An axiomatic approach of the discrete Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework

Jean-Luc Marichal*†

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Abstract

We present a model allowing to aggregate decision criteria when the available information is of a qualitative nature. The use of the Sugeno integral as an aggregation function is justified by an axiomatic approach. It is also shown that the mutual preferential independence of criteria reduces the Sugeno integral to a dictatorial aggregation.

Keywords: multicriteria decision making, ordinal scale, Sugeno (fuzzy) integral, preferential independence.

1 Introduction

Assume $A = \{a, b, c, \dots\}$ is a finite set of potential alternatives, among which the decision maker must choose. Consider also a finite set of criteria $N = \{1, \dots, n\}$ to be satisfied. Each criterion $i \in N$ is represented by a mapping g_i from the set of alternatives A to a given finite ordinal scale

$$X_i = \{r_1^{(i)} < \cdots < r_{k_i}^{(i)}\} \subset \mathbb{R},$$

that is, a scale where only order matters, and not numbers. For example, a scale of evaluation of importance of scientific papers by referees such as

1=Poor, 2=Below Average, 3=Average,
4=Very Good, 5=Excellent

is a finite ordinal scale. The coding by real numbers is used only to fix an order on the scale.

For each alternative $a \in A$ and each criterion $i \in N$, $g_i(a)$ represents the evaluation of a along criterion i. We assume that all the mappings g_i are given beforehand.

Our central interest is the problem of constructing a single comprehensive criterion from the given criteria. Such a criterion, which is supposed to be a representative of the original criteria, is modeled by a mapping g from A to a given finite ordinal scale

$$X = \{r_1 < \cdots < r_k\} \subset \mathbb{R}.$$

The value $g(a)$ then represents the global evaluation of alternative a expressed in the scale X. Without loss of generality, we can embed this scale in the unit interval $[0, 1]$ and fix the endpoints $r_1 := 0$ and $r_k := 1$.

In order to aggregate the partial evaluations of $a \in A$, we will assume that there exist n non-decreasing mappings $U_i : X_i \to X$ $(i \in N)$ and an aggregation function $M : X^n \to X$ such that

$$g(a) = M\big[U_1(g_1(a)), \dots, U_n(g_n(a))\big] \qquad (a \in A).$$

The mappings U_i, called *commensurateness mappings*, enable us to express all the partial evaluations in the common scale X, so that the function M aggregates commensurable evaluations. We will also make the assumption that $U_i(r_1^{(i)}) = 0$ and $U_i(r_{k_i}^{(i)}) = 1$ for all $i \in N$.

In this paper we present an axiomatic framework for defining a suitable aggregation model. As presented above, this model is determined by the mapping g, which can be constructed in two steps:

1. The aggregation function M can be identified by means of an axiomatic approach. The one we propose, which is mainly based on the ordinal nature of the evaluation scales, leads to the discrete Sugeno integral (cf. Definition 3.3 below).

2. Each mapping U_i $(i \in N)$ can be identified by asking appropriate questions to the decision maker. On this issue, Marichal and Roubens [10] proposed a procedure to obtain these mappings. This procedure will be discussed in Section 4.

Notice that other characterizations of the discrete Sugeno integral have already been proposed in the earlier literature (see [4, 5, 8]). However, most of the properties used in those characterizations do not have a clear interpretation in the framework of multicriteria decision making. An example is given in Proposition 3.1 below.

Another aim of this paper is to show that the aggregation of criteria by the Sugeno integral makes sense only

*Scientific Research Worker of FNRS (National Scientific Research Fund, Belgium)

†The author is with the Institute of Mathematics, University of Liège, Grande Traverse 12 - B37, Sart Tilman, B-4000 Liège, Belgium. Email: jl.marichal[at]ulg.ac.be

when criteria interact. More precisely, we show that when criteria are "mutually preferentially independent" then the Sugeno integral collapses into a projection function, that is, a dictatorial aggregation.

This paper is organized as follows. In Section 2 we present a first axiomatic-based aggregation model. In Section 3 we improve this model by taking into account the importance of the different criteria. This leads to using the Sugeno integral as an appropriate aggregation function. In Section 4 we deal with a practical method to identify the commensurateness mappings. In Section 5 we investigate the aggregation functions which are both Sugeno integrals and Choquet integrals. Finally, Section 6 deals with the interaction phenomena among criteria and the related concepts of preferential independence.

Throughout the paper, $\wedge$ and $\vee$ denote the minimum and maximum operations, respectively.

2 Meaningful aggregation functions

In this section we propose an axiomatic setting allowing to determine a suitable aggregation function $M : X^n \to X$.

First of all, since the scale $X \subset [0,1]$ is of ordinal nature, the numbers that are assigned to it are defined up to an increasing bijection φ from $[0,1]$ onto itself. A meaningful aggregation function should then satisfy the following property (see Orlov [14]):

Definition 2.1 *A function $M : [0,1]^n \to \mathbb{R}$ is comparison meaningful (from an ordinal scale) if for any increasing bijection $\varphi : [0,1] \to [0,1]$ and any n-tuples $x, x' \in [0,1]^n$, we have*

$$M(x) \leq M(x') \quad \Leftrightarrow \quad M(\varphi(x)) \leq M(\varphi(x')),$$

where the notation $\varphi(x)$ means $(\varphi(x_1), \ldots, \varphi(x_n))$.

Comparison meaningfulness is an essential condition. Indeed, numbers defined on an ordinal scale cannot be aggregated by means of usual arithmetic operations, unless these operations involve only order. For example, the arithmetic mean is forbidden, but the median or any order statistic is permitted. In illustration, let us consider the pairs of evaluations $(0.3, 0.5)$ and $(0.1, 0.8)$. Of course, we have

$$\frac{0.3 + 0.5}{2} < \frac{0.1 + 0.8}{2}.$$

Using a transformation $\varphi : [0,1] \to [0,1]$ such that $\varphi(0.1) = 0.1$, $\varphi(0.3) = 0.4$, $\varphi(0.5) = 0.7$, $\varphi(0.8) = 0.8$, we now have

$$\frac{0.4 + 0.7}{2} > \frac{0.1 + 0.8}{2},$$

which shows that the arithmetic mean is not comparison meaningful.

As for most of the aggregation functions, we will also assume that M is internal to the set of its arguments.

Definition 2.2 *A function $M : [0,1]^n \to \mathbb{R}$ is internal if*

$$\bigwedge_{i\in N} x_i \leq M(x_1, \ldots, x_n) \leq \bigvee_{i \in N} x_i \qquad (x \in [0,1]^n).$$

Definition 2.3 *A function $M : [0,1]^n \to \mathbb{R}$ is idempotent if $M(x, \ldots, x) = x$ for all $x \in [0,1]$.*

Obviously, any internal function is idempotent. Moreover, it was shown by Ovchinnikov [15, Sect. 4] that, for any internal and comparison meaningful function $M : [0,1]^n \to \mathbb{R}$, we have

$$M(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\} \qquad (x \in [0,1]^n). \tag{1}$$

This property is in accordance with the assumption that M ranges in X.

