} + \xi_j^* E_i^* E_j^* Y_{ij}\sin(\delta_{ij}^*+\alpha_{ij})\big] = \eta_i P_i > 0. \tag{18}$$

By continuity, the denominator of λ in (17) remains locally positive. Hence, **Assumption 2** holds locally.

Remark 4: As shown in (17), the expression for the ith element of the cross-term is independent of the jth element of the gradient vector, *i.e.* λ_i contains no $\frac{\partial\psi}{\partial\delta_j}$ for all $i \neq j$. This indicates that, with a suitable selection of ψ, the cross-term in (17) can be used with minor changes for the transient stability analysis of the n-machine model for any $n \geq 3$.

2) Auxiliary state design: We design the dynamics of the auxiliary state as

$$\begin{aligned}\dot\xi_1 &= \dot\lambda_1 + \kappa_1(\lambda_1-\xi_1) - \frac{1}{\rho_1}\Big(\eta_1\omega_1\lambda_1 G_{11}E_1^{*2} + \eta_2\omega_2\lambda_2E_2^*E_1^*Y_{21}\sin(\delta_{21}+\alpha_{21})\Big),\\ \dot\xi_2 &= \dot\lambda_2 + \kappa_2(\lambda_2-\xi_2) - \frac{1}{\rho_2}\Big(\eta_2\omega_2\lambda_2 G_{22}E_2^{*2} + \eta_1\omega_1\lambda_1E_1^*E_2^*Y_{12}\sin(\delta_{12}+\alpha_{12})\Big),\end{aligned} \tag{19}$$

where $\kappa \in \mathbb{R}^2_{>0}$ is a tunable coefficient.

Since $\dot\lambda(\delta,\xi)$ contains $\dot\xi$, the expression for $\dot\xi$ in (19) is basically given in an implicit form. Then, for any selection of ψ verifying **Assumption 2**, we can calculate $\dot\lambda$ and obtain

$$\dot\xi = [\mathcal{I}-\mathcal{A}]^{-1}\mathcal{B}, \tag{20}$$

where $\mathcal{I} \in \mathbb{R}^{2\times2}$ is the identity matrix; while $\mathcal{A} \in \mathbb{R}^{2\times2}$ and $\mathcal{B} \in \mathbb{R}^{2\times1}$ are two matrices, the elements of which are given by

$$\mathcal{A}_{ij} = \frac{\partial\lambda_i}{\partial\xi_j}, \tag{21}$$

$$\mathcal{B}_i = \sum_{j=1}^{2}\frac{\partial\lambda_i}{\partial\delta_j}\omega_j + \kappa_i(\lambda_i-\xi_i) - \frac{1}{\rho_i}\Big(\eta_i\omega_i\lambda_iG_{ii}E_i^{*2} + \sum_{\substack{j=1,\\ j\neq i}}^{2}\eta_j\omega_j\lambda_jE_j^*E_i^*Y_{ji}\sin(\delta_{ji}+\alpha_{ji})\Big), \tag{22}$$

for all i and j.

Assumption 3: The matrix $[\mathcal{I}-\mathcal{A}]$ is invertible along the trajectories of the system (2).

Remark 5: Through suitable selections of ψ and of $\dot\xi$, both **Assumptions 2** and **3** can be locally satisfied. In addition, the auxiliary state can be initialized at its desired equilibrium, *i.e.* $\xi(0) := \xi^* = \mathbf{1} \in \mathbb{R}^2$. This mechanism contributes to the local satisfactions of the assumptions.

