Enumeration of MOLS of small order*†

Judith Egan and Ian M. Wanless

School of Mathematical Sciences
Monash University
VIC 3800 Australia

{judith.egan, ian.wanless}@monash.edu

Abstract

We report the results of a computer investigation of sets of mutually orthogonal latin squares (MOLS) of small order. For $n \leqslant 9$ we

1. Determine the number of orthogonal mates for each species of latin square of order n.
2. Calculate the proportion of latin squares of order n that have an orthogonal mate, and the expected number of mates when a square is chosen uniformly at random.
3. Classify all sets of MOLS of order n up to various different notions of equivalence.

We also provide a triple of latin squares of order 10 that is the closest to being a set of MOLS so far found.

1 Introduction

A latin square of order n is an $n \times n$ matrix in which n distinct symbols are arranged so that each symbol occurs once in each row and column. Two latin squares $A = [a_{ij}]$ and $B = [b_{ij}]$ of order n are said to be *orthogonal* if the n^2 ordered pairs (a_{ij}, b_{ij}) are distinct. A set of MOLS (*mutually orthogonal latin squares*) is a set of latin squares in which each pair of latin squares is orthogonal. The primary aim of this paper is a thorough computational study of all sets of MOLS composed of latin squares of order at most 9.

We use k-MOLS(n) as shorthand for k MOLS of order n. If A and B are orthogonal then B is an *orthogonal mate* of A, and vice versa. A latin square with no orthogonal mate is called a *bachelor* square [33]. A set of k-maxMOLS(n) is a set of k-MOLS(n) that is *maximal* in the sense that it is not contained in any set of $(k+1)$-MOLS(n). The existence problem for 1-maxMOLS(n) (i.e. bachelor latin squares) was solved in [11, 37]. For the most recent progress on the existence of 2-maxMOLS(n), see [4]. For $k > 2$ our knowledge is quite

*2010 AMS Subject Classification 05B15 (62K99).

†Keywords: latin square; MOLS; transversal; plex; orthogonal mate

patchy; see [3] for a summary. However, for $1 \leqslant k < n \leqslant 9$, the question of whether or not there exists a set of k-maxMOLS(n) is completely answered due to the collective works of Drake [7], Jungnickel and Grams [15], and Drake and Myrvold [8]. In [3, p190] there is a table of values of k for which k-maxMOLS(n) are known to exist for $n \leqslant 61$, but missing from it is the case $(k, n) = (4, 9)$ due to [8].

A *transversal* in a latin square of order n is a selection of n distinct entries in which each row, column and symbol has exactly one representative. A partition of a latin square of order n into n disjoint transversals is called a *1-partition*. A latin square has an orthogonal mate if and only if it possesses a 1-partition [5, p155]. More generally, a p-plex is a selection of pn distinct entries in which each row, column and symbol has exactly p representatives [35]. A partition of a latin square into disjoint p-plexes is called a *p-partition*. We discuss algorithms for finding p-plexes and p-partitions in Section 3.

For a latin square L of order n, we define $\theta = \theta(L)$ to be the number of 1-partitions of L. Put another way, $\theta(L)$ is the number of orthogonal mates of L that have their first row equal to $[1, 2, \ldots, n]$. The number of transversals in a latin square is well known to be a species invariant and it follows from the same reasoning that $\theta(L)$ is also a species invariant. The definition of species (also known as "main class") will be given in Section 2.

The results in [10] include a classification of the species of orders up to 9 by whether or not they possess an orthogonal mate (see [35] for an earlier table, giving similar data for orders up to 8). In Section 5 we report data on the number of mates for all species of orders up to 9. We then calculate the expected value of $\theta(L)$ for L selected uniformly at random from the latin squares of order 9. For orders $n \leqslant 8$ this information is available in [9].

A set of $(n-1)$-MOLS(n) is also known as a *complete* set of MOLS and is equivalent to an affine plane of order n [1]. In 1896, Moore showed that the maximum cardinality of a set of MOLS of order n is $n-1$, and that this upper bound is achieved if n is a prime power [27]. The converse, whether $(n-1)$-MOLS(n) exist only if n is a prime power, is a prominent open problem. Further information and partial results can be found in [3, 5, 20]. We use the name *planar latin square* for any latin square that is a member of some set of $(n-1)$-MOLS(n). We will refer to the species of planar latin squares of order 9 by the labels $a, b, \ldots, k$ given to them by Owens and Preece [31]. In Section 6 we will investigate the role that these squares play in forming sets of maxMOLS that are not complete.

For excellent general references on enumeration problems of the type we undertake, see [16, 29]. For recent related work on enumerating mutually orthogonal latin cubes, see [18].

The outline of the paper is as follows. In Section 2 we define our basic terminology and establish the different notions of equivalence that we want to use when counting MOLS. In Section 3 we describe the basic algorithms that we used for counting MOLS, as well as providing the mathematical theory that underpins those algorithms. The case of 2-MOLS(9) is the most difficult that we treat and it requires some special considerations that are described in Section 4. In the process we give our first data from the computations, which is a classification of the 2-MOLS(9) according to how many symmetries they possess. In Section 5 we provide data on how many orthogonal mates each latin square of order up to 9 possesses and identify the squares with the most mates. We also calculate the probability that a random latin square will have a mate and the expected number of mates. The main data is provided in Section 6, where we provide counts of MOLS and maxMOLS classified according to the many notions of equivalence defined in Section 2. We also provide information on many other matters such as the number of disjoint common transversals, which species of latin squares are most prevalent in the sets of MOLS, how many MOLS contain planar latin squares and so on. In Section 7 we describe a number of ways in which we have crosschecked our data in order to reduce the chances of errors. Finally, in Section 8 we give three latin squares of order 10 that are closer to being a set of 3-MOLS(10) than any previously published.

There are numerous tables in this paper which report counts of different types of MOLS. In every table we use the convention that a blank entry should be interpreted as zero, meaning there are definitely no MOLS in that category.

2 Symmetries and notions of equivalence

The number of latin squares of order n grows rapidly as n grows and is only known [26, 14] for $n \leqslant 11$. Little is known about the number of sets of MOLS, although it is clear that it too increases very quickly [6]. To cope with this "combinatorial explosion" it is vital to use a notion of equivalence to classify the different possibilities. Several different notions of equivalence are outlined in this section. We used the weakest notion (that is, the one that considers the most things equivalent) in the first instance to compile a list of representatives from equivalence classes. From these we can then infer the number of equivalence classes using stronger notions of equivalence. With this strategy, the computational limit is the MOLS of order 9.

Taking care in our enumeration, we will sometimes need to distinguish between sets of MOLS and lists of MOLS (a list is an ordered set). The distinction will become important shortly. The definitions below are intended to apply to any number of MOLS, including the (arguably degenerate) case of MOLS that consist of a single latin square. In that case, of course, sets and lists are the same thing.

To introduce our various notions of equivalence, it is useful to discuss a well-known relationship between MOLS and orthogonal arrays. Let S be a set of cardinality s and let O be an $s^2 \times k$ array of symbols chosen from S. If, for any pair of columns of O, the ordered pairs in $S \times S$ each occur exactly once among the rows in those chosen columns, then O is an *orthogonal array* of strength 2 and index 1. We will omit further reference to the strength and index, since we will not need orthogonal arrays with other values of these parameters. See [3] for further details and background on orthogonal arrays.

A list $L_1, L_2, \dots, L_k$ of MOLS of order n can be used to build an $n^2 \times (k+2)$ orthogonal array as follows. For each row r and column c of the latin squares there is one row of the orthogonal array equal to

$$(r,\ c,\ L_1[r,c],\ L_2[r,c],\ \dots,\ L_k[r,c])\,,$$

where $L_i[r,c]$ is the symbol in row r, column c of the square L_i. Moreover, the process is reversible, so that any $n^2 \times (k+2)$ orthogonal array can be interpreted as a list of k MOLS of order n. In other words, orthogonal arrays correspond to lists of MOLS (the correspondence is not one-to-one, but only because permuting the rows of the orthogonal array changes the array but does not affect the MOLS). We will sometimes talk of an orthogonal array representing a set of MOLS. In such cases we will mean that any order can be imposed on the set to make it a list, and it makes no material difference which order is chosen.

We call two orthogonal arrays of the same dimensions *equivalent* if they are the same up to permutation of the rows and columns of the array and permutations of the symbols within each column of the array. We define two lists of MOLS to be *paratopic* if they define equivalent orthogonal arrays in this sense.

