The Variational Attitude Estimator in the Presence of Bias in Angular Velocity Measurements

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Abstract— **Estimation of rigid body attitude motion is a long-standing problem of interest in several applications. This problem is challenging primarily because rigid body motion is described by nonlinear dynamics and the state space is nonlinear. The extended Kalman filter and its several variants have remained the standard and most commonly used schemes for attitude estimation over the last several decades. These schemes are obtained as approximate solutions to the nonlinear optimal filtering problem. However, these approximate or near optimal solutions may not give stable estimation schemes in general. The variational attitude estimator was introduced recently to fill this gap in stable estimation of arbitrary rigid body attitude motion in the presence of uncertainties in initial state and unknown measurement noise. This estimator is obtained by applying the Lagrange-d'Alembert principle of variational mechanics to a Lagrangian constructed from residuals between measurements and state estimates with a dissipation term that is linear in the angular velocity measurement residual. In this work, the variational attitude estimator is generalized to include angular velocity measurements that have a constant bias in addition to measurement noise. The state estimates converge to true states almost globally over the state space. Further, the bias estimates converge to the true bias once the state estimates converge to the true states.**

1. INTRODUCTION

Estimation of attitude motion is essential in applications to spacecraft, unmanned aerial and underwater vehicles as well as formations and networks of such vehicles. In this work, we consider estimation of attitude motion of a rigid body from measurements of known inertial directions and angular velocity measurements with a constant bias, where all measurements are made with body-fixed sensors corrupted by sensor noise. The number of direction vectors measured by the body may vary over time. For the theoretical developments in this paper, it is assumed that at least two directions are measured at any given instant; this assumption ensures that the attitude can be uniquely determined from the measured directions at every instant. The attitude estimation scheme presented here follows the variational framework of the estimation scheme recently reported in [1], [2]. Like the estimation scheme in [1], the scheme presented here has the following important properties: (1) attitude is represented globally over

the configuration space of rigid body attitude motion without using local coordinates or quaternions; (2) no assumption is made on the statistics of the measurement noise; (3) unlike model-based estimation schemes (e.g., [3], [4], [5]), no knowledge of the attitude dynamics model is assumed; (4) the estimation scheme is obtained by applying the Lagrange-d'Alembert principle from variational mechanics [6], [7] to a Lagrangian constructed from the measurement residuals with a dissipation term linear in the angular velocity measurement residual; and (5) the estimation scheme is discretized for computer implementation by applying the discrete Lagrange-d'Alembert principle [8], [9]. It is assumed that measurements of direction vectors and angular velocity are available at sufficient frequency, such that a dynamics model is not needed to propagate state estimates between measurements.

The earliest solution to the attitude determination problem from two inertial vector measurements is the so-called "TRIAD algorithm" from the early 1960s [10]. This was followed by developments in the problem of attitude determination from a set of vector measurements, which was set up as an optimization problem called Wahba's problem [11]. This problem of instantaneous attitude determination has many different solutions in the prior literature, a sample of which can be obtained in [12], [13], [14]. Much of the published literature on estimation of attitude states use local coordinates or unit quaternions to represent attitude. Local coordinate representations, including commonly used quaternion-derived parameters like the Rodrigues parameters and the modified Rodrigues parameters (MRPs), cannot describe arbitrary or tumbling attitude motion, while the unit quaternion representation of attitude is known to be ambiguous. Each physical attitude corresponds to an element of the Lie group of rigid body rotations SO(3), and can be represented by a pair of antipodal quaternions on the hypersphere $\mathbb{S}^3$, which is often represented as an embedded submanifold of $\mathbb{R}^4$ in attitude estimation. For dynamic attitude estimation, this ambiguity in the representation could lead to instability of continuous state estimation schemes due to unwinding, as is described in [15], [16], [17].

Attitude observers and filtering schemes on SO(3) and SE(3) have been reported in, e.g., [14], [18], [19], [20], [21], [22], [23]. These estimators do not suffer from kinematic singularities like estimators using coordinate descriptions of attitude, and they do not suffer from the unstable unwinding phenomenon which may be encountered by estimators using unit quaternions. Many of these schemes are based on near optimal filtering and do not have provable stability. Related to Kalman filtering-type schemes is the maximum-likelihood

(minimum energy) filtering method of Mortensen [24], which was recently applied to attitude estimation, resulting in a nonlinear attitude estimation scheme that seeks to minimize the stored "energy" in measurement errors [25], [26]. This scheme is obtained by applying Hamilton-Jacobi-Bellman (HJB) theory [27] to the state space of attitude motion, as shown in [26]. Since the HJB equation can be only approximately solved with increasingly unwieldy expressions for higher order approximations, the resulting filter is only "near optimal" up to second order. Unlike the filtering schemes that are based on Kalman filtering or "near optimal" solutions of the HJB equation and do not have provable stability, the estimation scheme obtained here is shown to be almost globally asymptotically stable even in the case of biased angular velocity measurements. The special case of unbiased velocity measurements was dealt with in a prior version of this estimator that appeared recently [1]. Moreover, unlike filters based on Kalman filtering, the estimator proposed here does not require any knowledge about the statistics of the initial state estimate or the sensor noise.

