CHARACTERIZING INTERMEDIATE TENSE LOGICS IN TERMS OF GALOIS CONNECTIONS

WOJCIECH DZIK, JOUNI JÄRVINEN, AND MICHIRO KONDO

Abstract. We propose a uniform way of defining for every logic L intermediate between intuitionistic and classical logics, the corresponding intermediate tense logic LK_t. This is done by building the fusion of two copies of intermediate logic with a Galois connection LGC, and then interlinking their operators by two Fischer Servi axioms. The resulting system is called $\mathsf{L2GC{+}FS}$. In the cases of intuitionistic logic Int and classical logic Cl, it is noted that $\mathsf{Int2GC{+}FS}$ is syntactically equivalent to intuitionistic tense logic IK_t by W. B. Ewald and $\mathsf{Cl2GC{+}FS}$ equals classical tense logic K_t. This justifies calling $\mathsf{L2GC{+}FS}$ the L-tense logic LK_t for any intermediate logic L. We define H2GC+FS-algebras as expansions of HK1-algebras, introduced by E. Orłowska and I. Rewitzky. For each intermediate logic L, we show algebraic completeness of $\mathsf{L2GC{+}FS}$ and its conservativeness over L. We prove relational completeness of $\mathsf{Int2GC{+}FS}$ with respect to the models defined on IK-frames introduced by G. Fischer Servi. We also prove a representation theorem stating that every H2GC+FS-algebra can be embedded into the complex algebra of its canonical IK-frame.

1. Introduction

In this paper, we consider the following method of introducing unary operators to intuitionistic propositional logic:

(A) Building the fusion $\mathsf{IntGC}{\otimes}\mathsf{IntGC}$ of two copies of intuitionistic logic with a Galois connection IntGC, the first one with a Galois connection $(\Diamond, \boxdot)$ and the second one with $(\diamonddot, \Box)$, and adding Fischer Servi axioms to connect $(\Diamond, \Box)$ and $(\diamonddot, \boxdot)$.

Another method of introducing unary operators leading to intuitionistic tense logic was investigated by J. M. Davoren [13]:

(B) Building the fusion $\mathsf{IK}{\otimes}\mathsf{IK}$ of two copies of intuitionistic modal logic IK, the first one with modalities $(\Diamond, \Box)$ and the second one with $(\diamonddot, \boxdot)$, and adding Brouwerian axioms to connect $(\Diamond, \boxdot)$ and $(\diamonddot, \Box)$.

These two methods are shown here to be equivalent and the result is called $\mathsf{Int2GC{+}FS}$, according to (A). This name should be understood as "intuitionistic logic with two Galois connections combined using Fischer Servi axioms".

Note that for combinations of modal logics, we follow the notation of [13]. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are axiomatically presented modal logics in languages Λ_1 and Λ_2, respectively, then the fusion $\mathcal{L}_1{\otimes}\mathcal{L}_2$ is the smallest multi-modal logic in the language $\Lambda_1{\otimes}\Lambda_2$ containing $\mathcal{L}_1$ and $\mathcal{L}_2$, and closed under all the inference rules of $\mathcal{L}_1$ and $\mathcal{L}_2$, where $\Lambda_1{\otimes}\Lambda_2$ denotes the smallest common extension of the languages Λ_1 and Λ_2. If $\mathcal{L}$ is a logic in language Λ, and Γ is a finite set of schemes in Λ, then the extension $\mathcal{L} \oplus \Gamma$ is the smallest logic in Λ extending $\mathcal{L}$, containing the schemes in Γ as additional axioms, and closed under the rules of $\mathcal{L}$.

If $\Diamond$, $\Box$ and $\boxdot$-diamond, $\boxdot$ are identified with tense operators F, G (future) and P, H (past), respectively, the system $\mathsf{Int2GC{+}FS}$ is equivalent to the known system IK_t, called *intuitionistic tense logic*, introduced by W. B. Ewald [20].[1] The logic IK_t is generally taken as the intuitionistic counterpart of the *classical tense logic* K_t (see [13, 41], for instance) and we will neither discuss this fact here nor consider the philosophical issues raised by IK_t (for instance, its constructivity). We would also like to emphasize that this is not a matter of providing another list of axioms for IK_t that is much shorter than the Ewald's list of axioms. Note that the logic K_t is often in the literature called the *minimal tense logic*. Since we consider only the minimal tense (classical, intuitionistic, intermediate) logics, we will omit the word "minimal" in the rest of the paper. Methods (A) and (B) are visualized in Figure 1.

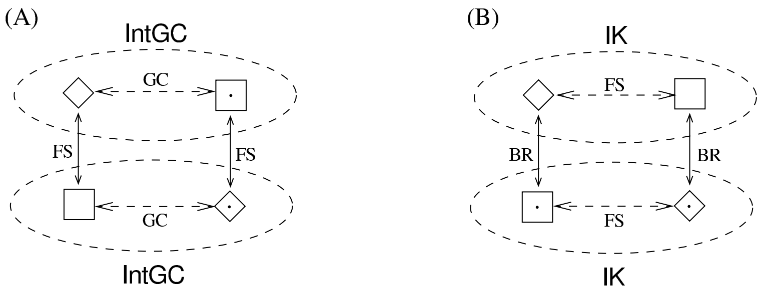

FIGURE 1. Two methods of building IK_t. Here FS stands for the Fischer Servi axioms, GC for Galois connections, and BR for the Browerian axioms.

The above equivalences also hold if one changes the basic logic from intuitionistic to classical, in which case one gets classical tense logic K_t. Adopting approach (A) from intuitionistic and classical logics to any intermediate logic L, we present a method to obtain the corresponding logic $\mathsf{L2GC{+}FS}$. This is done simply by adding to L two Galois connections (by means of the appropriate rules), or by building the fusion $\mathsf{LGC}{\otimes}\mathsf{LGC}$ of two copies of intermediate logic with a Galois connection LGC, and then interlinking their operators by two Fischer Servi axioms. We prove algebraic completeness for $\mathsf{L2GC{+}FS}$ and show that it is conservative over L. We also give facts justifying why $\mathsf{L2GC{+}FS}$ can be considered as an *intermediate* L*-tense logic* LK_t.

There are several advantages of approach (A) over (B). The L-tense logic LK_t (or equivalently $\mathsf{L2GC{+}FS}$) can be uniformly built for every intermediate logic L, without entering the problem of what is the modal version LK of L, since the "modal part" is provided solely by the Galois connections, and the Fischer Servi axioms make a duality-like connection between the operators. For a given intermediate logic L, it is often not clear what is its modal analogue LK (between IK and K). For instance, it took several years to find out, what is Gödel modal logic. In [8], strong completeness of the $\Box$-version

[1]This equivalence was proved already in [17].

and the $\Diamond$-version of Gödel modal logic were proved. Recent studies [9, 10] show that there are several Gödel modal logics of two modalities which are defined by a Kripke frame semantics. In particular, Gödel modal logics are different for "crisp" frames and for "fuzzy" frames. Moreover, approach (A) allows a uniform treatment of algebraic semantics.

Galois connections play a central role both in (A) and (B) – and in the whole paper, hence we recall some well-known properties of order-preserving Galois connections used here. They can be found in [19], for instance. Let $\varphi\colon P \to Q$ and $\psi\colon Q \to P$ be maps between ordered sets P and Q. The pair (φ, ψ) is a *Galois connection* between P and Q, if for all $p \in P$ and $q \in Q$,

$$\varphi(p) \leq q \iff p \leq \psi(q).$$

An equivalent characterisation states that a pair (φ, ψ) forms a Galois connection between P and Q if and only if

(1.1) $p \leq \psi(\varphi(p))$ for all $p \in P$ and $\varphi(\psi(q)) \leq q$ for all $q \in Q$;

(1.2) the maps φ and ψ are order-preserving.

It is well known that Galois connections can be created by any relational frame (U, R) by reversing the relation R. The operators $\Diamond$ and $\Box$ defined for all $X \subseteq U$ by $\Diamond X = \{x \in U \mid (\exists y \in U)\, x\, R\, y \;\&\; y \in X\}$ and $\Box X = \{x \in U \mid (\forall y \in U)\, x\, R\, y \Rightarrow y \in X\}$ are both part of a Galois connection. The Galois connections in question on the powerset lattice $\wp(U)$ are then $(\Diamond, \boxdot)$ and $(\diamonddot, \Box)$, where the operators $X \mapsto \diamonddot X$ and $X \mapsto \boxdot X$ are defined by inverting the relation R. However, the idea of extending propositional calculus with a Galois connection as modalities appears to be rather new, and mainly motivated by applications in computer science. There is a growing interest in the study of Galois connections as modalities, as can be seen in the recent surveys by M. Menni and C. Smith [33] and García-Pardo *et al.* [24]. The study of Galois connections can be traced back to the initial works of O. Ore [34] and B. Jónsson and A. Tarski [30]. More recent studies of Galois connections as modal operators in complete lattices can be found, for instance, in [31], where B. von Karger developed several temporal logics from the theory of complete lattices, Galois connections, and fixed points, and in [28], where Galois connections, conjugate functions, and their fixed points are considered in complete Boolean lattices.

In "syntactical side", Galois connections can be subsumed into a logic only either by including Galois connection rules (see page 7) or by introducing Browerian axioms (see page 9). However, in "semantical side", the situation is different in the sense that, for instance, for a complete lattice $(L, \leq)$, a mapping $f\colon L \to L$ is known to be a part of a Galois connection if and only if f is a complete join-morphism, that is, $f(\bigvee S) = \bigvee f(S)$ for all $S \subseteq L$. In such a case, the "other part" g is defined by $g(a) := \max\{a \in L \mid f(a) \leq b\}$. This means, for example, that in a finite lattice $(L, \vee, \wedge)$, every additive and normal map $L \to L$ induces a Galois connection. In relational settings, Galois connections are essentially related to inverting a relation; if a possibility-like operator is defined in terms of a relation (or a composition of relations) by an "exists"-condition, then its adjoint operator is defined simply by a "for all"-condition and the inverse the original relation (or the inverse of the composition of relations). Similar kind of situation can be observed for "categorical functors", and a functor is known to have a left adjoint if and only if it is continuous and a certain "smallness condition" is satisfied. Note that every partially ordered set can be viewed as a category in a natural way: there is a unique morphism from x to y if and only if $x \leq y$. Thus, an order-preserving Galois connection is a pair of adjoint functors between two categories that arise from partially ordered sets.

In the literature can be found several papers that consider modalities as adjoint pairs. Topos-theoretic approaches to modality are presented by G. E. Reyes and H. Zolfaghari in [39], with adjoint pair $(\Diamond, \Box)$, and $\mathsf{S4}$-like axioms satisfied by $\Box$ and $\Diamond$ separately. In [38], G. E. Reyes and M. W. Zawadowski developed this theory further in the context of locales, giving axiomatisation, completeness and decidability of modal logics arising in this context. More recently, M. Sadrzadeh and R. Dyckhoff studied in [40] positive logic whose nonmodal part has conjunction and disjunction as connectives, and whose modalities come in adjoint pairs.

In categorical models, propositions are interpreted as the objects of a category and proofs as morphisms. For instance, P. N. Benton considers in [3] so-called LNL-models, which are categorical models for intuitionistic linear logic as defined by Girard. Benton studies also rules for LNL which are similar to our Galois connection rules. In [4], G.M. Bierman and V. de Paiva consider an intuitionistic variant $\mathsf{IS4}$ of the modal logic $\mathsf{S4}$ and its models in the framework of category theory. Alechina *et al.* study in [1] two systems of constructive modal logic which are computationally motivated. These logics are "Constructive S4" and "Propositional Lax Logic". They prove duality results which show how to relate Kripke models to algebraic models, and these in turn to the appropriate categorical models. Our work is based on algebraic and Kripke semantics, and since we consider minimal intermediate tense logics, we do not assume additional modal axioms. Hence, we do not follow the categorical proposal of modelling constructive $\mathsf{S4}$-modalities that uses the additional axioms T and $\mathsf{4}$. The difference between systems applying constructive $\mathsf{S4}$-modalities and ours is similar to the difference between classical tense logic and classical $\mathsf{S4}$.