Let $\mathcal{S}_n$ denote the symmetric group of degree n. Viewed another way, paratopism is an action of the wreath product $\mathcal{S}_n \wr \mathcal{S}_{k+2}$ on lists of k-MOLS(n), where each copy of $\mathcal{S}_n$ permutes the symbols in one of the columns of the corresponding orthogonal array, while $\mathcal{S}_{k+2}$ permutes the columns themselves. An orbit under paratopism is known as a *species* of MOLS. The stabiliser of a list of MOLS M under paratopism will be called its *autoparatopism group*, which we denote by $\mathrm{par}(M)$. We say that a group is *trivial* if it has order 1, and *non-trivial* otherwise. Lists of MOLS that have trivial autoparatopism group are *rigid*, all other MOLS will be called *symmetric*.

We call two lists of MOLS *isotopic* if they define the same orthogonal array, up to permutation of the symbols within each column of the array and permutation of the rows of the array. In latin squares terminology, we are allowing uniform permutation of rows and columns of the squares as well as permutation of the symbols within each square. We call two lists of MOLS *trisotopic* if they are isotopic, or if swapping the first two columns of the orthogonal array for one of the lists makes it isotopic to the other. In latin squares terminology, trisotopism is the same as isotopism except that we also allow the squares to be transposed in the usual matrix sense.

Isotopism can be viewed as an action of the direct product of $k+2$ copies of $\mathcal{S}_n$ on lists of k-MOLS(n). The stabiliser of a list M of MOLS under isotopism is known as the *autotopism group* of M, which is denoted $\mathrm{atp}(M)$. The orbit of M under isotopism is known as its isotopism class – it is the set of all lists of MOLS that are isotopic to M. Similarly, the trisotopism class of M is the set of all lists of MOLS that are trisotopic to M.

We call two sets of MOLS isotopic (respectively, trisotopic, paratopic) if there is any way in which they can be ordered so that the resulting lists of MOLS are isotopic (respectively, trisotopic, paratopic). Again, we define the isotopism class (respectively, trisotopism class, species) of a set M of MOLS to be the set of all sets of MOLS that are isotopic (respectively, trisotopic, paratopic) to M. We will frequently discuss species of MOLS without specifying whether the MOLS are lists or sets. This is appropriate since species of lists of MOLS correspond one-to-one to species of sets of MOLS, simply by "forgetting" the order of the lists. Similar statements fail for isotopism classes and trisotopism classes – there are typically more of these for lists of MOLS than for the corresponding sets of MOLS.

For complete sets/lists of MOLS there are also geometric notions of equivalence. We define two complete sets/lists of MOLS to be *PP-equivalent* if they correspond to isomorphic projective planes, and to be *DPP-equivalent* if the projective planes they define are, up to isomorphism, either equal or dual.

The strongest possible notion of equivalence for MOLS is equality, when considered as lists or sets. A list of MOLS is *reduced* if all squares in the set have their first row in order and the first square has its first column in order. A set of MOLS is *reduced* if an ordering can be put on it to make it a reduced list of MOLS.

It should be clear from the definitions that equality is a refinement of isotopism equivalence which in turn is a refinement of trisotopism equivalence which in turn is a refinement of paratopism equivalence. For complete sets of MOLS, paratopism equivalence is a refinement of PP-equivalence which is a refinement of DPP-equivalence. The relationship between projective planes, MOLS and different notions of equivalence was studied by Owens and Preece [30, 31, 32]. Our enumerations confirm and extend a number of their results.

Some of our terminology follows the pioneers of the subject, such as Norton [28], who used "species" in our sense for single latin squares and also for larger sets of MOLS. Another term that we want to borrow from [28] is the notion of an *aspect*. An *aspect* of a list or set of MOLS is obtained by selecting 3 columns of the corresponding orthogonal array, then interpreting the result as a latin square. In our work we will only care about which species each aspect is in, so we will talk of there being $\binom{k+2}{3}$ aspects for a set or list of k MOLS. In other words, aspects will be considered to be the same if they use the same 3 columns of the orthogonal array, but in a different order.

The orthogonal array interpretation of a set of MOLS provides an easy mechanism for converting any set of k-MOLS containing a particular latin square L into another set of k-MOLS that contains L', where L' is any latin square in the same species as L. What is not so obvious is that the conversion may change the species of some or all of the $k-1$ latin squares in the sets of MOLS other than L. Variation of the species of latin squares among paratopic sets of MOLS was observed by Owens and Preece [31, 32] in their study of complete sets of MOLS of order 9 obtained from affine planes of order 9. See also [21] for an explicit example. However, with that caveat, to enumerate species of MOLS it is sufficient to start with a set of representatives of species of latin squares and find the sets of MOLS that they are contained in. The details of how we did this will be discussed in Section 3.

Suppose that M is a set of k-MOLS(n) and O is the corresponding orthogonal array. A *common transversal* for M is a selection of n of the rows of O in which no two rows share the same symbol in any column. In other words, in the $n \times (k+2)$ subarray of O formed by the chosen rows, each column is a permutation of the n symbols in O. A particularly important consideration is whether O can be partitioned into subarrays of this type. The set M of k-MOLS(n) has a set of n disjoint common transversals if and only if M is a subset of some set of $(k+1)$-MOLS(n), in other words, it is not maximal.

We finish the section with an example that illustrates why we need to carefully distinguish between sets and lists when enumerating MOLS. All calculations in this example will be in $\mathbb{Z}_5$. Define L_x to be the latin square of order 5 whose entry in cell (i, j) is $xi + j$. It is easy to see that L_1, L_2, L_3 and L_4 are mutually orthogonal. However, the list (L_1, L_4) is isotopic to (L_4, L_1), by applying the permutation $x \mapsto 4x$ to the rows of both squares. In contrast, (L_1, L_2) is not isotopic, as a list, to (L_2, L_1), even though the corresponding sets are clearly equal. Hence, the sets of MOLS $\{L_1, L_2\}$ and $\{L_1, L_4\}$ correspond to a total of three different lists, up to isotopism. All three lists are in the same species.

This example illustrates an interesting point regarding autotopism groups. We have been careful to define atp (and par) only for lists of MOLS, where the group actions that we have described are well-defined. It is tempting to define autotopisms of sets of MOLS by considering the autotopisms of a corresponding list of MOLS. If we do this in the above example, the set $\{L_1, L_4\}$ seems to have twice as many autotopisms as $\{L_1, L_2\}$, since there are the autotopisms that preserve the list (L_1, L_4), as well as those that map L_1 to L_4 and vice versa. This would mean that the number of autotopisms is not a species invariant for sets of MOLS. In any case, we do not need a notion of an autotopism group for sets of MOLS in this work.

3 Basic Algorithms

In this section we discuss the algorithms that we used for enumerating the MOLS of a given order and testing them for equivalence. The first task was to obtain a set of species representatives for the MOLS. Next we used these species representatives to count the isotopism classes and trisotopism classes for sets and lists of MOLS. Lastly, we calculated the number of reduced MOLS using two theorems that we prove at the end of this section. The techniques described in this section were feasible in most cases. The case $(k, n) = (2, 9)$ required some additional considerations, which are described in Section 4.

We began with a set of species representatives for latin squares of order n. For $n \leqslant 8$ these are available in many places, including [22]. For order 9 there are too many to store, so we generated the species representatives "on the fly", using a program written for [24]. Our first task reduces to the problem of finding a set of species representatives for sets of $(k+1)$-MOLS(n) given a set of species representatives for sets of k-MOLS(n). This requires us to find all possible 1-partitions of each k-MOLS(n) in turn. Except when $(n, k+1) = (9, 2)$, the resulting number of $(k+1)$-MOLS(n) was small enough to screen for isomorphism, in a way that we describe below, to select the required set of species representatives.

In [9] we conducted an exhaustive study of the indivisible partitions of latin squares of orders $\leqslant 8$. The algorithm used for finding partitions in that study included the special case of 1-partitions. Since it is almost as simple to describe how to find p-partitions for a general p, we describe this more general algorithm now.

The first step was to generate and store all of the p-plexes. This was possible for the cases encountered in [9] and in the present work, but for most larger squares the number of p-plexes would be too large to store. To generate all p-plexes we used a simple backtracking algorithm, aided by bit-arithmetic. Given a list $L_1, \ldots, L_k$ of k-MOLS(n) we first computed an $n \times n$ array U of bitstrings. The entry in cell (r, c) of U was $2^c + \sum_{i=1}^{k} 2^{in+L_i[r,c]}$, using $\{0, \ldots, n-1\}$ to index rows, columns and symbols. The backtracking worked row by row, adding to our plex all allowable choices of p cells from a row. To keep track of what is allowable, we maintained one bitstring for each $i = 1, 2, \ldots, p$ which recorded which symbols from each latin square and/or columns were already represented i times in our plex. These bitstrings were updated using the matrix U. Each plex that was found was stored as a bitstring of n^2 bits saying which cells were included in the plex. This allowed rapid pairwise comparison to see if two plexes were disjoint, or similarly, to check if one plex was a subset of another. The latter question was vital when testing divisibility of plexes in [9], but is not so important to us here.