This paper is structured as follows. Section 2 details the measurement model for measurements of inertially-known vectors and biased angular velocity measurements using body-fixed sensors. The problem of variational attitude estimation from these measurements in the presence of rate gyro bias is formulated and solved on SO(3) in Section 3. A Lyapunov stability proof of this estimator is given in Section 4, along with a proof of the almost global domain of convergence of the estimates in the case of perfect measurements. It is also shown that the bias estimate converges to the true bias in this case. This continuous estimation scheme is discretized in Section 5 in the form of a Lie group variational integrator (LGVI) using the discrete Lagrange-d'Alembert principle. Numerical simulations are carried out using this LGVI as the discrete-time variational attitude estimator in Section 5 with a fixed set of gains. Section 6 gives concluding remarks, contributions and possible future extensions of the work presented in this paper.

2. MEASUREMENT MODEL

For rigid body attitude estimation, assume that some inertially-fixed vectors are measured in a body-fixed frame, along with body angular velocity measurements having a constant bias. Let $k \in \mathbb{N}$ known inertial vectors be measured in a coordinate frame fixed to the rigid body. Denote these measured vectors as u_j^m for $j = 1, 2, \ldots, k$, in the body-fixed frame. Denote the corresponding known vectors represented in inertial frame as e_j; therefore $u_j = R^T e_j$, where R is the rotation matrix from the body frame to the inertial frame. This rotation matrix provides a coordinate-free, global and unique description of the attitude of the rigid body. Define the matrix composed of all k measured vectors expressed in the body-fixed frame as column vectors,

$$U^m = [u_1^m \ u_2^m \ u_1^m \times u_2^m] \text{ when } k = 2, \text{ and}$$
$$U^m = [u_1^m \ u_2^m \ ...u_k^m] \in \mathbb{R}^{3 \times k} \text{ when } k > 2, \quad (1)$$

and the corresponding matrix of all these vectors expressed in the inertial frame as

$$E = [e_1 \ e_2 \ e_1 \times e_2] \text{ when } k = 2, \text{ and}$$
$$E = [e_1 \ e_2 \ ...e_k] \in \mathbb{R}^{3 \times k} \text{ when } k > 2. \quad (2)$$

Note that the matrix of the actual body vectors u_j corresponding to the inertial vectors e_j, is given by

$$U = R^T E = [u_1 \ u_2 \ u_1 \times u_2] \text{ when } k = 2, \text{ and}$$
$$U = R^T E = [u_1 \ u_2 \ ...u_k] \in \mathbb{R}^{3 \times k} \text{ when } k > 2.$$

The direction vector measurements are given by

$$u_j^m = R^T e_j + \nu_j \text{ or } U^m = R^T E + N, \quad (3)$$

where $\nu_j \in \mathbb{R}^3$ is an additive measurement noise vector and $N \in \mathbb{R}^{3 \times k}$ is the matrix with ν_j as its jth column vector.

The attitude kinematics for a rigid body is given by Poisson's equation:

$$\dot{R} = R\Omega^\times, \quad (4)$$

where $\Omega \in \mathbb{R}^3$ is the angular velocity vector and $(\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3) \subset \mathbb{R}^{3 \times 3}$ is the skew-symmetric cross-product operator that gives a vector space isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3)$. The measurement model for angular velocity is

$$\Omega^m = \Omega + w + \beta, \quad (5)$$

where $w \in \mathbb{R}^3$ is the measurement error in angular velocity and $\beta \in \mathbb{R}^3$ is a vector of bias in angular velocity component measurements, which we consider to be a constant vector.

3. ATTITUDE STATE AND BIAS ESTIMATION BASED ON THE LAGRANGE-D'ALEMBERT PRINCIPLE

In order to obtain attitude state estimation schemes from continuous-time vector and angular velocity measurements, we apply the Lagrange-d'Alembert principle to an action functional of a Lagrangian of the state estimate errors, with a dissipation term linear in the angular velocity estimate error. This section presents an estimation scheme obtained using this approach. Let $\hat{R} \in SO(3)$ denote the estimated rotation matrix. According to [1], the potential "energy" function representing the attitude estimate error can be expressed as a generalized Wahba's cost function as

$$\mathcal{U}(\hat{R}, U^m) = \Phi\left(\frac{1}{2}\langle E - \hat{R}U^m, (E - \hat{R}U^m)W \rangle\right), \quad (6)$$

where U^m is given by equation (1), E is given by (2), and W is the positive diagonal matrix of the weight factors for the measured directions. Note that W may be generalized to be any positive definite matrix, not necessarily diagonal. Furthermore, $\Phi : [0, \infty) \mapsto [0, \infty)$ is a C^2 function that satisfies $\Phi(0) = 0$ and $\Phi'(x) > 0$ for all $x \in [0, \infty)$. Also $\Phi'(\cdot) \leq \alpha(\cdot)$ where $\alpha(\cdot)$ is a Class-$\mathcal{K}$ function. Let $\hat{\Omega} \in \mathbb{R}^3$ and $\hat{\beta} \in \mathbb{R}^3$ denote the estimated angular velocity and bias vectors, respectively. The "energy" contained in the vector error between the estimated and the measured angular velocity is then given by

