

# Awareness Logic: Kripke Lattices as a Middle Ground between Syntactic and Semantic Models

Gaia Belardinelli and Rasmus K. Rendsvig

Center for Information and Bubble Studies, University of Copenhagen  
{belardinelli,rasmus}@hum.ku.dk

**Abstract.** The literature on awareness modeling includes both syntax-free and syntax-based frameworks. Heifetz, Meier & Schipper (HMS) propose a lattice model of awareness that is syntax-free. While their lattice approach is elegant and intuitive, it precludes the simple option of relying on formal language to induce lattices, and does not explicitly distinguish uncertainty from unawareness. Contra this, the most prominent syntax-based solution, the Fagin-Halpern (FH) model, accounts for this distinction and offers a simple representation of awareness, but lacks the intuitiveness of the lattice structure. Here, we combine these two approaches by providing a lattice of Kripke models, induced by atom subset inclusion, in which uncertainty and unawareness are separate. We show our model equivalent to both HMS and FH models by defining transformations between them which preserve satisfaction of formulas of a language for explicit knowledge, and obtain completeness through our and HMS' results. Lastly, we prove that the Kripke lattice model can be shown equivalent to the FH model (when awareness is propositionally determined) also with respect to the language of the Logic of General Awareness, for which the FH model where originally proposed.

## 1 Introduction

Awareness has been intensively studied in logic and game theory since its first formal treatment by Fagin and Halpern [9]. In these fields, awareness is added as a complement to uncertainty in models for knowledge and rational interaction. In short, where uncertainty concerns an agent's ability to distinguish possible states of the world based on its available information, awareness concerns the agent's ability to even contemplate aspects of a state, where such inability stems from the *unawareness* of the concepts that constitute said aspects. Thereby, models that include awareness avoid problems of logical omniscience (at least partially) and allows modeling game theoretic scenarios where the possibility of some action may come as an utter surprise.

Several models of awareness have been proposed in the literature, which either follow the *semantic* (or *syntax-free*) or the *syntactic* (or *syntax-based*) tradition of awareness modeling. In the semantic tradition, awareness is usually represented in Aumann-like event structures, which are defined without appeal to atomic propositions or other syntax. The awareness notion presented in these

frameworks inherits the syntax-free definition and is thus captured by a specific subset of states.

An instance of this approach is given by Heifetz, Meier and Schipper (HMS), who propose a lattice-based conceptualization of awareness [17]. The backbone of HMS’ *unawareness frames* is a complete lattice of state-spaces  $(\mathcal{S}, \preceq)$ , with the intuition that the higher a space is, the richer the “vocabulary” it has to describe its states. Since the approach is syntax-free, this intuition is not modeled using a formal language. It is represented using  $\preceq$  and a family of maps  $r_S^{S'}$  which projects state-space  $S'$  down to  $S$ , with  $r_S^{S'}(s)$  interpreted as the representation of  $s$  in the more limited vocabulary available in  $S$ . Uncertainty and unawareness are captured *jointly* by a *possibility correspondence*  $\Pi_a$  for each  $a \in Ag$ , which maps a state weakly downwards to the set of states the agent considers possible. If the mapped-to space is strictly less expressive, this represents that the agent does not have full awareness of the mapped-from state.

That HMS keep their model syntax-free is motivated in part by its applicability in theoretical economics [17, p. 79]. We think that their lattice-based conceptualization of awareness is both elegant, interesting and intuitive, as it captures different levels of awareness in a suggestive way. However, we also find its formalization cumbersome. Exactly the choice to go fully syntax-free robs the model of the option to rely on formal language to induce lattices and to specify events, resulting in constructions which we find laborious to deal with. This may, of course, be an artifact of us being accustomed to non-syntax-free models used widely in epistemic logic.

Another artifact of our familiarity with epistemic logic models is that we find HMS’ joint definition of uncertainty and unawareness difficult to relate to other formalizations of knowledge. When HMS propose properties of their  $\Pi_a$  maps, it is not clear to us which aspects concern knowledge and which concern awareness. They merge two dimensions which, to us, would be clearer if left separated.<sup>1</sup>

Moreover, while the HMS model allows agents to reason about their unawareness, as possibility correspondences  $\Pi_a$  provide them with a subjective perspective, Halpern and Rêgo [15] point out that the model includes no objective state, and so no outside perspective.

Alternatively, the literature has proposed syntactic approaches to awareness modeling. The syntactic tradition has been initiated by the seminal [9], where Fagin and Halpern (FH) introduce the Logic of General Awareness ( $\mathcal{A}_{LGA}$ ). Models for this logic (FH models) are Kripke models  $M = (W, R, V)$  augmented with an *awareness function*  $\mathcal{A}_a$ , for each agent  $a \in Ag$ , that represents an agent

---

<sup>1</sup> As a reviewer of the short version of this paper [3] pointed out, then HMS take *explicit* knowledge as foundational, and derive awareness from it. This makes the one-dimensional representation justified, if not even desirable. In contrast, epistemic logic models are standardly interpreted as taking *implicit* knowledge as foundational. We think along the second line, and add awareness as a second dimension. We are not taking a stand on whether one interpretation is superior, but provide results to move between them.

$a$ 's awareness at state  $w$  by assigning to  $(a, w)$  a set of formulas—which is why these models are called *syntax-based*.

Since FH models represent uncertainty using the accessibility relation  $R$ , as in standard epistemic logic, FH explicitly distinguish the uncertainty and unawareness dimension. This allows for a versatile representation of awareness, as, when the awareness function is not otherwise restricted, an agent's awareness in a state can be any arbitrary set of formulas. The FH approach has thus been inherited by a multitude of models.

However, FH models lack the intuitiveness of the lattice structure, and while Halpern and Rêgo argue that HMS models lack the objective perspective, HMS [17,22] also argue that FH models only present an outside perspective, as the full model must be taken into account when assigning knowledge and awareness.<sup>2</sup>

In the present paper, we aim at combining the advantages of the HMS and FH approaches. We propose to model awareness through a syntactically induced lattice structure—primarily inspired by the HMS model—where the awareness notion is captured through an awareness map defined semantically. Roughly, we suggest to start from a Kripke model  $K$  for a set of atoms  $At$ , spawn a lattice containing restrictions of  $K$  to subsets of  $At$ , and finally add maps  $\pi_a$  on the lattice that take a world to a copy of itself in a restricted model. This keeps the epistemic and awareness dimensions separate: accessibility relations  $R_a$  of  $K$  encode uncertainty while maps  $\pi_a$  encode awareness.

In this *Kripke lattice model* both subjective and objective perspectives are present: the starting Kripke model provides an outside perspective on agents' knowledge and awareness, while the submodel obtained by following  $\pi_a$  presents agent  $a$ 's subjective perspective. We remark further on this below.

Beyond the introduction of Kripke lattice models,<sup>3</sup> the main contribution of the paper is a set of technical results situating these models with respect to the HMS and FH models. These comprise three results about the equivalences of model classes (see Figure 1), and as corollaries, two completeness results for Kripke lattice models.



**Fig. 1. Known equivalence results between HMS, FH and Kripke lattice (KL) models.** Left:  $\mathcal{L}$ -equivalence results between the model classes, two shown in this paper. Right:  $\mathcal{L}^{KA}$ -equivalence between FH and KL models shown in this paper, and the open issue of the correspondence between HMS and the other two model classes with respect to  $\mathcal{L}^{KA}$ .

<sup>2</sup> [15] argues that this boils down to a difference in philosophical interpretation.

<sup>3</sup> First introduced in the short version of this paper, [3].

First, we show that, under three assumptions on  $\pi_a$  and when each  $R_a$  is an equivalence relation, the Kripke lattice model is  $\mathcal{L}$ -equivalent to the HMS model, in the sense that the two satisfy the same formulas of the language of explicit knowledge and awareness  $\mathcal{L}$ , defined below. Through this result and the completeness of HMS logic  $\Lambda_{HMS}$  with respect to the class of HMS models, we obtain completeness of  $\Lambda_{HMS}$  with respect to the class  $\mathbf{KLM}_{EQ}$  of Kripke lattice models with equivalence relations.

Second, we show that  $\mathbf{KLM}_{EQ}$  is  $\mathcal{L}$ -equivalent to the class  $\mathbf{S}$  of *propositionally determined* FH models with equivalence relations, as again the two satisfy the same  $\mathcal{L}$  formulas.

Third, switching to use Kripke lattice models and FH models as semantics for the language  $\mathcal{L}^{KA}$  for implicit and explicit knowledge and awareness—for which FH models were originally conceived—we show that the class of Kripke lattices without restriction on the accessibility relation and propositionally determined FH models are  $\mathcal{L}^{KA}$ -equivalent. By FH’s completeness result and our model equivalence result, we show that the *Logic of General Awareness*  $\Lambda_{LGA}$ , which is based on  $\mathcal{L}^{KA}$ , is also complete with respect to Kripke lattice models.

Jointly, these results firmly situates Kripke lattice models for awareness with respect to the main existing models. Through detailed transformations between the model classes, the results directly show correspondences between the models’ elements, and show that for both languages  $\mathcal{L}$  and  $\mathcal{L}^{KA}$ , Kripke lattice models provide a rich semantic framework, axiomatically characterizable by existing logics.

As Kripke lattice models are a novel construction, the paper’s constructions and results are new. However, the second result mentioned may also be obtained through the first and an existing result by Halpern and Rêgo [15], that show that the class  $\mathbf{S}$  of partitioned, propositionally determined FH models is  $\mathcal{L}$ -equivalent to the class of HMS models. We provide a direct proof of this result as the involved transformation directly explicates the relationship between Kripke lattices and FH models, used further to establish the  $\mathcal{L}^{KA}$ -equivalence of these model classes.

The paper progresses as follows. Sections 2 and 3 present respectively the HMS model and our rendition, Kripke lattice models. Section 4 introduces transformations between the two models classes, and Section 5 shows that the transformations preserve formula satisfaction. Section 6 presents a logic due to HMS [16], and shows, as a corollary to our results, that it is complete with respect to our rendition. Section 7 introduces the FH model structure with respect to language  $\mathcal{L}$ . As for the HMS model class, the next two sections, Section 8 and 8.3, presents the transformations between FH and Kripke lattice models and show that they preserve formula satisfaction with respect to language  $\mathcal{L}$ , respectively. Section 9 presents the language  $\mathcal{L}^{KA}$  on which the  $\Lambda_{LGA}$  is based, and shows the equivalence of Kripke lattice and FH models with respect to it, again using transformations.  $\Lambda_{LGA}$  is introduced in Section 10, where soundness and completeness of  $\Lambda_{LGA}$  over Kripke lattice models is shown. Section 11 holds concluding remarks.

Throughout the paper, we assume that  $Ag$  is a finite, non-empty set of agents, and that  $At$  is a countable, non-empty set of atoms.

## 2 The HMS Model

This section presents HMS unawareness frames [17], their syntax-free notions of knowledge and awareness, and their augmentation with HMS valuations, producing HMS models [16]. For context, the HMS model is a multi-agent generalization of the Modica-Rustichini model [21] which is equivalent to Halpern’s model in [12], generalized by Halpern and Rêgo to multiple agents [15], resulting in a model equivalent to the HMS model, cf. [16]. See [22] for an extensive review.

The following definition introduces the basic structure underlying the HMS model, as well as the properties of the  $\Pi_a$  map that controls the to-be-defined notions of knowledge and awareness. The properties are described after Definition 1. Following Definition 4 of HMS models, Figure 2 illustrates a full HMS model, including its unawareness frame.

**Definition 1.** An **unawareness frame** is a tuple  $F = (\mathcal{S}, \preceq, \mathcal{R}, \Pi)$  where  $(\mathcal{S}, \preceq)$  is a complete lattice with  $\mathcal{S} = \{S, S', \dots\}$  a set of disjoint, non-empty **state-spaces**  $S = \{s, s', \dots\}$  s.t.  $S \preceq S'$  implies  $|S| \leq |S'|$ . Let  $\Omega_F := \bigcup_{S \in \mathcal{S}} S$  be the disjoint union of state-spaces in  $\mathcal{S}$ . For  $X \subseteq \Omega_F$ , let  $S(X)$  be the state-space containing  $X$ , if such exists (else  $S(X)$  is undefined). Let  $S(s)$  be  $S(\{s\})$ .

$\mathcal{R} = \{r_S^{S'} : S, S' \in \mathcal{S}, S \preceq S'\}$  is a family of **projections**  $r_S^{S'} : S' \rightarrow S$ . Each  $r_S^{S'}$  is surjective,  $r_S^S$  is  $Id$ , and  $S \preceq S' \preceq S''$  implies commutativity:  $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$ . Denote  $r_S^T(w)$  also by  $w_S$ .

$D^\uparrow = \bigcup_{S' \succeq S} (r_S^{S'})^{-1}(D)$  is the **upwards closure** of  $D \subseteq S \in \mathcal{S}$ .<sup>4</sup>

$\Pi$  assigns each  $a \in Ag$  a **possibility correspondence**  $\Pi_a : \Omega_F \rightarrow 2^{\Omega_F}$  satisfying

**Conf (Confinement)** If  $w \in S'$ , then  $\Pi_a(w) \subseteq S$  for some  $S \preceq S'$ .

**Gref (Generalized Reflexivity)**  $w \in (\Pi_a(w))^\uparrow$  for every  $w \in \Omega_F$ .

**Stat (Stationarity)**  $w' \in \Pi_a(w)$  implies  $\Pi_a(w') = \Pi_a(w)$ .

**PPI (Projections Preserve Ignorance)** If  $w \in S'$  and  $S \preceq S'$ , then  $(\Pi_a(w))^\uparrow \subseteq (\Pi_a(r_S^{S'}(w)))^\uparrow$ .

**PPK (Projections Preserve Knowledge)** If  $S \preceq S' \preceq S''$ ,  $w \in S''$  and  $\Pi_a(w) \subseteq S'$ , then  $r_S^{S'}(\Pi_a(w)) = \Pi_a(r_S^{S''}(w))$ .

Jointly call these five properties of  $\Pi_a$  the **HMS properties**.

Conf ensures that agents only consider possibilities within one fixed “vocabulary”; Gref induces factivity of knowledge and Stat yields introspection for knowledge and awareness. PPI entails that at down-projected states, agents neither “miraculously” know or become aware of something new, while PPK implies that at down-projected states, the agent can still “recall” all events she knew before, if they are still expressible. Jointly PPI and PPK imply that agents preserve awareness of all events at down-projected states, if they are still expressible.

<sup>4</sup> To avoid confusion, note that for  $d \in S$ ,  $(r_S^{S'})^{-1}(d) = \{s' \in S' : r_S^{S'}(s') = d\}$  and for  $D \subseteq S$ ,  $(r_S^{S'})^{-1}(D) = \bigcup_{d \in D} (r_S^{S'})^{-1}(d)$ .

*Remark 2.* Unawareness frames include no objective perspective, as agents do not—unless they are fully aware—have a range of uncertainty defined for the maximal state-space. Taking the maximal state-space to contain a designated ‘actual world’ and as providing a full and objective description of states, one can still not evaluate agents “true” uncertainty/implicit knowledge. See e.g. Figure 2 below: In  $(\neg i, \ell)$ , the dashed agent’s “true” uncertainty about  $\ell$  is not determined.

## 2.1 Syntax-Free Unawareness

Unawareness frames provide sufficient structure to define syntax-free notions of knowledge and awareness. These are defined directly as events on  $\Omega_F$ .

