

ESTIMATING $\pi(x)$ AND RELATED FUNCTIONS UNDER PARTIAL RH ASSUMPTIONS

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ABSTRACT. We give a direct interpretation of the validity of the Riemann hypothesis for all zeros with $\Im(\rho) \in (0, T]$ in terms of the prime-counting function $\pi(x)$, by proving that Schoenfeld's explicit estimates for $\pi(x)$ and the Chebyshev functions hold as long as $4.92\sqrt{x/\log(x)} \leq T$.

We also improve some of the existing bounds of Chebyshev type for the function $\psi(x)$.

1. INTRODUCTION

The Riemann hypothesis has been subject to numerous numerical verifications, which typically lead to statements of the form *the first n complex zeros of the Riemann zeta function are simple and lie on the critical line $\Re(s) = 1/2$* ; see e.g. [Bre79].

Whilst such results are used as an ingredient in many estimates for functions of prime numbers, it is the purpose of this paper to give a direct interpretation in terms of the prime-counting function $\pi(x)$. This is done by proving the well-known Schoenfeld bound

$$|\pi(x) - \text{li}(x)| \leq \frac{\sqrt{x}}{8\pi} \log(x) \quad \text{for } x > 2657,$$

which is implied by the Riemann hypothesis [Sch76], holds for $4.92\sqrt{x/\log(x)} \leq T$ conditional on the Riemann hypothesis being valid for $0 < \Im(\rho) \leq T$. We also prove equivalent statements for the Riemann prime-counting function and the Chebyshev functions.

These results also have practical relevance, since calculating the zeros up to height T with fast methods like the Odlyzko-Schönhage algorithm has expected run time $O(T^{1+\varepsilon})$ [OS88]. Therefore, one obtains strong bounds for $\pi(x)$ for $x \leq x_1$ in expected run time $O(x_1^{1/2+\varepsilon})$ if the Riemann hypothesis holds up to the according height.

Apart from this, we also improve part of the bounds for $\psi(x)$ given in [FK15].

2. A MODIFIED CHEBYSHOV FUNCTION

For $A \subset X$ let

$$\chi_A^*(x) = \begin{cases} 1 & x \in A \setminus \partial A \\ 1/2 & x \in \partial A \\ 0 & x \in X \setminus \overline{A}. \end{cases}$$

Date: May 26, 2022.

2010 Mathematics Subject Classification. Primary 11N05, Secondary 11M26.

denote the normalized characteristic function. We intend to construct a continuous approximation to the (normalized) Chebyshev function

$$\psi(x) = \sum_{p^m} \chi_{[0,x]}^*(\log p),$$

for which we will prove an explicit formula similar to the von Mangoldt explicit formula

$$(2.1) \quad \psi(x) = x - \sum_{\rho}^* \frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2} \log(1 - x^{-2}),$$

where the sum is taken over all non-trivial zeros (according to their multiplicity) of the Riemann zeta function and the * indicates that the sum is computed as

$$\lim_{T \rightarrow \infty} \sum_{|\Im(\rho)| < T} \frac{x^{\rho}}{\rho}$$

[vM95].

To this end, we use the Fourier transform of the Logan function

$$\ell_{c,\varepsilon}(\xi) = \frac{c}{\sinh c} \frac{\sin(\sqrt{(\xi\varepsilon)^2 - c^2})}{\sqrt{(\xi\varepsilon)^2 - c^2}},$$

a sharp cutoff filter kernel [Log88], which will allow us to flexibly control the truncation point and the size of the remainder term of the sum over zeros. The Fourier transform is given by

$$(2.2) \quad \eta_{c,\varepsilon}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} \ell_{c,\varepsilon}(\xi) d\xi = \chi_{[-\varepsilon,\varepsilon]}^*(t) \frac{c}{2\varepsilon \sinh c} I_0(c\sqrt{1 - (t/\varepsilon)^2})$$

where $I_0(t) = \sum_{n=0}^{\infty} (t/2)^{2n} / (n!)^2$ denotes the 0-th modified Bessel function of the first kind [FKBJar].