$$\mathcal{T}(\hat{\Omega}, \Omega^m, \hat{\beta}) = \frac{m}{2}(\Omega^m - \hat{\Omega} - \hat{\beta})^T(\Omega^m - \hat{\Omega} - \hat{\beta}), \quad (7)$$

where m is a positive scalar. One can consider the Lagrangian composed of these "energy" quantities, as follows:

$$\mathcal{L}(\hat{R}, U^m, \hat{\Omega}, \Omega^m, \hat{\beta}) = \mathcal{T}(\hat{\Omega}, \Omega^m, \hat{\beta}) - \mathcal{U}(\hat{R}, U^m)$$
$$= \frac{m}{2}(\Omega^m - \hat{\Omega} - \hat{\beta})^{\mathrm{T}}(\Omega^m - \hat{\Omega} - \hat{\beta})$$
$$- \Phi\left(\frac{1}{2}\langle E - \hat{R}U^m, (E - \hat{R}U^m)W\rangle\right). \quad (8)$$

If the estimation process is started at time t_0, then the action functional of the Lagrangian (8) over the time duration $[t_0, T]$ is expressed as

$$\mathcal{S}(\mathcal{L}(\hat{R}, U^m, \hat{\Omega}, \Omega^m)) = \int_{t_0}^{T}\left(\mathcal{T}(\hat{\Omega}, \Omega^m, \hat{\beta}) - \mathcal{U}(\hat{R}, U^m)\right)\mathrm{d}s$$
$$= \int_{t_0}^{T}\left\{\frac{m}{2}(\Omega^m - \hat{\Omega} - \hat{\beta})^{\mathrm{T}}(\Omega^m - \hat{\Omega} - \hat{\beta})\right.$$
$$\left. - \Phi\left(\frac{1}{2}\langle E - \hat{R}U^m, (E - \hat{R}U^m)W\rangle\right)\right\}\mathrm{d}s. \quad (9)$$

Define the angular velocity measurement residual and the dissipation term:

$$\omega := \Omega^m - \hat{\Omega} - \hat{\beta}, \quad \tau_D = D\omega, \quad (10)$$

where $D \in \mathbb{R}^{3\times3}$ is positive definite. Consider attitude state estimation in continuous time in the presence of measurement noise and initial state estimate errors. Applying the Lagrange-d'Alembert principle to the action functional $\mathcal{S}(\mathcal{L}(\hat{R}, U^m, \hat{\Omega}, \Omega^m))$ given by (9), in the presence of a dissipation term linear in ω, leads to the following attitude and angular velocity filtering scheme.

Theorem 3.1: The filter equations for a rigid body with the attitude kinematics (4) and with measurements of vectors and angular velocity in a body-fixed frame, are of the form

$$\begin{cases} \dot{\hat{R}} = \hat{R}\hat{\Omega}^{\times} = \hat{R}(\Omega^m - \omega - \hat{\beta})^{\times}, \\ \\ m\dot{\omega} = -m\hat{\Omega} \times \omega + \Phi'\left(\mathcal{U}^0(\hat{R}, U^m)\right)S_L(\hat{R}) - D\omega, \\ \\ \hat{\Omega} = \Omega^m - \omega - \hat{\beta}, \end{cases}$$
$$(11)$$

where D is a positive definite filter gain matrix, $\hat{R}(t_0) = \hat{R}_0$, $\omega(t_0) = \omega_0 = \Omega_0^m - \hat{\Omega}_0$, $S_L(\hat{R}) = \mathrm{vex}(L^{\mathrm{T}}\hat{R} - \hat{R}^{\mathrm{T}}L) \in \mathbb{R}^3$, $\mathrm{vex}(\cdot) : \mathfrak{so}(3) \to \mathbb{R}^3$ is the inverse of the $(\cdot)^{\times}$ map, $L = EW(U^m)^{\mathrm{T}}$ and W is chosen to satisfy the conditions in Lemma 2.1 of [1].

Proof: In order to find an estimation scheme that filters the measurement noise in the estimated attitude, take the first variation of the action functional (9) with respect to $\hat{R}$ and $\hat{\Omega}$ and apply the Lagrange-d'Alembert principle with the dissipative term in (10). Consider the potential term $\mathcal{U}^0(\hat{R}, U^m) = \frac{1}{2}\langle E - \hat{R}U^m, (E - \hat{R}U^m)W\rangle$. Taking the first variation of this function with respect to $\hat{R}$ gives

$$\delta\mathcal{U}^0 = \langle -\delta\hat{R}U^m, (E - \hat{R}U^m)W\rangle$$
$$= \frac{1}{2}\langle \Sigma^{\times}, U^m W E^{\mathrm{T}}\hat{R} - \hat{R}^{\mathrm{T}}EW(U^m)^{\mathrm{T}}\rangle,$$
$$= \frac{1}{2}\langle \Sigma^{\times}, L^{\mathrm{T}}\hat{R} - \hat{R}^{\mathrm{T}}L\rangle = S_L^{\mathrm{T}}(\hat{R})\Sigma. \quad (12)$$

Now consider $\mathcal{U}(\hat{R}, U^m) = \Phi\left(\mathcal{U}^0(\hat{R}, U^m)\right)$. Then,

$$\delta\mathcal{U} = \Phi'\left(\mathcal{U}^0(\hat{R}, U^m)\right)\delta\mathcal{U}^0 = \Phi'\left(\mathcal{U}^0(\hat{R}, U^m)\right)S_L^{\mathrm{T}}(\hat{R})\Sigma. \quad (13)$$