**Definition 3.** Let  $F = (S, \preceq, \mathcal{R}, \Pi)$  be an unawareness frame. An **event** in  $F$  is any pair  $(D^\uparrow, S)$  with  $D \subseteq S \in \mathcal{S}$  with  $S$  also denoted  $S(D^\uparrow)$ . Let  $\Sigma_F$  be the set of events of  $F$ .

The **negation** of the event  $(D^\uparrow, S)$  is  $\neg(D^\uparrow, S) = ((S \setminus D)^\uparrow, S)$ .

The **conjunction** of events  $\{(D_i^\uparrow, S_i)\}_{i \in I}$  is  $((\bigcap_{i \in I} D_i^\uparrow), \sup_{i \in I} S_i)$ .

The events that a **knows** event  $(D^\uparrow, S)$  and where  $a$  is **aware** of it are

$$\mathbf{K}_a((D^\uparrow, S)) = \begin{cases} (\{w \in \Omega_F: \Pi_a(w) \subseteq D^\uparrow\}, S(D)) & \text{if } \exists w \in \Omega_F. \Pi_a(w) \subseteq D^\uparrow \\ (\emptyset, S(D)) & \text{else} \end{cases}$$

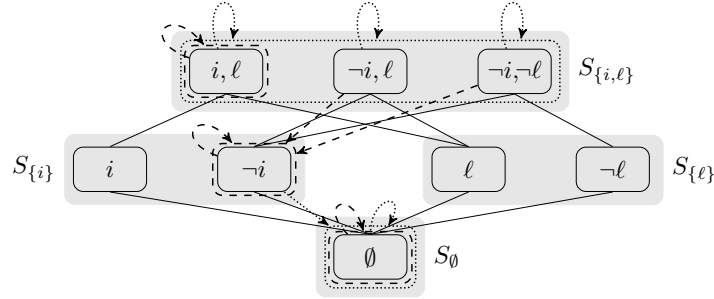
$$\mathbf{A}_a((D^\uparrow, S)) = \begin{cases} (\{w \in \Omega_F: \Pi_a(w) \subseteq S(D^\uparrow)^\uparrow\}, S(D)) & \text{if } \exists w \in \Omega_F. \Pi_a(w) \subseteq S(D^\uparrow)^\uparrow \\ (\emptyset, S(D)) & \text{else} \end{cases}$$

Negation, conjunction, knowledge and awareness events are well-defined [17,22]. To illustrate the definitions, some intuitions behind them: *i)* an event modeled as a pair  $(D^\uparrow, S)$  captures that  $a$ ) if the event is expressible in  $S$ , then it is also expressible in any  $S' \succeq S$ , hence  $D^\uparrow$  is the set of all states where the event is expressible and occurs, and  $b)$  the event is expressible in the “vocabulary” of  $S$ , but not the “vocabulary” of lower state-spaces:  $D \subseteq S$  are the states with the lowest “vocabulary” where the event is expressible and occurs. [22] remarks that for  $(D^\uparrow, S)$ , if  $D \neq \emptyset$ , then  $S$  is uniquely determined by  $D^\uparrow$ . *ii)* Events are given a non-binary understanding: an event  $(D^\uparrow, S)$  and its negation does not partition  $\Omega_F$ , as  $s \in S' \prec S$  is in neither, but they do partition every  $S'' \succeq S$ . *iii)* Conjunction defined using supremum captures that the state-space required to express the conjunction of two events is the least expressive state-space that can express both events. *iv)* Knowledge events are essentially defined as in Aumann structures/state-space models: the agent knows an event if its “information cell” is a subset of the event’s states. *v)* Awareness events captures that “an agent is aware of an event if she considers possible states in which this event is “expressible”.” [22, p. 97]

## 2.2 HMS Models

Though unawareness frames provide a syntax-free framework adequate for defining awareness, HMS [16] use them as a semantics for a formal language in order to identify their logic. The language and logic are topics of Sections 5 and 6.

Instead, the models we will later define are not syntax-free. As Kripke models, they include a valuation of atomic propositions. Therefore, they do not correspond to unawareness frames directly, but to the models that result by augmenting such frames with valuations. To compare the two model classes, we define such valuations here, postponing HMS syntax and semantics to Section 5. Figure 2 illustrates an HMS model, using an example inspired by [17, p. 87]



**Fig. 2.** An HMS model with four state-spaces (gray rectangles), ordered spatially as a lattice. States (smallest rectangles) are labeled with their true literals, over the set  $At = \{i, \ell\}$ . Thin lines between states show projections. There are two possibility correspondences (dashed and dotted): arrow-to-rectangle shows a mapping from state to set (information cell). Omitted arrows go to  $S_\emptyset$  and are irrelevant to the story.

**Story:** Buyer (dashed) and Owner (dotted) consider trading a firm, the price influenced by whether  $i$  (a value-raising innovation) and  $\ell$  (a value-lowering lawsuit) occurs. Assume both occur and take  $(i, \ell)$  as actual. Then Buyer has full information, while Owner has factual uncertainty and uncertainty about Buyer’s awareness and higher-order information, ultimately considering it possible that Buyer holds Owner fully unaware. *In detail:* Buyer’s  $(i, \ell)$  information cell has both  $i$  and  $\ell$  defined (and is also singleton), so Buyer is aware of them (and also knows everything). Owner is also aware of  $i$  and  $\ell$ , but their  $(i, \ell)$  information cell contains also  $\neg i$  and  $\neg \ell$  states, so Owner knows neither. Owner is also uncertain about Buyer’s information: Owner knows that either Buyer knows  $i$  and  $\ell$  (cf. Buyer’s  $(i, \ell)$  information cell), or Buyer knows  $\neg i$ , but is unaware of  $\ell$  (cf. the dashed arrows from  $\neg i$  states to the less expressive state space  $S_{\{i\}}$ ) and then only holds it possible that Owner is unaware of both  $i$  and  $\ell$  (cf. the dotted map to  $S_\emptyset$ ). See also Remark 6 concerning  $S_{\{\ell\}}$ .

**Definition 4.** Let  $F = (\mathcal{S}, \preceq, \mathcal{R}, \Pi)$  be an unawareness frame with events  $\Sigma_F$ . An **HMS valuation** for  $At$  and  $F$  is a map  $V_M : At \rightarrow \Sigma_F$ , assigning to every atom from  $At$  an event in  $F$ . An **HMS model** is an unawareness frame augmented with an HMS valuation, denoted  $M = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_M)$ .

*Remark 5.* HMS valuations only partially respect the intuitive interpretation of state-spaces lattices, where  $S \preceq S'$  represents that  $S'$  is at least as expressive as  $S$ . If  $S \preceq S'$ , then  $p \in At$  having defined truth value at  $S$  entails that it has defined truth value at  $S'$ , but if  $S$  is strictly less expressive than  $S'$ , then this

does not entail that there is some atom  $q$  with defined truth value in  $S'$ , but undefined truth value in  $S$ . Hence, there can exist two spaces defined for the same set of atoms, but where one is still “strictly more expressive” than the other.

*Remark 6.* Concerning Figure 2, then the state-space  $S_{\{\ell\}}$  is, in a sense, redundant: its presence does not affect the knowledge or awareness of agents in the state  $(i, \ell)$ , and its presence is not required by definition. This stands in contrast with the corresponding Kripke lattice model in Figure 3, cf. Remark 12.

### 3 Kripke Lattice Models

The models for awareness we construct start from Kripke models:

**Definition 7.** A *Kripke model* for  $At' \subseteq At$  is a tuple  $\mathbb{K} = (W, R, V)$  where  $W$  is a non-empty set of worlds,  $R : Ag \rightarrow \mathcal{P}(W^2)$  assigns to each agent  $a \in Ag$  an accessibility relation denoted  $R_a$ , and  $V : At' \rightarrow \mathcal{P}(W)$  is a valuation.

The *information cell* of  $a \in Ag$  at  $w \in W$  is  $I_a(w) = \{v \in W : wR_a v\}$ .

The term ‘information cell’ hints at an epistemic interpretation. For generality,  $R$  may assign non-equivalence relations. Some results explicitly assume otherwise.

As counterpart to the HMS state-space lattice, we build a lattice of restricted models. The below definition of the set of worlds  $W_X$  ensures that for any  $X, Y \subseteq At$ ,  $X \neq Y$ , the sets  $W_X$  and  $W_Y$  are disjoint, mimicking the same requirement for state-spaces. In the restriction  $\mathbb{K}_X$  of  $\mathbb{K}$ , it is required that  $(w_X, v_X) \in R_{aX}$  iff  $(w, v) \in R_a$ . Each direction bears similarity to an HMS property: left-to-right to PPK and right-to-left to PPI. They also remind us, resp., of the *No Miracles* and *Perfect Recall* properties from Epistemic Temporal Logic, cf. e.g., [4, 19].

**Definition 8.** Let  $\mathbb{K} = (W, R, V)$  be a Kripke model for  $At$ . The *restriction* of  $\mathbb{K}$  to  $X \subseteq At$  is the Kripke model  $\mathbb{K}_X = (W_X, R_X, V_X)$  for  $X$  where

$W_X = \{w_X : w \in W\}$  where  $w_X$  is the ordered pair  $(w, X)$ ,

$R_{Xa} = \{(w_X, v_X) : (w, v) \in R_a\}$  and

$V_X : X \rightarrow \mathcal{P}(W_X)$  such that, for all  $p \in X, w_X \in V_X(p)$  iff  $w \in V(p)$ .

For the  $R_{Xa}$  information cell of  $a$  at  $w_X$ , write  $I_a(w_X)$ .

To construct a lattice of restricted models, we simply order them in accordance with subset inclusion of the atoms. This produces a complete lattice.

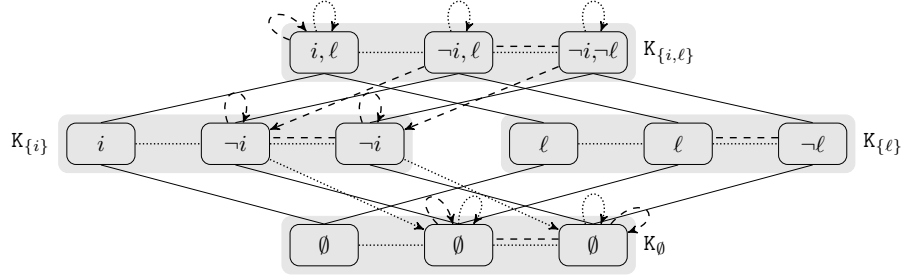
**Definition 9.** Let  $\mathbb{K}$  be a Kripke model for  $At$ . The *restriction lattice* of  $\mathbb{K}$  is  $(\mathcal{K}(\mathbb{K}), \trianglelefteq)$  where  $\mathcal{K}(\mathbb{K}) = \{\mathbb{K}_X\}_{X \subseteq At}$  is the set of restrictions of  $\mathbb{K}$ , and  $\mathbb{K}_X \trianglelefteq \mathbb{K}_Y$  iff  $X \subseteq Y$ .

Projections in unawareness frames are informally interpreted as mapping states to alternates of themselves in less expressive spaces. Restriction lattices offer the same, but implemented with respect to  $At$ : if  $Y \subseteq X \subseteq At$ , then  $w_Y$  is the alternate of  $w_X$  formally described by the smaller vocabulary of atoms,  $Y$ .

The accessibility relations of the Kripke models in a restriction lattice account for the epistemic dimension of the HMS possibility correspondence  $\Pi_a$ . For the



awareness dimension, each agent  $a \in Ag$  is assigned an *awareness map*  $\pi_a$  that maps a world  $w_X$  down to  $\pi_a(w_X) = w_Y$  for some  $Y \subseteq X$ . We think of  $\pi_a(w_X)$  as  $a$ 's *awareness image* of  $w_X$ —i.e.,  $w_X$  as it occurs to  $a$  given her (un)awareness; the submodel from  $\pi_a(w_X)$  is thus  $a$ 's subjective perspective.



**Fig. 3.** A Kripke lattice model of the Figure 2 example. See four restrictions (gray rectangles), ordered spatially as a lattice. States (smallest rectangles) are labeled with their true literals, over the set  $At = \{i, \ell\}$ . Horizontal dashed and dotted lines *inside restrictions* represent Buyer and Owner’s accessibility relations (omitted are links obtainable by reflexive-transitive closure), while dotted and dashed arrows *between restrictions* represent their awareness maps (some arrows are omitted: they go to states’ alternates in  $K_{\emptyset}$ , and are irrelevant from  $(i, \ell)$ ). Thin lines connect states with their alternate in lower restrictions. See also Remark 12 concerning  $K_{\{i\}}$ .

In the following definition, we introduce three properties of awareness maps, which we will assume. Intuitions follow the definition.

**Definition 10.** With  $L = (\mathcal{K}(K), \trianglelefteq)$  a restriction lattice, let  $\Omega_L = \bigcup \mathcal{K}(K)$  and let  $\pi$  assign to each agent  $a \in Ag$  an **awareness map**  $\pi_a : \Omega_L \rightarrow \Omega_L$  satisfying

- D (Downwards)** For all  $w_X \in \Omega_L$ ,  $\pi_a(w_X) = w_Y$  for some  $Y \subseteq X$ .
- II (Introspective Idempotence)** If  $\pi_a(w_X) = w_Y$ , then for all  $v_Y \in I_a(w_Y)$ ,  $\pi_a(v_Y) = u_Y$  for some  $u_Y \in I_a(w_Y)$ .
- NS (No Surprises)** If  $\pi_a(w_X) = w_Z$ , then for all  $Y \subseteq X$ ,  $\pi_a(w_Y) = w_{Y \cap Z}$ .

Call  $K = (\mathcal{K}(K), \trianglelefteq, \pi)$  the **Kripke lattice model** of  $K$ .

D ensures that an agent’s awareness image of a world is a restricted representation of that same world. Hence the awareness image does not conflate worlds, and does not allow the agent to be aware of a more expressive vocabulary than that which describes the world she views from. With II and accessibility assumed reflexive, it entails that  $\pi_a$  is idempotent: for all  $w_X$ ,  $\pi_a(\pi_a(w_X)) = \pi_a(w_X)$ . Alone, II states that in her awareness image, the agent knows, and is aware of, the atoms that she is aware of. Given that accessibility is distributed by inheritance through the Kripke models in restriction lattices, the property implies that the same holds for every such model. NS guarantees that awareness remains “consistent” down the lattice, so that awareness of an atom does not appear or

disappear without reason. Consider the consequent  $\pi_a(w_Y) = w_{Y \cap Z}$  and its two subcases  $\pi_a(w_Y) = w_{Y^*}$  with  $Y^* \subseteq Y \cap Z$  and  $Y^* \supseteq Y \cap Z$ . Colloquially, the first states that if atoms are removed from the description of the world from which the agent views, then they are also removed from her awareness. Oppositely, the second states that if atoms are removed from the description of the world from which the agent views, then no more than these should be removed from her awareness. Jointly, no awareness should “miraculously” appear, and all awareness should be “recalled”.<sup>5</sup>

*Remark 11.* Contrary to HMS models (cf. Remark 2), Kripke lattice models have an objective perspective: designating an ‘actual world’ in  $\mathbb{K}_{A_t}$  allows one to check agents’ uncertainty about the possible states of the world described by the maximal language, i.e., from  $\mathbb{K}_{A_t}$  we can read off their “actual implicit knowledge”. See e.g. Figure 3: In the  $(-i, \ell)$  state, the dashed agent’s “true” uncertainty about  $\ell$  is determined, contrary to the same state in the HMS model of Figure 2.