This paper continues our study of Galois connections in intuitionistic logic. In [15], we introduced intuitionistic propositional logic with a Galois connection $(\Diamond, \boxdot)$, called IntGC. We showed that $\Diamond$ and $\boxdot$ are *modal operators* in the sense that $\Diamond$ distributes over $\vee$ (that is, is *additive*) and preserves $\bot$ (that is, is *normal*) and $\boxdot$ distributes over $\wedge$ (i.e., is *multiplicative*) and preserves $\top$ (i.e., is *co-normal*). We gave both algebraic and relational semantics, and showed that IntGC is complete with respect to both of these semantics. We noted that IntGC is conservative over intuitionistic logic and that Glivenko's Theorem does not hold between propositional logic with a Galois connection [29] and IntGC. In addition, in [16] we proved that IntGC has the finite model property, which enabled us to state that a formula of IntGC is provable if and only if it is valid in any finite distributive lattice with an additive and normal operator, or equivalently, the formula is valid in any finite distributive lattice with a multiplicative and co-normal operator. With respect to relational semantics, this is equivalent to the validity in any finite relational models for IntGC. We also presented how IntGC is motivated by generalised fuzzy sets. In [18], we gave representations of expansions of bounded distributive lattices equipped with a Galois connection. We studied in [17] two Galois connections in intuitionistic logic and then with Fischer Servi axioms added, their algebraic and relational semantics. We announced there some results that are presented here. In a similar way, in this work we extend IntGC with a Galois connection $(\Diamond, \boxdot)$ by adding another Galois connection pair $(\diamonddot, \Box)$. Just adding another Galois connection does not change much, we have IntGC "doubled", called here Int2GC. Note that Int2GC is the same as the fusion IntGC$\otimes$IntGC. One of the motivating questions of this paper is: What axioms connecting two independent Galois connections $(\Diamond, \boxdot)$ and $(\diamonddot, \Box)$ should be added to obtain intuitionistic (or intermediate) tense logic?

In classical logic, the operators $\Diamond$ and $\Box$ may be defined as a shorthand of each other by using the following *De Morgan definitions*:

$$\Box A := \neg \Diamond \neg A \quad \text{and} \quad \Diamond A := \neg \Box \neg A. \tag{1.3}$$

Classical tense logic K_t can be obtained by adding to classical logic two Galois connections $(\Diamond, \boxdot)$ and $(\diamonddot, \Box)$, and then connecting them by the following *De Morgan axioms*:

$$\Box A \leftrightarrow \neg \Diamond \neg A \quad \text{and} \quad \boxdot A \leftrightarrow \neg \diamonddot \neg A, \tag{1.4}$$

or

$$\Diamond A \leftrightarrow \neg \Box \neg A \quad \text{and} \quad \diamonddot A \leftrightarrow \neg \boxdot \neg A \tag{1.5}$$

Note that in the case of classical logic, the formulas in (1.4) are equivalent to the ones in (1.5). In a more concise way, K_t may be determined by adding to classical logic one Galois connection $(\Diamond, \boxdot)$, and then defining the second one $(\diamonddot, \Box)$ in terms of (1.3), that is, by setting $\diamonddot A := \neg \boxdot \neg A$ and $\Box A := \neg \Diamond \neg A$. This approach is present in [42, Proposition 8.5(iii)] and also, in another, independent way, in [29].

However, if one changes the base logic from classical to intuitionistic, or algebraically from Boolean to Heyting algebras, these kinds of ways cannot be used, because they lead to serious faults and fallacies. In particular, having a Galois connection $(\Diamond, \boxdot)$, if one defines the operators $\Box$ and $\diamonddot$ by using (1.3) with intuitionistic negation, the resulting pair $(\diamonddot, \Box)$ does not form a Galois connection; see [15]. In another similar approach [12] (without using the term Galois connection, but providing the equivalent axiomatisation), the assertions (1.3) are used to define "possibility-like" tense operators F, P, over intuitionistic logic, from "necessity-like" tense operators G, H. It is claimed in [12] that the resulting logic is intuitionistic tense logic and that the "possibility-like" tense operators F, P are "existential quantifiers" (see [12, Remark 8]) meaning, in particular, that F, P preserve disjunctions (that is, lattice-joins). Showing that this is not true is the topic of [21]. Note also that in Example 3.6(c) we show that in Int2GC+FS = IK_t, formulas (1.4) and (1.5) are not provable.

By the above, it is clear that De Morgan axioms (1.4) and (1.5) are not appropriate for connecting modalities over intuitionistic logic due to the properties of intuitionistic negation. Our answer to the above question on intuitionistic logic level is to link the operators $\Diamond$, $\Box$ and $\diamonddot$, $\boxdot$, respectively, by using the "connecting axioms"

$$\Diamond(A \to B) \to (\Box A \to \Diamond B) \quad \text{and} \quad (\Diamond A \to \Box B) \to \Box(A \to B)$$

introduced by G. Fischer Servi in [23]. Note that in these axioms, negation is not involved. To define Int2GC+FS, the two Galois connections $(\Diamond, \boxdot)$ and $(\diamonddot, \Box)$ of Int2GC are interlinked with the axioms:

(FS1) $\Diamond(A \to B) \to (\Box A \to \Diamond B)$ (FS2) $\diamonddot(A \to B) \to (\boxdot A \to \diamonddot B)$
(FS3) $(\Diamond A \to \Box B) \to \Box(A \to B)$ (FS4) $(\diamonddot A \to \boxdot B) \to \boxdot(A \to B)$

We will show that in Int2GC, axioms (FS1) and (FS4) are equivalent, and the same holds for (FS2) and (FS3), meaning that we have some equivalent combinations of axioms to define Int2GC+FS, and thus also IK$_t$.

Another way of connecting two independent Galois connections, if one moves from Boolean to distributive lattices, is based on J. M. Dunn's axioms. These axioms connect modalities in positive modal logic. In [14], Dunn studied distributive lattices with two modal operators $\Box$ and $\Diamond$ and introduced conditions

$$\Diamond x \wedge \Box y \leq \Diamond(x \wedge y) \qquad \text{and} \qquad \Box(x \vee y) \leq \Box x \vee \Diamond y \tag{1.6}$$

for the interactions between $\Box$ and $\Diamond$. We use only the first of them, the second is false in IK$_t$. In fact, in Heyting algebras with two Galois connections, the conditions of (1.6) are independent of each other. One obtains a logic equivalent to IK$_t$ by adding to Int2GC axioms corresponding to the first condition of (1.6) applied to the pairs $(\Box, \Diamond)$ and $(\boxdot, \diamonddot)$.[2] The axioms are "positive" – negation is not present in distributive lattices. One may say that the role of linking two Galois connections played by De Morgan axioms in classical logic is taken by Fischer Servi axioms or by (positive) Dunn's axioms, in intuitionistic logic and, more general, in intermediate logics.

The next motivation of the paper is to show completeness of the logic for both algebraic and relational semantics, and to find a representation theorem for Heyting algebras with Galois connections via relational intuitionistic-modal frames. We consider H2GC+FS-algebras, which are algebras $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, \diamonddot, \boxdot)$ such that $(H, \vee, \wedge, \to, 0, 1, \Diamond, \boxdot)$ and $(H, \vee, \wedge, \to, 0, 1, \diamonddot, \Box)$ are HGC-algebras modelling IntGC [15], and $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box)$ and $(H, \vee, \wedge, \to, 0, 1, \diamonddot, \boxdot)$ are so-called HK1-algebras introduced by E. Orłowska and I. Rewitzky in [36]. We note that Int2GC+FS is complete with respect to H2GC+FS-algebras, and we generalise this result to completeness of the logic L2GC+FS for any intermediate logic L, with respect to L-Heyting algebras extended with two Galois connection pairs interlinked with Fischer Servi axioms. We also note that calculating using Heyting algebras with operators is much easier than calculating with categories, and calculating with algebras can be easily used in showing some of the non-theorems, for instance, that all Dunn's axioms (1.6) are not true in IK$_t$.

[2]Added in proof: after sending the first version of this paper to the editors in 2012 we learned that a similar result applying Dunn's axiom was presented in [33], appearing while our paper was in reviewing process, see also [17].

We recall IK-frames from [23] and show that Int2GC+FS is complete with respect to the models defined on IK-frames. In addition, we prove a representation theorem for H2GC+FS-algebras: every H2GC+FS-algebra can be embedded into the complex algebra of its canonical IK-frame. This is a non-classical generalisation of B. Jónsson and A. Tarski [30] representation of Boolean algebras with operators. Note that "complex algebra" is a commonly used name for the standard construction of an algebra of a certain type from a given frame, developed in [30]. Contrary to the case of algebraic semantics, relational semantics adequate for L2GC+FS does not necessarily exist for every intermediate logic L, because L may itself be Kripke-incomplete. Hence, relational completeness and the representation theorem for H_{L}2GC+FS-algebras for other intermediate logics L are left for a separate study.

This paper is structured as follows. In Section 2, we recall the logic IntGC introduced by the authors in [15]. We show that in the fusion of two IntGC logics with two independent Galois connection pairs $(\Diamond, \boxdot)$ and $(\diamonddot, \Box)$, axioms (FS1) and (FS4) are equivalent, and so are (FS2) and (FS3). Logic Int2GC+FS is then defined as a fusion of two IntGCs plus two axioms (FS1) and (FS2) added. We note that Int2GC+FS can be regarded as an intuitionistic bi-modal logic, and the pairs $\Diamond$, $\Box$ and $\diamonddot$, $\boxdot$ are intuitionistic modal connectives in the sense of Fischer Servi. In fact, Int2GC+FS extends the fusion IK$\otimes$IK by the Browerian axioms, and this gives us the procedure (B). We also prove that Int2GC+FS is syntactically equivalent to intuitionistic tense logic IK_t. Section 3 is devoted to H2GC+FS-algebras. In this section, also fuzzy modal operators on complete Heyting algebras are considered as another motivation. Section 4 contains a relational completeness results showing that Int2GC+FS is complete with respect to the models defined on IK-frames. We also give a representation theorem stating that any H2GC+FS-algebra can be embedded into the complex algebra of its canonical IK-frame. The paper ends with some concluding remarks.

2. Intuitionistic logic with two Galois connections and Fischer Servi axioms

We begin recalling the intuitionistic propositional logic with a Galois connection (IntGC) defined by the authors in [15]. The language of IntGC is constructed from an enumerable infinite set of propositional variables *Var*, the connectives $\neg$, $\vee$, $\wedge$, $\to$, and the unary operators $\Diamond$ and $\boxdot$. The constant *true* is defined by setting $\top := p \to p$ for some fixed propositional variable $p \in Var$, and the constant *false* is defined by $\bot := \neg\top$. We also set $A \leftrightarrow B := (A \to B) \wedge (B \to A)$. The logic IntGC is the smallest logic that contains intuitionistic propositional logic Int and is closed under modus ponens (MP), and rules (GC $\boxdot\Diamond$) and (GC $\Diamond\boxdot$):

(GC $\boxdot\Diamond$) $\dfrac{A \to \boxdot B}{\Diamond A \to B}$ (GC $\Diamond\boxdot$) $\dfrac{\Diamond A \to B}{A \to \boxdot B}$

It is known that the following rules are admissible in IntGC:

(RN$\boxdot$) $\dfrac{A}{\boxdot A}$

(RM$\boxdot$) $\dfrac{A \to B}{\boxdot A \to \boxdot B}$ (RM$\Diamond$) $\dfrac{A \to B}{\Diamond A \to \Diamond B}$

In addition, the following formulas are provable:

(i) $A \to \boxdot \Diamond A$ and $\Diamond \boxdot A \to A$;
(ii) $\Diamond A \leftrightarrow \Diamond \boxdot \Diamond A$ and $\boxdot A \leftrightarrow \boxdot \Diamond \boxdot A$;
(iii) $\boxdot \top$ and $\neg \Diamond \bot$;
(iv) $\boxdot(A \wedge B) \leftrightarrow \boxdot A \wedge \boxdot B$ and $\Diamond(A \vee B) \leftrightarrow \Diamond A \vee \Diamond B$;
(v) $\boxdot(A \to B) \to (\boxdot A \to \boxdot B)$.

Next we define Int2GC by adding another independent Galois connection pair to IntGC. The language of the logic Int2GC is thus the one of IntGC extended by two unary connectives $\dot{\Diamond}$ and $\Box$, and the logic Int2GC is the smallest logic extending IntGC by rules (GC $\Box\dot{\Diamond}$) and (GC $\dot{\Diamond}\Box$):

(GC $\Box\dot{\Diamond}$) $\dfrac{A \to \Box B}{\dot{\Diamond} A \to B}$ (GC $\dot{\Diamond}\Box$) $\dfrac{\dot{\Diamond} A \to B}{A \to \Box B}$

Obviously, in Int2GC also the rules:

(RN$\Box$) $\dfrac{A}{\Box A}$

(RM$\Box$) $\dfrac{A \to B}{\Box A \to \Box B}$ (RM$\dot{\Diamond}$) $\dfrac{A \to B}{\dot{\Diamond} A \to \dot{\Diamond} B}$

are admissible, and the following formulas are provable:

(i) $A \to \Box \dot{\Diamond} A$ and $\dot{\Diamond} \Box A \to A$;
(ii) $\dot{\Diamond} A \leftrightarrow \dot{\Diamond} \Box \dot{\Diamond} A$ and $\Box A \leftrightarrow \Box \dot{\Diamond} \Box A$;
(iii) $\Box \top$ and $\neg \dot{\Diamond} \bot$;
(iv) $\Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$ and $\dot{\Diamond}(A \vee B) \leftrightarrow \dot{\Diamond} A \vee \dot{\Diamond} B$;
(v) $\Box(A \to B) \to (\Box A \to \Box B)$.