Taking the first variation of the kinetic energy-like term (7) with respect to $\hat{\Omega}$ yields

$$\delta\mathcal{T} = -m(\Omega^m - \hat{\Omega} - \hat{\beta})^{\mathrm{T}}\delta\hat{\Omega}$$
$$= -m(\Omega^m - \hat{\Omega} - \hat{\beta})^{\mathrm{T}}(\dot{\Sigma} + \hat{\Omega} \times \Sigma)$$
$$= -m\omega^{\mathrm{T}}(\dot{\Sigma} + \hat{\Omega} \times \Sigma), \quad (14)$$

where ω is as given by (10). Applying the Lagrange-d'Alembert principle and integrating by parts leads to

$$\delta\mathcal{S} + \int_{t_0}^{T}\tau_D^{\mathrm{T}}\Sigma\mathrm{d}t = 0$$
$$\Rightarrow -m\omega^{\mathrm{T}}\Sigma\big|_{t_0}^{T} + \int_{t_0}^{T}m\dot{\omega}^{\mathrm{T}}\Sigma\mathrm{d}t \quad (15)$$
$$= \int_{t_0}^{T}\left\{m\omega^{\mathrm{T}}\hat{\Omega}^{\times} + \Phi'\left(\mathcal{U}^0(\hat{R}, U^m)\right)S_L^{\mathrm{T}}(\hat{R}) - \tau_D^{\mathrm{T}}\right\}\Sigma\mathrm{d}t,$$

where the first term in the left hand side vanishes, since $\Sigma(t_0) = \Sigma(T) = 0$. After substituting $\tau_D = D\omega$, one obtains the second equation in (11). $\square$

4. STABILITY AND CONVERGENCE OF VARIATIONAL ATTITUDE ESTIMATOR

The variational attitude estimator given by Theorem 3.1 can be used in the presence of bias in the angular velocity measurements given by the measurement model (5). The following analysis gives the stability and convergence properties of this estimator for the case that β in (5) is constant.

A. Stability of Variational Attitude Estimator

Prior to analyzing the stability of this attitude estimator, it is useful and instructive to interpret the energy-like terms used to define the Lagrangian in equation (8) in terms of state estimation errors. The following result shows that the measurement residuals, and therefore these energy-like terms, can be expressed in terms of state estimation errors.

Proposition 4.1: Define the state estimation errors

$$Q = R\hat{R}^{\mathrm{T}} \text{ and } \omega = \Omega - \hat{\Omega} - \tilde{\beta}, \quad (16)$$
$$\text{where } \tilde{\beta} = \beta - \hat{\beta}. \quad (17)$$

In the absence of measurement noise, the energy-like terms (6) and (7) can be expressed in terms of these state estimation errors as follows:

$$\mathcal{U}(Q) = \Phi\left(\langle I - Q, K\rangle\right) \text{ where } K = EWE^{\mathrm{T}}, \quad (18)$$
$$\text{and } \mathcal{T}(\omega) = \frac{m}{2}\omega^{\mathrm{T}}\omega. \quad (19)$$

Proof: The proof of this statement is obtained by first substituting $N = 0$ and $w = 0$ in (3) and (5), respectively. The resulting expressions for U^m and Ω^m are then substituted back into (6) and (7), respectively. Note that the same variable ω is used to represent the angular velocity estimation error for both cases: with and without measurement noise. Expression (18) is also derived in [1]. $\square$

The stability of this estimator, for the case of constant rate gyro bias vector β, is given by the following result.

Theorem 4.2: Let β in equation (5) be a constant vector. Then the variational attitude estimator given by equations (11), in addition to the following equation for update of the bias estimate:

$$\dot{\hat{\beta}} = \Phi'\big(\mathcal{U}^0(\hat{R}, U^m)\big) P^{-1} S_L(\hat{R}), \qquad (20)$$

is Lyapunov stable for $P \in \mathbb{R}^{3\times 3}$ positive definite.

Proof: To show Lyapunov stability, the following Lyapunov function is used:

$$V(U^m, \Omega^m, \hat{R}, \hat{\Omega}, \hat{\beta}) = \frac{m}{2}(\Omega^m - \hat{\Omega} - \hat{\beta})^\mathrm{T}(\Omega^m - \hat{\Omega} - \hat{\beta})$$
$$+ \Phi\big(\mathcal{U}^0(\hat{R}, U^m)\big) + \frac{1}{2}(\beta - \hat{\beta})^\mathrm{T} P (\beta - \hat{\beta}). \quad (21)$$

Now consider the case that there is no measurement noise, i.e., $N = 0$ and $w = 0$ in equations (3) and (5), respectively. In this case, the Lyapunov function (21) can be re-expressed in terms of the errors ω, Q and $\tilde{\beta}$ defined by equations (16)-(17), as follows:

$$V(Q, \omega, \tilde{\beta}) = \frac{m}{2}\omega^\mathrm{T}\omega + \Phi\big(\langle I - Q, K\rangle\big) + \frac{1}{2}\tilde{\beta}^\mathrm{T} P \tilde{\beta}. \quad (22)$$