*Remark 12.* In Remark 6, we mentioned that the HMS state-space  $S_{\{\ell\}}$  of Figure 2 is redundant. Similarly,  $\mathbb{K}_{\{\ell\}}$  is redundant in Figure 3 (from  $(i, \ell)$ ,  $\mathbb{K}_{\{\ell\}}$  is unreachable.) However, contrary to the HMS case, it is here required by definition, as a restriction lattice contains all restrictions of the original Kripke model. For simplicity of constructions, we have not here attempted to prune away redundant restrictions. A more general model class may be obtained by letting models be based on sub-orders of the restriction lattice. See also the concluding remarks.

## 4 Moving between HMS Models and Kripke Lattices: Transformations

To clarify the relationship between HMS models and Kripke lattice models, we introduce transformations between the two model classes, showing that a model from one class encodes the structure of a model from the other. The core idea is to think of a possibility correspondence  $\Pi_a$  as the composition of  $I_a$  and  $\pi_a$ :  $\Pi_a(w)$  is the information cell of the awareness image of  $w$ .

The propositions of this section show that the transformations produce models of the desired class. Additionally, their proofs shed partial light on the relationship between the HMS properties and those assumed for awareness maps  $\pi_a$  and accessibility relations  $R_a$ : we discuss this shortly in the concluding remarks.

### 4.1 From HMS Models to Kripke Lattice Models

Moving from HMS models to Kripke lattice models requires a somewhat involved construction as it must tease apart unawareness and uncertainty from the possibility correspondences, and track the distribution of atoms and their relationship to awareness. For an example, then the Kripke lattice model in Figure 3 is the HMS model of Figure 2 transformed.

---

<sup>5</sup> Again, we are reminded of *No Miracles* and *Perfect Recall*.

**Definition 13.** Let  $M = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_M)$  be an HMS model with maximal state-space  $T$ . For any  $O \subseteq \Omega_M$ , let  $At(O) = \{p \in At : O \subseteq V_M(p) \cup \neg V_M(p)\}$ .<sup>6</sup>

The *L-transform model* of  $M$  is  $L(M) = (\mathcal{K}(K), \trianglelefteq, \pi)$  where the Kripke model  $K = (W, R, V)$  for  $At$  given by

$$W = T;$$

$R$  maps each  $a \in Ag$  to  $R_a \subseteq W^2$  s.t.  $(w, v) \in R_a$  iff  $r_{S(\Pi_a(w))}^T(v) \in \Pi_a(w)$ ;

$V : At \rightarrow \mathcal{P}(W)$ , defined by  $V(p) \ni w$  iff  $w \in V_M(p)$ , for every  $p \in At$ ;

$\pi$  assigns each  $a \in Ag$  a map  $\pi_a : \Omega_{L(M)} \rightarrow \Omega_{L(M)}$  s.t. for all  $w_X \in \Omega_{L(M)}$ ,

$$\pi_a(w_X) = w_Y \text{ where } Y = At(S_Y) \text{ for the } S_Y \in \mathcal{S} \text{ with } S_Y \supseteq \Pi_a(r_{S_X}^T(w))$$

where  $S_X = \min\{S \in \mathcal{S} : At(S) = X\}$ .

The *state correspondence* between  $M$  and  $L(M)$  is the map  $\ell : \Omega_M \rightarrow 2^{\Omega_{L(M)}}$  s.t. for all  $s \in \Omega_M$

$$\ell(s) = \{w_X \in W_X : w \in (r_{S(s)}^T)^{-1}(s) \text{ for } X = At(S(s))\}.$$

Intuitively, in the  $L$ -transform model, a world  $v \in W$  is accessible from a world  $w \in W$  for an agent if, and only if,  $v$ 's restriction to the agent's vocabulary at  $w$  is one of the possibilities she entertains.<sup>7</sup> In addition, the awareness map  $\pi_a$  of agent  $a$  relates a world  $w_X$  to its less expressive counterpart  $w_Y$  if, and only if,  $Y$  is the vocabulary agent  $a$  adopts when describing what she considers possible.

*Remark 14.* The  $L$ -transform model  $L(M)$  of  $M$  is well-defined as the object  $K = (W, R, V)$  is in fact a Kripke model for  $At$ : *i)* By def. of HMS models,  $W = T \in \mathcal{S}$  is non-empty; *ii)* for each  $a$ ,  $R_a \subseteq W^2$  is well-defined: if  $w \in T = W$ , then by Conf,  $\Pi_a(w) \subseteq S$ , for some  $S \in \mathcal{S}$ . Hence,  $U = \{v \in T : r_S^T(v) \in \Pi_a(w)\}$  is well-defined, and so is  $\{(w, v) \in T^2 : v \in U\} = R_a$ ; *iii)* As  $V_M$  is an HMS valuation  $V_M : At \rightarrow \Sigma$  for  $At$ , clearly  $V$  is valuation for  $At$ . Hence  $K = (W, R, V)$  is a Kripke model for  $At$ .

*Remark 15.* The min used in defining  $S_X$  is due to the issue of Remark 5.

*Remark 16.* The state correspondence map  $\ell$  is also well-defined. That it maps each state in  $\Omega_M$  to a *set* of worlds in  $\Omega_{L(M)}$  points to a construction difference between HMS models and Kripke lattice models: in the former, the downwards projections of two states may 'merge' them, so state-spaces may shrink when moving down the lattice; in the latter, distinct worlds remain distinct, so all world sets in a restriction lattice share cardinality.

As unawareness and uncertainty are separated in Kripke lattice models, we show two results about  $L$ -transforms. The first shows that the Conf, Stat and PPK entail that  $\pi_a$  assigns awareness maps, and the second that the five HMS properties entail that  $R$  assigns equivalence relations. In showing the first, we make use of the following lemma, which intuitively shows that the information

<sup>6</sup>  $At(O)$  contains the atoms that have a defined truth value in every  $s \in O$ .

<sup>7</sup> We thank a reviewer of the short version of this paper [3] for this wording.

cell of an agent contains a state described with a certain vocabulary if, and only if, the agent considers possible the corresponding state described with the same vocabulary:

**Lemma 17.** *For every  $w_Y \in \Omega_K$ , if  $\Pi_a(w) \subseteq S$  and  $At(S) = Y$ , then  $v_Y \in I_a(w_Y)$  iff  $v_S \in \Pi_a(w)$ .*

*Proof.* Let  $w_Y \in \Omega_{L(M)}$ . Consider the respective  $w \in T = W$  and let  $\Pi_a(w) \subseteq S$ , with  $At(S) = Y$ . Assume that  $v_Y \in I_a(w_Y)$ . This is the case iff (def. of  $I_a$ )  $(w_Y, v_Y) \in R_{Y_a}$  iff (def. of restriction lattice)  $(w, v) \in R_a$  iff (Def. 13)  $v_S \in \Pi_a(w)$ .

**Proposition 18.** *For any HMS model  $M$ , its  $L$ -transform  $L(M)$  is a Kripke lattice model.*

*Proof.* Let  $M = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_M)$  be an HMS model with maximal state-space  $T$ . We show that  $L(M) = (\mathcal{K}(K), \trianglelefteq, \pi)$  is a Kripke lattice model by showing that  $\pi_a$  satisfies the three properties of an awareness map:

*D:* Consider an arbitrary  $w_X \in \Omega_{L(M)}$ . By def. of  $L$ -transform,  $X = At(S)$  for some  $S \in \mathcal{S}$ . Let  $S_X = \min\{S \in \mathcal{S} : At(S) = X\}$ . If  $w_X \in W_X$  then for some  $w \in W = T$ ,  $w_{S_X} \in S_X$ . By Conf,  $\Pi_a(w_{S_X}) \subseteq S_Y$ , for some  $S_Y \preceq S_X$ . Let  $Y = At(S_Y)$ . Then, by def. of  $\pi_a$ ,  $\pi_a(w_X) = w_Y$  and  $Y \subseteq X$ .

*II:* Let  $\pi_a(w_X) = w_Y$ . By def. of  $\pi_a$ , it holds that  $\Pi_a(r_{S_X}^T(w)) \subseteq S_Y$  with  $At(S_Y) = Y$  and  $S_X = \min\{S \in \mathcal{S} : At(S) = X\}$ . For a contradiction, suppose there exists a  $v_Y \in I_a(w_Y)$  such that for all  $u_Y \in I_a(w_Y)$ ,  $\pi_a(v_Y) \neq u_Y$ . Then  $\pi_a(v_Y) = t_Z$  for some  $Z \subseteq Y$  and  $t_Z \notin I_a(w_Y)$ . By def. of  $\pi_a$ ,  $\pi_a(v_Y) = t_Z$  iff  $\Pi_a(r_{S_Y}^T(v)) \subseteq S_Z$ , where  $Z = At(S_Z)$ . Then, by Lemma 17,  $t_Z \in I_a(v_Z)$  iff  $t_{S_Z} \in \Pi_a(r_{S_Y}^T(v))$ . Moreover, as  $\Pi_a(r_{S_X}^T(w)) \subseteq S_Y$  and  $At(S_X) = X$ , by Lemma 17, it also follows that  $v_Y \in I_a(w_Y)$  iff  $v_{S_Y} \in \Pi_a(r_{S_X}^T(w))$ . Since  $v_Y \in I_a(w_Y)$  then  $v_{S_Y} \in \Pi_a(r_{S_X}^T(w))$ . Hence, by Stat,  $\Pi_a(r_{S_X}^T(w)) = \Pi_a(r_{S_Y}^T(v))$ , which implies  $t_{S_Z} \in \Pi_a(r_{S_X}^T(w))$ . But then  $t_Z \in I_a(v_Z)$ , contradicting the assumption that  $t_Z \notin I_a(w_Y)$ . Thus, for all  $v_Y \in I_a(w_Y)$ ,  $\pi_a(v_Y) = u_Y$  for some  $u_Y \in I_a(w_Y)$ .

*NS:* Let  $\pi_a(w_X) = w_Y$ . By D (cf. item 1. above),  $Y \subseteq X$ . Consider an arbitrary  $Z \subseteq X$ . We have two cases: either *i*)  $Z \subseteq Y$  or *ii*)  $Y \subseteq Z$ . *i*): then  $Z \subseteq Y \subseteq X$ . Let  $Z = At(S_Z)$ ,  $Y = At(S_Y)$ , and  $X = At(S_X)$ . Then  $S_Z \preceq S_Y \preceq S_X$ . By PPK,  $(\Pi_a(r_{S_X}^T(w)))_Z = \Pi_a(r_{S_Z}^T(w))$ . As  $\pi_a(w_X) = w_Y$ , by def. of  $\pi_a$ ,  $\Pi_a(r_{S_X}^T(w)) \subseteq S_Y$ . Then  $(\Pi_a(r_{S_X}^T(w)))_Z = r_{S_Z}^{S_Y}(\Pi_a(r_{S_X}^T(w))) \subseteq S_Z$ . Hence  $\Pi_a(r_{S_Z}^T(w)) \subseteq S_Z$ , and by def. of  $\pi_a$ ,  $\pi_a(w_Z) = w_Z$ . As  $Z \subseteq Y$ ,  $\pi_a(w_Z) = w_Z = w_{Z \cap Y}$ . *ii*): then  $Y \subseteq Z \subseteq X$ . By analogous reasoning, we have  $\pi_a(w_Y) = w_Y = w_{Y \cap Z}$  as  $Y \subseteq Z$ . We can conclude that if  $\pi_a(w_X) = w_Y$ , then for all  $Z \subseteq X$ ,  $\pi_a(w_Z) = w_{Z \cap X}$ .

**Proposition 19.** *If  $L(M) = (\mathcal{K}(K = (W, R, V)), \trianglelefteq, \pi)$  is the  $L$ -transform of an HMS model  $M$ , then for every  $a \in Ag$ ,  $R_a$  is an equivalence relation.*

*Proof.* Let  $M = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_M)$  have maximal state-space  $T$ .

*Reflexivity:* Let  $w \in T$  and  $\Pi_a(w) \subseteq S$ , for some  $S \in \mathcal{S}$ . By def. of upwards closure,  $(\Pi_a(w))^\uparrow = \bigcup_{S' \succeq S} (r_S^{S'})^{-1}(\Pi_a(w))$ , and by Gref,  $w \in (\Pi_a(w))^\uparrow = \bigcup_{S' \succeq S} (r_S^{S'})^{-1}(\Pi_a(w))$ . Since  $T \succeq S$ , then  $r_S^T(w) \in \Pi_a(w)$ . Thus,  $(w, w) \in R_a$ , by def.  $L$ -transform. By def. of restriction lattices, this holds for all  $A \subseteq At$ , i.e.  $(w_A, w_A) \in R_{Aa}$ .

*Transitivity:* Let  $w, v, u$  be in  $T$ . By Conf, there are  $S, S' \in \mathcal{S}$  such that  $\Pi_a(w) \subseteq S$  and  $\Pi_a(v) \subseteq S'$ . Assume that  $(w, v) \in R_a$  and  $(v, u) \in R_a$ . By def. of  $R_a$ , then  $r_S^T(v) \in \Pi_a(w)$  and  $r_{S'}^T(u) \in \Pi_a(v)$ . By Stat,  $\Pi_a(w) = \Pi_a(r_S^T(v))$  and  $\Pi_a(v) = \Pi_a(r_{S'}^T(u))$ . As  $v \in T$  and  $S \preceq T$ , by PPI,  $\Pi_a(v)^\uparrow \subseteq \Pi_a(r_{S'}^T(u))^\uparrow = \Pi_a(w)^\uparrow$ . Hence, as  $r_{S'}^T(u) \in \Pi_a(v)^\uparrow$ , also  $r_{S'}^T(u) \in \Pi_a(w)^\uparrow$ . By def. of upwards closure,  $r_S^T(u) \in \Pi_a(w)$ . Finally,  $(w, u) \in R_a$  by def. of  $R_a$ .

*Symmetry:* Let  $w, v \in T$  be in  $T$ . Assume that  $(w, v) \in R_a$ . By Conf, there are  $S, S' \in \mathcal{S}$  such that  $\Pi_a(w) \subseteq S$  and  $\Pi_a(v) \subseteq S'$ . Then  $r_S^T(v) \in \Pi_a(w)$  (def. of  $L$ -transform), and by Stat,  $\Pi_a(w) = \Pi_a(r_S^T(v))$ . As  $v \in T$  and  $T \succeq S$ , by PPI, by  $\Pi_a(v)^\uparrow \subseteq \Pi_a(r_S^T(v))^\uparrow$ . Then, by def. of upwards closure,  $T \succeq S' \succeq S$ . As  $v \in T$ , by PPK,  $r_{S'}^{S'}(\Pi_a(v)) = \Pi_a(r_S^T(v))$ . By Gref,  $x \in \Pi_a(w)^\uparrow$ , and since  $\Pi_a(w) \subseteq S$  then  $r_S^T(w) \in \Pi_a(w)$ , by def. of upward closure. Then  $r_S^T(w) \in \Pi_a(w) = \Pi_a(r_S^T(v)) = r_{S'}^{S'}(\Pi_a(v))$ . So  $r_S^T(w) \in r_{S'}^{S'}(\Pi_a(v))$ , i.e.  $r_{S'}^T(w) \in \Pi_a(v)$ , by def. of  $r$ . Hence,  $(v, w) \in R_a$ , by def. of  $R_a$ .