In fact, Int2GC is just the fusion IntGC$\otimes$IntGC of two separate IntGCs having the Galois connections $(\Diamond, \boxdot)$ and $(\dot{\Diamond}, \Box)$, respectively.

Intuitionistic modal logic IK was introduced by G. Fischer Servi in [23]. The logic IK is obtained by adding two modal connectives $\Diamond$ and $\Box$ to intuitionistic logic satisfying the following axioms:

(IK1) $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$
(IK2) $\Box A \wedge \Box B \to \Box(A \wedge B)$
(IK3) $\neg \Diamond \bot$
(IK4) $\Diamond(A \to B) \to (\Box A \to \Diamond B)$
(IK5) $(\Diamond A \to \Box B) \to \Box(A \to B)$

In addition, the *monotonicity rules* for both $\Diamond$ and $\Box$ are admissible:

(RM$\Diamond$) $\dfrac{A \to B}{\Diamond A \to \Diamond B}$ (RM$\Box$) $\dfrac{A \to B}{\Box A \to \Box B}$

Note that axiom (IK4) is the same as (FS1) and (IK5) equals (FS3), and (FS2) and (FS4) are analogous axioms for $\dot{\Diamond}$ and $\boxdot$. Note also that in [41] it is argued that IK is the true intuitionistic analogue of "classical" K.

Proposition 2.1. *The following assertions hold in* Int2GC*.*

(a) *Axioms* (FS1) *and* (FS4) *are equivalent.*

(b) *Axioms* (FS2) *and* (FS3) *are equivalent.*

Proof. We prove only assertion (a), because (b) can be proved analogously. Here $\vdash A$ denotes that A is provable in Int2GC.

(FS1)⇒(FS4): Let us set $X := A$, $Y := \Box \diamonddot A$ and $Z := \Diamond \boxdot B$ in the provable formula $(X \to Y) \to ((Y \to Z) \to (X \to Z))$. We get $\vdash (\Box \diamonddot A \to \Diamond \boxdot B) \to (A \to \Diamond \boxdot B)$ by using also $\vdash A \to \Box \diamonddot A$. This is equivalent to $\vdash A \wedge (\Box \diamonddot A \to \Diamond \boxdot B) \to \Diamond \boxdot B$. Because $\vdash \Diamond \boxdot B \to B$, this means $\vdash A \wedge (\Box \diamonddot A \to \Diamond \boxdot B) \to B$ and $\vdash (\Box \diamonddot A \to \Diamond \boxdot B) \to (A \to B)$. If we set $A := \diamonddot A$ and $B := \boxdot B$ in (FS1), we obtain $\vdash \Diamond(\diamonddot A \to \boxdot B) \to (\Box \diamonddot A \to \Diamond \boxdot B)$, and so $\vdash \Diamond(\diamonddot A \to \boxdot B) \to (A \to B)$. This implies $\vdash (\diamonddot A \to \boxdot B) \to \boxdot(A \to B)$ by (GC $\Diamond\boxdot$).

(FS4)⇒(FS1): We set $X := \diamonddot \Box A$, $Y := A$ and $Z := B$ in $(X \to Y) \to ((Y \to Z) \to (X \to Z))$. This gives $\vdash (\diamonddot \Box A \to A) \to ((A \to B) \to (\diamonddot \Box A \to B))$, and $\vdash (A \to B) \to (\diamonddot \Box A \to \boxdot \Diamond B)$, since $\vdash \diamonddot \Box A \to A$ and $\vdash B \to \boxdot \Diamond B$. By monotonicity, $\vdash \Diamond(A \to B) \to \Diamond(\diamonddot \Box A \to \boxdot \Diamond B)$. By setting $A := \Box A$ and $B := \Diamond B$ in (FS4), we have $\vdash (\diamonddot \Box A \to \boxdot \Diamond B) \to \boxdot(\Box A \to \Diamond B)$ and $\vdash \Diamond(\diamonddot \Box A \to \boxdot \Diamond B) \to (\Box A \to \Diamond B)$ by (GC $\boxdot\Diamond$). Therefore, we obtain $\vdash \Diamond(A \to B) \to (\Box A \to \Diamond B)$. □

Logic Int2GC+FS is defined as the extension of Int2GC that satisfies (FS1) and (FS2). By Proposition 2.1, it is clear that we have several equivalent axiomatisations of Int2GC+FS given in the next corollary.

Corollary 2.2.

$$\mathsf{Int2GC{+}FS} = \mathsf{Int2GC} \oplus \{(\mathrm{FS1}) \textit{ or } (\mathrm{FS4})\} \oplus \{(\mathrm{FS2}) \textit{ or } (\mathrm{FS3})\}.$$

Logic Int2GC+FS satisfies the counterparts of axioms (IK1)–(IK5) of IK, so Int2GC+FS can be regarded as a intuitionistic bi-modal logic, and the pairs of operators $(\Diamond, \Box)$ and $(\diamonddot, \boxdot)$ can be regarded as intuitionistic modal connectives in the sense of Fischer Servi. Hence, Int2GC+FS can be seen as an extension of the fusion IK⊗IK of two copies of intuitionistic modal logic IK, the first one with the modalities $(\Diamond, \Box)$ and the second one with $(\diamonddot, \boxdot)$.

In ordered sets, there is another way of defining Galois connections presented in conditions (1.1) and (1.2). This gives us method (B) mentioned in Introduction. Let us considerer the fusion IK⊗IK of two copies of intuitionistic modal logic IK, the first one with the operators $(\Diamond, \Box)$ and the second one with $(\diamonddot, \boxdot)$. We extend IK⊗IK by the so-called *Brouwerian axioms*:

(BR1) $A \to \boxdot \Diamond A$ (BR2) $\Diamond \boxdot A \to A$

(BR3) $A \to \Box \diamonddot A$ (BR4) $\diamonddot \Box A \to A$

These axioms are also referred to as the *converse axioms*, since these axioms are needed to ensure that the accessibility relations for the operators F, G and P, H are each others converse in tense logics. We denote this logic by IK⊗IK+BR.

Proposition 2.3. Int2GC+FS = IK⊗IK+BR.

Proof. We have already noted that in Int2GC+FS axioms (IK1)–(IK5) are provable for the operator pairs $\Box$, $\Diamond$ and $\boxdot$, $\diamonddot$, and the operators $\Box$, $\Diamond$, $\boxdot$, $\diamonddot$ satisfy the monotonicity rule. Additionally, Brouwerian axioms are provable in Int2GC+FS.

On the other hand, $\vdash A \to \boxdot B$ implies $\vdash \Diamond A \to \Diamond \boxdot B$, which by (BR2) gives $\vdash \Diamond A \to B$. Similarly, $\vdash \Diamond A \to B$ implies $\vdash \boxdot \Diamond A \to \boxdot B$, and by (BR1) we get $\vdash A \to \boxdot B$. Thus, $(\Diamond, \boxdot)$ is a Galois connection, and similarly we can show the same for the pair $(\diamonddot, \Box)$. Fischer Servi axioms (FS1)–(FS4) hold trivially in $\mathsf{IK}{\otimes}\mathsf{IK}$. □

Proposition 2.3 means that there exist two ways to extend intuitionistic logic with two Galois connections such that these pairs are interlinked with Fischer Servi axioms.

Remark 2.4. The proof of Proposition 2.3 reveals also that it is possible to endow a Galois connection in two ways to any logic $\mathcal{L}$ having modus ponens and satisfying the so-called *law of syllogism* $(A \to B) \to ((B \to C) \to (A \to C))$. The first way is to add operators $\Diamond$ and $\boxdot$ to $\mathcal{L}$ and add rules (GC $\boxdot\Diamond$) and (GC $\Diamond\boxdot$). Or equivalently, we may add Brouwerian axioms (BR1), (BR2) and rules of monotonicity (RM$\diamonddot$) and (RM$\boxdot$) for $\Diamond$ and $\boxdot$. Hence, the following are equivalent:

(i) $\mathcal{L} \oplus \{(\mathrm{GC}\,\boxdot\Diamond), (\mathrm{GC}\,\Diamond\boxdot)\}$
(ii) $\mathcal{L} \oplus \{(\mathrm{BR1}), (\mathrm{BR2}), (\mathrm{RM}\,\Diamond), (\mathrm{RM}\,\boxdot)\}$

Our next aim is to show that Int2GC+FS is equivalent to IK_t. We need the following lemma.

Lemma 2.5. *The following formulas are* Int2GC+FS*-provable:*

(a) $\Box A \wedge \Diamond B \to \Diamond(A \wedge B)$ *and* $\boxdot A \wedge \diamonddot B \to \diamonddot(A \wedge B)$*;*
(b) $\Box(A \to B) \to (\Diamond A \to \Diamond B)$ *and* $\boxdot(A \to B) \to (\diamonddot A \to \diamonddot B)$*;*
(c) $\Box \neg A \to \neg \Diamond A$ *and* $\boxdot \neg A \to \neg \diamonddot A$.

Proof. We only prove the first formula of each statement.

(a) Axiom (FS1) is equivalent to $\Diamond(A \to B) \wedge \Box A \to \Diamond B$. If we set $B := A \wedge B$ in this formula, we have that $\vdash (\Diamond(A \to A \wedge B) \wedge \Box A) \to \Diamond(A \wedge B)$. Because $A \to A \wedge B$ is equivalent to $A \to B$, and $\vdash B \to (A \to B)$ and monotonicity of $\Diamond$ imply $\vdash \Diamond B \to \Diamond(A \to B)$, we obtain $\vdash \Diamond B \wedge \Box A \to \Diamond(A \wedge B)$.

(b) Because $\vdash \diamonddot \Box(A \to B) \to (A \to B)$, we have $\vdash \diamonddot \Box(A \to B) \wedge A \to B$ and $\vdash \Diamond(\diamonddot \Box(A \to B) \wedge A) \to \Diamond B$. Let us set $A := \diamonddot \Box(A \to B)$ and $B := A$ in (a). We obtain $\vdash \Box \diamonddot \Box(A \to B) \wedge \Diamond A \to \Diamond(\diamonddot \Box(A \to B) \wedge A)$. Thus, $\vdash \Box \diamonddot \Box(A \to B) \wedge \Diamond A \to \Diamond B$. Because $\vdash \Box(A \to B) \to \Box \diamonddot \Box(A \to B)$, we have $\vdash \Box(A \to B) \wedge \Diamond A \to \Diamond B$. This is equivalent to $\vdash \Box(A \to B) \to (\Diamond A \to \Diamond B)$.

(c) If we set $B := \bot$ in (b), we get $\vdash \Box(A \to \bot) \to (\Diamond A \to \Diamond \bot)$. Because $\Diamond \bot$ is equivalent to $\bot$, we obtain $\vdash \Box \neg A \to \neg \Diamond A$. □

Next, we show that IK_t and Int2GC+FS are syntactically equivalent. Logic IK_t is obtained by extending the language of intuitionistic propositional logic with the usual temporal expressions FA (A is true at some future time), PA (A was true at some past time), GA (A will be true at all future times), and HA (A has always been true in the past). The following Hilbert-style axiomatisation of IK_t is given by Ewald in [20, p. 171]:

(1) All axioms of intuitionistic logic
(2) $G(A \to B) \to (GA \to GB)$ (2′) $H(A \to B) \to (HA \to HB)$

(3)	$G(A \wedge B) \leftrightarrow GA \wedge GB$	(3′)	$H(A \wedge B) \leftrightarrow HA \wedge HB$
(4)	$F(A \vee B) \leftrightarrow FA \vee FB$	(4′)	$P(A \vee B) \leftrightarrow PA \vee PB$
(5)	$G(A \to B) \to (FA \to FB)$	(5′)	$H(A \to B) \to (PA \to PB)$
(6)	$GA \wedge FB \to F(A \wedge B)$	(6′)	$HA \wedge PB \to P(A \wedge B)$
(7)	$G\neg A \to \neg FA$	(7′)	$H\neg A \to \neg PA$
(8)	$FHA \to A$	(8′)	$PGA \to A$
(9)	$A \to HFA$	(9′)	$A \to GPA$
(10)	$(FA \to GB) \to G(A \to B)$	(10′)	$(PA \to HB) \to H(A \to B)$
(11)	$F(A \to B) \to (GA \to FB)$	(11′)	$P(A \to B) \to (HA \to PB)$

The rules of inference are modus ponens (MP), and

(RH) $\dfrac{A}{HA}$ (RG) $\dfrac{A}{GA}$

Our next theorem shows that if we identify $\Diamond$, $\Box$, $\Diamonddot$, $\boxdot$ with F, G, P, H, respectively, then Int2GC+FS and IK_t are syntactically equivalent.