Finally, we will assume that any $E \subseteq [0,1]^n$ is a closed subset whenever its image $\{M(x) \,|\, x \in E\}$ is a closed subset of $[0,1]$. This regularity condition simply expresses that M is a continuous function.

The class of all the aggregation functions fulfilling the properties above was described by the author [9, Sect. 4] as follows.

Theorem 2.1 *The function $M : [0,1]^n \to \mathbb{R}$ is continuous, idempotent, and comparison meaningful if and only if there exists a non-constant set function $c : 2^N \to \{0,1\}$, with $c(\emptyset) = 0$, such that*

$$M(x) = \bigvee_{\substack{T \subseteq N \\ c(T)=1}} \bigwedge_{i\in T} x_i \qquad (x \in [0,1]^n).$$

Theorem 2.1 provides the general form of functions $M : X^n \to X$ that seem appropriate to aggregate the given criteria. It represents all the possible *lattice polynomials* on X. It was also shown [9] that when replacing the continuity by the increasing monotonicity in Theorem 2.1, then the restriction of M to $]0,1[^n$ is again a lattice polynomial. A general discussion on this type of polynomials can be found in [8, 9, 16, 17].

Although the axiomatic that supports this aggregation model seems sensible and satisfactory, the corresponding functions present however the following major drawback. If e_S represents the characteristic vector in $\{0,1\}^n$ of a given subset of criteria $S \subseteq N$ then we have

$$M(e_S) \in \{0,1\}.$$

This means that the global evaluation of an alternative that fully satisfies criteria S and totally fails to satisfy the other criteria is always an extreme value of X. In particular, the compensation effects are not allowed. As the following result shows [9, Sect. 4], dropping the idempotency property does not enable to overcome completely this undesirable phenomenon.

Theorem 2.2 *The function $M : [0,1]^n \to \mathbb{R}$ is non-constant, continuous, and comparison meaningful if and* *only if there exists a non-constant set function* $c : 2^N \to \{0,1\}$, *with* $c(\emptyset) = 0$, *and a continuous and strictly monotonic function* $g : [0,1] \to \mathbb{R}$ *such that*

$$M(x) = g\Big(\bigvee_{\substack{T\subseteq N\\ c(T)=1}} \bigwedge_{i\in T} x_i\Big) \qquad (x \in [0,1]^n).$$

We also notice that the commensurability hypothesis (that is, the presence of the mappings $U_i : X_i \to X$ in the aggregation model) is essential to avoid a dictatorial aggregation. Indeed, suppose that the scales $X_i \subseteq [0,1]$ are independent (i.e., all different) and that g is defined as

$$g(a) = M[g_1(a), \ldots, g_n(a)] \qquad (a \in A),$$

with an aggregation function M from $X_1 \times \cdots \times X_n$ to X. In this case, M maps independent ordinal scales into an ordinal scale and thus should satisfy the following property.

Definition 2.4 *A function* $M : [0,1]^n \to \mathbb{R}$ *is comparison meaningful from independent ordinal scales if, for any increasing bijections* $\varphi_1, \ldots, \varphi_n : [0,1] \to [0,1]$ *and any* n*-tuples* $x, x' \in [0,1]^n$, *we have*

$$M(x) \le M(x') \quad \Leftrightarrow \quad M(\varphi(x)) \le M(\varphi(x')),$$

where the notation $\varphi(x)$ *means* $(\varphi_1(x_1), \ldots, \varphi_n(x_n))$.

The following results [9, Sect. 5] show that such an aggregation function leads to a dictatorial aggregation process.

Theorem 2.3 *The function* $M : [0,1]^n \to \mathbb{R}$ *is nonconstant, continuous, and comparison meaningful from independent ordinal scales if and only if there exists* $k \in N$ *and a continuous and strictly monotonic function* $g : [0,1] \to \mathbb{R}$ *such that*

$$M(x) = g(x_k) \qquad (x \in [0,1]^n).$$

Theorem 2.4 *The function* $M : [0,1]^n \to \mathbb{R}$ *is continuous, idempotent, and comparison meaningful from independent ordinal scales if and only if there exists* $k \in N$ *such that*

$$M(x) = x_k \qquad (x \in [0,1]^n).$$

3 The Sugeno integral as an aggregation function

The remark regarding Theorem 2.1 shows that it is necessary to enrich the aggregation model so that compensation effects are authorized. Whatever the function M considered, it seems natural to interpret the global evaluation

$$v(S) := M(e_S)$$

as the importance of the combination S of criteria. This importance should be expressed in X and not restricted to the extreme values.

It is clear that any mapping $v : 2^N \to X$ that represents the importance of combinations of criteria should fulfill the boundary conditions $v(\emptyset) = 0$ and $v(N) = 1$. In some practical applications, one might even demand that this set function is a fuzzy measure, a concept introduced by Sugeno [18].

Definition 3.1 *A fuzzy measure on* N *is a monotone set function* $\mu : 2^N \to [0,1]$ *such that* $\mu(\emptyset) = 0$ *and* $\mu(N) = 1$. *Monotonicity means that* $\mu(S) \le \mu(T)$ *whenever* $S \subseteq T$.

Now, a suitable aggregation function should take into consideration the importance of each combination of criteria. So, it is natural to define an aggregation function $M_v : X^n \to X$ for each set function $v : 2^N \to X$, with $v(\emptyset) = 0$ and $v(N) = 1$, which represents the importance of criteria. Moreover, since the partial evaluations and the importance coefficients are expressed in the same scale X, we assume that the mapping $(x, v) \mapsto M_v(x)$, viewed as a function from $[0,1]^{n+2^n-2}$ to $\mathbb{R}$, is comparison meaningful. We also assume that it is continuous.

A typical example of aggregation function fulfilling those properties is given by the *weighted max-min functions*, introduced by the author [8].

Definition 3.2 *For any set function* $v : 2^N \to [0,1]$ *such that* $v(\emptyset) = 0$ *and*

$$\bigvee_{T\subseteq N} v(T) = 1,$$

the weighted max-min function $\mathrm{W}_v^{\vee\wedge} : [0,1]^n \to \mathbb{R}$, *associated to* v, *is defined by*

$$\mathrm{W}_v^{\vee\wedge}(x) = \bigvee_{T\subseteq N} \Big[v(T) \wedge (\bigwedge_{i\in T} x_i)\Big] \qquad (x \in [0,1]^n).$$

It was proved [8, Proposition 3.1] that for any set function v defining $\mathrm{W}_v^{\vee\wedge}$, the coefficient $v(N)$ can always be replaced by one without altering $\mathrm{W}_v^{\vee\wedge}$. Thus, the weighted max-min functions fulfill the properties mentioned above.

Another example is given by the discrete Sugeno integral [18, 19], which will play a central role in this paper.