The time derivative of the attitude estimation error, $Q \in$ SO(3), is obtained as:

$$\dot{Q} = R(\Omega - \hat{\Omega})^\times \hat{R}^\mathrm{T} = Q\big(\hat{R}(\omega - \tilde{\beta})\big)^\times, \qquad (23)$$

after substituting for $\hat{\Omega}$ from the third of equations (11) in the case of zero angular velocity measurement noise (when $\Omega^m = \Omega + \beta$). The time derivative of the Lyapunov function expressed as in (22) can now be obtained as follows:

$$\dot{V}(Q, \omega, \tilde{\beta}) = \qquad (24)$$
$$m\omega^\mathrm{T}\dot{\omega} - \Phi'\big(\langle I - Q, K\rangle\big) S_L^\mathrm{T}(\hat{R})(\omega - \tilde{\beta}) - \tilde{\beta}^\mathrm{T} P \dot{\hat{\beta}}.$$

After substituting equation (20) and the second of equations (11) in the above expression, one can simplify the time derivative of this Lyapunov function along the dynamics of the estimator as

$$\dot{V}(Q, \omega, \tilde{\beta}) = -\omega^\mathrm{T} D \omega \leq 0. \qquad (25)$$

This time derivative is negative semi-definite in the estimate errors $(Q, \omega, \tilde{\beta}) \in \mathrm{TSO}(3) \times \mathbb{R}^3$. This proves the result. $\square$

B. Domain of Convergence of Variational Attitude Estimator

The domain of convergence of this estimator is given by the following result.

Theorem 4.3: The variational attitude estimator in the case of biased velocity measurements, given by eqs. (11) and (20), converges asymptotically to $(Q, \omega, \tilde{\beta}) = (I, 0, 0) \in$ $\mathrm{TSO}(3) \times \mathbb{R}^3$ with an almost global domain of convergence.

The proof of this result is similar to the proof of the domain of convergence of the variational attitude estimator for the bias-free case in [1]. The additional estimate error state $\tilde{\beta}$ converges to zero asymptotically for almost all initial (Q, ω) except those that lie on a set whose complement is dense and open in $\mathrm{TSO}(3) \simeq \mathrm{SO}(3) \times \mathbb{R}^3$.

5. DISCRETE-TIME ESTIMATOR

The "energy" in the measurement residual for attitude is discretized as:

$$\mathcal{U}(\hat{R}_i, U_i^m) = \Phi\Big(\mathcal{U}^0(\hat{R}_i, U_i^m)\Big) \qquad (26)$$
$$= \Phi\Big(\frac{1}{2}\langle E_i - \hat{R}_i U_i^m, (E_i - \hat{R}_i U_i^m) W_i\rangle\Big),$$

where W_i is a positive diagonal matrix of weight factors for the measured directions at time t_i, and $\Phi : [0, \infty) \mapsto [0, \infty)$ is a C^2 function that satisfies $\Phi(0) = 0$ and $\Phi'(x) > 0$ for all $x \in [0, \infty)$. Furthermore, $\Phi'(\cdot) \leq \alpha(\cdot)$ where $\alpha(\cdot)$ is a Class-$\mathcal{K}$ function. The "energy" in the angular velocity measurement residual is discretized as

$$\mathcal{T}(\hat{\Omega}_i, \Omega_i^m) = \frac{m}{2}(\Omega_i^m - \hat{\Omega}_i - \hat{\beta}_i)^\mathrm{T}(\Omega_i^m - \hat{\Omega}_i - \hat{\beta}_i), \quad (27)$$

where m is a positive scalar.

Similar to the continuous-time attitude estimator in [1], one can express these "energy" terms for the case that perfect measurements (with no measurement noise) are available. In this case, these "energy" terms can be expressed in terms of the state estimate errors $Q_i = R_i \hat{R}_i^\mathrm{T}$ and $\omega_i = \Omega_i - \hat{\Omega}_i - \hat{\beta}_i$:

$$\mathcal{U}(Q_i) = \Phi\Big(\frac{1}{2}\langle E_i - Q_i^\mathrm{T} E_i, (E_i - Q_i^\mathrm{T} E_i) W_i\rangle\Big) =$$
$$\Phi\big(\langle I - Q_i, K_i\rangle\big) \text{ where } K_i = E_i W_i E_i^\mathrm{T}, \qquad (28)$$
$$\text{and } \mathcal{T}(\omega_i) = \frac{m}{2}\omega_i^\mathrm{T}\omega_i \text{ where } m > 0.$$

The weights in W_i can be chosen such that K_i is always positive definite with distinct (perhaps constant) eigenvalues, as in the continuous-time estimator of [1]. Using these "energy" terms in the state estimate errors, the discrete-time Lagrangian is expressed as:

$$\mathcal{L}(Q_i, \omega_i) = \mathcal{T}(\omega_i) - \mathcal{U}(Q_i)$$
$$= \frac{m}{2}\omega_i^\mathrm{T}\omega_i - \Phi\big(\langle I - Q_i, K_i\rangle\big). \qquad (29)$$

The following statement gives a first-order discretization, in the form of a Lie group variational integrator, for the continuous-time estimator of Theorem 3.1.