Theorem 2.6. IK_t = Int2GC+FS.

Proof. First we will show that the IK_t-axioms are provable in Int2GC+FS, and all rules of IK_t are admissible in Int2GC+FS. As mentioned in Section 2, axioms (2), (2′), (3), (3′) (4), (4′), (8), (8′), (9), (9′) are provable even in Int2GC. Additionally, rules (MP), (RH), and (RG) are admissible in Int2GC. Axioms (10), (10′), (11), (11′) are Fischer Servi axioms (FS3), (FS4), (FS1), (FS2), so they are provable in Int2GC+FS. The provability of (5), (5′), (6), (6′), (7), and (7′) is shown in Lemma 2.5.

Because axioms (10), (10′), (11), (11′) are the Fischer Servi axioms, for the other direction it is enough to show the admissibility of rules (GC $\boxdot\Diamond$), (GC $\Diamond\boxdot$), (GC $\Box\Diamonddot$), (GC $\Diamonddot\Box$) in IK_t. First, we show the admissibility of the rules of monotonicity, that is, if $A \to B$ is provable, then $HA \to HB$, $PA \to PB$, $GA \to GB$, and $FA \to FB$ are provable.

Here $\vdash A$ denotes that the formula A is provable in IK_t. Assume $\vdash A \to B$. By (RG), $\vdash G(A \to B)$. Now $\vdash GA \to GB$ follows by (2), and from $\vdash G(A \to B)$, we obtain also $\vdash FA \to FB$ by (5). Similarly, $\vdash A \to B$ implies $\vdash HA \to HB$ and $\vdash PA \to PB$ by applying (RH), (2′), and (5′).

Next we prove the admissibility of (GC $\boxdot\Diamond$). Assume that $\vdash A \to HB$. Then, $FA \to FHB$ by the monotonicity of F. Because $\vdash FHB \to B$ by (8), we obtain $\vdash FA \to B$. Similarly, by (8′) and the monotonicity of P, $A \to GB$ implies $PA \to B$, that is, (GC $\Box\Diamonddot$) is admissible in IK_t. The monotonicity of H and axiom (9) yield that $FA \to B$ implies $A \to HB$, and monotonicity of G and (9′) give that $PA \to B$ implies $A \to BG$. Thus, rules (GC $\Diamond\boxdot$) and (GC $\Diamonddot\Box$) are admissible. □

By combining Proposition 2.3 and Theorem 2.6, we get the following corollary. Note that $\mathsf{IK}_t = \mathsf{IK}\otimes\mathsf{IK}$+BR is proved already by Davoren [13].

Corollary 2.7. IK_t = Int2GC+FS = $\mathsf{IK}\otimes\mathsf{IK}$+BR.

In [29], the logic extending classical logic Cl with a Galois connection $(\Diamond, \boxdot)$ was introduced and it is proved that if we add another two operators $\Diamonddot$ and $\Box$ that are connected to the Galois connection $(\Diamond, \boxdot)$ by the De Morgan axioms:

(DM1) $\Box A \leftrightarrow \neg \Diamond \neg A$ (DM2) $\Diamonddot A \leftrightarrow \neg \boxdot \neg A$,

then also the pair $(\Diamonddot, \Box)$ is a Galois connection, that is, rules (GC $\Box\Diamonddot$) and (GC $\Diamonddot\Box$) are admissible. Hence, in classical case, one Galois connection is defined by the other (obtained "for free"), which is not the case in intuitionistic logic; see [15]. It is proved in [29] that this logic is syntactically equivalent to classical tense logic K_t, when $\Diamond$, $\Box$, $\Diamonddot$, $\boxdot$ are identified with the tense operators F, G, P, H, respectively. Note that algebras corresponding to K_t are considered in [31, 42], for example, and these are generally called *tense algebras*.

As stated in [41, p. 54], it is routine to derive $\Diamond A \leftrightarrow \neg \Box \neg A$ in IK, together with the Law of the Excluded Middle. Since $\mathsf{Int2GC{+}FS}$ is an extension of the fusion $\mathsf{IK}{\otimes}\mathsf{IK}$ of two copies of intuitionistic modal logic IK, then it is clear that classical logic with two Galois connection pairs $(\Diamond, \boxdot)$ and $(\Diamonddot, \Box)$, which are interlinked with (FS1) and (FS2), denoted here $\mathsf{Cl2GC{+}FS}$, satisfies (DM1) and (DM2). On the other hand, in K_t, the pairs (F, H) and (P, G) form Galois connections, and axioms (FS1) and (FS2) are provable. Therefore, we can write:

$$\mathsf{K}_t = \mathsf{Cl2GC{+}FS}$$

Observe that K_t can be defined as the fusion $\mathsf{K}{\otimes}\mathsf{K}$ extended with Brouwerian axioms (BR1)–(BR4), denoted by $\mathsf{K}{\otimes}\mathsf{K}{+}\mathsf{BR}$. In summary, we have:

Corollary 2.8. $\mathsf{K}_t = \mathsf{Cl2GC{+}FS} = \mathsf{K}{\otimes}\mathsf{K}{+}\mathsf{BR}$.

In conclusion, if we add to intuitionistic logic Int two Galois connections $(\Diamond, \boxdot)$ and $(\Diamonddot, \Box)$ that are connected using Fischer Servi axioms (FS1) and (FS2), then we get the intuitionistic tense logic IK_t. Analogously, if two Galois connections combined with axioms (FS1) and (FS2) are added to classical logic, we obtain the classical tense logic K_t. Here we discuss how for each intermediate logic L, we can define the corresponding L-*tense logic* LK_t.

An *intermediate logic* is a propositional logic extending intuitionistic logic. Classical logic Cl is the strongest intermediate logic and it is obtained from Int by extending the axioms of Int by the "Law of the excluded middle" $A \vee \neg A$, or equivalently, by the "Double negation elimination" $\neg\neg A \to A$ or by "Peirce's law" $((A \to B) \to A) \to A$. There exists a continuum of different intermediate logics. For example, the Gödel–Dummett logic G is obtained from Int by adding the axiom $(A \to B) \vee (B \to A)$. For more examples of intermediate logics and their semantics; see [11, 26].

We denote by L any intermediate logic, that is, $\mathsf{Int} \subseteq \mathsf{L} \subseteq \mathsf{Cl}$. We can write that for any intermediate logic L,

$$\mathsf{IK}_t \subseteq \mathsf{L2GC{+}FS} \subseteq \mathsf{K}_t.$$

Because Ewald's IK_t is commonly accepted as the intuitionistic analogue of the classical tense logic K_t, taking into account the equivalences $\mathsf{IK}_t = \mathsf{Int2GC{+}FS}$ and $\mathsf{K}_t = \mathsf{Cl2GC{+}FS}$, the logic $\mathsf{L2GC{+}FS}$ can be regarded as the L-tense logic LK_t for any intermediate logic L. Then, as one of the main results of this work, we have a general uniform method (A) of building L-tense logic for any intermediate logic L by setting $\mathsf{LK}_t = \mathsf{L2GC{+}FS}$, that is, L added with two Galois connection pairs combined using Fischer Servi axioms.

Moreover, since the equivalences $\mathsf{K}_t = \mathsf{Cl2GC{+}FS}$ and $\mathsf{IK}_t = \mathsf{Int2GC{+}FS}$ were shown syntactically, L-tense logics exist independent of whether they are Kripke complete or canonical, or not. This can be presented as the following theorem.

Theorem 2.9. *For any intermediate logic* L*, there is an* L*-tense logic* LK_t*, which is* L *endowed with two independent Galois connections connected by Fischer Servi axioms* (FS1) *and* (FS2)*.*

3. Heyting algebras with Galois connections

E. Orłowska and I. Rewitzky [36] defined a *Heyting algebra with modal operators* as a Heyting algebra $(H, \vee, \wedge, \to, 0, 1)$ equipped with unary operators $\Diamond$ and $\Box$ satisfying for all $x, y \in H$:

$$\Diamond x \vee \Diamond y = \Diamond(x \vee y) \quad \text{and} \quad \Box x \wedge \Box y = \Box(x \wedge y). \tag{3.1}$$

These algebras are called *HM-algebras*, for short. In addition, they defined *HK-algebras* as HM-algebras satisfying

$$\Diamond 0 = 0 \quad \text{and} \quad \Box 1 = 1. \tag{3.2}$$

We introduced in [15] *HGC-algebras* as Heyting algebras provided with an order-preserving Galois connection $(\Diamond, \boxdot)$. Equationally HGC-algebras can be defined as algebras $(H, \vee, \wedge, \to, 0, 1, \Diamond, \boxdot)$ such that $(H, \vee, \wedge, \to, 0, 1)$ satisfies the identities for Heyting algebras (which can be found in e.g. [2,7]), the operators $\Diamond$ and $\boxdot$ satisfy (3.1), and for all $x \in H$,

$$x \le \boxdot \Diamond x \quad \text{and} \quad \Diamond \boxdot x \le x. \tag{3.3}$$

By definition, HGC-algebras are HM-algebras, but HGC-algebras are also HK-algebras, because $0 \le \boxdot 0$ implies $\Diamond 0 \le 0$ and $\Diamond 1 \le 1$ gives $1 \le \boxdot 1$. Thus, $\Diamond$ and $\boxdot$ satisfy (3.2).

In [15], we proved that IntGC is algebraizable in terms of HGC-algebras. More precisely, any valuation v assigning to propositional variables elements of an HGC-algebra can be extended to all formulas inductively by the following way:

$$\begin{aligned} v(A \wedge B) &= v(A) \wedge v(B) & \qquad v(A \vee B) &= v(A) \vee v(B) \\ v(A \to B) &= v(A) \to v(B) & v(\neg A) &= \neg v(A) \\ v(\Diamond A) &= \Diamond v(A) & v(\boxdot A) &= \boxdot v(A). \end{aligned}$$

Then, a formula A is provable in IntGC if and only if $v(A) = 1$ for all valuations v on any HGC-algebra.

We define *H2GC-algebras* as structures $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, \diamonddot, \boxdot)$ such that $(H, \vee, \wedge, \to, 0, 1, \Diamond, \boxdot)$ and $(H, \vee, \wedge, \to, 0, 1, \diamonddot, \Box)$ are HGC-algebras. Similarly as in case of IntGC, we can show, by applying Lindenbaum–Tarski algebras, that $\mathsf{Int2GC}$ is complete with respect to H2GC-algebras, that is, a formula $A \in \Phi$ is provable in $\mathsf{Int2GC}$ if and only if $v(A) = 1$ for all valuations v on any H2GC-algebra.

Orłowska and Rewitzky [36] studied also an extension of HK-algebras, called *HK1-algebras*, that are algebraic counterparts of the logic IK. They extended HK-algebras by the following two conditions that correspond to Fischer Servi axioms (FS1) and (FS2):

$$\Diamond(x \to y) \leq \Box x \to \Diamond y \quad \text{and} \quad \Diamond x \to \Box y \leq \Box(x \to y) \tag{3.4}$$

Let us denote for any H2GC-algebra $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, \Diamonddot, \boxdot)$ the corresponding conditions by (FS1)–(FS4), that is,

(FS1) $\Diamond(x \to y) \leq \Box x \to \Diamond y$ (FS2) $\Diamond x \to \Box y \leq \Box(x \to y)$
(FS3) $\Diamonddot(x \to y) \leq \boxdot x \to \Diamonddot y$ (FS4) $\Diamonddot x \to \boxdot y \leq \boxdot(x \to y)$

Note that we used (FS1)–(FS4) to denote also the corresponding Fischer Servi axioms in logic. This should not cause any confusion, because the context shows whether we are dealing with logic axioms or lattice-theoretical conditions. In addition, we denote by (D1) and (D2) the conditions corresponding the first condition of (1.6), that is,

(D1) $\Diamond x \wedge \Box y \leq \Diamond(x \wedge y)$ (D2) $\Diamonddot x \wedge \boxdot y \leq \Diamonddot(x \wedge y)$

Proposition 3.1. *Let* $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, \Diamonddot, \boxdot)$ *be an H2GC-algebra.*

(a) *Conditions* (FS1)*,* (D1)*, and* (FS4) *are equivalent.*
(b) *Conditions* (FS2)*,* (D2)*, and* (FS3) *are equivalent.*

Proof. We prove only assertion (a), because (b) can be proved in a similar way. Assume that (D1) holds, and set $x := a \to b$ and $y := a$ in it. We obtain $\Diamond(a \to b) \wedge \Box a \leq \Diamond(a \wedge (a \to b)) \leq \Diamond b$, because $a \wedge (a \to b) \leq b$. This gives directly $\Diamond(a \to b) \leq \Box a \to \Diamond b$, and thus (D1) implies (FS1). Conversely, if we set $x := b$ and $y := a \wedge b$ in (FS1), we have $\Diamond a \leq \Diamond(b \to a) = \Diamond(b \to a \wedge b) \leq \Box b \to \Diamond(a \wedge b)$, because $b \to a \wedge b = b \to a$ and $a \leq b \to a$. This is equivalent to $\Diamond a \wedge \Box b \leq \Diamond(a \wedge b)$. Hence, also (FS1) implies (D1). That (FS1) and (FS4) are equivalent can be shown as in Proposition 2.1. □

Proposition 3.1 together with the completeness of Int2GC with respect to H2GC-algebras implies that

$$\mathsf{IK}_t = \mathsf{Int2GC{+}FS} = \mathsf{Int2GC} \oplus \{(\mathrm{D1}), (\mathrm{D2})\},$$

where (D1) and (D2) denote the axioms:

(D1) $\Diamond A \wedge \Box B \to \Diamond(A \wedge B)$ (D2) $\Diamonddot A \wedge \boxdot B \to \Diamonddot(A \wedge B)$.