In the process of generating the plexes, we also computed a look-up table T which recorded, for each plex P and row r, the index of the plex $T[P, r]$ which was the first plex in the catalogue after P whose cells in row r were different (in at least one place) from the ones used in P. This look-up table greatly sped up the second stage, which was the finding of all p-partitions. Here again we used backtracking. We built each p-partition one p-plex at a time. However, if we found that a particular plex P could not be added to our partition, then we located the first row r in which P intersected with the plexes already chosen, then skipped forward in the catalogue to $T[P, r]$, the next plex that might have a chance of being compatible. As an example, consider the process of choosing transversals to make a 1-partition. We end up choosing the transversals in order of which cell they use in the first row. Skipping forward using T is one way to ensure that we do not waste time considering transversals that clash in the first row with a transversal that we have already chosen. Note that if we are only interested in finding 1-partitions then we may enforce that the i^{th} transversal that we choose uses the i^{th} cell in the first row. However, if we are looking for the largest number of disjoint transversals then we may only assume that each transversal that we choose uses a cell in the first row to the right of that used by the previous transversal.

With the above algorithm we were able to find all 1-partitions of a set of MOLS. In particular, of course, if there are no 1-partitions then the MOLS are maximal. It is worth making some comments on an alternative approach to finding 1-partitions. Finding all the transversals in a latin square can be viewed as an instance of the `exact cover` problem [16, 29]. Once the transversals have been generated and stored, finding all the 1-partitions is another instance of `exact cover`. A solver for `exact cover`, called `libexact`, is available at [17]. It uses what [16, p.149] describes as "an algorithm that lacks serious competitors". However, as often happens, we were able to beat the general purpose algorithm by exploiting the particularities of our setting. We found that our algorithm was faster than `libexact` by a factor of 2 for average latin squares of order 9, and faster by a factor of 7 for the latin squares with the most orthogonal mates (the group tables). The time taken to find the transversals was negligible compared to the time taken to find the 1-partitions. On a standard desktop PC, our code took roughly 17 seconds to find the 12445836 mates for the elementary abelian group of order 9, but could process over 1200 typical latin squares of order 9 per second.

Next we discuss the issue of equivalence testing for MOLS. For this task we used `nauty` [25] to canonically label our MOLS, which could then be compared pairwise to see if their canonical forms were equal. This is a standard way to employ `nauty`, but we needed to encode our MOLS as a graph so that `nauty` could be applied. This is easiest to describe by considering the orthogonal array representation for the MOLS.

Suppose that we have an $n^2 \times k$ orthogonal array O corresponding to a list of MOLS M.

We now define an undirected graph G_O corresponding to O. The vertices of G_O are of three types. There are:

- k type 1 vertices that correspond to the columns of O,
- kn type 2 vertices that correspond to the symbols in each of the columns of O, and
- n^2 type 3 vertices that correspond to the rows of O.

Each type 1 vertex is joined to the n type 2 vertices that correspond to the symbols in its column. Each type 3 vertex is connected to the k type 2 vertices that correspond to the symbols in its row. There are no more edges. Vertices are coloured according to their type so that isomorphisms are not allowed to change the type of a vertex. It is now routine to check the following key facts (that generalise observations from [24], which dealt with the case $k = 3$):

- the automorphism group of G_O is isomorphic to $\mathrm{par}(M)$.
- If $G_{O'}$ is the graph corresponding to another orthogonal array O' then G_O is isomorphic to $G_{O'}$ if and only if O is paratopic to O'.

This shows how we tested paratopism (of sets or lists) of MOLS. Moreover, we can test the other equivalence relations we need by altering the colouring of the type 1 vertices. Suppose that the first two type 1 vertices correspond to the rows and columns of the latin squares respectively. Then to test isotopism of lists of MOLS we give each type 1 vertex a different colour. To test isotopism of sets of MOLS we give the first two type 1 vertices different colours, then all remaining type 1 vertices are given a third colour. In both cases, trisotopism is the same as isotopism except that the first two type 1 vertices get the same colour. Since `nauty` looks only for colour preserving isomorphisms, this allowed us to test the different notions of equivalence that we needed. We could simply take each species representative, reorder the columns of their orthogonal array in all ways that might plausibly be inequivalent, then test with `nauty` which ones were in fact inequivalent.

One other point bears mentioning, which is that `nauty` can be dramatically quickened by use of vertex invariants [25]. We trialed several invariants of which the fastest was `cellfano2`, which is one of the invariants that ships in the current distribution of `nauty`.

By using `nauty` as described above, we were able to compile catalogues of representatives for species, trisotopism classes and isotopism classes of MOLS for all cases except when $(k, n) = (2, 9)$. We did not do any computations of PP or DPP equivalence, since classification of sets of MOLS under those notions is well known [3] for orders up to and including 9. So it only remains to discuss how we counted reduced MOLS. For this we employed the following theorems. In the next result, a *class* of MOLS should be interpreted as containing both lists and sets of MOLS, with a list being a member of the class if and only if the corresponding set is in the class.

Theorem 3.1. *Suppose* $1 \leqslant k < n$. *Let* $\mathscr{M}$ *be any class of* k-$\mathrm{MOLS}(n)$ *that is closed under isotopisms. Let* $\mathrm{RS}_{\mathscr{M}}$ *and* $\mathrm{RL}_{\mathscr{M}}$ *be the number of reduced sets and reduced lists, respectively,* *in $\mathscr{M}$. Let $\mathrm{AS}_{\mathscr{M}}$ and $\mathrm{AL}_{\mathscr{M}}$ be the corresponding numbers of arbitrary (that is, not necessarily reduced) sets and lists. These numbers are related by*

$$(k-1)!n!^k(n-1)!\,\mathrm{RS}_{\mathscr{M}} = n!^k(n-1)!\,\mathrm{RL}_{\mathscr{M}} = \mathrm{AL}_{\mathscr{M}} = k!\,\mathrm{AS}_{\mathscr{M}}\,.$$

Proof. Since orthogonal latin squares cannot be equal, the last equality is immediate. The first equality is similar, given that any reduced set or list of MOLS contains a unique reduced latin square.

To prove the middle inequality, we construct a bipartite multigraph where the two vertex parts are, respectively, the reduced and arbitrary lists of k-MOLS(n). Let R be any reduced list of k-MOLS(n). We add one edge from R for every isotopism, with the other end of the edge being the list that results from applying the isotopism to R. Hence the degree of R will be $n!^{k+2}$, the number of possible isotopisms.

Now consider A, an arbitrary list of k-MOLS(n). The degree of A will be the number of isotopisms that can be applied to A to produce a reduced list of MOLS. Such an isotopism is determined by the permutation it applies to the columns of the squares in A, and which row becomes the first row. Once these choices are made, there is a unique way to permute the symbols in each square to get the first row in order and a unique way to permute the remaining rows to get the first column of the first square in order. Hence there are $n!\,n$ possible choices, and each produces exactly one reduced list. In other words, the degree of A is $n!\,n$. Thus our multigraph is bi-regular, so the sizes of the two vertex parts are in the opposite ratio to the degrees of the vertices in those parts, yielding the claimed equality. □

Theorem 3.1 deals with classes of k-MOLS(n) that are closed under isotopisms. An important example is the class of all k-MOLS(n). In that case we will write $\mathrm{RS}_{k,n}$, $\mathrm{RL}_{k,n}$, $\mathrm{AL}_{k,n}$, $\mathrm{AS}_{k,n}$ instead of $\mathrm{RS}_{\mathscr{M}}$, $\mathrm{RL}_{\mathscr{M}}$, $\mathrm{AL}_{\mathscr{M}}$, $\mathrm{AS}_{\mathscr{M}}$, respectively.