## 4.2 From Kripke Lattice Models to HMS Models

Moving from Kripke lattice models to HMS models requires a less involved construction, as the restriction lattice almost encodes projections, and unawareness and uncertainty are simply composed to form possibility correspondences:

**Definition 20.** Let  $\mathbb{K} = (\mathcal{K}(\mathbb{K} = (W, R, V)), \triangleleft, \pi)$  be a Kripke lattice model for  $At$ . The  $H$ -transform of  $\mathbb{K}$  is  $H(\mathbb{K}) = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{H(\mathbb{K})})$  where

$$\mathcal{S} = \{W_X \subseteq \Omega_{\mathbb{K}} : \mathbb{K}_X \in \mathcal{K}(\mathbb{K})\};$$

$$W_X \preceq W_Y \text{ iff } \mathbb{K}_X \triangleleft \mathbb{K}_Y;$$

$$\mathcal{R} = \{r_{W_Y}^{W_X} : r_{W_Y}^{W_X}(w_X) = w_Y \text{ for all } w \in W, \text{ and all } X, Y \subseteq At\};$$

$$\Pi = \{\Pi_a \in (2^{\Omega_{\mathbb{K}}})^{\Omega_{\mathbb{K}}} : \Pi_a(w_X) = I_a(\pi_a(w_X)) \text{ for all } w \in W, X \subseteq At, a \in Ag\};$$

$$V_{H(\mathbb{K})}(p) = \{w_X \in \Omega_{\mathbb{K}} : X \ni p \text{ and } w_X \in V_X(p)\} \text{ for all } p \in At.$$

As HMS models lump together unawareness and uncertainty, we show only one result in this direction:

**Proposition 21.** For any Kripke lattice model  $\mathbb{K} = (\mathcal{K}(\mathbb{K} = (W, R, V)), \triangleleft, \pi)$  such that  $R$  assigns equivalence relations, the  $H$ -transform  $H(\mathbb{K})$  is an HMS model.

*Proof.* Let  $\mathbb{K}$  be as stated and let  $H(\mathbb{K}) = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{H(\mathbb{K})})$  be its  $H$ -transform.  $\mathcal{S} = \{W_X, W_Y, \dots\}$  is composed of non-empty disjoint sets by construction and  $(\mathcal{S}, \preceq)$  is a complete lattice as  $(\mathcal{K}(\mathbb{K}), \triangleleft)$  is so.  $\mathcal{R}$  is clearly a family of well-defined, surjective and commutative projections. As  $\Pi$  assigns to each  $a \in Ag$ ,  $\Pi_a(w_X) = I_a(\pi_a(w_X))$ , for all  $w \in W$ ,  $X \subseteq At$ , it assigns a map  $\Pi_a : \Omega_{H(\mathbb{K})} \rightarrow 2^{\Omega_{H(\mathbb{K})}}$ , which is a possibility correspondence as it satisfies the HMS properties:

*Conf:* For  $w_X \in W_X$ ,  $\Pi_a(w_X) = I_a(\pi_a(w_X))$ , by Def. 20. By D,  $\pi_a(w_X) = w_Y$  for some  $Y \subseteq X$ , and  $I_a(\pi_a(w_X)) = I_a(w_Y)$ . So,  $\Pi_a(w_X) \subseteq W_Y$  for some  $Y \subseteq X$ .

*Gref:* Let  $w_X \in \Omega_K$ ,  $X \subseteq At$ . By D,  $\pi_a(w_X) = w_Y$  for some  $Y \subseteq X$ . By def. of  $\Pi_a$  and  $I_a$ ,  $\Pi_a(w_X) = I_a(w_Y) = \{v_Y \in \Omega_K : (w_Y, v_Y) \in R_{Y_a}\}$ . Hence  $\Pi_a(w_X) \subseteq W_Y$ . By def. of upward closure,  $(\Pi_a(w_X))^\uparrow = (I_a(w_Y))^\uparrow = \{u_Z \in \Omega_K : Y \subseteq Z \text{ and } u_Y \in \{v_Y \in \Omega_K : (w_Y, v_Y) \in R_{Y_a}\}\}$ , with the last identity given by the def. of  $r_{W_Y}^{W_Z}$ . As  $R_a$  is an equivalence relation, so is  $R_{Y_a}$ , by def. So  $w_Y \in \{v_Y \in \Omega_K : (w_Y, v_Y) \in R_{Y_a}\}$ , and since  $Y \subseteq X$ , then  $w_X \in (\Pi_a(w_X))^\uparrow$ .

*Stat:* For  $w_X \in \Omega_K$ , assume  $v \in \Pi_a(w_X) = I_a(\pi_a(w_X))$ . By D,  $v \in I_a(w_Y)$ , for some  $Y \subseteq X$ . With  $R_{Y_a}$  an equivalence relation,  $v \in I_a(w_Y)$  iff  $w_Y \in I_a(v)$ , i.e.,  $I_a(v) = I_a(w_B)$ . II and D entails that for all  $u_Y \in I_a(w_Y)$ ,  $\pi_a(u_Y) = u_Y$ , so  $\pi_a(v) = v$ . Therefore  $\Pi_a(v) = I_a(\pi_a(v)) = I_a(v) = I_a(w_Y) = I_a(\pi_a(w_X)) = \Pi_a(w_X)$ . Thus, if  $v \in \Pi_a(w_X)$ , then  $\Pi_a(v) = \Pi_a(w_X)$ .

*PPF:* Let  $w_X \in W_X$  and  $W_Y \preceq W_X$ , i.e.  $Y \subseteq X \subseteq At$ . Let  $q_Q \in (\Pi_a(w_X))^\uparrow$  with  $Q \subseteq At$ . By def. of  $\Pi_a$  and D,  $\Pi_a(w_X) = I_a(\pi_a(w_X)) = I_a(w_Z)$  for some  $Z \subseteq X$ . By def. of upwards closure, it follows that  $q_Z \in I_a(w_Z) = \Pi_a(w_X)$ . Now let  $\pi_a(w_Y) = w_P$  for some  $P \subseteq Y$ . Then, by NS,  $P = Z \cap Y$ , so  $P \subseteq Z$ . As  $q_Z \in I_a(w_Z)$ , then  $q_P \in I_a(w_P) = I_a(\pi_a(w_Y)) = \Pi_a(w_Y)$ , by def. of restriction lattice. Since  $q_Q \in (\Pi_a(w_X))^\uparrow = (I_a(w_Z))^\uparrow$ , then  $Z \subseteq Q$ . It follows that  $P \subseteq Z \subseteq Q$ , which implies  $q_Q \in (\Pi_a(w_Y))^\uparrow$ . Hence, if  $q_Q \in (\Pi_a(w_X))^\uparrow$ , then  $q_Q \in (\Pi_a(w_Y))^\uparrow$ , i.e.,  $(\Pi_a(w_X))^\uparrow \subseteq (\Pi_a(w_Y))^\uparrow$ .

*PPK:* Suppose that  $W_Z \preceq W_Y \preceq W_X$ ,  $w_X \in W_X$  and  $\Pi_a(w_X) \subseteq W_Y$ , i.e.  $\Pi_a(w_X) = I_a(w_Y)$  and  $\pi_a(w_X) = w_Y$ . As  $Z \subseteq Y \subseteq X$ , NS implies  $\pi_a(w_Z) = w_{Z \cap Y} = w_Z$ . Hence,  $\Pi_a(w_Z) = I_a(w_Z) \subseteq W_Z$ . Hence PPK is established if  $(I_a(w_Y))_Z = I_a(w_Z)$ . As  $(I_a(w_Y))_Z = \{x_Z \in \Omega_K : x_Y \in I_a(w_Y)\}$ , then clearly  $(I_a(w_Y))_Z = I_a(w_Z)$ . Thus,  $(\Pi_a(w_X))_Z = \Pi_a(w_Z)$ .

Finally,  $V_{H(K)}$  is an HMS valuation as for each  $p \in At$ ,  $V_{H(K)}(p)$  is an event  $(D^\uparrow, S)$  with  $D = \{w_{\{p\}} \in W_{\{p\}} : w_{\{p\}} \in V_{\{p\}}(p)\}$  and  $S = W_{\{p\}}$ .

## 5 Language for Awareness and $\mathcal{L}$ -Equivalence

Multiple languages for knowledge and awareness exist. The Logic of General Awareness ( $\mathcal{A}_{LGA}$ , [9]) which we will see in Section 9, takes implicit knowledge and awareness as primitives, and define explicit knowledge as ‘implicit knowledge  $\wedge$  awareness’; other combinations are discussed in [5].

HMS [16] follow instead Modica-Rustichini [20, 21] and take explicit knowledge as primitive and awareness as defined: an agent is aware of  $\varphi$  iff she either explicitly knows  $\varphi$ , or explicitly knows that she does not explicitly know  $\varphi$ .

**Definition 22.** *With  $a \in Ag$  and  $p \in At$ , define the language  $\mathcal{L}$  by*

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi$$

*and define  $A_a\varphi := K_a\varphi \vee K_a\neg K_a\varphi$ .*

*Let  $At(\varphi) = \{p \in At : p \text{ is a subformula of } \varphi\}$ , for all  $\varphi \in \mathcal{L}$ .*

### 5.1 HMS Models as a Semantics for $\mathcal{L}$

The satisfaction of formulas over HMS models is defined as follows. The semantics are three-valued, so formulas may have undefined truth value: there may exist a  $w \in \Omega_M$  such that neither  $M, w \models \varphi$  nor  $M, w \models \neg\varphi$ . This happens if and only if  $\varphi$  contains atoms with undefined truth value in  $w$ .

**Definition 23.** Let  $M = (\mathcal{S}, \preceq, \mathcal{R}, II, V_M)$  be an HMS model and let  $w \in \Omega_M$ . Satisfaction of  $\mathcal{L}$  formulas is given by

$$\begin{array}{ll} M, w \models \top & \text{for all } w \in \Omega_M \\ M, w \models p & \text{iff } w \in V_M(p) \\ M, w \models \neg\varphi & \text{iff } w \in \neg\llbracket\varphi\rrbracket \end{array} \quad \begin{array}{ll} M, w \models \varphi \wedge \psi & \text{iff } w \in \llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket \\ M, w \models K_a\varphi & \text{iff } w \in \mathbf{K}_a(\llbracket\varphi\rrbracket) \end{array}$$

where  $\llbracket\varphi\rrbracket = \{v \in \Omega_M : M, v \models \varphi\}$  for all  $\varphi \in \mathcal{L}$ .

With the HMS semantics being three-valued, they adopt a non-standard notion of validity which requires only that a formula be always satisfied *if its has a defined truth value*. The below is equivalent to the definition in [16], but is stated so that it also works for Kripke lattice models:

**Definition 24.** A formula  $\varphi \in \mathcal{L}$  is valid over a class of models  $\mathcal{C}$  iff for all models  $M \in \mathcal{C}$ , for all states  $w$  of  $M$  which satisfy  $p$  or  $\neg p$  for all  $p \in At(\varphi)$ ,  $w$  also satisfies  $\varphi$ .

### 5.2 Kripke Lattice Models as a Semantics for $\mathcal{L}$

We define semantics for  $\mathcal{L}$  over Kripke lattice models. Like the HMS semantics, the semantics are three-valued, as it is possible that a pointed Kripke lattice model  $(M, w_X)$  satisfies neither  $\varphi$  nor  $\neg\varphi$ . This happens exactly when  $\varphi$  contains atoms not in  $X$ .

**Definition 25.** Let  $K = (\mathcal{K}(K = (W, R, V)), \trianglelefteq, \pi)$  be a Kripke lattice model with  $w_X \in \Omega_K$ . Satisfaction of  $\mathcal{L}$  formulas is given by

$$\begin{array}{ll} K, w_X \Vdash \top & \text{for all } w_X \in \Omega_K \\ K, w_X \Vdash p & \text{iff } w_X \in V_X(p) \quad \text{and } p \in X \\ K, w_X \Vdash \neg\varphi & \text{iff not } K, w_X \Vdash \varphi \quad \text{and } At(\varphi) \subseteq X \\ K, w_X \Vdash \varphi \wedge \psi & \text{iff } K, w_X \Vdash \varphi \text{ and } K, w_X \Vdash \psi \quad \text{and } At(\varphi \wedge \psi) \subseteq X \\ K, w_X \Vdash K_a\varphi & \text{iff } \pi_a(w_X)R_Y a v_Y \text{ implies } K, v_Y \Vdash \varphi, \\ & \text{for } Y \subseteq At \text{ s.t. } \pi_a(w_X) \in W_Y \quad \text{and } At(\varphi) \subseteq X \end{array}$$

### 5.3 The $\mathcal{L}$ -Equivalence of HMS and Kripke Lattice Models

$L$ - and  $H$ -transforms not only produce models of the correct class, but also preserve finer details, as any model and its transform satisfy the same  $\mathcal{L}$  formulas.

**Proposition 26.** For any HMS model  $M$  with  $L$ -transform  $L(M)$ , for all  $\varphi \in \mathcal{L}$ , for all  $w \in \Omega_M$ , and for all  $v \in \ell(w)$ ,

$$M, w \models \varphi \text{ iff } L(M), v \Vdash \varphi.$$

*Proof.* Let  $\Sigma_M$  be the events of  $M = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_M)$  with maximal state-space  $T$  and let  $L(M) = (\mathcal{K}(K = (W, R, V)), \leq, \pi)$ . The proof is by induction on formula complexity. Let  $\varphi \in \mathcal{L}$  and let  $w \in \Omega_M$  with  $At(S(w)) = X$ .

*Base:* *i)*  $\varphi := p \in At$  or *ii)*  $\varphi := \top$ . *i)*  $M, w \models p$  iff  $w \in V_M(p)$ . As  $V_M(p) \in \Sigma_M$ ,  $(r_{S(w)}^T)^{-1}(w) \subseteq V_M(p)$ . By def. of  $L(M)$ , if  $v \in T = W$ , then  $v \in V_M(p)$  iff  $v \in V(p)$ , so  $v \in (r_{S(w)}^T)^{-1}(w)$  iff  $v \in V(p)$  iff  $v_X \in V_X(p)$ , with  $p \in X$  (def. of Kripke lattice models). Hence, by def. of  $\ell$ ,  $v \in \ell(w) = \{u_X \in W_X : u \in (r_{S(w)}^T)^{-1}(w) \text{ for } X = At(S(w))\}$  iff  $v \in V_X(p)$ , i.e., iff  $L(M), v \Vdash p$  for all  $v \in \ell(w)$ . *ii)* is trivial.

*Step.* Assume  $\psi, \chi \in \mathcal{L}$  satisfy Prop. 26.

$\varphi := \neg\psi$ . There are two cases: *i)*  $At(\psi) \subseteq At(S(w))$  or *ii)*  $At(\psi) \not\subseteq At(S(w))$ . *i)*  $M, w \models \neg\psi$  iff (def. of  $\models$ )  $w \in \neg[[\psi]]$  iff (def. of  $V_M$ )  $(r_{S(w)}^T)^{-1}(w) \subseteq \neg[[\psi]]$  iff (def. of  $[[\psi]]$ ) for all  $v \in (r_{S(w)}^T)^{-1}(w)$ ,  $M, v \not\models \psi$  iff (Def. 13) for all  $v \in (r_{S(w)}^T)^{-1}(w)$ , not  $L(M), v \Vdash \psi$  iff (def. of  $\ell(w)$ ) for all  $v_X \in \ell(w)$ , not  $L(M), v_X \Vdash \psi$ , with  $At(\psi) \subseteq X$  iff (def. of  $\Vdash$ ) for all  $v_X \in \ell(w)$ ,  $L(M), v_X \Vdash \neg\psi$ . *ii)* is trivial:  $\varphi$  is undefined in  $(M, w)$  iff it is so in  $(L(M), w_X)$ .