Proposition 5.1: Let discrete-time measurements for two or more inertial vectors along with angular velocity be available at a sampling period of h. Further, let the weight matrix W_i for the set of vector measurements E_i be chosen such that $K_i = E_i W_i E_i^\mathrm{T}$ satisfies Lemma 2.1 in [1]. A discrete-time estimator obtained by applying the discrete Lagrange-d'Alembert principle to the Lagrangian (29) is:

$$\hat{R}_{i+1} = \hat{R}_i \exp\big(h(\Omega_i^m - \omega_i - \hat{\beta}_i)^\times\big), \qquad (30)$$
$$\hat{\beta}_{i+1} = \hat{\beta}_i + h\Phi'\big(\mathcal{U}^0(\hat{R}_i, U_i^m)\big) P^{-1} S_{L_i}(\hat{R}_i), \qquad (31)$$
$$\hat{\Omega}_i = \Omega_i^m - \omega_i - \hat{\beta}_i, \qquad (32)$$
$$m\omega_{i+1} = \exp(-h\hat{\Omega}_{i+1}^\times)\Big\{(mI_{3\times 3} - hD)\omega_i \qquad (33)$$
$$+ h\Phi'\big(\mathcal{U}^0(\hat{R}_{i+1}, U_{i+1}^m)\big) S_{L_{i+1}}(\hat{R}_{i+1})\Big\},$$

where $S_{L_i}(\hat{R}_i) = \mathrm{vex}(L_i^\mathrm{T}\hat{R}_i - \hat{R}_i^\mathrm{T} L_i) \in \mathbb{R}^3$, $L_i = E_i W_i (U_i^m)^\mathrm{T} \in \mathbb{R}^{3\times 3}$, $\mathcal{U}^0(\hat{R}_i, U_i^m)$ is defined in (26) and $(\hat{R}_0, \hat{\Omega}_0) \in \mathrm{SO}(3) \times \mathbb{R}^3$ are initial estimated states.

The proof is very similar to the proof of the discrete-time variational attitude estimator presented in [1]. Note that the discrete-time total energy corresponding to the discrete Lagrangian (29) is dissipated with time, as with the continuous time estimator.

6. NUMERICAL SIMULATION

This section presents numerical simulation results of the discrete estimator presented in Section 5, in the presence of constant bias in angular velocity measurements. In order to validate the performance of this estimator, "true" rigid body attitude states are generated using the rotational kinematics and dynamics equations. The rigid body moment of inertia is selected as $J_v = \text{diag}([2.56 \quad 3.01 \quad 2.98]^{\mathrm{T}})$ kg.m^2. Moreover, a sinusoidal external torque is applied to this body, expressed in body fixed frame as

$$\varphi(t) = [0 \quad 0.028\sin(2.7t - \frac{\pi}{7}) \quad 0]^{\mathrm{T}} \text{ N.m.} \quad (34)$$

The true initial attitude and angular velocity are given by,

$$R_0 = \text{expm}_{\text{SO}(3)}\left(\left(\frac{\pi}{4} \times [\frac{3}{7} \quad \frac{6}{7} \quad \frac{2}{7}]^{\mathrm{T}}\right)^{\times}\right)$$
$$\text{and } \Omega_0 = \frac{\pi}{60} \times [-2.1 \quad 1.2 \quad -1.1]^{\mathrm{T}} \text{ rad/s.} \quad (35)$$

A set of at least two inertial sensors and three gyros perpendicular to each other are assumed to be onboard the rigid body. The true states generated from the kinematics and dynamics of this rigid body are also used to generate the observed directions in the body fixed frame. We assume that there are at most nine inertially known directions which are being measured by the sensors fixed to the rigid body at a constant sample rate. Bounded zero mean noise is added to the true direction vectors to generate each measured direction. A summation of three sinusoidal matrix functions is added to the matrix U, to generate a measured U^m with measurement noise. The frequency of the noises are 1, 10 and 100 Hz, with different phases and different amplitudes, which are up to $2.4°$ based on coarse attitude sensors like sun sensors and magnetometers. Similarly, two sinusoidal noises of 10 Hz and 200 Hz frequencies are added to Ω to form the measured Ω^m. These signals also have different phases and their magnitude is up to $0.97°/s$, which corresponds to a coarse rate gyro. Besides, the gyro readings are assumed to contain a constant bias in three directions, as follows:

$$\beta = [-0.01 \quad -0.005 \quad 0.02]^{\mathrm{T}} \text{ rad/s.} \quad (36)$$

The estimator is simulated over a time interval of $T = 40$s, with a time stepsize of $h = 0.01$s. The scalar inertia-like gain is $m = 5$ and the dissipation matrix is selected as:

$$D = \text{diag}([17.4 \quad 18.85 \quad 20.3]^{\mathrm{T}}). \quad (37)$$

As in [1], $\Phi(x) = x$. The weight matrix W is also calculated using the conditions in [1]. The positive definite matrix for bias gain is selected as $P = 2 \times 10^3 I$. The initial estimated
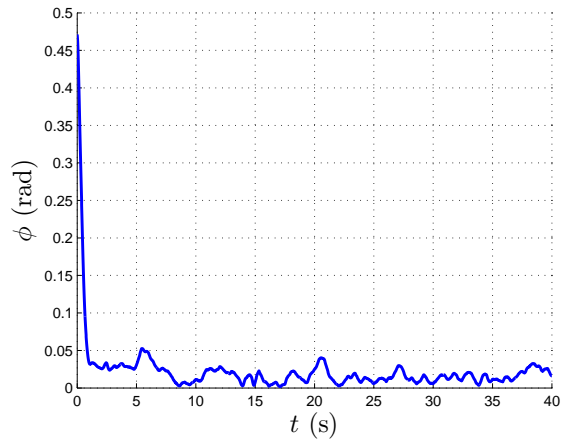