Theorem 3.2. *Suppose $1 \leqslant k < n$. Let $\mathscr{O}$ be a set of sets of k-MOLS(n) such that no two elements of $\mathscr{O}$ are paratopic. The number of reduced sets of k-MOLS(n) that are paratopic to some member of $\mathscr{O}$ is*

$$n!\,n(k+2)(k+1)k\sum_{M\in\mathscr{O}}\frac{1}{|\operatorname{par}(M)|}.$$

Proof. It suffices to prove the case when $\mathscr{O}$ contains a single set M of k-MOLS(n). Let A be an orthogonal array representation of M. By the Orbit-Stabiliser Theorem, the number of orthogonal arrays equivalent to A is

$$\frac{|\mathcal{S}_n \wr \mathcal{S}_{k+2}|}{|\operatorname{par}(M)|} = \frac{n!^{k+2}(k+2)!}{|\operatorname{par}(M)|}. \tag{1}$$

The number of reduced sets of k-MOLS(n) paratopic to M is obtained by dividing (1) by $n!^k(n-1)!(k-1)!$, by Theorem 3.1. □

In particular, Theorem 3.2 can be used to find $\mathrm{RS}_{k,n}$ from a set of species representatives for sets of k-MOLS(n), using `nauty` to find $|\operatorname{par}(M)|$ for each representative M. We can then employ Theorem 3.1 to find $\mathrm{RL}_{k,n}$, $\mathrm{AL}_{k,n}$ and $\mathrm{AS}_{k,n}$.

4 Pairs of MOLS of order 9

In this section we explain the most difficult part of our computations, namely finding the number of pairs of MOLS of order 9 modulo each of the equivalences defined in Section 2. Throughout this section, MOLS will mean an ordered pair (list) of reduced MOLS, and all latin squares will have order 9.

Each pair of MOLS has exactly four aspects. We use the notation $P[i]$ to denote the aspect that results from deleting the i^{th} column of the orthogonal array corresponding to a pair P of MOLS.

Unsurprisingly, symmetry plays a crucial role in our counting. For this reason, one task was to find sets Γ_1 and Γ_2 of species representatives for the symmetric latin squares and symmetric MOLS, respectively (we stress that symmetric is used throughout in the sense defined in Section 2, not in the usual matrix sense). The authors of [24] collated Γ_1, which contains 2523159 latin squares. We will explain below how we found Γ_2, and then analysed it to deduce data on the rigid MOLS from the overall totals.

Let Λ be a set of species representatives of reduced latin squares of order 9. Let Ω be the set of all pairs (A, B) of reduced MOLS for which $A \in \Lambda$. Using the method discussed in Section 3, we generated Ω and found that

$$|\Omega| = \sum_{A \in \Lambda} \theta(A) = 390255632. \tag{2}$$

We did not store all of Ω but kept statistics from the generation as well as a list (which will be defined shortly) of candidates for members of Γ_2. In any MOLS with a non-trivial autotopism group both latin squares have a non-trivial autotopism group and hence are paratopic to a member of Γ_1. Such MOLS are relatively easy to generate directly from Γ_1. Hence, while generating Ω, we only needed to find all MOLS that have an autoparatopism that is not an autotopism. Such MOLS necessarily have two paratopic aspects. Rather than the relatively time-consuming task of calculating the autoparatopism group of each set of MOLS in Ω we computed two species invariants for each aspect. First we calculated the number of intercalates, and if that did not discriminate between the aspects, we counted the number of transversals. If any two of the four aspects agreed on both statistics then we stored the MOLS as a candidate for being in Γ_2. These candidates, together with the 25382851 MOLS (A, B) for which $A \in \Gamma_1$, were subsequently screened to produce Γ_2. As it turned out, $|\Gamma_2| = 257442$. A by-product of our method for finding Γ_2 is that we were also able to identify MOLS with two paratopic aspects even if there was no symmetry that mapped one to the other. Data on this issue will be presented in Table 15.

We next consider how many times a given species of MOLS will appear in Ω.

Lemma 4.1. *The number of MOLS in Ω that are paratopic to a given pair P is*

$$\frac{1}{|\operatorname{par}(P)|} \sum_{i=1}^{4} \big|\operatorname{par}(P[i])\big|.$$

$\lvert \mathrm{par}(A) \rvert$	#Species	#Pairs	#Symmetric	χ
1	19268330382	364872781	70240	364802541
2	2497877	2620967	654163	983402
3	15618	77434	42211	11741
4	6890	923949	166421	189382
5	12			
6	2237	1010064	65304	157460
7	5	7		1
8	151	149780	47940	12730
9	10	677	434	27
10	21			
12	196	1807096	122512	140382
14	1			
16	10	25392	8224	1073
18	43	93779	12923	4492
20	3			
21	4			
24	28	555467	74291	20049
30	4			
32	1	284	124	5
36	23	685034	79838	16811
48	1	197	149	1
54	2	187657	16693	3166
60	1			
72	6	541584	105192	6061
96	2	14568	10152	46
108	4	260888	27392	2162
162	1	3124	2314	5
168	1	84	84	
216	2	544264	105136	2033
324	1	139968	81972	179
432	1	4171	3739	1
972	1	1241361	225621	1045
2916	1	2049219	375435	574
23328	1	12445836	4071084	359
Total	19270853541	390255632	6369588	366355728

Table 1: Data for counting pairs (A, B) of MOLS(9)

Proof. Let G denote the paratopism group $\mathcal{S}_n \wr \mathcal{S}_4$ and let $H = \mathcal{S}_n \wr (\mathcal{S}_3 \times S_1)$ be the subgroup of G that preserves the species of $P[4]$, the first square in the pair P. For $g \in G$, let P^g denote the image of P under the action of g and let P^G denote the orbit of P under the action of G. The quantity we seek is $|\Omega \cap P^G|$. From the action of H we see that each choice of A from a species of latin squares has the same number of choices for B for which $(A, B) \in P^G$. Hence

$$\left|\Omega \cap P^G\right| = \left|\{(A,B) \in P^G : A \in \Lambda\}\right| = \sum_{(A,B)\in P^G} \frac{1}{|(A,B)^H|} = \sum_{(A,B)\in P^G} \frac{|\operatorname{par}(A)|}{|H|},$$

by the Orbit-Stabiliser Theorem. Now

$$\sum_{(A,B)\in P^G} |\operatorname{par}(A)| = \sum_{g\in G} \frac{|\operatorname{par}(P^g[4])|}{|\operatorname{par}(P)|} = \sum_{i=1}^{4} \frac{|H||\operatorname{par}(P^g[i])|}{|\operatorname{par}(P)|},$$

from which the result follows. $\square$

Table 1 shows some of the data that was used to calculate the number of pairs of MOLS of order 9. In it, MOLS are classified according to $g = |\operatorname{par}(A)|$, the order of the autoparatopism group of the first latin square in the pair. The value of g is listed in the first column. The second column counts how many species of latin squares have autoparatopism group of size g (this data was first calculated in [24]). The third column records the number of MOLS, in other words, $\sum \theta(A)$ over all $A \in \Lambda$ with $|\operatorname{par}(A)| = g$. The value for $g = 1$ was deduced from (2) and the values for larger g. The fourth column lists how many symmetric MOLS were counted in the third column. This information was obtained by applying Lemma 4.1 to Γ_2. The last column of Table 1 is headed χ. It is calculated by subtracting the fourth column from the third column, then dividing by g (the first column). By Lemma 4.1, the total χ, namely 366355728, is four times the number of rigid MOLS, which must therefore be 91588932. Together with the $|\Gamma_2| = 257442$ species of symmetric MOLS, this shows that there are a total of 91846374 species of 2-MOLS(9).

Table 2 shows the 91846374 species of MOLS categorised by the sizes of their autoparatopism group and autotopism group ($|\operatorname{par}|$ and $|\operatorname{atp}|$, respectively). For each combination of these group sizes, the table lists the total number of species with groups of those sizes and also the number of species of non-maximal MOLS. There are only 433 species of non-maximal MOLS. These were identified by screening Ω as it was produced, with all non-maximal MOLS that we encountered then being stored in a separate file for later analysis, including construction of all larger sets of MOLS. We stress that the values of $|\operatorname{par}|$ and $|\operatorname{atp}|$ given in Table 2 are for lists rather than sets of MOLS (cf. the example at the end of Section 2).