$\varphi := \psi \wedge \chi$ . The case follows by tracing *iffs* through the definitions of  $\models$ ,  $V_M$ ,  $[[\cdot]]$ ,  $(r_{S(w)}^T)^{-1}$ ,  $L$ -transform,  $\ell$ , and  $\Vdash$ .

$\varphi := K_a\psi$ .  $M, w \models K_a\psi$  iff (def. of  $\models$ )  $w \in \mathbf{K}_a([[ \psi ]])$  iff (def. of  $\mathbf{K}_a$ )  $\Pi_a(w) \subseteq [[\psi]]$ . Let  $\Pi_a(w) \subseteq S$ , for some  $S \in \mathcal{S}$ , and let  $X = At(S(w))$  and  $Y = At(S)$ . Then  $v_S \in \Pi_a(w) \subseteq [[\psi]]$  iff (def. of  $[[\psi]]$ ) for all  $v_S \in \Pi_a(w)$ ,  $M, v_S \models \psi$  iff (def. of  $V_M$ ) for all  $(r_S^T)^{-1}(v_S)$  with  $v_S \in \Pi_a(w)$ ,  $M, v_T \models \psi$  iff (def. of  $L$ -transform) for all  $v_{At}$  with  $r_S^T(v) \in \Pi_a(w)$ ,  $L(M), v_{At} \Vdash \psi$  and  $At(\psi) \subseteq At$  iff (def. of  $L$ -transform) for all  $v_{At}$  with  $(w_{At}, v_{At}) \in R_{Ata}$ ,  $L(M), v_{At} \Vdash \psi$  and  $At(\psi) \subseteq At$  iff (def. of restriction lattice) for all  $v_Y$  with  $(w_Y, v_Y) \in R_{Ya}$ ,  $L(M), v_Y \Vdash \psi$  and  $At(\psi) \subseteq Y$  iff (def. of  $\pi_a$  and  $\pi_a(w_X) = w_Y$ ), for all  $v_Y$  with  $(\pi_a(w_X), v_Y) \in R_{Ya}$ ,  $L(M), v_Y \Vdash \psi$  and  $At(\psi) \subseteq Y$  iff (def. of  $\Vdash$ )  $L(M), w_X \Vdash K_a\psi$  and  $At(\psi) \subseteq Y$ .

**Proposition 27.** *For any Kripke lattice model  $K$  with  $H$ -transform  $H(K)$ , for all  $\varphi \in \mathcal{L}$ , for all  $w_X \in \Omega_K$ ,*

$$K, w_X \Vdash \varphi \text{ iff } H(K), w_X \models \varphi.$$

*Proof.* Let  $K = (\mathcal{K}(K = (W, R, V)), \leq, \pi)$  with  $w_X \in \Omega_K$ ,  $\pi_a(w_X) \in W_Y$  with  $Y \subseteq At$ , and let  $H(K) = (\mathcal{S}, \preceq, \mathcal{R}, \Pi, V_{H(K)})$ . Let  $\varphi \in \mathcal{L}$  and proceed by induction on formula complexity.

*Base:* *i)*  $\varphi := p \in At$  or *ii)*  $\varphi := \top$ . *i)*  $K, w_X \Vdash p$  iff (def. of  $\Vdash$ )  $w_X \in V_X(p)$  with  $p \in X$  iff (def. of  $H$ -transform)  $w_X \in V_{H(K)}(p)$  iff (def. of  $\models$ )  $H(K), w_X \models p$ . *ii)* is trivial.

*Step.* Assume  $\psi, \chi \in \mathcal{L}$  satisfy Prop. 27.

$\varphi := \neg\psi$ . There are two cases: *i)*  $At(\psi) \subseteq X$  or *ii)*  $At(\psi) \not\subseteq X$ . *i)*  $K, w_X \Vdash \neg\psi$  iff (def. of  $\Vdash$ ) not  $K, w_X \Vdash \psi$  iff (def. of  $[[\psi]]$ )  $w_X \notin [[\psi]]$  iff (def. of  $[[\psi]]$  and  $At(\psi) \subseteq X$ )  $w_X \in \neg[[\psi]]$  iff (def. of  $\models$ )  $H(K), w_X \models \neg\psi$ . *ii)* is trivial:  $\varphi$  is undefined in  $(K, w_X)$  iff it is so in  $(H(K), w_X)$ .

$\varphi := \psi \wedge \chi$ . The case follows by tracing *iffs* through the definitions of  $\Vdash$ ,  $H$ -transform, and  $\models$ .



$\varphi := K_a\psi$ .  $\mathbf{K}, w_X \Vdash K_a\psi$  iff (def. of  $\Vdash$ )  $\pi_a(w_X)R_{Y_a}v_Y$  implies  $\mathbf{K}, v_Y \Vdash \varphi$  iff (def. of  $\pi_a$ , i.e.  $\pi_a(w_X) = w_Y$  and def. of  $I_a$ ), for all  $v_Y$  such that  $(w_Y, v_Y) \in R_{Y_a}$ , i.e. for all  $v_Y \in I_a(w_Y)$ ,  $\mathbf{K}, v_Y \Vdash \varphi$  iff (def. of  $\Pi_a$ , i.e.  $\Pi_a(w_X) = I_a(\pi_a(w_X)) = I_a(w_Y)$ )  $\Pi_a(w_X) \subseteq \llbracket \psi \rrbracket$  iff (def. of  $\mathbf{K}_a$ )  $w \in \mathbf{K}_a(\llbracket \psi \rrbracket)$  iff (def. of  $\models$ )  $H(\mathbf{K}), w_X \models K_a\psi$ .

## 6 The HMS Logic of Kripke Lattice Models with Equivalence Relations

As we may transition back-and-forth between HMS models and Kripke lattice models with equivalence relations in a manner that preserve satisfaction of formula of  $\mathcal{L}$ , soundness and completeness of a  $\mathcal{L}$ -logic is also transferable between the model classes. We thereby show such results for Kripke lattice models with equivalence relations as a corollary to results by HMS [16].

**Definition 28.** *The logic  $\Lambda_{HMS}$  is the smallest set of  $\mathcal{L}$  formulas that contains the axioms in, and is closed under the inference rules of, Table 1.*

All substitution instances of propositional logic, including the formula $\top$	
$A_a\neg\varphi \leftrightarrow A_a\varphi$	(Symmetry)
$A_a(\varphi \wedge \psi) \leftrightarrow A_a\varphi \wedge A_a\psi$	(Awareness Conjunction)
$A_a\varphi \leftrightarrow A_aK_b\varphi$ , for all $b \in Ag$	(Awareness Knowledge Reflection)
$K_a\varphi \rightarrow \varphi$	(T, Axiom of Truth)
$K_a\varphi \rightarrow K_aK_a\varphi$	(4, Positive Introspection Axiom)
From $\varphi$ and $\varphi \rightarrow \psi$ , infer $\psi$	(Modus Ponens)
For $\varphi_1, \varphi_2, \dots, \varphi_n, \varphi$ that satisfy $At(\varphi) \subseteq \bigcup_{i=1}^n At(\varphi_i)$ , from $\bigwedge_{i=1}^n \varphi_i \rightarrow \varphi$ , infer $\bigwedge_{i=1}^n K_a\varphi_i \rightarrow K_a\varphi$	(RK-Inference)

**Table 1.** Axioms and inference rules of the HMS logic of unawareness,  $\Lambda_{HMS}$ .

As the the  $L$ -transform of an HMS model has equivalence relations, one may be surprised by the lack of the standard negative introspection axiom 5 :  $(\neg K_a\varphi \rightarrow K_a\neg K_a\varphi)$  among the axioms of  $\Lambda_{HMS}$ . However, including 5 would make collapse awareness [20]. In [16], HMS remarks that  $\Lambda_{HMS}$  implies the weakened version  $K_a\neg K_a\neg K_a\varphi \rightarrow (K_a\varphi \vee K_a\neg K_a\varphi)$ , which by the Modica-Rustichini definition of awareness is  $K_a\neg K_a\neg K_a\varphi \rightarrow A_a\varphi$ . Defining unawareness by  $U_a\varphi := \neg A_a\varphi$ , this again equates  $U_a\varphi \rightarrow \neg K_a\neg K_a\neg K_a\varphi$ . Additionally, HMS notes that if  $\varphi$  is a theorem, then  $A_a\varphi \rightarrow K_a\varphi$  is a theorem, that 4 implies introspection of awareness  $(A_a\varphi \rightarrow K_aA_a\varphi)$ , while  $\Lambda_{HMS}$  entails that *awareness is generated by primitives propositions*, i.e., that  $A_a\varphi \leftrightarrow \bigwedge_{p \in At(\varphi)} A_ap$  is a theorem. The latter two properties entails that HMS awareness is *propositionally determined*, in the terminology of [15].

Using the above given notion of validity and standard notions of proof, soundness and completeness, HMS [16] state that, as standard,

**Lemma 29.** *The logic  $\Lambda_{HMS}$  is complete with respect to a class of structures  $\mathfrak{S}$  iff every set of  $\Lambda_{HMS}$  consistent formulas is satisfied in some  $\mathfrak{s} \in \mathfrak{S}$ .*

Let  $\mathbf{M}$  be the class of HMS modes. Using a canonical model, HMS show:

**Theorem 30 ([16]).**  $\Lambda_{HMS}$  is sound and complete with respect to  $M$ .

Let  $KLM_{EQ}$  be the class of Kripke lattice models where all accessibility relations are equivalence relations. As a corollary to Theorem 30 and our transformation and equivalence results, we obtain

**Theorem 31.**  $\Lambda_{HMS}$  is sound and complete with respect to  $KLM_{EQ}$ .

*Proof.* Soundness: The axioms of  $\Lambda_{HMS}$  are valid in  $KLM_{EQ}$ . We show the contrapositive. Let  $\varphi \in \mathcal{L}$ . If  $\varphi$  is not valid in  $KLM_{EQ}$ , then for some  $K \in KLM_{EQ}$  and some  $w$  from  $K$ ,  $K, w \Vdash \neg\varphi$ . Then its  $H$ -transform  $H(K)$  is an HMS model cf. Prop. 21, and  $H(K), w \vDash \neg\varphi$  cf. Prop. 27. Hence  $\varphi$  is not valid in the class of HMS models. The same reasoning implies that the  $\Lambda_{HMS}$  inference rules preserve validity.

Completeness: Assume  $\Phi \subseteq \mathcal{L}$  is a consistent set, and let  $\mathfrak{M}$  be the canonical model of HMS, with  $\mathfrak{w}$  a state in  $\mathfrak{M}$  that satisfies  $\Phi$ . This exists, cf. [16]. By Prop.s 18 and 19,  $L(\mathfrak{M})$  is in  $KLM_{EQ}$ . By Prop. 26, for all  $v \in \ell(\mathfrak{w})$ ,  $L(\mathfrak{M}), v \Vdash \Phi$ . By Lemma 29,  $\Lambda_{HMS}$  is thus complete with respect to  $KLM_{EQ}$ .

## 7 The FH Model

We next turn to the syntax-based FH model, the first model for awareness in the field of logic, introduced in [9]. In [9] the models are referred to as *awareness structures*. We propose transformations between these structures and Kripke lattice models, to show the relations between the two model classes. The transformations preserve formula satisfaction.

In the literature, the FH model is said to adopt a *syntactic approach*, as it models awareness by adding a syntactic *awareness function*  $\mathcal{A}_a$  to standard Kripke models  $(W, R, V)$  for  $At' \subseteq At$ .<sup>8</sup>

The language on which FH originally defined the awareness function—call it  $\mathcal{L}^{KA}$ —includes both an awareness and an implicit knowledge operators as primitives, as well as an explicit knowledge operator definable as the conjunction of the two [9].

As we seek to directly establish the Figure 1's promised equivalence between FH models and Kripke lattice models with respect to the HMS language  $\mathcal{L}$  (containing only the explicit knowledge operator  $K_a$ ), in this section we use FH models as a semantics for  $\mathcal{L}$ . This entails letting  $\mathcal{A}_a$  assign formulas from  $\mathcal{L}$ , and not  $\mathcal{L}^{KA}$ . Additionally, to establish equivalence, we must focus on the special case of FH models that in which awareness is *propositionally determined* (cf. Def.

<sup>8</sup> In [15],  $R$  is not defined as assigning to each agent  $a \in Ag$  a relation  $R_a$  between states, as we do above, but as providing a *possibility correspondence*  $R'_a : W \rightarrow 2^W$ . As Halpern and Rêgo write, the approaches are equivalent:  $R_a$  is definable from a possibility correspondence, and *vice versa*, by taking  $v \in R'_a(w)$  iff  $(w, v) \in R_a$ .

Similarly for the valuation function, which FH defines as  $V' : W \times At' \rightarrow \{0, 1\}$  and we define as  $V : At' \rightarrow \mathcal{P}(W)$ . The two definitions are equivalent, as we can define one in terms of the other by taking  $V'(w, p) = 1$  iff  $w \in V(p)$ .

32). We introduce  $\mathcal{L}^{KA}$  in Section 9, where we show that the FH and Kripke lattice models are equivalent with respect to that language as well.

**Definition 32.** An **FH model** for  $At' \subseteq At$  is a tuple  $S = (W, R, V, \mathcal{A})$  where  $(W, R, V)$  is a Kripke model for  $At'$ , and  $\mathcal{A}$  is an **awareness function**  $\mathcal{A} : Ag \times W \rightarrow 2^{\mathcal{L}}$  that assigns to each agent  $a \in Ag$  and world,  $w \in W$  a set of formula denoted  $\mathcal{A}_a(w)$ .

The function  $\mathcal{A}$  satisfies

- PP (Awareness is Generated by Primitive Propositions)** if for all  $a \in Ag$  and  $\varphi \in \mathcal{L}$ ,  $\varphi \in \mathcal{A}_a(w)$  iff for all  $p \in At(\varphi)$ ,  $p \in (\mathcal{A}_a(w) \cap At')$ .
- KA (Agents Know What They are Aware of)** if or all  $a \in Ag$ ,  $(w, v) \in R_a$  implies  $\mathcal{A}_a(w) = \mathcal{A}_a(v)$ .

If  $\mathcal{A}_a$  satisfies PP and KA, then  $S$  is **propositionally determined**.  $S$  is called **partitional** (resp. **reflexive**, **transitive**) iff for each  $a \in Ag$ ,  $R_a$  is an equivalence relation (resp. reflexive, transitive).

If no restrictions are applied to  $\mathcal{A}_a$ , then an agent can be aware of an arbitrary set of formulas. For example, for  $w \in W$ , we may have both  $\neg\varphi \in \mathcal{A}_a(w)$  and  $\varphi \in \mathcal{A}_a(w)$ , or  $\varphi \wedge \psi \in \mathcal{A}_a$  without having  $\psi \wedge \varphi \in \mathcal{A}_a(w)$  [9]. That awareness is generated by primitive proposition ensures that, at every state, the agent is aware of all and only the formulas that are formed from some subset of the set of atoms  $At$ .