Definition 3.3 *Let* μ *be a fuzzy measure on* N. *The (discrete) Sugeno integral of* $x \in [0,1]^n$ *with respect to* μ *is defined by*

$$\mathcal{S}_\mu(x) := \bigvee_{i=1}^{n} \Big[x_{(i)} \wedge \mu(\{(i), \ldots, (n)\})\Big],$$

where $(\cdot)$ *indicates a permutation on* N *such that* $x_{(1)} \le \cdots \le x_{(n)}$.

It was proved in [18, Theorem 3.1] (see also [6, 8]) that the Sugeno integral can also be put in the form:

$$\mathcal{S}_\mu(x) = \bigvee_{T\subseteq N} \Big[\mu(T) \wedge (\bigwedge_{i\in T} x_i)\Big] \qquad (x \in [0,1]^n). \qquad (2)$$

This shows that any Sugeno integral on $[0,1]^n$ is a weighted max-min function. Conversely, for any set function $v : 2^N \to [0,1]$ defining $\mathrm{W}_v^{\vee\wedge}$, we have $\mathrm{W}_v^{\vee\wedge} = \mathrm{W}_\mu^{\vee\wedge} = \mathcal{S}_\mu$, where μ is a fuzzy measure on N defined by

$$\mu(S) = \bigvee_{T\subseteq S} v(T) \qquad (S \subseteq N).$$

Thus, any weighted max-min function is also a Sugeno integral. Consequently, the class of Sugeno integrals on $[0,1]^n$ coincides with that of weighted max-min functions. The following result (Theorem 4.2 in [8]) gives an axiomatic characterization of this class.

Proposition 3.1 *The aggregation function $F : [0,1]^n \to \mathbb{R}$ is increasing in each variable and fulfills the following two conditions:*

$$\begin{aligned} F(x_1 \vee r, \dots, x_n \vee r) &= F(x_1, \dots, x_n) \vee r, \\ F(x_1 \wedge r, \dots, x_n \wedge r) &= F(x_1, \dots, x_n) \wedge r, \end{aligned}$$

for all $x \in [0,1]^n$ and all $r \in [0,1]$, if and only if there exists a fuzzy measure μ on N such that $F = \mathcal{S}_\mu$.

We also note that any Sugeno integral $\mathcal{S}_\mu$ satisfies the following property:

$$\mathcal{S}_\mu(e_S) = \mu(S) \qquad (S \subseteq N), \tag{3}$$

which corresponds to our definition of the importance of combinations of criteria.

Now, let $\mathcal{V}_N$ denote the family of set functions $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$ and $v(N) = 1$, and let $\mathcal{F}_N$ denote the set of fuzzy measures on N. We then have the following result.

Theorem 3.1 *Let Σ be a set of functions $M_v : [0,1]^n \to \mathbb{R}$ $(v \in \mathcal{V}_N)$ fulfilling the following three properties:*

- *there exist $v, v' \in \mathcal{V}_N$ and $x, x' \in [0,1]^n$ such that $M_v(x) \neq M_{v'}(x')$,*
- *$M_v(x, \dots, x) = M_{v'}(x, \dots, x)$ for all $x \in [0,1]$ and all $v, v' \in \mathcal{V}_N$,*
- *the mapping $(x, v) \mapsto M_v(x)$, viewed as a function from $[0,1]^{n+2^n-2}$ to $\mathbb{R}$, is continuous and comparison meaningful.*

Then there exists a continuous and strictly monotonic function $g : [0,1] \to \mathbb{R}$ such that

$$\Sigma \subseteq \{g \circ \mathcal{S}_\mu \mid \mu \in \mathcal{F}_N\} = \{g \circ \mathrm{W}_v^{\vee\wedge} \mid v \in \mathcal{V}_N\}.$$

Conversely, for any such function g, the set

$$\{g \circ \mathrm{W}_v^{\vee\wedge} \mid v \in \mathcal{V}_N\}$$

is a candidate for Σ.

Proof. Set $P(N) := 2^N \setminus \{\emptyset, N\}$ and consider the function $M^* : [0,1]^{n+2^n-2} \to \mathbb{R}$, defined by $M^*(x, v) = M_v(x)$ for all $(x, v) \in [0,1]^n \times \mathcal{V}_N$. Since M^* fulfills the assumptions of Theorem 2.2, there exists a non-constant set function $c : 2^N \times 2^{P(N)} \to \{0,1\}$, with $c(\emptyset, \emptyset) = 0$, and a continuous and strictly monotonic function $g : [0,1] \to \mathbb{R}$ such that

$$g^{-1}(M^*(x,v)) = \bigvee_{\substack{(T_1,T_2)\subseteq N\times P(N)\\ c(T_1,T_2)=1}} \Big[\bigwedge_{i\in T_1} x_i \wedge \bigwedge_{I\in T_2} v(I) \Big]$$

for all $(x, v) \in [0,1]^n \times \mathcal{V}_N$. By the second hypothesis of the theorem, we have, using a simplified notation,

$$\begin{aligned} 0 &= g^{-1}(M^*(0,0)) \\ &= g^{-1}(M^*(0,v)) = \bigvee_{\substack{T_2\subseteq P(N)\\ c(\emptyset,T_2)=1}} \bigwedge_{I\in T_2} v(I), \end{aligned}$$

which implies

$$g^{-1}(M^*(x,v)) = \bigvee_{\substack{(T_1,T_2)\subseteq N\times P(N)\\ T_1\neq\emptyset \\ c(T_1,T_2)=1}} \Big[\bigwedge_{i\in T_1} x_i \wedge \bigwedge_{I\in T_2} v(I) \Big]$$

for all $(x, v) \in [0,1]^n \times \mathcal{V}_N$. Similarly, we have

$$\begin{aligned} 1 &= g^{-1}(M^*(1,1)) \\ &= g^{-1}(M^*(1,v)) = \bigvee_{\substack{(T_1,T_2)\subseteq N\times P(N)\\ T_1\neq\emptyset \\ c(T_1,T_2)=1}} \bigwedge_{I\in T_2} v(I). \end{aligned}$$