5 The number of orthogonal mates

In this section we provide data on the number of orthogonal mates for latin squares of order up to and including 9. Data on the number of species of bachelor latin squares of order

|par|	|atp|	all	non-max
1	1	91588932	3
2	1	72273	12
2	2	156009	18
3	1	1859	3
3	3	17346	40
4	1	25	1
4	2	4923	32
4	4	411	1
6	1	302	7
6	2	275	
6	3	1522	28
6	6	1074	90
8	1	2	2
8	2	111	10
8	4	123	3
8	8	1	
9	3	103	3
9	9	256	18
12	2	51	4
12	3	1	1
12	4	6	
12	6	228	32
12	12	4	
16	2	9	2
16	4	37	
16	8	5	2
18	3	75	6
18	6	43	6
18	9	101	18
18	18	50	24
24	1	1	1
24	6	10	1
24	12	6	
27	27	12	
32	4	4	1
32	8	4	3
36	6	27	6

|par|	|atp|	all	non-max
36	12	2	
36	18	40	15
48	2	3	2
48	4	1	
48	6	7	
48	8	3	2
48	12	1	
54	9	19	2
54	27	15	1
54	54	2	1
64	8	4	1
72	12	1	
72	18	7	7
72	36	1	1
81	27	2	
96	12	2	
108	18	9	1
108	54	7	3
144	36	1	1
162	27	4	3
162	54	1	
162	162	1	
216	36	1	
216	54	3	2
288	48	1	1
324	54	1	
384	48	1	1
432	54	1	1
432	72	2	2
486	81	1	
3888	486	1	
576	72	1	1
972	162	4	4
5184	648	1	1
11664	486	1	1
93312	3888	1	1
	Total	91846374	433

Table 2: Species of 2-MOLS(9) categorised by symmetry

order	Proportion of species that have a mate	Probability of a random latin square having a mate	Expected number of mates
3	1	1	1
4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$
5	$\frac{1}{2}$	$\frac{3}{28} \approx 0.107143$	$\frac{9}{28} \approx 0.321429$
6	0	0	0
7	$\frac{6}{147} \approx 0.040816$	$\frac{5891}{564736} \approx 0.010431$	$\frac{1427}{70592} \approx 0.020215$
8	$\frac{2024}{283657} \approx 0.007135$	$\frac{103065585}{22303391744} \approx 0.004621$	$\frac{40888485}{2787923968} \approx 0.014666$
9	$\frac{348498052}{19270853541} \approx 0.018084$	$\frac{23706924145915}{1311102676959232} \approx 0.018082$	$\frac{24960190907155}{1311102676959232} \approx 0.019038$

Table 3: Data for random latin squares of order $3 \leqslant n \leqslant 9$

r	7	8	9
0	1	1223	336634416
1	3	329	11654552
2		175	123054
3	1	90	38700
4		67	20131
5		49	10913
6		31	7672
7		17	4552
8		15	2141
9	1	7	902
10		4	341

r	7	8	9
11		6	379
12		5	217
13		1	30
14		3	6
15		1	
16		1	31
17			10
18			2
20			2
23			1
Total	6	2024	348498052

Table 4: Non-bachelor species of order $7 \leqslant n \leqslant 9$ grouped by $r = \lfloor \log_2(\theta) \rfloor$

$n \leqslant 9$ was first published in [10]. Here we calculate the probability of a uniformly random latin square having an orthogonal mate, and the expected number of mates. This is a simple calculation where each species is weighted by the number of latin squares in that species in order to calculate statistics across the whole set of latin squares of a given order. The results are given in Table 3. It is noteworthy that in [24] it is estimated that around 60% of latin squares of order 10 have mates and the expected number of mates in a random latin square of order 10 exceeds 1. The values for orders in the range $5 \leqslant n \leqslant 9$ are clearly a lot smaller than this.

The only latin squares of order less than 7 that have orthogonal mates are isotopic to the cyclic group of order 3 (which has 1 mate), the elementary abelian group of order 4 (2 mates) or the cyclic group of order 5 (3 mates). Hence for the remainder of this section we concentrate on the range $7 \leqslant n \leqslant 9$.

For the latin squares of order $7 \leqslant n \leqslant 9$ that have $\theta > 0$ mates, we provide a summary of the number of mates in Table 4. Since θ takes many different values for these squares and the distribution is distinctly skewed towards smaller values, we have grouped the counts according to the value of $r = \lfloor \log_2(\theta) \rfloor$. In other words, for each r the table reports the number of different species for which the number of mates lies in the interval $[2^r, 2^{r+1})$.

It is not surprising that the latin squares with the most orthogonal mates tend to have nice algebraic structure. The two species of order 9 with the most mates contain the elementary abelian group (12445836 mates) and the cyclic group (2049219 mates). The species with the third highest number of mates (1241361) contains the 3 non-associative conjugacy-closed loops of order 9 (see [19] for a definition of these loops). Below that, the sequence of the number of mates continues 403056, 277788, 253276, 242832, 237786, 226822, 207297,.... There are 74 species with at least 10000 mates and every one of them has a non-trivial autotopism group and at least 4 subsquares of order 3. The species with the largest number of mates and a trivial autotopism group (in fact, it is rigid) has 8226 mates, 6 subsquares of order 3 and 371 transversals. A representative of this species is

$$\begin{bmatrix}
0&1&2&3&4&5&6&7&8\\
1&2&0&4&5&3&7&8&6\\
2&0&1&5&3&4&8&6&7\\
5&8&3&2&7&6&0&1&4\\
6&4&7&8&1&0&5&3&2\\
8&7&6&1&0&2&3&4&5\\
4&3&5&6&8&7&2&0&1\\
7&6&8&0&2&1&4&5&3\\
3&5&4&7&6&8&1&2&0
\end{bmatrix},$$

where shading indicates the subsquares of order 3 other than those formed by the first 3 rows. Among the species with no subsquares of order 3, the one with the most mates (4171) is the planar species d, which has 72 subsquares of order 2, the maximum possible number [23].

For order 8 the species with the three highest numbers of mates contain the elementary abelian group (70272 mates), dihedral group (33408 mates) and quaternion group (32256 mates), respectively. In fourth place, with 23232 mates, is a species containing a loop that is nearly a group in the sense that it has a large nucleus (isomorphic to the Klein 4-group). The species of the group $\mathbb{Z}_4 \times \mathbb{Z}_2$ is in fifth place (23040 mates). The top five places are occupied by the only latin squares with 384 transversals, which is the most that any latin square of order 8 has. The next highest number of mates is 12048. The cyclic group, of course, has no transversals and hence no mates.

For order 7 the species with the highest numbers of mates contain the cyclic group (133 transversals, 63 mates), the Steiner quasigroup (63 transversals, 8 mates) and the pan-hamiltonian latin square that is not atomic (25 transversals, 3 mates) (see A_7 in [34]).

n	k	Equality	Isotopism	Trisotopism	Paratopism	PP	DPP
2	1	1	1	1	1	1	1
3	2	1	1	1	1	1	1
4	1	3	1	1	1		
4	3	1	1	1	1	1	1
5	1	50	1	1	1		
5	4	6	1	1	1	1	1
6	1	9408	22	17	12		
7	1	16765350	549	314	141		
7	2	341880	17	11	5		
7	6	120	1	1	1	1	1
8	1	532807827816	1665394	836595	281633		
8	2	7832534400	23005	11704	2127		
8	3	14923440	221	147	38		
8	7	240	1	1	1	1	1
9	1	370769976810235296	113527931950	56764991345	18922355489		
9	2	7188529229970480	1101731294	550905816	91845941		
9	3	7648799760	2943	1578	232		
9	4	665884800	371	203	22		
9	5	222499200	318	200	36		
9	8	7728840	19	15	7	4	3

Table 5: Number of reduced sets of k-maxMOLS(n)

6 Number of sets of MOLS and maxMOLS

For $1 \leqslant k < n \leqslant 9$, Table 5 gives the number of reduced sets of k-maxMOLS(n) modulo each of the different notions of equivalence defined in Section 2. The column headed "Equality" gives the total number of reduced sets of k-maxMOLS(n), in other words, $\mathrm{RS}_{k,n}$. These numbers were calculated from a list of species representatives using Theorem 3.2. The number of (not necessarily reduced) sets of k-maxMOLS(n) can be found from $\mathrm{RS}_{k,n}$ using Theorem 3.1, as can the number of lists of k-maxMOLS(n) (reduced or otherwise).