Halpern [12] shows that if  $\mathcal{A}_a$  satisfies this property, then in a partitional awareness structures  $S$ , the awareness operator can be characterized as Modica-Rustichini and HMS suggest [17, 20, 21], i.e. so that any FH model validates  $A_i \leftrightarrow (K_i \vee (\neg K_i \wedge K_i \neg K_i))$ , when employing the following semantics:

**Definition 33.** Let  $S = (W, R, V, \mathcal{A})$  be an FH model and let  $w \in W$ . Satisfaction of  $\mathcal{L}$  formulas is given by

$$\begin{array}{ll} S, w \models \top & \text{for all } w \in W; \\ S, w \models p & \text{iff } w \in V(p); \\ S, w \models \neg\varphi & \text{iff } S, w \not\models \varphi; \end{array} \quad \begin{array}{l} S, w \models \varphi \wedge \psi \text{ iff } S, w \models \varphi \text{ and } S, w \models \psi; \\ S, w \models K_a \varphi \text{ iff } \varphi \in \mathcal{A}_a(w) \text{ and for all } v \in W \\ \text{s.t. } (w, v) \in R_a, S, v \models \varphi. \end{array}$$

The FH semantics for  $\mathcal{L}$  over FH models is defined as standard in epistemic logic, except for the knowledge operator  $K_a$ , with  $a \in Ag$ . In standard epistemic logic,  $K_a$  represents *implicit* knowledge, semantically defined as the formulas that are satisfied in all the worlds the agent has access to. In the FH semantics,  $K_a$  represents *explicit* knowledge, namely the formulas that  $a$  implicitly knows and that belong to  $a$ 's awareness set.

## 8 Moving between FH Models and Kripke Lattices: Transformations and $\mathcal{L}$ -Equivalence

To clarify the relationship between FH models and Kripke lattice models, we introduce transformations between the two model classes, showing that a model

from one class encodes the structure of a model from the other. As both structure types are based on Kripke models (FH models are Kripke models augmented with an awareness function, and Kripke lattices are spawned from a Kripke model), the main task in moving from FH models to Kripke lattices is to compose the awareness map  $\pi_a$  by extracting semantic information from the syntactically defined awareness function  $\mathcal{A}_a$ . Conversely, moving from Kripke lattices to FH models requires to compose  $\mathcal{A}_a$  by extracting syntactic information from the semantically defined  $\pi_a$ .

### 8.1 From FH Models to Kripke Lattice Models

**Definition 34.** Let  $S = (W, R, V, \mathcal{A})$  be an FH model for *At*. The ***K-transform model of S*** is  $K(S) = (\mathcal{K}(K), \triangleleft, \pi)$  with Kripke model  $K = (W', R', V')$  for *At* given by

$$\begin{aligned} W' &= W; \\ R' &= R; \\ V'(p) &= V(p), \text{ for every } p \in \text{At}; \\ \pi &\text{ assigns to each } a \in \text{Ag a map } \pi_a : \Omega_{K(S)} \rightarrow \Omega_{K(S)} \text{ s.t., for all } w_X \in \Omega_{K(S)}, \\ \pi_a(w_X) &= w_Z \text{ with } Z = X \cap Y \text{ and } Y = \{p \in \text{At} : p \in \bigcup_{\varphi \in \mathcal{A}_a(w)} \text{At}(\varphi)\}. \end{aligned}$$

The *K*-transform takes the Kripke model on which the FH model is based and spawns a lattice from there. Then, it constructs the awareness map  $\pi_a$  by extracting, for every world  $w$ , the set  $Y$  of atoms occurring in formulas in  $\mathcal{A}_a(w)$ , and relating each world  $w_X$  in the Kripke lattice to its weakly less expressive counterpart  $w_Z$  if, and only if, the vocabulary  $Z$  is the subset of  $Y$  that is defined in  $X$  (and thus expressible in  $w_X$ ).

*Remark 35.* The *K*-transform model  $K(S)$  is well-defined as the object  $K = (W', R', V')$  is clearly a Kripke model for *At*.

**Proposition 36.** For any FH model  $S$  where agents know what they are aware of, its *K*-transform  $K(S)$  is a Kripke lattice model.

*Proof.* Let  $S = (W, R, V, \mathcal{A})$  be an FH model. We show that  $K(S) = (\mathcal{K}(K = (W', R', V'), \triangleleft, \pi)$  is a Kripke lattice model by showing that  $\pi_a$  satisfies the three properties of an awareness map:

*D:* Consider some  $w_X \in \Omega_{K(S)}$ . By def. of *K*-transform,  $w \in W' = W$  and, for all  $a \in \text{Ag}$ ,  $\pi_a(w_X) = w_Z$ , with  $Z = X \cap Y$  and  $Y = \{p \in \text{At} : p \in \text{At}(\varphi), \varphi \in \mathcal{A}_a(w)\}$ . Thus,  $Z \subseteq X$ , i.e. D holds for  $\pi_a$ .

*II:* Let  $\pi_a(w_X) = w_Z$ , and consider some  $v_Z \in \Omega_{K(S)}$  such that  $v_Z \in I_a(w_Z)$ , with  $I_a(w_Z) = \{v_Z \in \Omega_{K(S)} : (w_Z, v_Z) \in R'_{Za}\}$ . Then,  $(w, v) \in R'_a$ , by def. of Kripke lattice model, and  $(w, v) \in R_a$ , by construction of  $K(S)$ . By KA, it follows that  $\mathcal{A}_a(w) = \mathcal{A}_a(v)$ , and so  $\{p \in \text{At} : p \in \bigcup_{\varphi \in \mathcal{A}_a(w)} \text{At}(\varphi)\} = Y = \{p' \in \text{At} : p' \in \bigcup_{\varphi' \in \mathcal{A}_a(v)} \text{At}(\varphi')\}$ . Then, by construction of  $\pi_a$  in  $K(S)$ ,  $\pi_a(v_Z) = v_{Z'}$  with  $Z' = Z \cap Y = (X \cap Y) \cap Y = Z$ . Hence,  $\pi_a(v_Z) = v_Z$ , and since by assumption  $v_Z \in I_a(w_Z)$ , then II holds for  $\pi_a$ .

NS: Let  $\pi_a(w_X) = w_Z$ . Then,  $Z = X \cap Y$  with  $Y = \{p \in At : p \in \bigcup_{\varphi \in \mathcal{A}_a(w)} At(\varphi)\}$ , by construction of  $K(S)$ . Consider some  $X' \subseteq X$ . By def. of  $K(S)$ ,  $\pi_a(w_{X'}) = w_{Z''}$  with  $Z'' = X' \cap Y$ . We have two cases: *i*)  $X' \subseteq Z$ ; *ii*)  $Z \subset X'$ . *i*) As  $Z = X \cap Y$ , then by  $X' \subseteq Z$ ,  $X' \subseteq (X \cap Y)$  and so  $X' \subseteq Y$ . Then,  $Z'' = X'$ , and since  $X' \subseteq Z$ , then  $Z'' = X' \cap Z$ . Hence,  $\pi_a(w_{X'}) = w_{X' \cap Z}$  and NS holds for  $\pi_a$ . *ii*) As  $Z \subset X'$ , then  $X' \cap Z = Z$ . So to show that  $\pi_a(w_{X'}) = w_{X' \cap Z}$ , we need to show that  $Z = Z'' = X' \cap Y$ , i.e. 1)  $Z \subset (X' \cap Y)$  and 2)  $(X' \cap Y) \subseteq Z$ . 1) By assumption  $Z \subset X'$ , so  $(X \cap Y) \subset X'$ , and clearly  $(X \cap Y) \subseteq Y$ . Thus,  $(X \cap Y) \subset (X' \cap Y)$ , i.e.  $Z \subset (X' \cap Y)$ ; 2) By assumption,  $X' \subseteq X$ , so  $(X' \cap Y) \subseteq X$ , and clearly  $(X' \cap Y) \subseteq Y$ . Thus,  $(X' \cap Y) \subseteq (X \cap Y)$ , i.e.  $(X' \cap Y) \subseteq Z$ . Hence,  $Z = X' \cap Y$  and  $\pi_a(w_{X'}) = w_{X' \cap Z}$ , so NS holds for  $\pi_a$ .

*Remark 37.* In Prop. 36, the requirement that agents know what they are aware of is necessary to match the Introspective Idempotency property of  $\pi_a$ .

*Remark 38.* Prop. 36 does not require that awareness is generated by primitive propositions, as the  $K$ -transform extracts atomic information from by checking subformulas of  $\mathcal{A}_a$ . Hence  $\mathcal{A}_a$  need not itself contain atoms.

*Remark 39.* Prop. 36 further does not require assuming that the FH model is partitional, reflexive, or transitive. These properties are however clearly preserved by  $K$ -transforms: for any FH model  $S = (W, R, V, \mathcal{A})$ , where, for all  $a \in Ag$ ,  $R_a$  satisfies  $C \subseteq \{\text{partitional, reflexive, or transitive}\}$ , and agents are aware of their own awareness, its  $K$ -transform  $K(S) = (\mathcal{K}(K = (W', R', V'), \leq, \pi)$  is a Kripke lattice model, where  $R'_a$  satisfies  $C$  as well.

## 8.2 From Kripke Lattice Models to FH Models

In the following, we define the  $FH$ -transform, which encodes a Kripke lattice model as an FH model. The core idea is to take the top model of the lattice and augment it with an awareness function. The latter assigns, for each agent, the set of all formulas from  $\mathcal{L}$  that mention any of the atoms appearing in the model of the lattice where the awareness image of the agent resides.

**Definition 40.** Let  $K = (\mathcal{K}(K = (W, R, V)), \leq, \pi)$  be a Kripke lattice model for  $At$ . The  $FH$ -transform of  $K$  is  $FH(K) = (W', R', V', \mathcal{A})$  where  
 $W' = W$ ;  
 $R' = R$ ;  
 $V'(p) = V(p)$  for all  $p \in At$ ;  
 $\mathcal{A}$  is such that, for all  $a \in Ag$ ,  $\mathcal{A}_a \in (2^{\mathcal{L}})^W$  with  $\mathcal{A}_a(w) = \{\varphi \in \mathcal{L} : At(\varphi) \subseteq Y \subseteq At \text{ for the } Y \text{ such that } \pi_a(w) = w_Y\}$ .

We show that the  $FH$ -transform produces a model of the FH class.

**Proposition 41.** For any Kripke lattice model  $K$ , the  $FH$ -transform  $FH(K)$  is an FH model where awareness is generated by primitive propositions.

*Proof.* Let  $\mathbb{K}$  be  $\mathbb{K} = (\mathcal{K}(\mathbb{K} = (W, R, V)), \triangleleft, \pi)$ , where  $\mathbb{K}$  is a Kripke model for  $At$ . Let  $FH(\mathbb{K}) = (W', R', V', \mathcal{A})$  be its  $FH$ -transform. Clearly,  $(W', R', V')$  is a Kripke model for  $At$ , as  $\mathbb{K}$  is so. Moreover,  $\mathcal{A}$  is an awareness function, as for each  $a \in Ag$  and  $w \in W'$ ,  $\mathcal{A}_a(w)$  is a set of formulas from  $\mathcal{L}$ . Lastly, we show that  $\mathcal{A}_a(w)$  is generated by primitive propositions. Let  $\pi_a(w) = w_Y$  for some  $Y \subseteq At$ ,  $a \in Ag$ ,  $w \in W'$  and consider  $\varphi \in \mathcal{L}$ .

( $\Rightarrow$ ) Suppose that  $\varphi \in \mathcal{A}_a(w)$ . Then, by construction of  $\mathcal{A}$ , for all  $p \in At(\varphi)$ ,  $p \in \mathcal{A}_a(w)$ , and since  $At(\varphi) \subseteq At$  then  $p \in (\mathcal{A}_a(w) \cap At)$ .

( $\Leftarrow$ ) Suppose that for all  $p \in At(\varphi)$ ,  $p \in (\mathcal{A}_a(w) \cap At)$ . Then, by construction of  $\mathcal{A}$ ,  $At(\varphi) \subseteq Y \subseteq At$ , where  $Y$  is the unique set such that  $\pi_a(w) = w_Y$ , and so  $\varphi \in \mathcal{A}_a(w)$ . Hence, for all  $a \in Ag$ ,  $\varphi \in \mathcal{A}_a(w)$  iff for all  $p \in At(\varphi)$ ,  $p \in (\mathcal{A}_a(w) \cap At)$ . Thus,  $FH(\mathbb{K})$  is an FH model where awareness is generated by primitive propositions.

*Remark 42.* The requirement that awareness is generated by primitive propositions is needed as the  $FH$ -transform constructs  $\mathcal{A}_a$  by collecting atomic information from  $a$ 's awareness image and then setting  $\mathcal{A}_a$  to be exactly the  $\mathcal{L}$  sublanguage built from these atoms. The resulting awareness notion is thus propositionally generated.

*Remark 43.* As with  $K$ -transforms (cf. Remark 39), also  $FH$ -transforms preserve relation properties: for any Kripke lattice model  $\mathbb{K} = (\mathcal{K}(\mathbb{K} = (W, R, V), \triangleleft, \pi)$  where, for all  $a \in Ag$ ,  $R_a$  satisfies  $C \subseteq \{\text{partitional, reflexive, or transitive}\}$  and awareness is generated by primitive propositions, the  $FH$ -transform  $FH(\mathbb{K}) = (W', R', V', \mathcal{A})$  is an FH model where  $R'_a$  satisfies  $C$  as well.

### 8.3 The $\mathcal{L}$ -Equivalence of FH and Kripke Lattice Models

$K$ - and  $FH$ -transforms not only produce models of the correct class, but also preserve finer details, as any model and its transform satisfy the same  $\mathcal{L}$  formulas.

**Proposition 44.** *For any FH model  $\mathbb{S}$  satisfying  $KA$ , with  $K$ -transform  $K(\mathbb{S})$ , for all  $\varphi \in \mathcal{L}$ , for all  $w \in W$  and for all  $w_X \in \Omega_{K(\mathbb{S})}$  with  $X \supseteq At(\varphi)$ ,*

$$\mathbb{S}, w \models \varphi \text{ iff } K(\mathbb{S}), w_X \Vdash \varphi.$$

*Proof.* Let  $\mathbb{S} = (W, R, V, \mathcal{A})$  be an FH model and let  $K(\mathbb{S}) = (\mathcal{K}(\mathbb{K}), \triangleleft, \pi)$  be its  $K$ -transform, with  $\mathbb{K} = (W', R', V')$ . The proof is by induction on formula complexity. Let  $\varphi \in \mathcal{L}$ ,  $w \in W$  and  $w_X \in \Omega_{K(\mathbb{S})}$ , with  $At(\varphi) \subseteq X$  (clearly at least one such  $w_X$  exists:  $\mathbb{S}$  and  $K(\mathbb{S})$  are defined for the same set of atoms,  $K(\mathbb{S})$  is spawned from a Kripke model  $\mathbb{K}$  that is identical to  $(W, R, V)$ , and there is a model for every  $X' \subseteq At$ ).

*Base:* *i)*  $\varphi := p \in At$  or *ii)*  $\varphi := \top$ . *i)*  $\mathbb{S}, w \models p$  iff (def. of  $\models$ )  $w \in V(p)$  iff (def. of  $K(\mathbb{S})$ )  $w \in V'(p)$  and  $w \in V'_X(p)$  such that  $At(\varphi) \subseteq X$  iff (def. of  $\Vdash$ )  $K(\mathbb{S}), w_X \Vdash \varphi$  with  $At(\varphi) \subseteq X$ . *ii)* is trivial.

*Step.* Assume  $\psi, \chi \in \mathcal{L}$  satisfy Prop. 44.

The cases in which  $\varphi := \neg\psi$  or  $\varphi := \psi \wedge \chi$  follow by tracing *iff*s through the definitions of  $\models$ ,  $V$ ,  $K$ -transform,  $\Vdash$ , and by inductive hypothesis.