Now, for each fixed $v \in \mathcal{V}_N$, the function $F := g^{-1} \circ M_v : [0,1]^n \to [0,1]$ is increasing in each argument. Moreover, for any $x \in [0,1]^n$ and any $r \in [0,1]$, we have

$$\begin{aligned} &F(x_1 \vee r, \dots, x_n \vee r) \\ &= \bigvee_{\substack{(T_1,T_2)\subseteq N\times P(N)\\ T_1\neq\emptyset \\ c(T_1,T_2)=1}} \Big[\bigwedge_{i\in T_1} (x_i \vee r) \wedge \bigwedge_{I\in T_2} v(I) \Big] \\ &= \bigvee_{\substack{(T_1,T_2)\subseteq N\times P(N)\\ T_1\neq\emptyset \\ c(T_1,T_2)=1}} \Bigg[\Big[(\bigwedge_{i\in T_1} x_i) \vee r \Big] \wedge \Big[\bigwedge_{I\in T_2} v(I) \Big] \Bigg] \\ &= \bigvee_{\substack{(T_1,T_2)\subseteq N\times P(N)\\ T_1\neq\emptyset \\ c(T_1,T_2)=1}} \Bigg[\Big[\bigwedge_{i\in T_1} x_i \wedge \bigwedge_{I\in T_2} v(I) \Big] \\ &\qquad \vee \Big[(\bigwedge_{I\in T_2} v(I)) \wedge r \Big] \Bigg] \\ &= F(x_1, \dots, x_n) \vee \Bigg[\underbrace{\Big[\bigvee_{\substack{(T_1,T_2)\subseteq N\times P(N)\\ T_1\neq\emptyset \\ c(T_1,T_2)=1}} \bigwedge_{I\in T_2} v(I) \Big]}_{=1} \wedge r \Bigg] \\ &= F(x_1, \dots, x_n) \vee r \end{aligned}$$

and

$$\begin{aligned} &F(x_1 \wedge r, \ldots, x_n \wedge r) \\ &= \bigvee_{\substack{(T_1,T_2)\subseteq N\times P(N)\\ T_1\neq\emptyset\\ c(T_1,T_2)=1}} \Big[\bigwedge_{i\in T_1}(x_i\wedge r)\wedge\bigwedge_{I\in T_2} v(I)\Big] \\ &= \Bigg[\bigvee_{\substack{(T_1,T_2)\subseteq N\times P(N)\\ T_1\neq\emptyset\\ c(T_1,T_2)=1}} \Big[\bigwedge_{i\in T_1}x_i\wedge\bigwedge_{I\in T_2} v(I)\Big]\Bigg]\wedge r \\ &= F(x_1,\ldots,x_n)\wedge r. \end{aligned}$$

By Proposition 3.1, there exists $\mu \in \mathcal{F}_N$ such that $F = \mathcal{S}_\mu$, which proves the first part of Theorem 3.1. The second part is immediate. ∎

The second property mentioned in Theorem 3.1 can be interpreted as follows. When the partial evaluations of a given alternative do not depend on criteria, then they do not depend on their importance either. Note however that this property was used in the proof only at $x = 0$ and $x = 1$.

Regarding idempotent functions, we have the following result, which follows immediately from Theorem 3.1.

Theorem 3.2 *Let Σ be a set of functions $M_v : [0,1]^n \to \mathbb{R}$ $(v \in \mathcal{V}_N)$ fulfilling the following two properties:*

- *M_v is idempotent for all $v \in \mathcal{V}_N$,*
- *the mapping $(x, v) \mapsto M_v(x)$, viewed as a function from $[0,1]^{n+2^n-2}$ to $\mathbb{R}$, is continuous and comparison meaningful.*

Then

$$\Sigma \subseteq \{\mathcal{S}_\mu \mid \mu \in \mathcal{F}_N\} = \{\mathrm{W}_v^{\vee\wedge} \mid v \in \mathcal{V}_N\}.$$

Conversely, the set $\{\mathrm{W}_v^{\vee\wedge} \mid v \in \mathcal{V}_N\}$ is a candidate for Σ.

Theorems 3.2 brings a rather natural motivation for the use of the Sugeno integral as an appropriate aggregation function. Nevertheless, continuity may seem to be a questionable hypothesis in the sense that its classical definition uses a distance between aggregated values and makes use of the cardinal properties of the arguments. Though continuity and comparison meaningfulness are not contradictory, coupling these two axioms can be somewhat awkward since the latter one implies that the cardinal properties of the partial evaluations should not be used. Suppressing the continuity property or replacing it by a natural property such as increasing monotonicity remains a quite interesting open problem.

Before closing this section, we present a result showing that the Sugeno integral is a very natural concept despite its rather strange definition. First, from the variables $x_1, \ldots, x_n \in [0,1]$ and any constants $r_1, \ldots, r_m \in [0,1]$, we can form a lattice polynomial

$$P_{r_1,\ldots,r_m}(x_1, \ldots, x_n)$$

in a usual manner using $\wedge$, $\vee$, and, of course, parentheses. Now, we claim that if such a polynomial fulfills

$$P_{r_1,\ldots,r_m}(0,\ldots,0) = 0 \quad \text{and} \quad P_{r_1,\ldots,r_m}(1,\ldots,1) = 1,$$

then it is a Sugeno integral on $[0,1]^n$. The proof can be easily adapted from that of Theorem 3.1. Indeed, the mapping $(x, r) \mapsto P_r(x)$, viewed as a function from $[0,1]^{n+m}$ to $\mathbb{R}$, is continuous, idempotent, and comparison meaningful. By Theorem 2.1, there exists a non-constant set function $c : 2^N \times 2^{[m]} \to \{0,1\}$ ($[m] := \{1,\ldots,m\}$), with $c(\emptyset,\emptyset) = 0$, such that

$$P_r(x) = \bigvee_{\substack{(T_1,T_2)\subseteq N\times[m]\\ c(T_1,T_2)=1}} \Big[\bigwedge_{i\in T_1} x_i \wedge \bigwedge_{j\in T_2} r_j\Big]$$

for all $(x,r) \in [0,1]^{n+m}$. Using the same reasoning as in the proof of Theorem 3.1, we prove that there exists $\mu \in \mathcal{F}_N$ such that $P_r = \mathcal{S}_\mu$.

For example,

$$P_{r_1,r_2}(x_1,x_2,x_3) = ((x_1 \vee r_2) \wedge x_3) \vee (x_2 \wedge r_1)$$

is a Sugeno integral on $[0,1]^3$. The corresponding fuzzy measure can be identified by (3).

4 Identification of the commensurateness mappings

Of course, the aggregation by means of the Sugeno integral cannot be made if the mappings U_i are not known. On this issue, Marichal and Roubens [10] proposed a method to learn those mappings by asking appropriate questions to the decision maker. A slightly improved version of that method is given in this section.

Firstly, the Sugeno integral $\mathcal{S}_\mu$ is uniquely determined by the knowledge of the corresponding fuzzy measure μ, that is, the importance coefficients

$$\mu(S) = \mathcal{S}_\mu(e_S) \qquad (S \subseteq N). \tag{4}$$

These coefficients can be provided directly by the decision maker. Of course, this consists of $(2^n - 2)$ questions. However, in practical problems the total violation of at least two criteria often lead to the lowest global evaluation, that is 0. Combining this with the monotonicity of the fuzzy measure, the number of coefficients to appraise can be reduced significantly.