Table 6 gives, for each $1 \leqslant k < n \leqslant 9$, the number of non-equivalent reduced sets of k-MOLS(n) under each of the different notions of equivalence defined in Section 2. We stress that the difference between Table 5 and Table 6 is that the former counts only *maximal* sets, while the latter also includes sets that are not maximal. Following those tables, we give Table 7 and Table 8 which provide the same information, except for lists of MOLS rather than sets of MOLS. In Tables 5 to 8, the stipulation that the MOLS should be reduced only affects the counts in the column headed "Equality". Every isotopism class contains reduced

n	k	Equality	Isotopism	Trisotopism	Paratopism	PP	DPP
2	1	1	1	1	1	1	1
3	1	1	1	1	1		
3	2	1	1	1	1	1	1
4	1	4	2	2	2		
4	2	2	1	1	1		
4	3	1	1	1	1	1	1
5	1	56	2	2	2		
5	2	18	2	2	1		
5	3	18	1	1	1		
5	4	6	1	1	1	1	1
6	1	9408	22	17	12		
7	1	16942080	564	324	147		
7	2	342480	20	14	7		
7	3	1200	4	3	1		
7	4	1200	3	3	1		
7	5	600	1	1	1		
7	6	120	1	1	1	1	1
8	1	535281401856	1676267	842227	283657		
8	2	7850589120	23362	11887	2165		
8	3	14927040	224	149	39		
8	4	4800	3	2	1		
8	5	3600	1	1	1		
8	6	1440	1	1	1		
8	7	240	1	1	1	1	1
9	1	377597570964258816	115618721533	57810418543	19270853541		
9	2	7188534981260640	1101734942	550907773	91846374		
9	3	9338177520	4428	2408	371		
9	4	1526884800	1096	642	96		
9	5	493008600	454	293	56		
9	6	162305640	82	62	15		
9	7	54101880	38	29	11		
9	8	7728840	19	15	7	4	3

Table 6: Number of reduced sets of k-MOLS(n)

n	k	Equality	Isotopism	Trisotopism	Paratopism	PP	DPP
2	1	1	1	1	1	1	1
3	2	1	1	1	1	1	1
4	1	3	1	1	1		
4	3	2	1	1	1	1	1
5	1	50	1	1	1		
5	4	36	6	3	1	1	1
6	1	9408	22	17	12		
7	1	16765350	549	314	141		
7	2	341880	29	17	5		
7	6	14400	120	60	1	1	1
8	1	532807827816	1665394	836595	281633		
8	2	7832534400	45222	23005	2127		
8	3	29846880	1217	616	38		
8	7	172800	240	120	1	1	1
9	1	370769976810235296	113527931950	56764991345	18922355489		
9	2	7188529229970480	2203304036	1101731294	91845941		
9	3	15297599520	15963	8228	232		
9	4	3995308800	8150	4111	22		
9	5	5339980800	18060	9030	36		
9	8	38953353600	56700	28350	7	4	3

Table 7: Number of reduced lists of k-maxMOLS(n)

MOLS so counting reduced MOLS up to isotopism is the same as counting isotopism classes. Similar statements apply to trisotopism classes and species.

For the remainder of this discussion we count all MOLS by species. Hedayat, Parker and Federer [12] showed how sets of disjoint common transversals of a set of MOLS can be used to design successive experiments. In Table 9, the 5 sets of 2-maxMOLS(7) are classified according to their number of common transversals and maximum number of disjoint common transversals. We present similar tables for the 2127 sets of 2-maxMOLS(8) (Table 10), the 232 sets of 3-maxMOLS(9) (Table 11) and the 22 sets of 4-maxMOLS(9) (Table 12). We do not provide tables for the 38 sets of 3-maxMOLS(8) or the 36 sets of 5-maxMOLS(9), each of which has no common transversal. We also do not provide a table for the 91845941 sets of 2-maxMOLS(9), since we did not collect data on their common transversals. However, in Table 13 we do summarise the *symmetric* 2-maxMOLS(9) according to their common transversals.

Our next aim is to examine how prevalent planar species of latin squares are in MOLS. We say that a latin square L is *involved* in MOLS M if at least one aspect of M is paratopic

n	k	Equality	Isotopism	Trisotopism	Paratopism	PP	DPP
2	1	1	1	1	1	1	1
3	1	1	1	1	1		
3	2	1	1	1	1	1	1
4	1	4	2	2	2		
4	2	2	1	1	1		
4	3	2	1	1	1	1	1
5	1	56	2	2	2		
5	2	18	3	2	1		
5	3	36	6	3	1		
5	4	36	6	3	1	1	1
6	1	9408	22	17	12		
7	1	16942080	564	324	147		
7	2	342480	34	20	7		
7	3	2400	20	10	1		
7	4	7200	60	30	1		
7	5	14400	120	60	1		
7	6	14400	120	60	1	1	1
8	1	535281401856	1676267	842227	283657		
8	2	7850589120	45927	23362	2165		
8	3	29854080	1227	621	39		
8	4	28800	40	20	1		
8	5	86400	120	60	1		
8	6	172800	240	120	1		
8	7	172800	240	120	1	1	1
9	1	377597570964258816	115618721533	57810418543	19270853541		
9	2	7188534981260640	2203310919	1101734942	91846374		
9	3	18676355040	23677	12264	371		
9	4	9161308800	21705	10944	96		
9	5	11832206400	27510	13800	56		
9	6	19476676800	28350	14220	15		
9	7	38953353600	56700	28350	11		
9	8	38953353600	56700	28350	7	4	3

Table 8: Number of reduced lists of k-MOLS(n)

#Common transversals	#Disjoint		Total
	0	1	
0	1		1
1		1	1
2		1	1
4		2	2
Total	1	4	5

Table 9: 2-maxMOLS(7) according to their common transversals

#Common transversals	#Disjoint				Total
	0	1	2	4	
0	1980				1980
1		23			23
2		10	60		70
3		1			1
4			16	26	42
8			1	7	8
12			1	1	2
19			1		1
Total	1980	34	79	34	2127

Table 10: 2-maxMOLS(8) according to their common transversals

to L. We say that a set of MOLS has type P (respectively N) if every latin square in the set of MOLS is planar (respectively, non-planar). A set of MOLS is of type M (for mixed) if it is neither of type P or N. In Table 14 we classify the species of k-maxMOLS(9) according to which types of MOLS they contain. Types of MOLS that are not listed are assumed to be not present. So, for example, the column headed "PM" counts species of MOLS that contain at least one set of MOLS of type P, at least one set of MOLS of type M, and no sets of MOLS of type N. It is worth remarking that there are no columns headed "M" or "PN" because no k-maxMOLS(9) fell in those categories. There seems to be no obvious reason why "M" is impossible, but we now describe an obstacle that prevents "PN" occurring. Suppose that we have an $n^2 \times k$ orthogonal array O. Let O_{ij} be the set of MOLS obtained by taking column i of O to index the rows of our MOLS, and column j of O to index the columns of our MOLS. Suppose that O_{12} is of type P and O_{ab} is of type N, for some $1 \leqslant a < b \leqslant k$. Then O_{1b} is of type M since it contains one latin square paratopic to an element of O_{12} and another latin square paratopic to an element of O_{ab}.

Clearly, each of the 7 species of 8-maxMOLS(9) involve only planar latin squares. Planar latin squares are also involved in many of the k-maxMOLS(9) for $k \in \{3, 4, 5\}$. In particular, we can see from Table 14 that all 36 species of 5-maxMOLS(9) involve at least one planar latin

#Common transversals	#Disjoint 0	1	3	5	Total
0	188				188
1		8			8
2		5			5
3		7	5		12
4		6			6
5		6		1	7
6			3		3
10		1			1
11			1		1
12			1		1
Total	188	33	10	1	232

Table 11: 3-maxMOLS(9) according to their common transversals

#Common transversals	#Disjoint 0	3	Total
0	21		21
6		1	1
Total	21	1	22

Table 12: 4-maxMOLS(9) according to their common transversals

square and seven of them involve only planar latin squares. All 22 species of 4-maxMOLS(9) involve at least one planar latin square and at least one non-planar latin square. There are three species of 3-maxMOLS(9) for which there is only one species of latin square involved; in one case the sole species is the planar species e, in the other two cases the species is not planar. There is one species of 5-maxMOLS(9) that involves only one species of latin square (namely, the planar species a, the elementary abelian group). All other 5-maxMOLS(9) involve at least two distinct planar species and between 3 and 9 (inclusive) species of latin squares in total.

We next consider the possibility that a latin square L may be in a set of $\theta(L)+1$ MOLS. In other words, the set of all orthogonal mates for L itself forms a set of MOLS. This is automatically true if $\theta(L) = 1$ but we would expect it to be rare for larger values of θ. For order 9 we have the following data:

- There are exactly 11222874 species of order 9 that possess exactly two mates. Of these, 27 species appear in a set of 3-maxMOLS(9).
- Of the 431678 species with $\theta = 3$, none occur in a set of 4-maxMOLS(9).
- Of the 74741 species with $\theta = 4$, precisely one species is in a set of 5-maxMOLS(9). A

#Common transversals	#Disjoint 0	1	2	3	4	5	6	7	Total
0	183793								183793
1		14079							14079
2		32580	1244						33824
3–4		8051	3672	1605	9				13337
5–8		2397	3128	1756	74	47	22	1	7425
9–16		483	1023	1328	253	69	67	6	3229
17–32		140	210	457	157	79	75	2	1120
33–60		13	20	97	13	2	9	2	156
66–120		20	3	21	2	1		1	48
216				1					1
Total	183793	57763	9300	5265	508	198	173	12	257012

Table 13: Symmetric 2-maxMOLS(9) according to their common transversals

k	P	N	PM	NM	PNM	Total
1		18922355489				18922355489
2	3	91835638	6	10224	70	91845941
3	1	39	3	186	3	232
4			3	4	15	22
5	7		19	6	4	36
8	7					7

Table 14: Species of k-maxMOLS(9) classified by planarity type

representative of that species is

$$\begin{bmatrix} 0&1&2&3&4&5&6&7&8 \\ 1&2&0&7&5&4&8&3&6 \\ 2&0&1&6&8&7&3&4&5 \\ 3&8&6&4&0&1&2&5&7 \\ 4&5&7&0&3&8&1&6&2 \\ 5&4&8&1&7&6&0&2&3 \\ 6&7&3&2&1&0&5&8&4 \\ 7&6&5&8&2&3&4&0&1 \\ 8&3&4&5&6&2&7&1&0 \end{bmatrix}.$$

It has 242 transversals, 3 subsquares of order 3 (all including the entry in the top left corner) and an autoparatopism group of order 4.