$\varphi := K_a\psi$ .  $\mathbf{S}, w \models K_a\psi$  iff (def. of  $\models$ ) (i)  $\psi \in \mathcal{A}_a(w)$  and (ii) for all  $v \in W$  such that  $(w, v) \in R_a$ ,  $\mathbf{S}, v \models \psi$ . By def. of  $K$ -transform and assumption that  $At(\varphi) \subseteq X$ , (i) is the case iff  $At(\psi) \subseteq Y = \{p \in At : p \in \bigcup_{\varphi' \in \mathcal{A}_a(w)} At(\varphi')\}$  and  $\pi_a(w_X) = w_Z$  with  $Z = X \cap Y$ , and by  $At(\psi) \subseteq X$ , then  $At(\psi) \subseteq Z$ . By def. of Kripke lattice model and  $At(\psi) \subseteq Z$ , (ii) is the case iff for all  $v_Z \in W$  such that  $(w_Z, v_Z) \in R_{Za}$ ,  $\mathbf{S}, v_Z \models \psi$ . Hence, by def. of  $\Vdash$  and assumption that  $At(\varphi) \subseteq X$ ,  $K(\mathbf{S}), w_X \Vdash K_a\psi$ .

**Proposition 45.** *For any Kripke lattice model  $\mathbf{K}$  with  $FH$ -transform  $FH(\mathbf{K})$ , for all  $\varphi \in \mathcal{L}$ , for all  $w_X \in \Omega_{K(\mathbf{S})}$  with  $X \supseteq At(\varphi)$ ,*

$$\mathbf{K}, w_X \Vdash \varphi \text{ iff } FH(\mathbf{K}), w \models \varphi.$$

*Proof.* Let  $\mathbf{K} = (\mathcal{K}(K = (W, R, V)), \triangleleft, \pi)$  with  $w_X \in \Omega_{\mathbf{K}}$ ,  $\pi_a(w_X) \in W_Y$  with  $Y \subseteq At$ , and let  $FH(\mathbf{K}) = (W', R', V', \mathcal{A}_a)$ . Let  $\varphi \in \mathcal{L}$ ,  $w_X \in \Omega_{\mathbf{K}}$ , with  $At(\varphi) \subseteq X$ ,  $w \in W$ , and proceed by induction on formula complexity.

*Base:* i)  $\varphi := p \in At$  or ii)  $\varphi := \top$ . i)  $\mathbf{K}, w_X \Vdash p$  iff (def. of  $\Vdash$ )  $w_X \in V_X(p)$  with  $p \in X$  iff (def. of Kripke lattice)  $w \in V(p)$  iff (def. of  $FH$ -transform)  $w \in V'(p)$  iff (def. of  $\models$ )  $FH(\mathbf{K}), w \models p$ . ii) is trivial.

*Step.* Assume  $\psi, \chi \in \mathcal{L}$  satisfy Prop. 45.

The cases in which  $\varphi := \neg\psi$  or  $\varphi := \psi \wedge \chi$  follow by tracing *iff*s through the definitions of  $\models$ ,  $V$ ,  $K$ -transform,  $\Vdash$ , and by inductive hypothesis.

$\varphi := K_a\psi$ . Suppose  $\pi_a(w_X) = w_Y$  for  $Y \subseteq At$ .  $\mathbf{K}, w_X \Vdash K_a\psi$  iff (def. of  $\Vdash$ ) for all  $v_Y \in \Omega_{\mathbf{K}}$ , such that  $w_Y R_{Y_a} v_Y$ ,  $\mathbf{K}, v_Y \Vdash \psi$ . By def. of Kripke lattice,  $\mathbf{K}, v_Y \Vdash \psi$  iff  $At(\psi) \subseteq Y$ .

*Claim.*  $\pi_a(w_X) = w_Y$  with  $At(\psi) \subseteq Y$  iff  $\pi_a(w) = w_Z$  with  $At(\psi) \subseteq Z$ .

We prove the two directions separately.

( $\Rightarrow$ ) Suppose not. Then  $\pi_a(w_X) = w_Y$  with  $At(\psi) \subseteq Y$ , and  $\pi_a(w) = w_Z$  with  $At(\psi) \not\subseteq Z$ . As  $X \subseteq At$ , then by NS  $\pi_a(w_X) = w_{X \cap Z}$ , and  $At(\psi) \not\subseteq X \cap Z$ . But  $\pi_a(w_X) = w_Y$ , so  $Y = X \cap Z$ , and since  $At(\psi) \subseteq Y$ , then  $At(\psi) \subseteq Z$ , which contradicts our initial assumption. Hence,  $\pi_a(w) = w_Z$  with  $At(\psi) \subseteq Z$ .

( $\Leftarrow$ ) Suppose that  $\pi_a(w) = w_Z$  with  $At(\psi) \subseteq Z$ . As  $X \subseteq At$ , then by NS,  $\pi_a(w_X) = w_{X \cap Z}$ . By assumption  $At(\varphi) \subseteq X$ , and so  $At(\varphi) \subseteq (X \cap Z)$ . Hence,  $\pi_a(w_X) = w_Y$  with  $At(\psi) \subseteq Y$ .

So by Claim,  $\pi_a(w_X) = w_Y$  iff  $\pi_a(w) = w_Z$  with  $At(\psi) \subseteq Z$  iff (def. of  $FH$ -transform)  $\psi \in \mathcal{A}_a(w)$ . By def. of Kripke lattice, for all  $v_Y \in \Omega_{\mathbf{K}}$ , such that  $w_Y R_{Y_a} v_Y$ ,  $\mathbf{K}, v_Y \Vdash \psi$  iff for all  $v \in W$  such that  $(w, v) \in R_a$ ,  $\mathbf{K}, v \models \psi$  iff (def. of  $FH$ -transform) for all  $v \in W'$  such that  $(w, v) \in R'_a$ ,  $FH(\mathbf{K}), v \models \psi$  iff (def. of  $\models$  and  $\psi \in \mathcal{A}_a(w)$ )  $FH(\mathbf{K}), w \models K_a\psi$ .

*Remark 46.* Prop. 44 and Prop. 45 provide us with another path to prove soundness and completeness of the HMS logic  $\Lambda_{HMS}$  over the class  $\mathbf{KLM}_{EQ}$  of Kripke lattice models with equivalence relations. Soundness follows by the same proof structure used in the soundness proof of Theorem 31 (this time using Prop.

41 and Prop. 45). Completeness follows by using Halpern and Rêgo [15] completeness results of a logic which we call  $\Lambda_{FH}$  over partial and propositionally determined FH models. The logic  $\Lambda_{FH}$  is based on  $\mathcal{L}$  and an axiom system which Halpern and Rêgo show to be equivalent to that of  $\Lambda_{HMS}$  from Table 1 (see [15] for details). Therefore, as a corollary of this and our transformation results Prop. 36 and Prop. 44, one can show that  $\Lambda_{HMS}$  is complete with respect to  $\mathbf{KLM}_{EQ}$ .

*Remark 47.* These proofs “close the triangle” of Figure 1, as we have shown that partitioned Kripke lattice models, HMS models, and partitioned propositionally determined FH models are all equivalent with respect to language  $\mathcal{L}$ .

## 9 $\mathcal{L}^{KA}$ -Equivalence of FH and Kripke Lattice Models

As we mentioned, the FH model and the awareness function  $\mathcal{A}_a$  were originally designed for the logic  $\Lambda_{LGA}$  based on the language  $\mathcal{L}^{KA}$ , which contains both an implicit knowledge and an awareness operators as primitive, with an explicit knowledge operator definable [9]. Multiple variations of  $\Lambda_{LGA}$  exist in the literature, some including quantification over objects [6], formulas [1, 13, 14], and even unawareness [7], alternative operators informed through cognitive science [2], and dynamic extensions [5, 7, 11, 18].

In this section, we show that Kripke lattice models are equivalent to FH models also with respect to  $\mathcal{L}^{KA}$ . To show this, we present the language and semantics of  $\mathcal{L}^{KA}$  over FH and Kripke lattice models. From this, the  $K$ - and  $FH$ -transformations allow us to show  $\mathcal{L}^{KA}$ -equivalence.

**Definition 48.** *With  $a \in Ag$  and  $p \in At$ , define the language  $\mathcal{L}^{KA}$  by*

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid A_a\varphi$$

*Define  $X_i\varphi := A_a\varphi \wedge K_a\varphi$ .*

*Let  $At(\varphi) = \{p \in At : p \text{ is a subformula of } \varphi\}$ , for all  $\varphi \in \mathcal{L}^{KA}$ .*

### 9.1 FH Models as Semantics for $\mathcal{L}^{KA}$

The semantics for  $\mathcal{L}^{KA}$  over FH models is defined as the semantics for  $\mathcal{L}$  given in Def. 33, except for the knowledge operator  $K_a$ , which now represents *implicit* knowledge, and for the awareness operator  $A_a$ , which is now taken as primitive.

**Definition 49.** *Let  $S = (W, R, V, \mathcal{A})$  be an FH model for  $At$  and let  $w \in W$ . Satisfaction of  $\mathcal{L}^{KA}$  formulas on  $S$  is given by Def. 33 for all formulas except*

$$\begin{aligned} S, w \models K_a\varphi & \text{ iff for all } v \text{ s.t. } (w, v) \in R_a, S, v \models \varphi; \\ S, w \models A_a\varphi & \text{ iff } \varphi \in \mathcal{A}_a(w). \end{aligned}$$

Semantics for explicit knowledge  $X_a$  is then given by the conjunction of the semantics for  $K_a$  and  $A_a$ , with  $a \in Ag$ .

$K_a$  behaves as a classical knowledge operator in epistemic logic, as it captures formulas that are satisfied in the information cell of agent  $a$ . This notion is closed



under implication, whereas explicit knowledge is not necessarily so: an agent  $a$  knows something explicitly only if  $a$  is aware of it, so  $X_a p \wedge ((X_a p \rightarrow X_a q) \wedge \neg X_a q)$  is satisfiable at  $w \in W$  when  $q \notin \mathcal{A}_a(w)$  [9]. However, the kind of FH models considered below are such that awareness is propositionally generated, i.e. they satisfy PP. In this restricted class of models, explicit knowledge is closed under implication as well.

## 9.2 Kripke Lattice Models as Semantics for $\mathcal{L}^{KA}$

As for FH models, also the semantics for  $\mathcal{L}^{KA}$  over Kripke lattice models are defined as the semantics for  $\mathcal{L}$  given in Def. 25, except for  $K_a$  and  $A_a$ .

**Definition 50.** *Let  $\mathbb{K} = (\mathcal{K}(\mathbb{K} = (W, R, V)), \triangleleft, \pi)$  be a Kripke lattice model with  $w_X \in \Omega_{\mathbb{K}}$ . Satisfaction of  $\mathcal{L}^{KA}$  formulas on  $\mathbb{K}$  is given by Def. 33 for all formulas except*

$$\begin{aligned} \mathbb{K}, w_X \Vdash K_a \varphi & \text{ iff for all } v \in W \text{ s.t. } (w, v) \in R_a, \mathbb{K}, v \vDash \varphi \\ \mathbb{K}, w_X \Vdash A_a \varphi & \text{ iff } \pi_a(w_X) = w_Y \quad \text{and } At(\varphi) \subseteq Y \end{aligned}$$

Since the top model in a Kripke lattice model represents the objective perspective, then implicit knowledge  $K_a$  is defined as the information cell of agent  $a$  in that model. The awareness operator semantics gives rise to a propositionally generated awareness notion, as it states that agent  $a$  is aware of all the formulas that mention any of the atoms belonging to the vocabulary that describes  $a$ 's awareness image.

## 9.3 The $\mathcal{L}^{KA}$ -Equivalence of FH and Kripke Lattice Models

To show the equivalence of FH and Kripke lattice models with respect to  $\mathcal{L}^{KA}$ , the definition of  $K$ - and  $FH$ -transforms must be adapted to the language  $\mathcal{L}^{KA}$ , by replacing  $\mathcal{L}$  with  $\mathcal{L}^{KA}$  in Definitions 34 and 40. The results showing that the transformed models are of the proper classes are straightforward given the proofs of Section 8, and are therefore omitted to the effect that we only state the results showing that  $K$ - and  $FH$ -transforms not only produce models of the correct class, but also preserve finer details, as any model and its transform satisfy the same  $\mathcal{L}^{KA}$  formulas.

**Proposition 51.** *For any FH model  $\mathbb{S}$  satisfying  $KA$ , with  $K$ -transform  $K(\mathbb{S})$ , for all  $\varphi \in \mathcal{L}^{KA}$ , for all  $w \in W$  and for all  $w_X \in \Omega_{K(\mathbb{S})}$  with  $X \supseteq At(\varphi)$ ,*

$$\mathbb{S}, w \vDash \varphi \text{ iff } K(\mathbb{S}), w_X \Vdash \varphi.$$

*Proof.* Let  $\mathbb{S} = (W, R, V, \mathcal{A})$  be an FH model and let  $K(\mathbb{S}) = (\mathcal{K}(\mathbb{K}), \triangleleft, \pi)$  be its  $K$ -transform, with  $\mathbb{K} = (W', R', V')$ . The proof is by induction on formula complexity. Let  $\varphi \in \mathcal{L}^{KA}$ ,  $w \in W$  and  $w_X \in \Omega_{K(\mathbb{S})}$ .

*Base:* *i)*  $\varphi := p \in At$  or *ii)*  $\varphi := \top$ . *i)*  $\mathbb{S}, w \vDash p$  iff (def. of  $\vDash$ )  $w \in V(p)$  iff (def. of  $K(\mathbb{S})$ )  $w \in V'(p)$  and  $w \in V'_X(p)$  such that  $At(\varphi) \subseteq X$  iff (def. of  $\Vdash$ )  $K(\mathbb{S}), w_X \Vdash \varphi$  with  $At(\varphi) \subseteq X$ . *ii)* is trivial.

*Step.* Assume  $\psi, \chi \in \mathcal{L}$  satisfy Prop. 51.

The cases in which  $\varphi := \neg\psi$  or  $\varphi := \psi \wedge \chi$  follow by tracing *iffs* through the definitions of  $\vDash, V, K$ -transform,  $\Vdash$ , and by inductive hypothesis.

$\varphi := K_a\psi$ .  $\mathbf{S}, w \vDash K_a\psi$  iff (def. of  $\vDash$ ) for all  $v \in W$  such that  $(w, v) \in R_a$ ,  $\mathbf{S}, v \vDash \psi$  iff (def. of  $K(\mathbf{S})$ ) for all  $v \in W'$  such that  $(w, v) \in R'_a$ ,  $K(\mathbf{S}), v \Vdash \psi$  iff (def. of Kripke lattice) for all  $v_X \in W'_X$  such that  $(w_X, v_X) \in R'_{X_a}$  and  $At(\psi) \subseteq X$ ,  $K(\mathbf{S}), v_X \Vdash \psi$  iff (def. of  $\Vdash$ )  $K(\mathbf{S}), w_X \Vdash \psi$ .

$\varphi := A_a\psi$ .  $\mathbf{S}, w \vDash A_a\psi$  iff (def. of  $\vDash$ )  $\psi \in \mathcal{A}_a(w)$  iff (def. of  $K(\mathbf{S})$ )  $\pi_a(w_X) = w_Z$  with  $Z = X \cap Y$  and  $Y = \{p \in At : p \in \bigcup_{\varphi \in \mathcal{A}_a(w)} At(\varphi)\}$  iff (assumption  $At(\varphi) \subseteq X$ )  $\pi_a(w_X) = w_Z$  and  $At(\psi) \subseteq Z$  iff (def. of  $\Vdash$ )  $K(\mathbf{S}), w_X \Vdash A_a\psi$ .