Now, note that (20) is such that

$$(9) + (12) + (15) = -\sum_{i=1}^{2}\rho_i\kappa_i(\lambda_i-\xi_i)^2 \leq 0. \tag{23}$$

3) Control law design: We propose the dynamic control law as

$$\begin{aligned}u_1 &= a_1\lambda_1E_1^* - b_{12}E_2\cos(\delta_{12}+\alpha_{12}) - e_{f1}^* + \dot\lambda_1E_1^* - \mathrm{Sat}_1(E_1-\lambda_1E_1^*)\\ &\quad + \frac{1}{\mu_1}\Big(\eta_1\omega_1\big(G_{11}(E_1+\lambda_1E_1^*) + E_2Y_{12}\sin(\delta_{12}+\alpha_{12})\big) + \eta_2\omega_2\lambda_2E_2^*Y_{21}\sin(\delta_{21}+\alpha_{21})\Big),\\ u_2 &= a_2\lambda_2E_2^* - b_{21}E_1\cos(\delta_{21}+\alpha_{21}) - e_{f2}^* + \dot\lambda_2E_2^* - \mathrm{Sat}_2(E_2-\lambda_2E_2^*)\\ &\quad + \frac{1}{\mu_2}\Big(\eta_2\omega_2\big(G_{22}(E_2+\lambda_2E_2^*) + E_1Y_{21}\sin(\delta_{21}+\alpha_{21})\big) + \eta_1\omega_1\lambda_1E_1^*Y_{12}\sin(\delta_{12}+\alpha_{12})\Big),\end{aligned} \tag{24}$$

in which the saturation term is set as[1]

$$\mathrm{Sat}_i(E_i-\lambda_iE_i^*) = L_i\tanh\Big(\frac{k_i}{L_i}(E_i-\lambda_iE_i^*)\Big), \tag{25}$$

with $L \in \mathbb{R}^2_{>0}$ and $k \in \mathbb{R}^2_{>0}$ tunable constants. Note that the inclusion of such a saturation term is motivated by the desire to limit the control input range.

[1] Any differentiable, monotonically increasing saturation function can be used.

Therefore, (24) is such that

$$\begin{aligned}(8)+(13)+(16) = &-\sum_{i=1}^{2}\mu_i a_i(E_i-\lambda_i E_i^*)^2 \\ &-\sum_{i=1}^{2}\mu_i(E_i-\lambda_i E_i^*)\mathrm{Sat}_i(E_i-\lambda_i E_i^*) \leq 0.\end{aligned} \tag{26}$$

As a result, the selections in (17), (20) and (24), provided **Assumptions 2** and **3** hold, are such that

$$\begin{aligned}\dot{V} = &-\sum_{i=1}^{2}\eta_i D_i\omega_i^2 \\ &-\sum_{i=1}^{2}\mu_i\Big(a_i(E_i-\lambda_i E_i^*)^2 \\ &\quad+(E_i-\lambda_i E_i^*)\mathrm{Sat}_i(E_i-\lambda_i E_i^*)\Big) \\ &-\sum_{i=1}^{2}\rho_i\kappa_i(\lambda_i-\xi_i)^2 \leq 0,\end{aligned} \tag{27}$$

from which we conclude stability of the desired closed-loop operating equilibrium x^*. Furthermore, a direct application of *LaSalle's invariance principle* shows that x^* is also attractive, hence it is asymptotically stable. The results are summarized in the following statement.

Proposition 1: Consider the lossy two-machine system model (2) and a desired operating equilibrium x^*. Select $\psi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, such that $\psi \in \mathcal{C}^2$, $\delta_{ij}^* = \arg\min \psi$, $\frac{\partial \psi}{\partial \delta_i} = 0$, $\forall\ i, j \in [1, 2]$. Assume **Assumptions 2** and **3** hold. Then, the dynamic control law (24) is such that x^* is locally asymptotically stable.

IV. Case Study

A. Selection of $\psi(\delta)$

The potential-energy-like term must satisfy the conditions stated in **Proposition 1**. In this case study, we select

$$\psi(\delta) := \sigma\Bigg(1-\cos\big(w(\delta_{21}-\delta_{21}^*)\big)\Bigg), \tag{28}$$

where $\sigma \in \mathbb{R}_{>0}$ and $w \in \mathbb{R}_{>0}$ are tunable parameters.