A set of k-maxMOLS(n) has $\binom{k+2}{3}$ different aspects that may potentially belong to different species. In Table 15 we show, for $2 \leqslant k < n \leqslant 9$, how many different species of latin

n	3	4	5	7		8			9				
#LS k	2	3	4	2	6	2	3	7	2	3	4	5	8
1	1	1	1	2	1	4	1	1	116	3		1	2
2				2		82	6		5953	10	1		2
3				1		512	13		100971	22	1	12	2
4						1529	16		91738901	44	5	12	
5							2			30	5	8	1
6										62	5	1	
7										38	1	1	
8										18	2		
9										1	2	1	
10										4			
Total	1	1	1	5	1	2127	38	1	91845941	232	22	36	7

Table 15: Number of species of LS involved in the species of k-maxMOLS(n)

squares are involved in each species of k-maxMOLS(n). The number, say s, of species of latin squares is listed in the first column of Table 15, while other columns list the number of species of k-maxMOLS(n) which involve exactly s different species of latin squares. It seems from the table that it is fairly common for pairs of MOLS to have aspects in 4 different species. However, for $k > 2$ the theoretical bound of $\binom{k+2}{3}$ different species is rarely attained among the cases covered by Table 15.

In Table 16 we record statistics on a selection of species of order 9. The species $\{a, b, \ldots, k\}$ are the planar species according to their alphabetic label given in [32]. The other two species referred to in Table 16 are the species of the Cayley table of $\mathbb{Z}_9$ and a species we call $\mathscr{T}$, which occurs with high frequency among k-maxMOLS(9) for $k \in \{3, 4, 5\}$. Each square in species $\mathscr{T}$ has 18 subsquares of order 3 (and none of order 2). A representative of $\mathscr{T}$ is

$$\begin{bmatrix} 0\,1\,2\,3\,4\,5\,6\,7\,8 \\ 1\,2\,0\,4\,5\,3\,7\,8\,6 \\ 2\,0\,1\,5\,3\,4\,8\,6\,7 \\ 3\,5\,4\,6\,7\,8\,0\,2\,1 \\ 4\,3\,5\,7\,8\,6\,1\,0\,2 \\ 5\,4\,3\,8\,6\,7\,2\,1\,0 \\ 6\,8\,7\,0\,2\,1\,3\,4\,5 \\ 7\,6\,8\,1\,0\,2\,4\,5\,3 \\ 8\,7\,6\,2\,1\,0\,5\,3\,4 \end{bmatrix}$$

When considered as a loop, it has the antiautomorphic inverse property. This means that it satisfies the law $(xy)^\star = y^\star x^\star$, for all x and y, where * denotes the left inverse. In Table 16 we give the number of transversals for each species. Next we give the value of α, which is the smallest number of transversals in a maximal set of disjoint transversals (see [10]). After that, we give θ, the number of orthogonal mates. The remaining columns count how many

Species	Transversals	α	θ	$k=2$	$k=3$	$k=4$	$k=5$	$k=8$
a	2241	5	12445836	935	69	16	34	5
b	417	4	11448	265	1			2
c	489	4	197	9	3		1	1
d	801	4	4171	20	9	1	7	2
e	553	4	3120	87	9			1
f	405	3	8928	200	69	7	22	1
g	1620	4	1241361	1816	94	18	30	1
h	861	4	242832	4248	9	2	1	1
i	351	4	2886	424	1	1		1
j	369	4	59	12		1	1	1
k	855	4	403056	2335	2	1	2	1
non-planar:								
$\mathbb{Z}_9$	2025	5	2049219	932	6	5	6	
$\mathscr{T}$	819	4	141208	863	77	16	20	

Table 16: Statistics on selected species of order 9 including the number of species of k-maxMOLS(9) in which they occur

species of k-maxMOLS(9) include the given species of latin square.

An interesting feature of Table 16 is that planar species a has an order of magnitude more mates than any other latin square, but is a long way from being involved in the most species of 2-maxMOLS(9). In fact that honour does not go to any of the species covered in the table, but rather to the species represented by

$$\begin{bmatrix}
0\,1\,2\,3\,4\,5\,6\,7\,8\\
1\,2\,3\,4\,5\,6\,7\,8\,0\\
2\,0\,1\,5\,3\,7\,8\,6\,4\\
3\,4\,8\,6\,7\,2\,0\,1\,5\\
4\,8\,0\,7\,2\,3\,1\,5\,6\\
5\,6\,4\,8\,0\,1\,2\,3\,7\\
6\,7\,5\,0\,1\,8\,3\,4\,2\\
7\,5\,6\,1\,8\,0\,4\,2\,3\\
8\,3\,7\,2\,6\,4\,5\,0\,1
\end{bmatrix}.$$

This square has 755 transversals and an autoparatopism group of order 2. It has 121330 mates and is in 58296 different species of 2-MOLS(9), all of them maximal.

7 Crosschecking

Any computation runs the risk of errors, with the risk increasing with the length and complexity of the computation. The following precautions and crosschecks have been implemented to try to minimise the risk of errors affecting our results.

- Data in all of the tables was computed at least twice. There was some common code used, most notably the generator of species representatives from [24] and the code for screening MOLS for isomorphism. Both of these programs have been previously used for multiple tasks, reducing the likelihood that bugs would have been undetected. With the caveat that this code was common, the main computations were performed independently. For example, both authors found their own versions of the set Γ_2, which were then compared to check that each set contained the same species of MOLS.
- After we generated our catalogues, Brendan McKay kindly gave us code he had written for canonically labelling MOLS and calculating their autoparatopism and autotopism groups. With this code we were able to verify that MOLS in our catalogues of representatives really were from distinct species or isotopism classes, as appropriate. We also checked that our code agreed with his on all group sizes, including those in Table 2.
- We found k-maxMOLS(n) exist exactly when the prior literature (see Section 1) said they should.
- The number of species of 2-MOLS(n) had previously been computed by Brendan McKay [22] for $n \leqslant 8$. His results agree with ours in Table 6 and Table 8.
- Norton [28] manually enumerated lists of MOLS of order 7. His enumeration of species of latin squares of order 7 was incomplete, but the single species that he missed contains bachelor latin squares, so this did not affect his results on MOLS. His values for the number of species, isotopism classes and reduced latin squares agree with ours in Table 8 for $2 \leqslant k < 7 = n$. He also calculated that $\mathrm{AL}_{2,7} = 6263668776960000$, which agrees with the value given by Theorem 3.1 from our value of $\mathrm{RL}_{2,7}$.
- A number of our computations confirm results obtained by Owens and Preece for sets of 8-MOLS(9). It was reported in [31] that there are 19 isotopism classes (in 7 species) of sets of 8-MOLS(9). This agrees with our results in the final line of Table 5 and Table 6. Also, the last column of Table 15 tallies with [32, Table 4].
- The total of the χ column in Table 1 evaluated to a multiple of four, as it should. If that total had been corrupted by one or more errors, it is quite likely that the result would not be divisible by 4.
- For each n, the number of isotopism classes of sets of 2-MOLS(n) equals the number of trisotopism classes of lists of 2-MOLS(n). A similar equality holds if attention is restricted to 2-maxMOLS(n). Thus there are several cases of equinumerous objects being counted in Tables 5 to 8. The reason can be seen by considering the corresponding orthogonal arrays and which operations result in equivalence. For isotopism of sets we allow the last two columns of the orthogonal array to be exchanged, whereas for trisotopism of lists we allow the first two columns to be exchanged. In other respects the two cases are identical. Hence, reversing the order of the columns of the orthogonal arrays provides a bijection between the objects that we claimed are equinumerous.