Let us now turn to the evaluation of the commensurateness mappings. First, we introduce the following notation. For any $\mu \in \mathcal{F}_N$, any $k \in N$, and any $x \in [0,1]^n$, we set

$$\begin{aligned} \mathcal{S}_\mu^{(k,0)}(x) &:= \mathcal{S}_\mu(x_1,\ldots,x_{k-1},0,x_{k+1},\ldots,x_n), \\ \mathcal{S}_\mu^{(k,1)}(x) &:= \mathcal{S}_\mu(x_1,\ldots,x_{k-1},1,x_{k+1},\ldots,x_n). \end{aligned}$$

Recall also the classical definition of the median of three numbers $z_1, z_2, z_3 \in [0,1]$:

$$\mathrm{median}(z_1,z_2,z_3) := (z_1 \wedge z_2) \vee (z_2 \wedge z_3) \vee (z_3 \wedge z_1)\,.$$

We then have the following lemma.

Lemma 4.1 *For any $\mu \in \mathcal{F}_N$, any $k \in N$, and any $x \in [0,1]^n$, we have*

$$\mathcal{S}_\mu(x) = \text{median}(\mathcal{S}_\mu^{(k,0)}(x), \mathcal{S}_\mu^{(k,1)}(x), x_k). \qquad (5)$$

Proof. Let us fix $\mu \in \mathcal{F}_N$, $k \in N$, and $x \in [0,1]^n$. If $\mathcal{S}_\mu^{(k,0)}(x) = \mathcal{S}_\mu^{(k,1)}(x)$ then, since $\mathcal{S}_\mu$ is an increasing function, we have $\mathcal{S}_\mu(x) = \mathcal{S}_\mu^{(k,0)}(x)$, which is sufficient.

Assume now that $\mathcal{S}_\mu^{(k,1)}(x) > \mathcal{S}_\mu^{(k,0)}(x)$. By (2), we have

$$\mathcal{S}_\mu^{(k,0)}(x) = \bigvee_{\substack{T\subseteq N\\ T\not\ni k}} \Big[\mu(T) \wedge (\bigwedge_{i\in T} x_i)\Big]$$

and

$$\begin{aligned}\mathcal{S}_\mu^{(k,1)}(x) &= \Bigg[\bigvee_{\substack{T\subseteq N\\ T\ni k}} \Big[\mu(T) \wedge (\bigwedge_{i\in T\setminus\{k\}} x_i)\Big]\Bigg] \\ &\quad \vee \underbrace{\Bigg[\bigvee_{\substack{T\subseteq N\\ T\not\ni k}} \Big[\mu(T) \wedge (\bigwedge_{i\in T} x_i)\Big]\Bigg]}_{\mathcal{S}_\mu^{(k,0)}(x)} \\ &= \bigvee_{\substack{T\subseteq N\\ T\ni k}} \Big[\mu(T) \wedge (\bigwedge_{i\in T\setminus\{k\}} x_i)\Big].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathcal{S}_\mu(x) &= \Bigg[\bigvee_{\substack{T\subseteq N\\ T\ni k}} \Big[\mu(T) \wedge x_k \wedge (\bigwedge_{i\in T\setminus\{k\}} x_i)\Big]\Bigg] \\ &\quad \vee \Bigg[\bigvee_{\substack{T\subseteq N\\ T\not\ni k}} \Big[\mu(T) \wedge (\bigwedge_{i\in T} x_i)\Big]\Bigg] \\ &= \Big[x_k \wedge \mathcal{S}_\mu^{(k,1)}(x)\Big] \vee \mathcal{S}_\mu^{(k,0)}(x),\end{aligned}$$

which completes the proof. ∎

Now, from Eq. (4) and (5), we have

$$\mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S) = \text{median}\Big(U_i(r_j^{(i)}), \mu(S), \mu(S \cup \{i\})\Big)$$

for all $i \in N$, all $S \subseteq N \setminus \{i\}$, and all $j \in \{1, \dots, k_i\}$. Of course, the case where $\mu(S) = \mu(S \cup \{i\})$ is not of interest. On the other hand, if $\mu(S) < \mu(S \cup \{i\})$ then the following implications hold:

1. $\mu(S) < \mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S) < \mu(S \cup \{i\})$
   $\Rightarrow U_i(r_j^{(i)}) = \mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S)$
2. $\mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S) = \mu(S)$
   $\Rightarrow U_i(r_j^{(i)}) \leq \mu(S)$
3. $\mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S) = \mu(S \cup \{i\})$
   $\Rightarrow U_i(r_j^{(i)}) \geq \mu(S \cup \{i\})$.

Let us fix $i \in N$. To determine $U_i : X_i \to X$, we choose $S \subseteq N \setminus \{i\}$ such that the gap between $\mu(S)$ and $\mu(S \cup \{i\})$ is maximum. Often, the subset $S = N \setminus \{i\}$ will be chosen. Next, we ask the decision maker to appraise in

$$X \cap [\mu(S), \mu(S \cup \{i\})]$$

the following global evaluations

$$\mathcal{S}_\mu(U_i(r_j^{(i)})\, e_i + e_S) \qquad (j \in \{1, \dots, k_i\}).$$

Then, the implications above can be used to determine $U_i(r_j^{(i)})$.

If all the commensurateness mappings are not uniquely determined, we can go further by repeating the procedure with another subset S.

The following example [10] deals with the ranking of candidates that apply for a permanent position in a given university. The evaluations are done on three criteria: 1) Scientific value, 2) Teaching effectiveness, and 3) Interview by evaluation committee. The ordinal scales are given as follows:

Scientific value:
Weak < Sat. < Good < Very Good < Exc.
Teaching effectiveness:
Very Weak < Weak < Sat. < Very Good < Exc.
Interview:
Negative < Medium < Positive
Global evaluation:
$C < B < A_2 < A_1$

The decision maker gives the following global evaluations:

$$\begin{aligned}&\mu(\{1,2,3\}) = A_1\\ &\mu(\{1,2\}) = A_2\\ &\mu(\{1,3\}) = \mu(\{1\}) = B\\ &\mu(\{2,3\}) = C\end{aligned}$$

Recall that we have made the assumption that $U_i(r_1^{(i)}) = 0$ and $U_i(r_{k_i}^{(i)}) = 1$ for all $i \in N$. Thus we have

$$\begin{aligned}U_1(W) = U_2(VW) = U_3(N) = C,\\ U_1(E) = U_2(E) = U_3(P) = A_1.\end{aligned}$$

To determine U_1, the decision maker proposes the following evaluations:

$$\begin{aligned}\mathcal{S}_\mu\big(U_1(VG), 1, 1\big) &= A_1\\ \mathcal{S}_\mu\big(U_1(G), 1, 1\big) &= A_2\\ \mathcal{S}_\mu\big(U_1(S), 1, 1\big) &= B.\end{aligned}$$

Since $\mathcal{S}_\mu\big(U_1, 1, 1\big) = \text{median}(U_1, C, A_1) = U_1$, these three evaluations determine completely U_1. We then have $U_1(W) = C$, $U_1(S) = B$, $U_1(G) = A_2$, $U_1(VG) = A_1$, $U_1(E) = A_1$.