Note that $\psi(\delta_{12}^*) = 0$, and that ψ is positive definite around the local minimizer at the desired post-fault operating equilibrium.

B. Scenario Specifications

The simulations conducted are based on the two-machine system model (2). As shown in Fig. 1, the short circuit fault is assumed to occur at the location denoted by **F**.

Consider now the classical test scenario whereby the fault occurs at $t_f = 1s$, resulting in the switching of the considered model from the pre-fault mode to the fault-on mode; then, the fault is cleared at $t_c = 1.080s$ by switching-off the faulty transmission line, resulting in the switching from the fault-on mode to the post-fault mode.

The values of the parameters and of the coefficients in the pre-fault mode, the fault-on mode and the post-fault mode, respectively, are listed in Table I. In addition, the values of the desired operating equilibrium are also given in Table I. Note that there exists no operating equilibrium in the fault-on mode. The values of the constant parameters in (2), (28) and (24), and of the weighting coefficients in (5) are listed in Table II.

TABLE I
PARAMETERS, COEFFICIENTS AND OPERATING EQUILIBRIUM

Mode	pre-fault	fault-on	post-fault
G_{11}	21.9365	8.0846	24.7152
G_{22}	16.1875	0.0008	15.6692
Y_{12}	41.0925	0.0006	34.7940
Y_{21}	58.2485	0.0008	49.3205
α_{12}	0.3123	1.5980	0.3057
α_{21}	0.3123	1.5980	0.3057
a_1	0.6844	0.9535	0.6634
a_2	0.8321	1.2291	0.7344
b_{12}	0.2819	0	0.2387
b_{21}	0.4187	0	0.3545
δ_{12}^*	−0.2042	–	−0.2294
δ_{21}^*	0.2042	–	0.2294
E_1^*	1.1590	–	1.1433
E_2^*	1.4959	–	1.5996

TABLE II
CONSTANT PARAMETERS AND WEIGHTING COEFFICIENTS

D	$[0.3770, 0.5344]^\top$
P	$[37.1558, 86.0933]^\top$
e_f^*	$[0.3740, 0.8228]^\top$
σ	1.7684
w	0.8000
κ	$[20, 20]^\top$
L	$[5, 5]^\top$
k	$[20, 20]^\top$
η	$[0.1, 0.1]^\top$
μ	$[3.7699, 5.3438]^\top$
ρ	$[15.0796, 21.3754]^\top$

C. Transient Performance

Fig. 2 illustrates the transient performance yielded by two control approaches, *i.e.* the solid lines yielded by the proposed control law in (24) and the dash-dotted lines yielded by an existing control law in [11, equation (23)], respectively. Note that each approach has its own merits. On one hand, since the control law in [11] acts directly on the internal transient voltage, it yields better performance in terms of the transient dynamics of this state. On the other hand, the proposed control law yields better performance in terms of the transient dynamics of the rotor angular separation and of the angular speed deviation.

Fig. 3 illustrates the time histories of the auxiliary state and of the cross term. It can be observed that the auxiliary state converges to the equilibrium of the cross-term as desired.

Fig. 4 illustrates the time histories of the control input (24), limited inside a feasible operating range. Notably, with proper weighting coefficients, such a limitation does not destroy closed-loop performance.