- Hicks et al. [13] show that there are exactly $(p^d-2)!/d$ reduced sets of (p^d-1)-MOLS(p^d) that define the Desarguesian projective plane of order p^d. Our computations agreed for $p^d \leqslant 9$.
- The method outlined in Section 4 for counting the pairs of MOLS of order 9 was replicated for order 8 and agreed with the results of our more direct computations. Smaller orders do not provide useful test cases, since there are no rigid MOLS of order $n \leqslant 7$.

Data from our enumerations is available online at [36], including species representatives for the MOLS that we generated.

8 Order 10

For orders 10 and higher there are simply too many latin squares to attempt the sorts of comprehensive enumerations of the sort undertaken in the previous sections. However, given the tremendous interest in the existence or otherwise of a triple of MOLS of order 10 (see [24] and the references therein), we did use our programs to investigate the latin squares with autoparatopism groups of order at least 3. A catalogue of these squares was produced by the authors of [24]. It was already established in [24] that none of these squares is in any triple of MOLS. However, some of them come much closer than any previously known examples, as we discovered. Consider the following three squares

$$A = \begin{bmatrix} 0897564231 \\ 9146273805 \\ 7425138690 \\ 8653921047 \\ 6218409573 \\ 4932750168 \\ 5371086924 \\ 3509842716 \\ 1760395482 \\ 2084617359 \end{bmatrix}, \quad B = \begin{bmatrix} 0789123456 \\ 9061832547 \\ 7204391865 \\ 8530217694 \\ 6953074218 \\ 4176508932 \\ 5428960371 \\ 3617485029 \\ 1842659703 \\ 2395746180 \end{bmatrix}, \quad C = \begin{bmatrix} 0789123456 \\ 6428951370 \\ 4953276018 \\ 5176438902 \\ 3290715684 \\ 1037682549 \\ 2801349765 \\ 9542860137 \\ 7365094821 \\ 8614507293 \end{bmatrix}.$$

Square A is orthogonal to both B and C. When B and C are overlayed, 91 different pairs are produced out of a possible 100. Moreover, the only duplicated pairs involve symbols 7,8,9 in C. We conclude that A and B have 7 common transversals. The previously best published result showed a pair of MOLS of order 10 with 4 common transversals [2].

Note that A is semisymmetric and A, B and C all have the automorphism $(0)(123)(456)(789)$.

Acknowledgments. This research was supported in part by an Australian Mathematical Society Lift-Off Fellowship and by ARC grant FT110100065. Computations were performed mainly on the Monash Sun Grid and Monash Green SPONGE computing facilities. The authors are deeply indebted to Darcy Best for carefully checking a number of the tables. The authors are also grateful to Brendan McKay for making available his code for canonically labelling orthogonal arrays, and to Patric Ostergård and Petteri Kaski for helpful discussions. We would also like to thank Wendy Myrvold for independently confirming the results of Section 8 and for providing the program to generate species representatives from [24].

References

[1] R. C. Bose. On the application of the properties of Galois fields to the problem of construction of hyper-Graeco-Latin squares. *Sankhyā* **3** (1938), 323–338.

[2] J. W. Brown, A. S. Hedayat, and E. T. Parker. A pair of orthogonal Latin squares of order 10 with four shared parallel transversals, *J. Combin. Inform. System Sci.* **18** (1993), 113–115.

[3] C. J. Colbourn and J. H. Dinitz (eds). *Handbook of Combinatorial Designs, Second Edition.* Chapman & Hall/CRC, Boca Raton, 2007.

[4] P. Danziger, I. M. Wanless, and B. S. Webb. Monogamous latin squares. *J. Combin. Theory Ser. A* **118** (2011), 796–807.

[5] J. Dénes and A. D. Keedwell. *Latin squares and their applications.* Akadémiai Kiadó, Budapest, 1974.

[6] D. M. Donovan and M. J. Grannell, On the number of transversal designs, *J Combin. Theory, Ser. A* **120** (2013), 1562–1574.

[7] D. A. Drake. Maximal sets of latin squares and partial transversals. *J. Statist. Plann. Inference* **1** (1977), 143–149.

[8] D. A. Drake and W. J. Myrvold. The non-existence of maximal sets of four mutually orthogonal latin squares of order 8. *Des. Codes Cryptogr.* **194** (2004), 87–94.

[9] J. Egan and I. M. Wanless. Indivisible partitions of latin squares. *J. Statist. Plann. Inference* **141** (2011), 402–417.

[10] J. Egan and I. M. Wanless. Latin squares with restricted transversals. *J. Combin. Designs* **20** (2012), 344–361.

[11] A. B. Evans, Latin squares without orthogonal mates, *Des. Codes Cryptogr.* **40** (2006), 121–130.

[12] A. Hedayat, E. T. Parker, and W. T. Federer. The existence and construction of two families of designs for two successive experiments. *Biometrika* **57** (1970), 351–55.

[13] K. Hicks, G. L. Mullen, L. Storme and J. Vanpoucke, The number of different reduced complete sets of mutually orthogonal latin squares corresponding to the Desarguesian projective planes, submitted for publication.

[14] A. Hulpke, P. Kaski and P. R. J. Östergård, The number of Latin squares of order 11, *Math. Comp.* **80** (2011) 1197–1219.

[15] D. Jungnickel and G. Grams. Maximal difference matrices of order $\leqslant 10$. *Discrete Math.* **58** (1986), 199–203.

[16] P. Kaski and P. R. J. Östergård, Classification algorithms for codes and designs, Algorithms and Computation in Mathematics **15**, Springer-Verlag, Berlin, 2006.

[17] P. Kaski and O. Pottonen, `http://koti.kapsi.fi/pottonen/libexact.html` `libexact` software.

[18] J. I. Kokkala and P. R. J. Östergård, Classification of Graeco-Latin cubes, *J. Combin. Des.*, to appear.

[19] K. Kunen, The structure of conjugacy closed loops, *Trans. Amer. Math. Soc.* **352** (2000), 2889-2911.

[20] C. F. Laywine and G. L. Mullen. *Discrete Mathematics Using Latin Squares*. Wiley, New York, 1998.

[21] B. M. Maenhaut and I. M. Wanless. Atomic latin squares of order eleven. *J. Combin. Des.* **12** (2004), 12–34.

[22] B. D. McKay, Latin squares, `http://cs.anu.edu.au/~bdm/data/latin.html`.

[23] B. D. McKay, J. C. McLeod, and I. M. Wanless. The number of transversals in a latin square. *Des. Codes Cryptogr.* **40** (2006), 269–284.

[24] B. D. McKay, A. Meynert, and W. Myrvold. Small latin squares, quasigroups, and loops. *J. Combin. Des.* **15** (2007), 98–119.

[25] B. D. McKay and A. Piperno, Practical Graph Isomorphism II, *J. Symbolic Comput.*, **60** (2014), 94–112.

[26] B. D. McKay and I. M. Wanless, On the number of Latin squares, *Ann. Comb.* **9** (2005), 335–344.

[27] E. H. Moore. Tactical memoranda I-III. *Amer. J. Math.* **18** (1896), 264–303.

[28] H. W. Norton, The 7×7 squares, *Ann. Eugenics* **9** (1939), 269–307.

[29] P. R. J. Östergård, "Constructing combinatorial objects via cliques," Surveys in Combinatorics, B. S. Webb (ed.), Cambridge University Press, Cambridge, 2005, pp. 57–82.

[30] P. J. Owens, Complete sets of pairwise orthogonal Latin squares and the corresponding projective planes. *J. Combin. Theory Ser. A* **59** (1992), 240–252.

[31] P. J. Owens and D. A. Preece. Complete sets of pairwise orthogonal Latin squares of order 9. *J. Combin. Math. Combin. Comput.* **18** (1995), 83–96.

[32] P. J. Owens and D. A. Preece. Aspects of complete sets of 9×9 pairwise orthogonal Latin squares. *Discrete Math.* **167/168** (1997), 519–525.

[33] G. H. J. van Rees. Subsquares and transversals in latin squares. *Ars Combin.* **29** (1990), 193–204.

[34] I. M. Wanless, Perfect factorisations of bipartite graphs and Latin squares without proper subrectangles, *Electron. J. Combin.* **6** (1999), #R9.

[35] I. M. Wanless. A generalisation of transversals for latin squares. *Electron. J. Combin.* **9** (2002), #R12.

[36] I. M. Wanless, `http://users.monash.edu.au/~iwanless/data/MOLS/`, author's homepage.

[37] I. M. Wanless and B. S. Webb, The existence of latin squares without orthogonal mates, *Des. Codes Cryptogr.* **40** (2006), 131–135.