For U_2, the following evaluations are proposed:

$$\begin{aligned}\mathcal{S}_\mu\big(1, U_2(VG), 1\big) &= A_1\\ \mathcal{S}_\mu\big(1, U_2(S), 1\big) &= A_1\\ \mathcal{S}_\mu\big(1, U_2(W), 1\big) &= A_2.\end{aligned}$$

Since $\mathcal{S}_\mu\big(1, U_2, 1\big) = \text{median}(U_2, B, A_1) = U_2 \vee B$, these evaluations determine completely U_2. We then have $U_2(VW) = C$, $U_2(W) = A_2$, $U_2(S) = A_1$, $U_2(VG) = A_1$, $U_2(E) = A_1$.

Finally, the decision maker gives:

$$\mathcal{S}_\mu\big(1, 1, U_3(M)\big) = A_2.$$

Since $\mathcal{S}_\mu\big(1, 1, U_3\big) = \text{median}(U_3, A_2, A_1) = U_3 \vee A_2$, this evaluation only indicates that $U_3(M) \leq A_2$. We then have: $U_3(N) = C$, $U_3(M) \in \{C, B, A_2\}$, $U_3(P) = A_1$.

Although $U_3(M)$ is not known, the Sugeno integral is completely determined. To see this, let us use Eq. (2). We then have

$$\begin{aligned} & \mathcal{S}_\mu\big(U_1, U_2, U_3(M)\big) \\ = \; & C \vee \big(B \wedge U_1\big) \vee \big(C \wedge U_2\big) \vee \big(C \wedge U_3(M)\big) \\ & \vee \big(A_2 \wedge U_1 \wedge U_2\big) \vee \big(B \wedge U_1 \wedge U_3(M)\big) \\ & \vee \big(C \wedge U_2 \wedge U_3(M)\big) \vee \big(A_1 \wedge U_1 \wedge U_2 \wedge U_3(M)\big) \\ = \; & (B \wedge U_1) \vee (A_2 \wedge U_1 \wedge U_2). \end{aligned}$$

For instance, suppose that a candidate presents the profile (E, S, M). The global evaluation of this candidate will then be given by

$$\begin{aligned} \mathcal{S}_\mu\big(U_1(E), U_2(S), U_3(M)\big) &= (B \wedge A_1) \vee (A_2 \wedge A_1 \wedge A_1) \\ &= A_2. \end{aligned}$$

5 Boolean max-min functions

When the fuzzy measure μ is $\{0,1\}$-valued, the Sugeno integral $\mathcal{S}_\mu$ becomes a Boolean max-min function [8], also called a lattice polynomial [16]. Its definition, already encountered in Theorem 2.1, is the following.

Definition 5.1 *For any non-constant set function $c : 2^N \to \{0,1\}$ such that $c(\emptyset) = 0$, the Boolean max-min function $\mathrm{B}_c^{\vee\wedge} : [0,1]^n \to [0,1]$, associated to c, is defined by*

$$\mathrm{B}_c^{\vee\wedge}(x) := \bigvee_{T \subseteq N} \Big[c(T) \bigwedge_{i \in T} x_i\Big] = \bigvee_{\substack{T \subseteq N \\ c(T)=1}} \bigwedge_{i \in T} x_i.$$

In this section we investigate this particular Sugeno integral. First, we can readily see that any Boolean max-min function always provides one of its arguments, see Eq. (1). On the other hand, it is *unanimously increasing*, that is, it strictly increases whenever all its arguments strictly increase.

Definition 5.2 *A function $M : [0,1]^n \to \mathbb{R}$ is unanimously increasing if, for any $x, x' \in [0,1]^n$, we have*

i) $x_i \leq x'_i$ *for all* $i \in N \;\Rightarrow\; M(x) \leq M(x')$

ii) $x_i < x'_i$ *for all* $i \in N \;\Rightarrow\; M(x) < M(x')$.

The following result (Theorem 5.1 in [8]) shows that the Boolean max-min functions are exactly those Sugeno integrals which are unanimously increasing.

Proposition 5.1 *Consider a function $M : [0,1]^n \to \mathbb{R}$. The following two assertions are equivalent.*

i) There exists a set function $c : 2^N \to \{0,1\}$ such that $M = \mathrm{B}_c^{\vee\wedge}$.

ii) There exists $\mu \in \mathcal{F}_N$ such that $M = \mathcal{S}_\mu$ and M is unanimously increasing.

As we will prove below, any Boolean max-min function is also a particular Choquet integral [2].

Definition 5.3 *Let μ be a fuzzy measure on N. The (discrete) Choquet integral of $x \in [0,1]^n$ with respect to μ is defined by*

$$\mathcal{C}_\mu(x) := \sum_{i=1}^n x_{(i)} \Big[\mu(\{(i), \ldots, (n)\}) - \mu(\{(i+1), \ldots, (n)\})\Big],$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq \cdots \leq x_{(n)}$.

Murofushi and Sugeno [12, Sect. 2] proved that the Sugeno and Choquet integrals associated to $\{0,1\}$-valued fuzzy measures are Boolean max-min functions.

Proposition 5.2 *If μ is a $\{0,1\}$-valued fuzzy measure on N then $\mathcal{S}_\mu = \mathcal{C}_\mu = \mathrm{B}_\mu^{\vee\wedge}$.*

We now prove a stronger result. The common part between the class of Choquet integrals and that of Sugeno integrals coincides with the class of Boolean max-min functions. This result as well as some others are stated in the following theorem.

Theorem 5.1 *Consider a function $M : [0,1]^n \to \mathbb{R}$. The following seven assertions are equivalent.*

i) There exists a set function $c : 2^N \to \{0,1\}$ such that $M = \mathrm{B}_c^{\vee\wedge}$.

ii) There exists $\mu \in \mathcal{F}_N$ such that $M = \mathcal{S}_\mu$ and $M(x) \in \{x_1, \ldots, x_n\}$ $\forall\, x \in [0,1]^n$.

iii) There exists a $\{0,1\}$-valued $\mu \in \mathcal{F}_N$ such that $M = \mathcal{S}_\mu$.

iv) There exists $\mu \in \mathcal{F}_N$ such that $M = \mathcal{C}_\mu$ and $M(x) \in \{x_1, \ldots, x_n\}$ $\forall\, x \in [0,1]^n$.

v) There exists a $\{0,1\}$-valued $\mu \in \mathcal{F}_N$ such that $M = \mathcal{C}_\mu$.

vi) There exist $\mu, \nu \in \mathcal{F}_N$ such that $M = \mathcal{S}_\mu = \mathcal{C}_\nu$.

vii) There exists $\mu \in \mathcal{F}_N$ such that $M = \mathcal{S}_\mu$ and M is unanimously increasing.

Proof. *i)* $\Rightarrow$ *ii)* By Proposition 5.1, any Boolean max-min function is a Sugeno integral. The second part is trivial.

ii) $\Rightarrow$ *iii)* For any $S \subseteq N$, we have $\mu(S) = M(e_S) \in \{0,1\}$.

iii) $\Rightarrow$ *iv)* See Proposition 5.2.

iv) $\Rightarrow$ *v)* For any $S \subseteq N$, we have $\mu(S) = M(e_S) \in \{0,1\}$.

v) $\Rightarrow$ *vi)* See Proposition 5.2.

vi) $\Rightarrow$ *vii)* Evident, since any Choquet integral is unanimously increasing.

vii) $\Rightarrow$ *i)* See Proposition 5.1. ■

A very particular case of Boolean max-min function is given by the projection functions, already encountered in Theorem 2.4.