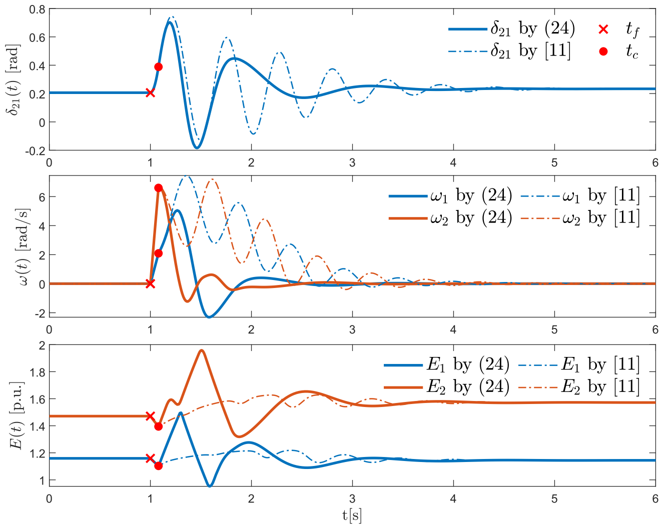

Fig. 2. Time histories of the states of the two-machine system model: rotor angular separation (top), angular speed deviation (middle) and internal transient voltage (bottom)

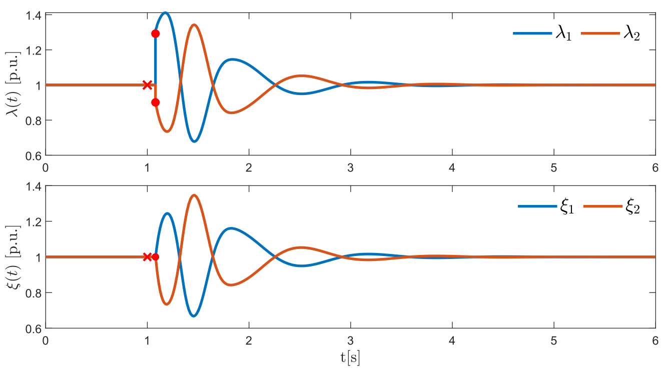

Fig. 3. Time histories of the auxiliary state (top) and of the cross-term (bottom)

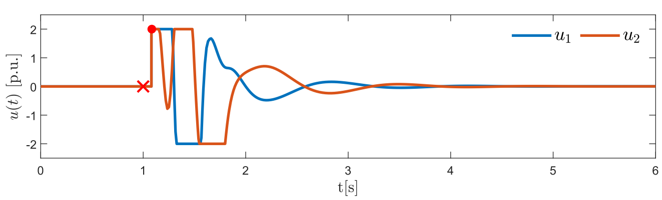

Fig. 4. Time histories of the bounded control input u

Fig. 5 illustrates the time histories of the CLF in (5), which serves as a monitor for the stable transient performance with respect to the post-fault mode. Note that, after the fault-clearing time t_c, the CLF decreases monotonically to zero, consistent with the fact that the post-fault trajectory converges asymptotically to the desired post-fault operating equilibrium.

V. Conclusion

The paper has presented a new analytical control solution to the long-standing problem of transient stabilization for lossy multi-machine power systems. A new form of control Lyapunov functions including a flexible potential-energy-like term has been proposed. The introduction of an auxiliary state allows computing the cross-term without relying on

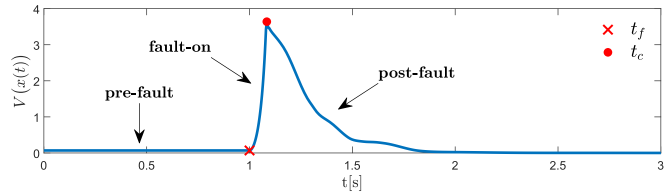

Fig. 5. Time history of the control Lyapunov function V

any integration. Therefore, the proposed control Lyapunov function can be extended to the study of the transient stability of lossy multi-machine models with any number of synchronous generators. Finally, the proposed dynamic control law ensures asymptotic stability of the desired post-fault operating equilibrium.

One natural direction for future work is to verify the performance of such a Lyapunov-based transient stability analysis tool for benchmark n-machine model with $n \geq 3$. Note that the proposed control law relies on the information exchange between all the synchronous generators, which might be hard to implement. Hence, another possible direction is to render the proposed control law decentralized by using local information available from the ith synchronous generator only.

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