Let us define *H2GC+FS-algebras* as H2GC-algebras satisfying (FS1) and (FS2). Proposition 3.1 has the following corollary.

Corollary 3.2. *Let* $\mathbb{H} = (H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, \Diamonddot, \boxdot)$ *be an H2GC-algebra.*

(a) $\mathbb{H}$ *is an H2GC+FS-algebra if and only if* $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box)$ *and* $(H, \vee, \wedge, \to, 0, 1, \Diamonddot, \boxdot)$ *are HK1-algebras.*
(b) $\mathbb{H}$ *is an H2GC+FS-algebra if and only if it satisfies* (D1) *and* (D2)*.*

For any H2GC+FS-algebra $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, \Diamonddot, \boxdot)$, a valuation v is a function $v \colon \mathit{Var} \to H$, which is inductively extended to all formulas in Φ as is done above in the case of HGC-algebras. A formula $A \in \Phi$ is *H2GC+FS-valid* if $v(A) = 1$ for every valuation v on any H2GC+FS-algebra.

We have shown in [15] that rules (GC $\boxdot\Diamond$) and (GC $\Diamond\boxdot$) preserve validity, and obviously the same holds for (GC $\Box\Diamonddot$) and (GC $\Diamonddot\Box$). In addition, axioms (FS1) and (FS2) are also valid, because H2GC+FS-algebras are defined by using analogous conditions. Thus, $\mathsf{Int2GC{+}FS}$-provable formulas are H2GC+FS-valid.

To obtain algebraic completeness, we apply Lindenbaum–Tarski algebras. We denote by $\mathcal{F}(\Phi)$ the *algebra of* Φ*-formulas*, that is, the abstract algebra

$$\mathcal{F}(\Phi) = (\Phi, \vee, \wedge, \to, \bot, \diamond, \Box, \diamondsuit, \boxdot).$$

We define an equivalence $\equiv$ on Φ by

$$A \equiv B \iff A \leftrightarrow B \text{ is } \mathsf{Int2GC{+}FS}\text{-provable}.$$

It is easy to observe that $\equiv$ is a congruences on $\mathcal{F}(\Phi)$. Let $[A]$ denote the $\equiv$-class of A. We define the *quotient algebra* $\mathcal{F}(\Phi)/{\equiv}$ by introducing the operations:

$$[A] \vee [B] = [A \vee B], \quad [A] \wedge [B] = [A \wedge B], \quad [A] \to [B] = [A \to B],$$
$$\diamond[A] = [\diamond A], \quad \Box[A] = [\Box A], \quad \diamondsuit[A] = [\diamondsuit A], \quad \boxdot[A] = [\boxdot A]$$

Because H2GC+FS-algebras form an equational class, $\mathcal{F}(\Phi)/{\equiv}$ forms an H2GC+FS-algebra. Note that $[\bot]$ and $[\top]$ are the zero and the unit in this algebra. We define a valuation $v\colon \mathit{Var} \to \Phi/{\equiv}$ by $v(p) = [p]$. By straightforward formula induction, we see that $v(A) = [A]$ for all formulas $A \in \Phi$. If now $A \in \Phi$ is H2GC+FS-valid, then $v(A) = [\top]$ in $\mathcal{F}(\Phi)/{\equiv}$. This means $A \leftrightarrow \top$ and thus A is $\mathsf{Int2GC{+}FS}$-provable. Therefore, we can write the following completeness theorem.

Theorem 3.3. *A formula $A \in \Phi$ is* $\mathsf{Int2GC}$*-provable if and only if A is H2GC+FS-valid.*

If we change the underlying logic from intuitionistic to classical, we have that $\mathsf{Cl2GC{+}FS}$ is complete with respect to tense algebras – this is due to the standard algebraic completeness theorem of temporal logic K_t with respect to tense algebras.

Results of this section can be equally applied to intermediate logics. It is well known that intuitionistic logic and all intermediate logics are algebraizable; see, for example, [6]. For instance, the specific axiom $(A \to B) \vee (B \to A)$ of Gödel–Dummett logic G translates into in the identity $(x \to y) \vee (y \to x) = 1$ extending Heyting algebras. For every intermediate logic L, there exists a corresponding equational class of L-*algebras*. For each L-algebra $(H_{\mathsf{L}}, \vee, \wedge, \to, 0, 1)$, we define the corresponding *H_{L}2GC+FS-algebra* as an algebra $(H_{\mathsf{L}}, \vee, \wedge, \to, 0, 1, \diamond, \Box, \diamondsuit, \boxdot)$ by using the same identities as in the case of defining H2GC+FS-algebras from Heyting ones. Clearly, the class of H_{L}2GC-algebras is equational. Since the method of Lindenbaum–Tarski algebras is applicable to any $\mathsf{L2GC{+}FS}$-logic in a straightforward way, we get the algebraic completeness.

Corollary 3.4. *For every intermediate logic* L*, a formula $A \in \Phi$ is* $\mathsf{L2GC{+}FS}$*-provable if and only if A is valid in every H_{L}2GC–algebra.*

Very often completeness for an intermediate logic L is stated for a narrower class than the class of all L-algebras. For instance, Gödel–Dummett logic G is complete with respect to the class of finite chains, and in [16], we showed the finite model property of IntGC. However, here we will not consider algebraic completeness of L2GC+FS-logics with respect to these kinds of narrower classes.

Let Φ_0 denote the set of propositional formulas of intuitionistic logic only (thus not containing $\Diamond, \Box, \blacklozenge, \boxdot$). In [15, Prop. 4.6], we proved that IntGC is conservative over Int, and analogously we can prove the following theorem.

Theorem 3.5. *For every intermediate logic* L*, a formula* $A \in \Phi_0$ *is* L2GC+FS*-provable if and only if* A *is provable in intermediate propositional logic* L*.*

Example 3.6. (a) As a motivating example for H2GC+FS-algebras, we consider fuzzy modal operators on complete Heyting algebras. These are also closely connected to fuzzy Galois connections (see e.g. [5, 25]).

A *complete Heyting algebra* is a Heyting algebra $(H, \vee, \wedge, \to, 0, 1)$ such that its underlying lattice $(H, \leq)$ is complete. It is well known [27, 37] that a complete Heyting algebra H satisfies the *join-infinite distributive law*: for any $S \subseteq H$ and $x \in H$, $x \wedge (\bigvee S) = \bigvee\{x \wedge y \mid y \in S\}$.

Fuzzy sets on complete Heyting algebras generalise fuzzy sets on the unit interval $[0, 1]$. Let U be some universe of discourse. Each map $\varphi\colon U \to H$ is called a *fuzzy set* on U. For any object x, $\varphi(x)$ is the *grade of membership.* We denote by H^U the set of all fuzzy sets on U. Also H^U forms a complete Heyting algebra in which the operations are defined pointwise.

Let R be a fuzzy relation on U, that is, R is a mapping from $U \times U$ to H. For a fuzzy set $\varphi \in H^U$, we may define the fuzzy sets $\Diamond\,\varphi$, $\Box\,\varphi$, $\blacklozenge\,\varphi$, $\boxdot\,\varphi$ by setting for all $x \in U$:

$$\Diamond\,\varphi(x) = \bigvee_{y \in U} \{R(x,y) \wedge \varphi(y)\} \qquad \Box\,\varphi(x) = \bigwedge_{y \in U} \{R(x,y) \to \varphi(y)\}$$

$$\blacklozenge\,\varphi(x) = \bigvee_{y \in U} \{R(y,x) \wedge \varphi(y)\} \qquad \boxdot\,\varphi(x) = \bigwedge_{y \in U} \{R(y,x) \to \varphi(y)\}$$

We show first that $(\Diamond, \boxdot)$ and $(\blacklozenge, \Box)$ are Galois connections on H^U. Indeed, suppose φ and ψ are fuzzy sets such that $\varphi \leq \psi$. Then, for all $y \in U$, $R(x,y) \wedge \varphi(y) \;\leq\; R(x,y) \wedge \psi(y)$ and this implies

$$\Diamond\,\varphi(x) = \bigvee_{y \in U} \{R(x,y) \wedge \varphi(y)\} \leq \bigvee_{y \in U} \{R(x,y) \wedge \psi(y)\} = \Diamond\,\psi(x).$$

Similarly, $R(y,x) \to \varphi(y) \;\leq\; R(y,x) \to \psi(y)$ for all $y \in U$. Thus,

$$\boxdot\,\varphi(x) = \bigwedge_{y \in U} \{R(y,x) \to \varphi(y)\} \leq \bigwedge_{y \in U} \{R(y,x) \to \psi(y)\} = \boxdot\,\psi(x).$$

So, $\Diamond$ and $\boxdot$ are order-preserving. By definition, for all $x \in U$,

$$\begin{aligned} \Diamond\,\boxdot\,\varphi(x) &= \bigvee_{y \in U} \{R(x,y) \wedge \boxdot\,\varphi(y)\} = \bigvee_{y \in U} \Big\{R(x,y) \wedge \bigwedge_{z \in U} \{R(z,y) \to \varphi(z)\}\Big\} \\ &\leq \bigvee_{y \in U} \{R(x,y) \wedge (R(x,y) \to \varphi(x)\,)\,\} \leq \bigvee_{y \in U} \{\varphi(x)\} = \varphi(x). \end{aligned}$$

This means that $\Diamond \boxdot \varphi \leq \varphi$. Analogously, for any $x \in U$,

$$\begin{aligned}\boxdot \diamonddot \varphi(x) &= \bigwedge_{y\in U}\{R(y,x) \to \Diamond \varphi(y)\} = \bigwedge_{y\in U}\Big\{R(y,x) \to \bigvee_{z\in U}\{R(y,z) \wedge \varphi(z)\}\Big\} \\ &\geq \bigwedge_{y\in U}\{R(y,x) \to (\, R(y,x) \wedge \varphi(x)\,)\} \geq \bigwedge_{y\in U}\{\varphi(x)\} = \varphi(x).\end{aligned}$$

Thus, also $\varphi \leq \boxdot \Diamond \varphi$. We have that $(\Diamond, \boxdot)$ is a Galois connection. Similarly, we can show that $(\diamonddot, \Box)$ is a Galois connection.

Next we show that (D1) holds. For all $x, y \in U$, we have

$$\begin{aligned}R(x,y) \wedge \varphi(y) \wedge \Box \psi(x) &= R(x,y) \wedge \varphi(y) \wedge \bigwedge_{z\in U}\{R(x,z) \to \psi(z)\} \\ &\leq R(x,y) \wedge \varphi(y) \wedge (R(x,y) \to \psi(y)) \\ &= (R(x,y) \wedge (R(x,y) \to \psi(y))) \wedge \varphi(y) \\ &= R(x,y) \wedge \psi(y) \wedge \varphi(y) \\ &= R(x,y) \wedge (\varphi \wedge \psi)(y) \\ &\leq \bigvee_{z\in U}\{R(x,z) \wedge (\varphi \wedge \psi)(z)\} \\ &= \Diamond(\varphi \wedge \psi)(x).\end{aligned}$$

Hence, for all $x, y \in U$, $R(x,y) \wedge \varphi(y) \wedge \Box \psi(x) \leq \Diamond(\varphi \wedge \psi)(x)$. Because complete Heyting algebras satisfy the join-infinite distributive law, we have

$$\begin{aligned}(\Diamond \varphi \wedge \Box \psi)(x) = \Diamond \varphi(x) \wedge \Box \psi(x) &= \bigvee_{y\in U}\{R(x,y) \wedge \varphi(y)\} \wedge \Box \psi(x) \\ = \bigvee_{y\in U}\{R(x,y) \wedge \varphi(y) \wedge \Box \psi(x)\} &\leq \Diamond(\varphi \wedge \psi)(x).\end{aligned}$$

Thus, $\Diamond \varphi \wedge \Box \psi \leq \Diamond(\varphi \wedge \psi)$. Assertion (D2) can be proved similarly.