Definition 5.4 *For any $k \in N$, the projection function $\mathrm{P}_k : [0,1]^n \to \mathbb{R}$, associated to the kth argument, is defined by*

$$\mathrm{P}_k(x) = x_k \qquad (x \in \mathbb{R}^n).$$

The projection function P_k consists in projecting $x \in [0,1]^n$ onto the kth axis. As a particular aggregation function, it corresponds to a dictatorial aggregation.

6 Sugeno integral and preferential independence

In this final section, we deal with the problem of dependence between criteria when aggregated by the Sugeno integral.

Consider a Sugeno integral $\mathcal{S}_\mu$ defined on X^n. The associated fuzzy measure μ, which gives the relative importance of each subset of criteria, enables us to observe possible interaction phenomena between criteria. For example, two criteria $i, j \in N$ such that $\mu(\{i\}) = \mu(\{i,j\})$ are clearly dependent since in this case j is redundant in the presence of i.

Since the fuzzy measure μ takes its values in the ordinal scale X, the independence of criteria by means of the identity

$$\mu(S \cup T) = \mu(S) + \mu(T) \qquad (S \cap T = \emptyset),$$

makes sense only when μ ranges in $\{0,1\}$. In that case, by Proposition 5.2, the Sugeno integral becomes an additive Boolean max-min function. Since it is also an additive Choquet integral (that is, a weighted arithmetic mean), it corresponds to a projection function.

Another type of independence between criteria is the preferential independence, well-known in multiattribute utility theory (MAUT), see e.g. [3, 7, 20]. Suppose that the preferences over A (the set of alternatives) of the decision maker are known and expressed by a weak order $\succeq$ (i.e., a complete and transitive binary relation). Through the natural identification of alternatives with their profiles in $[0,1]^n$, this preference relation can be considered as a preference relation on $[0,1]^n$.

To define the preferential independence condition, we introduce the following notation. For any subset $S \subseteq N$ and any $x, y \in [0,1]^n$, xSy denotes the vector of $[0,1]^n$ whose ith component is x_i if $i \in S$, and y_i if $i \notin S$.

Definition 6.1 *The subset S of criteria is said to be preferentially independent of $N \setminus S$ if, for all $x, x', y, z \in [0,1]^n$, we have*

$$xSy \succeq x'Sy \quad \Leftrightarrow \quad xSz \succeq x'Sz. \tag{6}$$

The whole set of criteria N is said to be mutually preferentially independent if S is preferentially independent of $N \setminus S$ for every $S \subseteq N$.

When N is mutually preferentially independent, the weak order $\succeq$ is said to satisfy *independence of equal alternatives* [20]. A weaker property of independence for $\succeq$, called *weak separability*, corresponds to the restriction of (6) to $S = \{k\}$ for all $k \in N$. We will use this concept at the end of this section.

Now, let us assume the existence of an aggregation function $M : [0,1]^n \to \mathbb{R}$ which represents $\succeq$, that is such that

$$a \succeq b \quad \Leftrightarrow \quad M(x^a) \geq M(x^b) \qquad (a, b \in A),$$

where $x_i^a := U_i(g_i(a))$ for all $i \in N$ and all $a \in A$. Such a function M is called a utility function in MAUT.

Murofushi and Sugeno [11, 13] proved a fundamental result relating preferential independence and additivity of the fuzzy measure associated to the Choquet integral. To present it, we need a definition.

Definition 6.2 *A criterion $k \in N$ is called essential if there exist $x, x', y \in [0,1]^n$ such that*

$$x\{k\}y \succ x'\{k\}y.$$

Theorem 6.1 *Assume that the utility function is the Choquet integral $\mathcal{C}_\mu$ on $[0,1]^n$. If there are at least three essential criteria in N then the following assertions are equivalent:*

i) The criteria are mutually preferentially independent.

ii) μ is additive.

We now investigate the case where the utility function M is the Sugeno integral. We then have the following lemma.

Lemma 6.1 *Assume that the utility function is the Sugeno integral $\mathcal{S}_\mu$ on X^n. Then the criterion $k \in N$ is preferentially independent of $N \setminus \{k\}$ if and only if either $\mathcal{S}_\mu = \mathrm{P}_k$ or $\mathcal{S}_\mu = \mathcal{S}_\mu^{(k,0)}$.*

Proof. (Sufficiency) Trivial.

(Necessity) By definition, $k \in N$ is preferentially independent of $N \setminus \{k\}$ if, for all $x, x', y, z \in X^n$, we have

$$\mathcal{S}_\mu(x\{k\}y) \geq \mathcal{S}_\mu(x'\{k\}y)$$
$$\Updownarrow$$
$$\mathcal{S}_\mu(x\{k\}z) \geq \mathcal{S}_\mu(x'\{k\}z),$$

or, equivalently, by Lemma 4.1,

$$\begin{gathered}\text{median}(\mathcal{S}_\mu^{(k,0)}(y), \mathcal{S}_\mu^{(k,1)}(y), x_k) \\ \geq \text{median}(\mathcal{S}_\mu^{(k,0)}(y), \mathcal{S}_\mu^{(k,1)}(y), x_k') \\ \Updownarrow \\ \text{median}(\mathcal{S}_\mu^{(k,0)}(z), \mathcal{S}_\mu^{(k,1)}(z), x_k) \\ \geq \text{median}(\mathcal{S}_\mu^{(k,0)}(z), \mathcal{S}_\mu^{(k,1)}(z), x_k').\end{gathered} \tag{7}$$

Of course, this equivalence holds if $\mathcal{S}_\mu^{(k,1)} = \mathcal{S}_\mu^{(k,0)}$ on X^n. In this case, we have $\mathcal{S}_\mu = \mathcal{S}_\mu^{(k,0)}$ on X^n.

Assume now that there exists $y \in X^n$ such that $\mathcal{S}_\mu^{(k,1)}(y) > \mathcal{S}_\mu^{(k,0)}(y)$. Then, for any $z \in X^n$, we should have

$$\mathcal{S}_\mu^{(k,0)}(z) \leq \mathcal{S}_\mu^{(k,0)}(y) \quad \text{and} \quad \mathcal{S}_\mu^{(k,1)}(y) \leq \mathcal{S}_\mu^{(k,1)}(z).$$

Indeed, suppose for example that there exists $z \in X^n$ such that $\mathcal{S}_\mu^{(k,1)}(y) > \mathcal{S}_\mu^{(k,1)}(z)$. Then, setting

$$\begin{cases} x_k := \mathcal{S}_\mu^{(k,1)}(y), \\ x_k' := \max(\mathcal{S}_\mu^{(k,1)}(z), \mathcal{S}_\mu^{(k,0)}(y)), \end{cases}$$

we have

$$\mathcal{S}_\mu^{(k,1)}(y) = x_k > x_k' = \max(\mathcal{S}_\mu^{(k,1)}(z), \mathcal{S}_\mu^{(k,0)}(y)),$$

which violates the preferential independence condition (7).