Fig. 1.   Principal angle of the attitude estimate error

states and bias are set to:

$$\hat{R}_0 = \text{expm}_{\text{SO}(3)}\left(\left(\frac{\pi}{2.5} \times [\frac{3}{7} \quad \frac{6}{7} \quad \frac{2}{7}]^{\mathrm{T}}\right)^{\times}\right),$$
$$\hat{\Omega}_0 = [-0.26 \quad 0.1725 \quad -0.2446]^{\mathrm{T}} \text{ rad/s,} \quad (38)$$
$$\text{and } \hat{\beta}_0 = [0 \quad -0.01 \quad 0.01]^{\mathrm{T}} \text{ rad/s.}$$

In order to integrate the implicit set of equations in (30)-(33) numerically, the first two equations are solved at each sampling step. Using (32), $\hat{\Omega}_{i+1}$ in (33) is written in terms of ω_{i+1} next. The resulting implicit equation is solved with respect to ω_{i+1} iteratively to a set tolerance applying the Newton-Raphson method. The root of this nonlinear equation along with $\hat{R}_{i+1}$ and $\hat{\beta}_{i+1}$ are used for the next sampling time instant. This process is repeated till the end of the simulated duration.

Results from this numerical simulation are shown here. The principal angle corresponding to the rigid body's attitude estimation error is depicted in Fig. 1, and estimation errors in the angular velocity components are shown in Fig. 2. Finally, Fig. 3 portrays estimate errors in bias components. Estimation errors are seen to converge to a neighborhood of $(Q, \omega, \tilde{\beta}) = (I, 0, 0)$, where the size of this neighborhood depends on the bounds of the measurement noise.

7. CONCLUSION

The framework of variational attitude estimation is generalized to include bias in angular velocity measurements and estimate a constant bias vector. The continuous-time state estimator is obtained by applying the Lagrange-d'Alembert principle from variational mechanics to a Lagrangian consisting of the energies in the measurement residuals, along with a dissipation term linear in angular velocity measurement residual. The update law for the bias estimate ensures that the total energy content in the state and bias estimation errors is dissipated as in a dissipative mechanical system. The resulting generalization of the variational attitude estimator is almost globally asymptotically stable, like the variational attitude estimator for the bias-free case reported in [1].
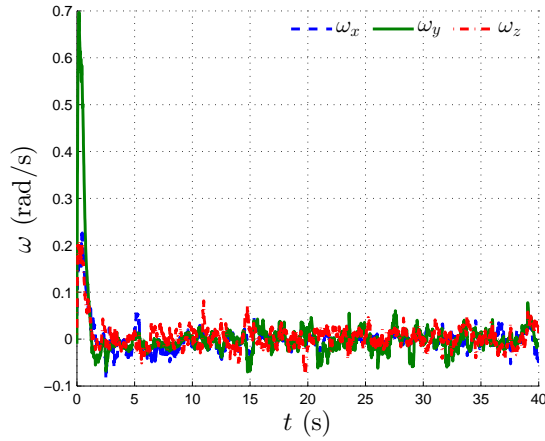
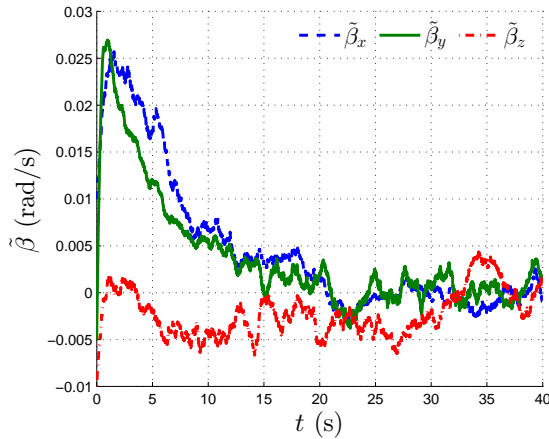Fig. 2.   Angular velocity estimate error

Fig. 3.   Bias estimate error

A discretization of this estimator is obtained in the form of an implicit first order Lie group variational integrator, by applying the discrete Lagrange-d'Alembert principle to the discrete Lagrangian with the dissipation term linear in the angular velocity estimation error. This discretization preserves the stability of the continuous estimation scheme. Using a realistic set of data for rigid body rotational motion, numerical simulations show that the estimated states and estimated bias converge to a bounded neighborhood of the true states and true bias when the measurement noise is bounded. Future planned extensions of this work are to develop an explicit discrete-time implementation of this attitude estimator, and implement it in real-time with optical and inertial sensors.