(b) The instances

$$\Box(a \vee b) \leq \Box a \vee \Diamond b \quad \text{and} \quad \boxdot(a \vee b) \leq \boxdot a \vee \diamonddot b \tag{3.5}$$

of Dunn's second axiom of (1.6) are false in some H2GC+FS-algebras of fuzzy modalities.

Namely, let $U = \{x, y\}$ and consider the finite (and hence complete) Heyting algebra $\mathbf{2}^2 \oplus 1$, that is, $\mathbb{H} = \{0, a, b, c, 1\}$ is the Heyting algebra with the order $0 < a, b < c < 1$, where a and b are incomparable. Note that $\neg a = b$ and $\neg b = a$.

We define two fuzzy sets φ, ψ on U by setting $\varphi(u) = 0$ and $\psi(u) = 1$ for all $u \in U$. A fuzzy relation $R\colon U \times U \to H$ is defined by $R(x,x) = R(y,y) = a$ and $R(x,y) = R(y,x) = b$. Then,

$$\Box(\varphi \vee \psi)(x) = \bigwedge_{u\in U}(R(x,u) \to (\varphi \vee \psi)(u)) = 1,$$

but

$$\begin{aligned}\Box \varphi(x) \vee \Diamond \psi(x) &= \bigwedge_{u\in U}(R(x,u) \to \varphi(u)) \vee \bigvee_{u\in U}(R(x,u) \wedge \psi(u)) \\ &= (\neg a \wedge \neg b) \vee (a \vee b) = c.\end{aligned}$$

Hence, $\Box(\varphi \vee \psi) \leq \Box\varphi \vee \Diamond\psi$ is not satisfied. Similarly, $\boxdot(\varphi \vee \psi)(y) = 1$ and $\boxdot\varphi(y) \vee \Diamonddot\psi(y) = c$, that is, $\boxdot(\varphi \vee \psi) \leq \boxdot\varphi \vee \Diamonddot\psi$ is not satisfied.

(c) In Lemma 2.5(c), we showed the provability of $\Box\neg A \to \neg\Diamond A$ and $\boxdot\neg A \to \neg\Diamonddot A$. This implies that also $\Diamond A \to \neg\Box\neg A$ and $\Diamonddot A \to \neg\boxdot\neg A$ are provable. Here we show that De Morgan axioms $\Diamond A \leftrightarrow \neg\Box\neg A$ and $\Box A \leftrightarrow \neg\Diamond\neg A$ discussed in Introduction are not provable in $\mathsf{Int2GC{+}FS}$.

Let us consider a linear Heyting algebra (that is, a Gödel algebra)

$$\mathbb{H} = \left\{ \frac{1}{n+1} \,\middle|\, n \in \mathbb{N} \right\} \cup \{0,1\},$$

where $\mathbb{N} = \{1,2,3,\ldots\}$. Let us set $U = \mathbb{N}$ and define a fuzzy set $\varphi\colon U \to H$ by setting $\varphi(n) = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. We also define a fuzzy relation R on U simply by setting $R(m,n) = 1$ for all $m,n \in \mathbb{N}$.

For all $n \in \mathbb{N}$,

$$\Diamond\varphi(n) = \bigvee_{m\in\mathbb{N}} \{R(n,m) \wedge \varphi(m)\} = \bigvee_{m\in\mathbb{N}} \left\{\frac{1}{m+1}\right\} = \frac{1}{2},$$

but

$$\Box\neg\varphi(n) = \bigwedge_{m\in\mathbb{N}} \{R(n,m) \to \neg\varphi(m)\} = \bigwedge_{m\in\mathbb{N}} \{1 \to 0\} = 0,$$

which means $\neg\Box\neg\varphi(n) = 1$ for all $n \in \mathbb{N}$. Thus, $\Diamond\varphi(n) \neq \neg\Box\neg\varphi(n)$ for all $n \in \mathbb{N}$. Similarly, we can show $\Box\varphi \neq \neg\Diamond\neg\varphi$. Indeed, for $n \in \mathbb{N}$,

$$\Box\varphi(n) = \bigwedge_{m\in\mathbb{N}} \left\{R(n,m) \to \frac{1}{m+1}\right\} = \bigwedge_{m\in\mathbb{N}} \left\{\frac{1}{m+1}\right\} = 0.$$

On the other hand,

$$\Diamond\neg\varphi(n) = \bigvee_{m\in\mathbb{N}} \{R(n,m) \wedge \neg\varphi(m)\} = \bigvee_{m\in\mathbb{N}} \{1 \wedge 0\} = 0.$$

So, $\neg\Diamond\neg\varphi(n) = 1$ for all $n \in \mathbb{N}$.

Note that by the way the relation R is defined, these examples show that $\Diamonddot A \leftrightarrow \neg\boxdot\neg A$ and $\boxdot A \leftrightarrow \neg\Diamonddot\neg A$ cannot be proved in $\mathsf{Int2GC{+}FS}$. This example also shows that De Morgan axioms (1.4) and (1.5) cannot be proved in $\mathsf{G2GC{+}FS}$ neihter, where G stands for Gödel–Dummett logic.

Example 3.7. As we saw in Example 3.6(b), the first condition of (1.6) does not imply the second one. Here we show that the converse implication is not true neither. Thus, in Heyting algebras with two Galois connections, the conditions of (1.6) are independent.

Let $H = \{0,a,b,c,1\}$ with $0 < c < a,b < 1$, but a and b are not comparable. We define $\Diamond, \Box\colon H \to H$ by

$$\Diamond 0 = \Diamond b = \Diamond c = 0 \text{ and } \Diamond a = \Diamond 1 = a;$$
$$\Box 0 = \Box b = \Box c = b \text{ and } \Box a = \Box 1 = 1.$$

Then, we set $\Diamonddot = \Diamond$ and $\boxdot = \Box$. It is easy to verify that the pairs $(\Diamond, \boxdot)$ and $(\Diamonddot, \Box)$ are Galois connections.

In addition, the second condition of (1.6) holds for both $(\Diamond, \Box)$ and $(\Diamonddot, \boxdot)$, that is,

$$\Box(x \vee y) \leq \Box x \vee \Diamond y \text{ and } \boxdot(x \vee y) \leq \boxdot x \vee \Diamonddot y.$$

But now the first conditions of (1.6) does not hold, because, for instance, $\Diamond a \wedge \Box b = a \wedge b = c$, but $\Diamond(a \wedge b) = \Diamond c = 0$.

We end this section by noting that the above examples show that calculating using Heyting algebras with operators is much easier than calculating with categories, and calculating with algebras can be easily used in showing some of the non-theorems, for instance.

4. Representation theorem of H2GC+FS-algebras and relational completeness

We introduced in [15] relational frames and models for IntGC. An IntGC-frame $(X, \leq, R)$ is a relational structure such that $(X, \leq)$ is a quasiordered set and R is a relation on X such that

$$(\geq \circ R \circ \geq) \subseteq R. \tag{4.1}$$

An IntGC-model $(X, \leq, R, \models)$ is such that $(X, \leq, R)$ is an IntGC-frame and the satisfiability relation $\models$ is a binary relation from X to the set of propositional variables *Var* such that $x \models p$ and $x \leq y$ imply $y \models p$, For any $x \in X$ and $A \in \Phi$, we define the satisfiability relation inductively by the following way:

$$\begin{aligned}
x \models A \wedge B &\iff x \models A \text{ and } x \models B \\
x \models A \vee B &\iff x \models A \text{ or } x \models B \\
x \models A \to B &\iff \text{ for all } y \geq x,\ y \models A \text{ implies } y \models B \\
x \models \neg A &\iff \text{ for no } y \geq x \text{ does } y \models A \\
x \models \Diamond A &\iff \text{ exists } y \text{ such that } x\,R\,y \text{ and } y \models A \\
x \models \boxdot A &\iff \text{ for all } y, y\,R\,x \text{ implies } y \models A
\end{aligned}$$

We proved in [15] that IntGC is relationally complete, meaning that an IntGC-formula A is provable if and only if A is valid in all IntGC-models, that is, for any IntGC-model $(X, \leq, R, \models)$ and for all $x \in X$, we have $x \models A$.

It is clear that since Int2GC is a fusion of two independent IntGCs, relational frames for Int2GC are of the form $(X, \leq, R_1, R_2)$ such that $(X, \leq, R_1)$ and $(X, \leq, R_2)$ are IntGC-frames. Relational completeness can then be proved by defining complex frames for Int2GC and by applying the results of complex frames of IntGC (cf. [32]).

Fischer Servi described relational frames and models for IK in [23]. We will apply the same frames for Int2GC+FS. An IK-frame is a triple $(X, \leq, R)$, where $(X, \leq)$ is a quasiordered set and R is a relation on X such that

$$(R \circ \leq) \subseteq (\leq \circ R) \quad \text{and} \quad (\geq \circ R) \subseteq (R \circ \geq). \tag{4.2}$$

Note that in [35,36] these frames are called HK1-frames. Because the second condition of (4.2) is equivalent to $(R^{-1} \circ \leq) \subseteq (\leq \circ R^{-1})$, we have that $(X, \leq, R)$ is an IK-frame if and only if $(X, \leq, R^{-1})$ is an IK-frame. The following relationship holds between IntGC-frames and IK-frames.

Lemma 4.1. *A frame $(X, \leq, R)$ is an* IK*-frame if and only if $(X, \leq, R \circ \geq)$ and $(X, \leq, R^{-1} \circ \geq)$ are* IntGC*-frames.*

Proof. Let $(X, \leq, R)$ be an IK-frame. Then,

$$(\geq \circ (R \circ \geq) \circ \geq) \subseteq (\geq \circ R \circ \geq) \subseteq (R \circ \geq \circ \geq) \subseteq (R \circ \geq),$$

Thus, (4.1) holds for $(R \circ \geq)$. Similarly, since $(X, \leq, R^{-1})$ is an IK-frame,

$$\begin{aligned}(\geq \circ (R^{-1} \circ \geq) \circ \geq) &\subseteq (\geq \circ (R^{-1} \circ \geq)) = ((\geq \circ R^{-1}) \circ \geq) \\ &\subseteq ((R^{-1} \circ \geq) \circ \geq) \subseteq (R^{-1} \circ \geq).\end{aligned}$$

Hence, (4.1) holds for $(R^{-1} \circ \geq)$ also.

Conversely, suppose $(X, \leq, R \circ \geq)$ and $(X, \leq, (R^{-1} \circ \geq))$ are IntGC-frames. Then,

$$(R \circ \leq) = (\geq \circ R^{-1})^{-1} \subseteq (\geq \circ (R^{-1} \circ \geq) \circ \geq)^{-1} \subseteq (R^{-1} \circ \geq)^{-1} = (\leq \circ R).$$

In addition,

$$(\geq \circ R) \subseteq ((\geq \circ R) \circ \geq \circ \geq) = (\geq \circ (R \circ \geq) \circ \geq) \subseteq (R \circ \geq).$$

Therefore, R satisfies (4.2). $\square$

In IK-models $(X, \leq, R, \models)$, the satisfiability relation $\models$ for $\vee$, $\wedge$, $\to$, and $\neg$ are defined as earlier, and satisfiability for $\Diamond\, A$, and $\Box\, A$ are defined by:

$$\begin{aligned} x \models \Diamond\, A &\iff \text{exists } y \text{ such that } x\,(R \circ \geq)\,y \text{ and } y \models A \\ x \models \Box\, A &\iff \text{for all } y, x\,(\leq \circ R)\,y \text{ implies } y \models A \end{aligned}$$

For the remaining $\Diamonddot\, A$ and $\boxdot\, A$, we define the satisfiability relation by:

$$\begin{aligned} x \models \Diamonddot\, A &\iff \text{exists } y \text{ such that } y\,(\leq \circ R)\,x \text{ and } y \models A \\ x \models \boxdot\, A &\iff \text{for all } y, y\,(R \circ \geq)\,x \text{ implies } y \models A \end{aligned}$$

In the sequel, we call these models IK^2-models. The idea is that the models are based on IK-frames, but satisfiability is defined twice: both for the pairs $(\Diamond, \Box)$ and $(\Diamonddot, \boxdot)$. A formula A is *relationally valid* if it is valid in every IK^2-model, that is, $x \models A$ holds for all elements x in the model. Note that:

$$\begin{aligned} x \models \Diamond\, A &\iff \text{exists } y \text{ such that } x\,R\,y \text{ and } y \models A \\ x \models \Diamonddot\, A &\iff \text{exists } y \text{ such that } y\,R\,x \text{ and } y \models A \end{aligned}$$

We may now give the following soundness result.