Now, for any $z \in X^n$, we should have

$$\mathcal{S}_\mu^{(k,0)}(z) = \mathcal{S}_\mu^{(k,0)}(y) \quad \text{and} \quad \mathcal{S}_\mu^{(k,1)}(y) = \mathcal{S}_\mu^{(k,1)}(z).$$

Indeed, suppose for example that there exists $z \in X^n$ such that $\mathcal{S}_\mu^{(k,1)}(y) < \mathcal{S}_\mu^{(k,1)}(z)$. Then, applying again the previous reasoning, we have

$$\mathcal{S}_\mu^{(k,1)}(y) < \mathcal{S}_\mu^{(k,1)}(z) \leq \mathcal{S}_\mu^{(k,1)}(u)$$

for all $u \in X^n$, a contradiction.

Therefore, $\mathcal{S}_\mu^{(k,0)}$ and $\mathcal{S}_\mu^{(k,1)}$ are constant functions on X^n. Hence, we have

$$\mathcal{S}_\mu^{(k,0)} = \mathcal{S}_\mu^{(k,0)}(0, \ldots, 0) = 0$$

and

$$\mathcal{S}_\mu^{(k,1)} = \mathcal{S}_\mu^{(k,1)}(1, \ldots, 1) = 1.$$

By Lemma 4.1, we then have $\mathcal{S}_\mu = \mathrm{P}_k$, which completes the proof. ■

Theorem 6.2 *Assume that the utility function is the Sugeno integral $\mathcal{S}_\mu$ on X^n. The following assertions are equivalent:*

i) The criteria are mutually preferentially independent.

ii) $\succeq$ is a weakly separable weak order.

iii) There exists $k \in N$ such that $\mathcal{S}_\mu = \mathrm{P}_k$.

Proof. *iii)* ⇒ *i)* ⇒ *ii)* Trivial.

ii) ⇒ *iii)* By definition of weak separability, each $i \in N$ is preferentially independent of $N \setminus \{i\}$. By Lemma 6.1, we then have

$$\mathcal{S}_\mu \in \{\mathrm{P}_i, \mathcal{S}_\mu^{(i,0)}\} \qquad (i \in N).$$

Consequently, there exists $k \in N$ such that $\mathcal{S}_\mu = \mathrm{P}_k$. Indeed, otherwise we would have $\mathcal{S}_\mu = \mathcal{S}_\mu^{(i,0)}$ for all $i \in N$, implying that $\mathcal{S}_\mu$ is constant, which is impossible. ■

Note. It is worth comparing Theorems 2.4 and 6.2. Indeed, for the Sugeno integral, both the mutual preferential independence and the comparison meaningfulness from independent ordinal scales seem to be close forms of independence and each of them leads to a dictatorial aggregation model. A comparison with the Arrow's Theorem [1] in social choice theory seems interesting as well.

7 Concluding remarks

We have presented an axiomatic-based model to aggregate criteria measured on qualitative scales. First, it has been observed that aggregating non-commensurable evaluations leads to a dictatorial aggregation. Next, assuming the commensurability of the evaluations over all criteria, we have observed that the importance of each group of criteria is always an extreme value of the common scale. Finally, assuming commensurability between the evaluations together with the importance coefficients, we were able to point out a rather suitable aggregation function, namely, the Sugeno integral.

Thus, the Sugeno integral is now axiomatized for its use in multicriteria decision making. The next step will be to drop the continuity hypothesis from this axiomatization. A research is now in progress along this line.

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References

[1] K.J. Arrow, Social choice and individual values (John Wiley & Sons, New York, 1951).

[2] G. Choquet, Theory of capacities, *Annales de l'Institut Fourier* **5** (1953) 131–295.

[3] P.C. Fishburn, *Utility theory for decision making* (New York, Wiley, 1970).

[4] M. Grabisch, Fuzzy integral in multicriteria decision making, *Fuzzy Sets and Systems*, **69** (1995) 279–298.

[5] M. Grabisch, H.T. Nguyen and E.A. Walker, *Fundamentals of uncertainty calculi with applications to fuzzy inference* (Kluwer Academic, Dordrecht, 1995).

[6] G.H. Greco, On L-fuzzy integrals of measurable functions, *J. Math. Anal. Appl.* **128** (1987) 581–585.

[7] R.L. Keeney and H. Raiffa, *Decision with multiple objectives* (Wiley, New York, 1976).

[8] J.-L. Marichal, On Sugeno integral as an aggregation function, *Fuzzy Sets and Systems* **114** (3) (2000) 347–365.

[9] J.-L. Marichal, On order invariant synthesizing functions. Preprint 9917, GEMME, Faculty of Economics, University of Liège, Belgium, 1999.

[10] J.-L. Marichal and M. Roubens, Consensus with ordinal data, *7th Eur. Congr. on Intelligent Techniques and Soft Computing (EUFIT'99)*, Aachen, Germany, September 13–16, 1999.

[11] T. Murofushi and M. Sugeno, Non-additivity of fuzzy measures representing preferential dependence, *2nd Int. Conf. on Fuzzy Systems and Neural Networks*, Iizuka, Japan, July 1992, 617–620.

[12] T. Murofushi and M. Sugeno, Some quantities represented by the Choquet integral, *Fuzzy Sets & Systems* **56** (1993) 229–235.

[13] T. Murofushi and M. Sugeno, Choquet integral models and independence concepts in multiattribute utility theory, *Int. J. of Uncertainty and Knowledge-Based Systems* **8** (4) (2000) 385–415.

[14] A. Orlov, The connection between mean quantities and admissible transformations, *Mathematical Notes* **30** (1981) 774–778.

[15] S. Ovchinnikov, Means on ordered sets, *Mathematical Social Sciences* **32** (1996) 39–56.

[16] S. Ovchinnikov, Invariant functions on simple orders, *ORDER* **14** (1998) 365–371.

[17] S. Ovchinnikov, On ordinal OWA functions, *7th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'98)*, Paris, 1998.

[18] M. Sugeno, *Theory of fuzzy integrals and its applications*, Ph.D. Thesis, Tokyo Institute of Technology, Tokyo, 1974.

[19] M. Sugeno, Fuzzy measures and fuzzy integrals: a survey, in: M.M. Gupta, G.N. Saridis, and B.R. Gaines (eds.), *Fuzzy Automata and Decision Processes*, (North-Holland, Amsterdam, 1977) 89–102.

[20] P. Wakker, *Additive representations of preferences. A new foundation of decision analysis* (Kluwer Academic Publishers, 1989).