REFERENCES

[1] M. Izadi and A. Sanyal, "Rigid body attitude estimation based on the Lagrange-d'Alembert principle," *Automatica*, vol. 50, no. 10, pp. 2570 – 2577, 2014.

[2] M. Izadi, A. Sanyal, E. Samiei, and V. Kumar, "Comparison of an attitude estimator based on the Lagrange-d'Alembert principle with some state-of-the-art filters," in *2015 IEEE International Conference on Robotics and Automation, May 26 - 30, 2015*, Seattle, Washington, 2015.

[3] R. Leishman, J. Macdonald, R. Beard, and T. McLain, "Quadrotors and accelerometers: State estimation with an improved dynamic model," *Control Systems, IEEE*, vol. 34, no. 1, pp. 28–41, 2014.

[4] S. Brás, M. Izadi, C. Silvestre, A. Sanyal, and P. Oliveira, "Nonlinear observer for 3D rigid body motion," in *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*.   IEEE, 2013, pp. 2588–2593.

[5] M. Morgado, P. Oliveira, C. Silvestre, and J. Vasconcelos, "Embedded vehicle dynamics aiding for USBL/INS underwater navigation system," *Control Systems Technology, IEEE Transactions on*, vol. 22, no. 1, pp. 322–330, 2014.

[6] H. Goldstein, *Classical Mechanics*, 2nd ed.   Reading, MA: Addison-Wesley, 1980.

[7] D. Greenwood, *Classical Dynamics*.   Englewood Cliffs, NJ: Prentice Hall, 1987.

[8] J. Marsden and M. West, "Discrete mechanics and variational integrators," *Acta Numerica*, vol. 10, pp. 357–514, 2001.

[9] E. Hairer, C. Lubich, and G. Wanner, *Geometric Numerical Integration*.   New York: Springer Verlag, 2002.

[10] H. Black, "A passive system for determining the attitude of a satellite," *AIAA Journal*, vol. 2, no. 7, pp. 1350–1351, 1964.

[11] G. Wahba, "A least squares estimate of satellite attitude, Problem 65-1," *SIAM Review*, vol. 7, no. 5, p. 409, 1965.

[12] J. Farrell, J. Stuelpnagel, R. Wessner, J. Velman, and J. Brock, "A least squares estimate of satellite attitude, Solution 65-1," *SIAM Review*, vol. 8, no. 3, pp. 384–386, 1966.

[13] F. Markley, "Attitude determination using vector observations and the singular value decomposition," *The Journal of the Astronautical Sciences*, vol. 36, no. 3, pp. 245–258, 1988.

[14] A. Sanyal, "Optimal attitude estimation and filtering without using local coordinates, Part 1: Uncontrolled and deterministic attitude dynamics," in *American Control Conference, 2006*, Minneapolis, MN, 2006, pp. 5734–5739.

[15] S. P. Bhat and D. S. Bernstein, "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon," *Systems & Control Letters*, vol. 39, no. 1, pp. 63–70, 2000.

[16] N. A. Chaturvedi, A. K. Sanyal, and N. H. McClamroch, "Rigid body attitude control—Using rotation matrices for continuous, singularity-free control laws," *IEEE Control Systems Magazine*, vol. 31, no. 3, pp. 30–51, 2011.

[17] A. Sanyal and N. Nordkvist, "Attitude state estimation with multi-rate measurements for almost global attitude feedback tracking," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 35, no. 3, pp. 868–880, 2012.

[18] J. F. Vasconcelos, C. Silvestre, and P. Oliveira, "A nonlinear GPS/IMU based observer for rigid body attitude and position estimation," in *IEEE Conf. on Decision and Control*, Cancun, Mexico, Dec. 2008, pp. 1255–1260.

[19] C. Lageman, J. Trumpf, and R. Mahony, "Gradient-like observers for invariant dynamics on a Lie group," *IEEE Transaction on Automatic Control*, vol. 55, pp. 367 – 377, 2010.

[20] F. Markley, "Attitude filtering on SO(3)," *The Journal of the Astronautical Sciences*, vol. 54, no. 4, pp. 391–413, 2006.

[21] R. Mahony, T. Hamel, and J.-M. Pflimlin, "Complementary filters on the special orthogonal group," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1217, 2008.

[22] S. Bonnabel, P. Martin, and P. Rouchon, "Nonlinear symmetry-preserving observers on Lie groups," *IEEE Transactions on Automatic Control*, vol. 54, no. 7, pp. 1709–1713, 2009.

[23] J. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira, "A nonlinear position and attitude observer on SE(3) using landmark measurements," *Systems & Control Letters*, vol. 59, pp. 155–166, 2010.

[24] R. Mortensen, "Maximum-likelihood recursive nonlinear filtering," *Journal of Optimization Theory and Applications*, vol. 2, no. 6, pp. 386–394, 1968.

[25] A. Aguiar and J. Hespanha, "Minimum-energy state estimation for systems with perspective outputs," *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 226–241, 2006.

[26] M. Zamani, "Deterministic attitude and pose filtering, an embedded Lie groups approach," Ph.D. dissertation, Australian National University, Canberra, Australia, Mar. 2013.

[27] D. Kirk, *Optimal Control Theory: An Introduction*.   Englewood Cliffs, NJ: Prentice Hall, 1970.