Proposition 4.2. *Every* $\mathsf{Int2GC{+}FS}$*-provable formula is relationally valid.*

Proof. We need to show that the axioms of $\mathsf{Int2GC{+}FS}$ are valid in all IK^2-models, and that the Galois connection rules preserve validity.

In [23], it is proved that axioms (FS1) are (FS3) are valid in all IK-frames. As an example, we consider (FS2). Validity of (FS4) can be proved in a similar way. If (FS2) is not valid, then there exists $x \in X$ such that (i) $x \models \Diamonddot(A \to B)$, but (ii) $x \not\models \boxdot\, A \to \Diamonddot\, B$. By (i), there is $y\,R\,x$ such that (iii) $y \models A \to B$, and (ii) means that there is $z \geq x$ such that (iv) $z \models \boxdot\, A$, but (v) $z \not\models \Diamonddot\, B$. We have $y\,(R \circ \leq)\,z$, which implies by (4.2) that $y\,(\leq \circ R)\,z$, meaning that there is $v \geq y$ such that $v\,R\,z$. By (v), we get $v \not\models B$ and (iii) gives $v \not\models A$. Now $v\,(R \circ \geq)\,z$ implies $z \not\models \boxdot\, A$, a contradiction to (iv).

Because the validity of $\Diamond$ and $\boxdot$ are defined in terms of $R \circ \geq$ and its inverse, it is clear that the pair $(\Diamond, \boxdot)$ is a Galois connection on Φ, that is, the rules (GC $\Diamond\boxdot$) and (GC $\boxdot\Diamond$) preserve validity, and the same holds for the pair $(\Diamonddot, \Box)$. $\square$

Lemma 4.3. *For all* IK^2*-models* $(X, \leq, R, \models)$ *and formulas* $A \in \Phi$*:*

$$x \models A \quad and \quad x \leq y \quad imply \quad y \models A.$$

Proof. We need to show the persistency of $⟐$ and $⊡$, because other connectives are considered in [23]. Suppose $x \models ⟐ A$ and $x \leq y$. Then, there is $z\,R\,x$ such that $z \models A$. Now $z\,(R \circ \leq)\,y$ imply $z\,(\leq \circ R)\,y$. Thus, $y \models ⟐ A$.

Assume that $x \models ⊡ A$ and $x \leq y$. If $z\,(R \circ \geq)\,y$, then also $z\,(R \circ \geq)\,x$ and $z \models A$. So, $y \models ⊡ A$. □

Orłowska and Rewitzky studied canonical frames of HK1-algebras in [36]. For a HK1-algebra $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box)$, let us denote for any $A \subseteq H$,

$$\Box^{-1} A = \{x \in H \mid \Box x \in A\} \quad \text{and} \quad \Diamond^{-1} A = \{x \in H \mid \Diamond x \in A\}.$$

Let $X(H)$ be the set of all prime lattice filters of H. A relation R^c is defined on $X(H)$ by

$$F\,R^c\,G \iff \Box^{-1} F \subseteq G \subseteq \Diamond^{-1} F. \tag{4.3}$$

Orłowska and Rewitzky showed that this frame is an IK-frame. For an H2GC+FS-algebra $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, ⟐, ⊡)$, its *canonical frame* is $(X(H), \subseteq, R^c)$. So, we are using the same canonical frames as for HK1-algebras. Let us denote:

$$⊡^{-1} A = \{x \in H \mid ⊡ x \in A\} \quad \text{and} \quad ⟐^{-1} A = \{x \in H \mid ⟐ x \in A\}.$$

Lemma 4.4. *Let* $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, ⟐, ⊡)$ *be an H2GC+FS-algebra. Then, for all* $F, G \in X(H)$*,*

$$F\,R^c G \iff ⊡^{-1} G \subseteq F \subseteq ⟐^{-1} G. \tag{4.4}$$

Proof. Suppose $F\,R^c G$. If $x \in ⊡^{-1} G$, then $⊡ x \in G$. Now $\Diamond ⊡ x \leq x$. Assume $x \notin F$. Then $\Diamond ⊡ x \notin F$, which by the definition of R^c gives $⊡ x \notin G$, a contradiction. Similarly, if $x \in F$, then $\Box ⟐ x \geq x$ and $\Box ⟐ x \in F$. This implies $⟐ x \in G$ by the definition of R^c. Conversely, if $⊡^{-1} G \subseteq F \subseteq ⟐^{-1} G$, then $F\,R^c G$ can be proved in an analogous manner. □

Lemma 4.4 means that we have two ways to define the relation R^c, either by using $\Diamond$ and $\Box$, or by using $⟐$ and $⊡$. Let $(X(H), \subseteq, R^c)$ be the canonical frame of some H2GC+FS-algebra on H. To obtain the canonical model, we define the relation $\models_c$ from $X(H)$ to *Var* by $F \models_c p$ if and only if $v(p) \in F$.

In the book [35], it is shown that in the canonical frame $(X(H), \subseteq, R^c)$ of an HK1-algebra $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box)$, for all $F, G \in X(H)$,

(i) $F\,(\subseteq \circ R^c)\,G \iff \Diamond^{-1} F \subseteq G$;
(ii) $F\,(R^c \circ \supseteq)\,G \iff G \subseteq \Box^{-1} F$.

We extend this result to H2GC+FS-algebras.

Lemma 4.5. *Let* $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, ⟐, ⊡)$ *be an H2GC+FS-algebra. Then in the canonical frame* $(X(H), \subseteq, R^c)$*, for all* $F, G \in X(H)$*,*

(a) $F\,(\subseteq \circ R^c)\,G \iff F \subseteq ⟐^{-1} G$*;*
(b) $F\,(R^c \circ \supseteq)\,G \iff ⊡^{-1} G \subseteq F$*.*

Proof. (a) Suppose that $F\,(\subseteq \circ R^c)\,G$. Then, there is $J \in X(H)$ such that $F \subseteq J$ and $J\,R^c\,G$, that is, $\boxdot^{-1} G \subseteq J \subseteq \diamonddot^{-1} G$. Then, $F \subseteq \diamonddot^{-1} G$. For the other direction, assume that $F \subseteq \diamonddot^{-1} G$. Consider the filter K generated by $F \cup \boxdot^{-1} G$. Suppose $K \cap -\diamonddot^{-1} G \neq \emptyset$. Then, there are $a \in -\diamonddot^{-1} G$, $b \in F$ and $c \in \boxdot^{-1} G$ such that $b \wedge c \leq a$. Note that $\boxdot^{-1} G$ is a filter, so it is closed under finite meets. Hence, $b \leq c \to a$, and $\diamonddot b \leq \diamonddot(c \to a) \leq (\boxdot c \to \diamonddot a)$ by (FS3). Because $b \in F$ and $F \subseteq \diamonddot^{-1} G$, we have $\diamonddot b \in G$ and so $\boxdot c \to \diamonddot a \in G$. Now $\boxdot c \in G$ implies $\diamonddot a \in G$, a contradiction. Thus, $K \cap -\diamonddot^{-1} G = \emptyset$. It is easy to see that $-\diamonddot^{-1} G$ is an ideal. By the Prime Filter Theorem of distributive lattices, there is $J \in X(H)$ such that $K \subseteq J$ and $J \cap -\diamonddot^{-1} G = \emptyset$, that is, $J \subseteq \diamonddot^{-1} G$. Now $F \subseteq K \subseteq J$. Also $\boxdot^{-1} G \subseteq K$ by the definition of K. So, $\boxdot^{-1} G \subseteq J \subseteq \diamonddot^{-1} G$, that is, $J\,R^c\,G$ by Lemma 4.4. Thus, $F\,(\subseteq \circ R^c)\,G$.

(b) Assume that $F\,(R^c \circ \supseteq)\,G$. Then, there exists $J \in X(H)$ such that $F\,R^c\,J$ and $G \subseteq J$. Hence, $\boxdot^{-1} J \subseteq F$ and $\boxdot^{-1} G \subseteq \boxdot^{-1} J$ imply $\boxdot^{-1} G \subseteq F$. On the other hand, assume $\boxdot^{-1} G \subseteq F$. Let K be the filter generated by $G \cup \diamonddot F$, where $\diamonddot F = \{\diamonddot x \mid x \in F\}$. Since F is a prime filter, its complement $-F$ is a prime ideal. So, $-F \neq \emptyset$ and $\boxdot(-F) = \{\boxdot x \mid x \notin F\} \neq \emptyset$. Let I be an ideal generated by $\boxdot(-F)$. Assume for contradiction that $K \cap I \neq \emptyset$. Then, there exist $a \in G$, $b_1, \cdots, b_m \in F$, and $d \in I$ such that $a \wedge \diamonddot b_1 \wedge \cdots \wedge \diamonddot b_m \leq d$. Take $b = b_1 \wedge \cdots \wedge b_m \in F$. We note that $\diamonddot b = \diamonddot(b_1 \wedge \cdots \wedge b_m) \leq \diamonddot b_1 \wedge \cdots \wedge \diamonddot b_m$. Since $d \in I$, there are $c_1, c_2, \ldots, c_n \notin F$ such that $d \leq \boxdot c_1 \vee \boxdot c_2 \vee \cdots \vee \boxdot c_n \leq \boxdot(c_1 \vee c_2 \vee \cdots \vee c_n)$. Because F is a prime filter, $c = c_1 \vee c_2 \vee \cdots \vee c_n \notin F$. Since $a \wedge \diamonddot b \leq d \leq \boxdot c$, we have $a \leq \diamonddot b \to \boxdot c \leq \boxdot(b \to c)$ by (FS4). Now $a \in G$ implies $\boxdot(b \to c) \in G$. Because $\boxdot^{-1} G \subseteq F$, we have $b \to c \in F$. But now $b \in F$ implies $c \in F$, a contradiction. Hence, $K \cap I = \emptyset$. By the Prime Filter Theorem of distributive lattices, there is $J \in X(H)$ such that $K \subseteq J$ and $J \cap I = \emptyset$. By the definition of K, we have $G, \diamonddot F \subseteq K \subseteq J$. In addition, $J \subseteq -\boxdot(-F)$. Therefore, $\boxdot^{-1} J \subseteq \boxdot^{-1}(-\boxdot(-F)) = -\boxdot^{-1}(\boxdot(-F)) \subseteq F$. Because $\diamonddot F \subseteq J$, we obtain $F \subseteq \diamonddot^{-1} J$. Thus, $\boxdot^{-1} J \subseteq F \subseteq \diamonddot^{-1} J$ and $F\,R^c\,J$. Since $G \subseteq J$, we have $F\,(R^c \circ \supseteq)\,G$. □

For an IK-frame $(X, \leq, R)$, let $\mathcal{T}_\leq$ be the set of $\leq$-closed subsets of X, that is,

$$\mathcal{T}_\leq = \{A \subseteq X \mid (\forall x, y \in X)\, x \in A \;\&\; x \leq y \Rightarrow y \in A\}. \tag{4.5}$$

Then, $\mathcal{T}_\leq$ is an *Alexandrov topology*, that is, it is a topology closed also under arbitrary intersections. Another common name used for an Alexandrov topology is a *complete ring of sets*. Let us denote by $\mathcal{I}_\leq \colon \wp(X) \to \wp(X)$ the interior operator of the topology $\mathcal{T}_\leq$, that is, for all $A \subseteq X$,

$$\mathcal{I}_\leq(A) = \bigcup\{B \in \mathcal{T}_\leq \mid B \subseteq A\}.$$

This means that $\mathcal{T}_\leq = \{\mathcal{I}_\leq(A) \mid A \subseteq X\}$. The lattice $(\mathcal{T}_\leq, \subseteq)$ forms a Heyting algebra such that for all $A, B \in \mathcal{T}_\leq$,

$$A \to^c B = \mathcal{I}_\leq(-A \cup B).$$

Let us define for a relational frame $(X, \leq, R)$ the following four operators $\wp(X) \to \wp(X)$:

$$\Box^c A = \{x \in X \mid x\,(\leq \circ R)\,y \Rightarrow y \in A\}$$
$$\Diamond^c A = \{x \in X \mid (\exists y)\, x\,R\,y \;\&\; y \in A\}$$
$$\boxdot^c A = \{x \in X \mid y\,(R \circ \geq)\,x \Rightarrow y \in A\}$$
$$\Diamonddot^c A = \{x \in X \mid (\exists y)\, y\,R\,x \;\&\; y \in A\}.$$

Orłowska and Rewitzky [36] proved that

$$(\mathcal{T}_{\leq}, \cup, \cap, \to^c, \emptyset, X, \Diamond^c, \Box^c)$$

is an HK1-algebra. It is clear that since also $(X, \leq, R^{-1})$ is an IK-frame, and $\Diamonddot$ is defined in terms of the inverse R^{-1} of R and $\boxdot$ is defined in terms of the inverse of $(R \circ \geq)$ and $(R \circ \geq)^{-1} = (\leq \circ R^{-1})$, the algebra

$$(\mathcal{T}_{\leq}, \cup, \cap, \to^c, \emptyset, X, \Diamonddot^c, \boxdot^c)$$

is an HK1-algebra. Obviously, the pairs $(\Diamond, \boxdot)$ and $(\Diamonddot, \Box)$ are Galois connections on $(\mathcal{T}_{\leq}, \subseteq)$. Note that:

$$\Diamond^c A = \{x \in X \mid (\exists y)\, x\,(R \circ \geq)\,y \;\&\; y \in A\}$$
$$\Diamonddot^c A = \{x \in X \mid (\exists y)\, y\,(\leq \circ R)\,x \;\&\; y \in A\}.$$

This then implies that

$$C(X) = (\mathcal{T}_{\leq}, \cup, \cap, \to^c, \emptyset, X, \Diamond^c, \Box^c, \Diamonddot^c, \boxdot^c)$$

is an H2GC+FS-algebra, called the *complex H2GC+FS-algebra* of the IK-frame $(X, \leq, R)$.