**Proposition 52.** *For any Kripke lattice model  $\mathbf{K}$  with FH-transform  $FH(\mathbf{K})$ , for all  $\varphi \in \mathcal{L}^{KA}$ , for all  $w_X \in \Omega_{\mathbf{K}}$  with  $X \supseteq At(\varphi)$ ,*

$$\mathbf{K}, w_X \Vdash \varphi \text{ iff } FH(\mathbf{K}), w \vDash \varphi.$$

*Proof.* Let  $\mathbf{K} = (\mathcal{K}(K = (W, R, V)), \leq, \pi)$  with  $w_X \in \Omega_{\mathbf{K}}$ ,  $\pi_a(w_X) \in W_Y$  with  $Y \subseteq At$ , and let  $FH(\mathbf{K}) = (W', R', V', \mathcal{A}_a)$ . Let  $\varphi \in \mathcal{L}^{KA}$ ,  $w_X \in \Omega_{\mathbf{K}}$ , with  $At(\varphi) \subseteq X$ ,  $w \in W$ , and proceed by induction on formula complexity.

*Base:* *i)*  $\varphi := p \in At$  or *ii)*  $\varphi := \top$ . *i)*  $\mathbf{K}, w_X \Vdash p$  iff (def. of  $\Vdash$ )  $w_X \in V_X(p)$  with  $p \in X$  iff (def. of Kripke lattice)  $w \in V(p)$  iff (def. of FH-transform)  $w \in V'(p)$  iff (def. of  $\vDash$ )  $FH(\mathbf{K}), w \vDash p$ . *ii)* is trivial.

*Step.* Assume  $\psi, \chi \in \mathcal{L}$  satisfy Prop. 52.

The cases in which  $\varphi := \neg\psi$  or  $\varphi := \psi \wedge \chi$  follow by tracing *iffs* through the definitions of  $\vDash, V, K$ -transform,  $\Vdash$ , and by inductive hypothesis.

$\varphi := K_a\psi$ .  $\mathbf{K}, w_X \Vdash K_a\psi$  iff (def. of  $\Vdash$ ) for all  $v \in W$  such that  $(w, v) \in R_a$ ,  $\mathbf{K}, v \vDash \psi$  iff (def. of FH-transform) for all  $v \in W'$  such that  $(w, v) \in R'_a$ ,  $FH(\mathbf{K}), v \vDash \psi$  iff (def. of  $\vDash$ )  $FH(\mathbf{K}), w \vDash K_a\psi$ .

$\varphi := A_a\psi$ .  $\mathbf{K}, w_X \Vdash A_a\psi$  iff (def. of  $\Vdash$ )  $\pi_a(w_X) = w_Y$  and  $At(\psi) \subseteq Y$  iff (Claim in Prop. 45) iff  $\pi_a(w) = w_Z$  with  $At(\psi) \subseteq Z$  iff (def. of FH-transform)  $At(\psi) \subseteq \mathcal{A}_a(w)$  and  $\psi \in \mathcal{A}_a(w)$  iff (def. of  $\vDash$ )  $FH(\mathbf{K}), w \vDash A_a\psi$ .

## 10 The Logic of General Awareness of Kripke Lattice Models

The Logic of General Awareness ( $\Lambda_{LGA}$ ) is built on the language  $\mathcal{L}^{KA}$  and an axiom system for implicit knowledge, awareness and explicit knowledge which is presented in Table 2. Using the  $\mathcal{L}^{KA}$ -equivalence results from Section 9.3, and the transformations results provided by Prop. 36 and Prop. 41, we show that the class of Kripke lattice models **KLM** is sound and complete with respect to  $\Lambda_{LGA}$ .

**Definition 53.** *The logic  $\Lambda_{LGA}$  is the smallest set of  $\mathcal{L}^{KA}$  formulas that contains the axioms in, and is closed under the inference rules of, Table 2.*

All substitution instances of propositional logic, including the formula $\top$	
$(K_a\varphi \wedge (K_a\varphi \rightarrow K_a\psi)) \rightarrow K_a\psi$	(K, Distribution)
$X_a\varphi \leftrightarrow (K_a\varphi \wedge A_a\varphi)$	(Explicit Knowledge)
$A_a(\varphi \wedge \psi) \leftrightarrow (A_a\varphi \wedge A_a\psi)$	(A1, Awareness Distribution)
$A_a\neg\varphi \leftrightarrow A_a\varphi$	(A2, Symmetry)
$A_aX_b\varphi \leftrightarrow A_a\varphi$	(A3, Awareness of Explicit Knowledge)
$A_aA_b\varphi \leftrightarrow A_a\varphi$	(A4, Awareness Reflection)
$A_aK_b\varphi \leftrightarrow A_a\varphi$	(A5, Awareness of Implicit Knowledge)
$A_a\varphi \rightarrow K_aA_a\varphi$	(A11, Awareness Introspection)
$\neg A_a\varphi \rightarrow K_a\neg A_a\varphi$	(A12, Unawareness Introspection)
From $\varphi$ and $\varphi \rightarrow \psi$ , infer $\psi$	(Modus Ponens)
From $\varphi$ infer $K_a\varphi$	(K-Inference)

**Table 2.** Axioms and inference rules of  $\Lambda_{LGA}$ , for a propositionally determined notion of awareness.

The axiom system of Table 2 is sound and complete with respect to propositionally determined FH models, i.e. FH models that satisfy PP and KA. In particular, A1-A5 capture an awareness notion that is generated by primitive propositions, while A11-A12 are required if agents are to know what they are aware of [10, 12] (the numbering of the awareness axioms is taken from [12]). These two properties are needed to establish the transformations results of Prop. 36 and Prop. 41, and therefore in the later soundness and completeness proofs.

Let  $\mathcal{S}$  be the class of propositionally determined FH models. FH [9, 10] argue that:<sup>9</sup>

**Theorem 54 ([9, 10]).**  $\Lambda_{LGA}$  is sound and complete with respect to  $\mathcal{S}$ .

Let  $\mathcal{KLM}$  be the class of all Kripke lattice models, i.e., without special properties assumed of the accessibility relations. As a corollary to Theorem 54, our transformation, and  $\mathcal{L}^{KA}$ -equivalence results, we obtain

**Theorem 55.**  $\Lambda_{LGA}$  is sound and complete with respect to  $\mathcal{KLM}$ .

*Proof.* For both soundness and completeness, the reasoning is analogous to that provided in 31. Soundness uses Prop. 41 adapted to language  $\mathcal{L}^{KA}$  and Prop. 51. Completeness uses Prop. 36 adapted to language  $\mathcal{L}^{KA}$ , Prop. 51, and the existence result of the canonical model construction assumed as existing by [10].

<sup>9</sup> We say that these works *argue* for soundness and completeness of  $\Lambda_{LGA}$  with respect to FH models, where  $\Lambda_{LGA}$  is based on  $\mathcal{L}^{KA}$  which is a language for *knowledge* (not belief) and awareness, as they do not explicitly provide the proof. They only state that it is straightforward to provide. The relevant results argued for in the literature are:

1. In [9]: soundness and completeness for KD45+Explicit Knowledge with respect to FH models. It does not specify any such proof about FH models with the PP and KA properties.
2. In [12]: soundness and completeness for the single agent version of  $\Lambda_{LGA}$  with respect to FH models (also models satisfying PP and KA) is claimed a straightforward generalization of the soundness and completeness proof for the logic formed on language  $\mathcal{L}$  and the  $K$  axiom.
3. In [15]: says that soundness and completeness of  $\Lambda_{LGA}$  with respect to FH models with PP and KA is given by FH. Supposedly, they refer to [9], where such construction is not provided—see point 1 in this list.

*Remark 56.* The same result can clearly be obtained for the logic generated by the axioms in Table 2 and the axiom system S5.

## 11 Concluding Remarks

This paper has introduced Kripke lattice models as a model class for epistemic logics with awareness. This model is a Kripke model-based rendition of the syntax-free HMS model of awareness, and we have shown that the two model classes are equally general with respect to  $\mathcal{L}$ , by defining transformations between the two that preserve formula satisfaction. A corollary to this result is completeness of the HMS logic for the introduced model class. Moreover, we have shown that Kripke lattice models and the syntax-based FH models of awareness are equally general with respect to  $\mathcal{L}$ , as well as with respect to the language  $\mathcal{L}^{KA}$ . As a corollary, we obtain that the Logic of General Awareness is complete with respect to the introduced model class.

There are several issues we would like to study in future work:

In recasting HMS models as a Kripke lattice models, we teased apart the epistemic and awareness dimensions merged in the HMS possibility correspondences, and Propositions 18, 19 and 21 about  $L$ - and  $H$ -transforms show that the HMS properties are satisfied iff each  $\pi_a$  satisfies D, II and NS, and each  $R_a$  is an equivalence relation. For a more fine-grained property correspondence, the propositions' proofs show that each property of one model is entailed by a strict subset of the properties of the other. In some cases, the picture emerging is fairly clear: e.g., HMS' Conf is shown only using the restrictions lattice construction (RLC) plus D and *vice versa*; PPK uses only NS and RLC, while PPK and Conf entail NS. In other cases, the picture is more murky, e.g., when we use Stat, PPI and PPK to show the seemingly simple symmetry of  $R_a$ . We think it would be interesting to decompose properties on both sides to see if clearer relationships arise.

There are two issues with redundant states in Kripke lattice models. One concerns redundant restrictions, cf. Remark 12, which may be solved by working with a more general model class, where models may also be based on sub-orders of the restriction lattice. A second one concerns redundant states. For example, in Figure 3,  $K_\emptyset$  contains three 'identical' states where no atoms have defined truth values— $K_\emptyset$  is bisimilar to a one-state Kripke model. As bisimulation contracting each  $K_X$  may collapse states from which awareness maps differ, one must define a notion of bisimulation that takes awareness maps into consideration (notions of bisimulation for other awareness models exists, e.g. [8]). Together with a more general model class definition, this could hopefully solve the redundancy issues.

Kripke lattice models are  $\mathcal{L}^{KA}$ -equivalent to FH models, but it is an open issue how HMS models relate to both Kripke lattices and FH models with respect to that language, cf. the question marks in Figure 1 in the introduction. As  $\mathcal{L}^{KA}$  contains an implicit knowledge operator, but HMS models contain no objective perspective, studying that relation would seemingly mainly entail exploring how to capture the objective perspective in HMS models. It is an open question if

and how HMS may serve as a semantics for  $\mathcal{L}^{KA}$  in a manner that will entail  $\mathcal{L}^{KA}$ -equivalence with FH models and Kripke lattices.

The HMS logic is complete for HMS models and for Kripke lattice models with equivalence relations. [15] prove completeness for HMS models using a standard validity notion, a ‘ $\varphi$  is at least as expressive as  $\psi$ ’ operator and variants of axioms  $T$ , 4 and 5. We are very interested in considering this system and its weaker variants for Kripke lattice models, also with less assumptions on the relations.

Finally, issues of dynamics spring forth: first, whether existing awareness dynamics may be understood on Kripke lattice models; second, whether DEL action models may be applied lattice-wide with reasonable results, and how they compare with other action models for awareness in the literature [5, 7, 8, 18]; and third, whether the  $\pi_a$  maps may be thought in dynamic terms, as they map between models.

**Acknowledgments.** We thank the organizers of the 3rd DaLi Workshop for the opportunity to present our work there, and the participants and reviewers of the conference for their useful and productive comments. The Center for Information and Bubble Studies is funded by the Carlsberg Foundation. RKR was partially supported by the DFG-ANR joint project *Collective Attitude Formation* [RO 4548/8-1].

## References

1. Ågotnes, T., Alechina, N.: A Logic for reasoning about knowledge of unawareness. *Journal of Logic, Language and Information* **23**(2), 197–217 (2014)
2. A.Pietarinen: Awareness in Logic and Cognitive Neuroscience. In: Proceedings of IEEE International Conference on Cognitive Informatics. pp. 155–162 (2002)
3. Belardinelli, G., Rendsvig, R.K.: Awareness Logic: A Kripke-based Rendition of the Heifetz-Meier-Schipper. In: Martins, M.A., Sedlár, I. (eds.) *Dynamic Logic. New Trends and Applications (DaLi 2020)*. pp. 33–50. Springer (2020)
4. van Benthem, J., Gerbrandy, J., Hoshi, T., Pacuit, E.: Merging Frameworks for Interaction. *Journal of Philosophical Logic* **38**(5), 491–526 (2009)
5. van Benthem, J., Velázquez-Quesada, F.R.: The dynamics of awareness. *Synthese* **177**, 5–27 (2010)
6. Board, O., Chung, K.S.: Object-Based Unawareness. In: G. Bonanno, W. van der Hoek, M.W. (ed.) *Proceedings of LOFT 7*. pp. 35–41 (2006)
7. van Ditmarsch, H., French, T.: Semantics for Knowledge and Change of Awareness. *Journal of Logic, Language and Information* **23**(2), 169–195 (2014)
8. van Ditmarsch, H., French, T., Velázquez-Quesada, F.R., Wang, Y.N.: Knowledge, Awareness, and Bisimulation. In: *TARK 2013 - Proceedings of the 14th Conference on Theoretical Aspects of Rationality and Knowledge*. vol. 1, pp. 61–70 (2013)
9. Fagin, R., Halpern, J.Y.: Belief, Awareness, and Limited Reasoning. *Artificial Intelligence* **34**, 39–76 (1988)
10. Fagin, R., Halpern, J.Y., Moses, Y., Vardi, M.Y.: *Reasoning about Knowledge*. MIT press (1995)

11. Grossi, D., Velázquez-Quesada, F.R.: Syntactic awareness in logical dynamics. *Synthese* **192**(12), 4071–4105 (2015)
12. Halpern, J.Y.: Alternative Semantics for Unawareness. *Games and Economic Behavior* **37**(2), 321–339 (2001)
13. Halpern, J.Y., Rêgo, L.C.: Reasoning About Knowledge of Unawareness. *Games and Economic Behavior* **67**(2), 503–525 (2009)
14. Halpern, J.Y., Rêgo, L.C.: Reasoning about knowledge of unawareness revisited. *Mathematical Social Sciences* **65**(2), 73–84 (2013)
15. Halpern, J.Y., Rêgo, L.C.: Interactive unawareness revisited. *Games and Economic Behavior* **62**(1), 232–262 (2008)
16. Heifetz, A., Meier, M., Schipper, B.: A canonical model for interactive unawareness. *Games and Economic Behavior* (62), 304–324 (2008)
17. Heifetz, A., Meier, M., Schipper, B.C.: Interactive unawareness. *Journal of Economic Theory* **130**(1), 78–94 (2006)
18. Hill, B.: Awareness Dynamics. *Journal of Philosophical Logic* **39**(2), 113–137 (2010)
19. van Lee, H.S., Rendsvig, R.K., van Wijk, S.: Intensional Protocols for Dynamic Epistemic Logic. *Journal of Philosophical Logic* **48**, 1077–1118 (2019)
20. Modica, S., Rustichini, A.: Awareness and partitional information structures. *Theory and Decision* **37**(1), 107–124 (1994)
21. Modica, S., Rustichini, A.: Unawareness and Partitional Information Structures. *Games and Economic Behavior* **27**(2), 265–298 (1999)
22. Schipper, B.C.: Awareness. In: van Ditmarsch, H., Halpern, J.Y., van der Hoek, W., Kooi, B.P. (eds.) *Handbook of Epistemic Logic*. College Publications (2014)