Let $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, \Diamonddot, \boxdot)$ be an H2GC+FS-algebra. We define a mapping $h\colon H \to C(X(H))$ from H to the complex algebra of its canonical frame by

$$h(x) = \{F \in X(H) \mid x \in F\}.$$

It is proved in [35] that

$$h(x \vee y) = h(x) \cup h(y) \qquad h(x \wedge y) = h(x) \cap h(y)$$
$$h(x \to y) = h(x) \to^c h(y)$$
$$h(0) = \emptyset \qquad h(1) = X(H)$$
$$h(\Diamond\, x) = \Diamond^c h(x) \qquad h(\Box\, x) = \Box^c h(x)$$

Note that the proof of Lemma 4.4 in [36] contains some mistakes, but the proof is corrected in [35]. We can extend this result to H2GC+FS-algebras.

Lemma 4.6. *Let $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, \Diamonddot, \boxdot)$ be an H2GC+FS-algebra.*
(a) $h(\Diamonddot\, x) = \Diamonddot^c h(x)$;
(b) $h(\boxdot\, x) = \boxdot^c h(x)$.

Proof. (a) Suppose that $F \in \Diamonddot^c h(x)$. This means that there is $G \in X(H)$ such that $G\,R^c\,F$ and $G \in h(x)$. Now, $x \in G \subseteq \Diamonddot^{-1} F$ and $\Diamonddot\, x \in F$, that is, $F \in h(\Diamonddot\, x)$. On the other hand, assume $F \in h(\Diamonddot\, x)$, that is, $\Diamonddot\, x \in F$. Suppose ${\uparrow}x \cap - \Diamonddot^{-1} F \neq \emptyset$. Then, there exists $y \in H$ such that $x \leq y$ and $y \notin \Diamonddot^{-1} F$. We have $\Diamonddot\, y \notin F$ and $\Diamonddot\, x \leq \Diamonddot\, y$, which give $\Diamonddot\, x \notin F$, a contradiction. Therefore, ${\uparrow}x \cap - \Diamonddot^{-1} F = \emptyset$ and ${\uparrow}x \subseteq \Diamonddot^{-1} F$. Now ${\uparrow}x$ is a filter and $- \Diamonddot^{-1} F$ is an ideal, as we already noted. Then, by the Prime

Filter Theorem of distributive lattices, there is a prime filter $K \in X(H)$ such that $\uparrow x \subseteq K$ and $K \cap - \Diamond^{-1} F = \emptyset$, that is, $K \subseteq \Diamond^{-1} F$. Lemma 4.5(a) implies $K\,(\subseteq \circ R^c)\,F$. Since $x \in K$, we have $K \in h(x)$ and so $F \in \Diamond^c h(x)$.

(b) Suppose $F \in h(\boxdot x)$, that is, $\boxdot x \in F$. If $G\,(R^c \circ \supseteq)\,F$, then there is $K \supseteq F$ such that $\boxdot^{-1} K \subseteq G \subseteq \Diamond^{-1} K$. Then, $\boxdot x \in F \subseteq K$ and $x \in \boxdot^{-1} K \subseteq G$. We get $G \in h(x)$ and $F \in \boxdot^c h(x)$, as required. Conversely, assume that $F \notin h(\boxdot x)$. Then, $x \notin \boxdot^{-1} F$ and $\downarrow x \cap \boxdot^{-1} F = \emptyset$. Note that $\boxdot^{-1} F$ is a filter and $\downarrow x$ is an ideal. This means that by the Prime Filter Theorem of distributive lattices, there is a prime filter $G \in X(H)$ such that $\boxdot^{-1} F \subseteq G$ and $\downarrow x \cap G = \emptyset$. Then, $x \notin G$, $G \notin h(x)$, and by Lemma 4.5(b), $G\,(R^c \circ \supseteq)\,F$. Thus, $F \notin \boxdot^c h(x)$. □

As a corollary, we write the following representation theorem for H2GC+FS-algebras.

Theorem 4.7. *Every H2GC+FS-algebra can be embedded into the complex algebra of its canonical* IK*-frame.*

Remark 4.8. In general, the embedding h is not an isomorphism, but in some cases it can be also surjective. For instance, in [15, Theorem 7.2], we showed that every finite HGC-algebra is isomorphic to the complex algebra of its canonical frame, and a similar proof could be presented here. More generally, in [18, Theorem 18], we showed that for every spatial HGC-algebra $\mathbb{H}$, there exists an IntGC-frame $\mathcal{F}$ such that $\mathbb{H}$ is isomorphic to the complex algebra of $\mathcal{F}$. The same idea could be applied for H2GC+FS-algebras, because it is known that spatial (and thus complete) Heyting algebras are order-isomorphic to some Alexandrov topologies.

Note also that a representation theorem for tense symmetric Heyting algebras is given in [22], but their algebras differ essentially from ours.

In terms of Theorem 4.7, we can prove the Key Lemma.

Lemma 4.9. *Let* $(H, \vee, \wedge, \to, 0, 1, \Diamond, \Box, \Diamond, \boxdot)$ *be an H2GC+FS-algebra. Then, for all* $A \in \Phi$ *and* $F \in X(H)$,

$$F \models_c A \iff v(A) \in F. \tag{4.6}$$

Proof. We consider the operators $\Diamond$ and $\boxdot$ only, because for connectives $\vee$, $\wedge$, $\to$ the claim is well known, and for $\Diamond$ and $\Box$ the proof is analogous.

($\Diamond$) Suppose $A = \Diamond B$ for some $B \in \Phi$ and B satisfies (4.6). If $F \models_c \Diamond B$, then there is a prime filter G such that $F\,R^c\,G$ and $G \models_c B$, that is, $v(B) \in G$. Now, $G \subseteq \Diamond^{-1} F$ implies $v(B) \in \Diamond^{-1} F$ and $v(A) = v(\Diamond B) = \Diamond v(B) \in F$. Conversely, suppose $v(A) \in F$, that is, $F \in h(v(A)) = h(\Diamond v(B)) = \Diamond^c h(v(B))$. Then, there exists $G \in X(H)$ such that $G \in h(v(B))$ and $F\,R^c\,G$, that is, $v(B) \in G$, or equivalently $G \models_c B$. Hence, $F \models_c \Diamond B$.

($\boxdot$) Suppose that $A = \boxdot B$ and B satisfies (4.6). Assume that $v(\boxdot B) = \boxdot v(B) \in F$. If $G\,(R^c \circ \supseteq)\,F$, then by Lemma 4.5(b), $\boxdot^{-1} F \subseteq G$. Then, $\boxdot v(B) \in F$ gives $v(B) \in G$, $G \models_c B$, and $F \models_c \boxdot B$. On the other hand, if $F \models_c \boxdot B$, then $G\,(R^c \circ \supseteq)\,F$ implies $G \models_c B$, that is, $v(B) \in G$ and $G \in h(v(B))$. Hence, $F \in \boxdot^c h(v(B)) = h(\boxdot v(B)) = h(v(\boxdot B))$ and $v(\boxdot B) \in F$. □

We are now able to prove relational completeness.

Theorem 4.10. *Every formula $A \in \Phi$ is* $\mathsf{Int2GC{+}FS}$*-provable if and only if A is relationally valid.*

Proof. We have already noted that every $\mathsf{Int2GC{+}FS}$-provable formula is relationally valid. On the other hand, if A is not $\mathsf{Int2GC{+}FS}$-provable, there exists an H2GC+FS-algebra on some set H and a valuation v such that $v(A) \neq 1$. Let $(X(H), \subseteq, R, \models_c)$ be the corresponding canonical frame. Now, $h(v(A)) \neq X(H)$, which implies that there is a prime filter F such that $v(A) \notin F$. Using the Key Lemma, this implies $F \not\models_c A$, and thus A is not relationally valid. □

Kripke completeness of K_t is provided by Kripke frames (X, R), where R is an arbitrary binary relation on X. Thus, relational completeness for $\mathsf{Cl2GC{+}FS}$ is standard. Note also that the frames (X, R) can be considered as relational IK-frames $(X, R, =)$ and then the satisfiability relation is the same in both settings. The result analogous to Theorem 4.7 but for classical logic follows from the fundamental representation theorem for Boolean algebras with operators by Jónsson and Tarski [30], which says that every Boolean algebra with additive and normal operators can be embedded into the complex of its canonical frame. Therefore, if we have a Boolean algebra with two Galois connections $(\Diamond, \boxdot)$ and $(\diamonddot, \Box)$, then the operators $\Diamond$ and $\diamonddot$ are additive and normal. Since $\Box$ and $\boxdot$ are connected to $\Diamond$ and $\diamonddot$ by Fischer Servi axioms, this means that they are completely determined by De Morgan dualities.

As far as relational semantics is concerned, moving from intuitionistic logic (and classical logic) to intermediate logics, as a "base logic", is no longer as straightforward as in the case of algebraic semantics. Some intermediate logics are Kripke-incomplete, that is, they do not have adequate relational semantics. Since canonicity proofs for intermediate logics have different patterns (there are various kinds of canonicity like hypercanonicity, ω-canonicity, extensive canonicity; see [26], for instance), a uniform approach to completeness for intermediate logics seems unlikely. Therefore, the relational completeness and the representation theorems in case of particular intermediate logic are left for separate studies.

5. Some concluding remarks

We have introduced method (A), which for each intermediate logic L uniformly defines the corresponding tense logic LK_t. Method (B) introduced by Davoren [13] cannot be applied to every intermediate logic L, because this method first builds the fusion $\mathsf{LK}{\otimes}\mathsf{LK}$ of two copies of intuitionistic modal logic LK and then adds Brouwerian axioms interlinking the modalities – but in many cases, it is unclear, what the modal logic LK actually is.

Approach (A) allows a uniform treatment of algebraic semantics and we have shown algebraic completeness of $\mathsf{L2GC{+}FS}$ for any intermediate logic L. It is well known that there are many intermediate logics that are proved to be Kripke-incomplete and also there are several intermediate logics for which even their frames are not known. Here, we have presented a completeness theorem for $\mathsf{Int2GC{+}FS} = \mathsf{IK}_t$ in a way that uses IK-frames introduced by Fischer Servi [23]. This hints that in an analogous way, Kripke completeness of $\mathsf{L2GC{+}FS} = \mathsf{LK}_t$ can be proved for several intermediate modal logics L that are known to be at least Kripke-complete. For instance, Gödel–Dummett logic is characterized by the frames $(X, \leq)$ such that $(x \leq y$ and $x \leq z) \Rightarrow (y \leq z$ or $z \leq y)$. For other examples, see [11]. Notice also that different intermediate logics may need different tools to carry out a completeness proof, and these "apparatuses" are considered in [26].

We have also presented a representation theorem stating that every H2GC+FS-algebra can be embedded into the complex algebra of its canonical IK-frame. A similar proof for some intermediate logic algebras probably can be obtained, but this requires careful study of complex algebras and canonical frames. These will be studied in the future.

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