Now let $\lambda_{c,\varepsilon} = \ell_{c,\varepsilon}(i/2)$ and let

$$\varphi_{x,c,\varepsilon} = \frac{1}{\lambda_{c,\varepsilon}} (\chi_{[0,\log x]} \exp(\cdot/2)) * \eta_{c,\varepsilon},$$

where, as usual,

$$f * g(x) := \int_{\mathbb{R}} f(y)g(x-y) dy$$

denotes the convolution of two functions. Then we define the modified Chebyshev function by

$$\psi_{c,\varepsilon}(x) = \sum_{p^m} \frac{\log p}{p^{m/2}} \varphi_{x,c,\varepsilon}(m \log p).$$

Proposition 1. *Let $\varepsilon < 1/10$ and let*

$$(2.3) \quad M_{x,c,\varepsilon}(t) = \frac{\log t}{\lambda_{c,\varepsilon}} \left[\chi_{[x,\exp(\varepsilon)x]}^*(t) \int_{-\varepsilon}^{\log(t/x)} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau \right. \\ \left. - \chi_{[\exp(-\varepsilon)x,x]}^*(t) \int_{\log(t/x)}^{\varepsilon} \eta_{c,\varepsilon}(\tau) e^{-\tau/2} d\tau \right].$$

Then we have

$$(2.4) \quad \psi(x) = \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x < p^m < e^{\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m).$$

Moreover, we have

$$(2.5) \quad \psi(e^{-\alpha\varepsilon}x) \leq \psi_{c,\varepsilon}(x) - \sum_{e^{-\varepsilon}x < p^m \leq e^{-\alpha\varepsilon}x} \frac{1}{m} M_{x,c,\varepsilon}(p^m)$$

and

$$(2.6) \quad \psi(e^{\alpha\varepsilon}x) \geq \psi_{c,\varepsilon}(x) - \sum_{e^{\alpha\varepsilon}x \leq p^m < e^\varepsilon x} \frac{1}{m} M_{x,c,\varepsilon}(p^m)$$

for every $\alpha > 0$.

Proof. The identity (2.4) follows directly from

$$\exp(\cdot/2) * \eta_{c,\varepsilon}(t) = \lambda_{c,\varepsilon} \exp(t/2).$$

and from $\eta_{c,\varepsilon}$ being compactly supported on $[-\varepsilon, \varepsilon]$. The inequalities (2.5) and (2.6) then follow from (2.4), since (2.2) implies $\eta_{c,\varepsilon}(t) \geq 0$. \square

3. THE EXPLICIT FORMULA

The modified Chebyshev function satisfies an explicit formula similar to (2.1), of which we prove an approximate version.

Proposition 2. *Let $0 < \varepsilon < 1/10$ and let $\log(x) > 2/|\log \varepsilon|$. We define*

$$C_1 = -\gamma/2 - 1 - \log(\pi)/2$$

and

$$a_{c,\varepsilon}(\rho) = \frac{1}{\lambda_{c,\varepsilon}} \ell_{c,\varepsilon} \left(\frac{\rho}{i} - \frac{1}{2i} \right).$$

Then we have

$$(3.1) \quad \psi_{c,\varepsilon}(x) = x - \sum_{\rho} a_{c,\varepsilon}(\rho) \frac{x^\rho - 1}{\rho} + C_1 - \frac{1}{2} \log(1 - x^{-2}) + \Theta(8\varepsilon|\log \varepsilon|).$$

Proof. Let

$$f_x(t) = \chi_{[0, \log x]}^*(t) \exp(t/2)$$

so that we have $\varphi_{x,c,\varepsilon} = \lambda_{c,\varepsilon}^{-1} f_x * \eta_{c,\varepsilon}$. The assertion of the proposition follows by applying the Weil-Barner explicit formula [Bar81]

$$w_s(\hat{f}) = w_f(f) + w_\infty(f),$$

where

$$w_s(\hat{f}) = \sum_{\rho}^* \hat{f}(i/2 - i\rho) - \hat{f}(i/2) - \hat{f}(i/2),$$

$$w_f(f) = - \sum_p \sum_{m=0}^{\infty} \frac{\log p}{p^{m/2}} (f(m \log p) + f(-m \log p)),$$

$$w_\infty(f) = \left(\frac{\Gamma'}{\Gamma}(1/4) - \log \pi \right) f(0) - \int_0^\infty \frac{f(t) + f(-t) - 2f(0)}{1 - e^{-2t}} e^{-t/2} dt,$$

and where

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{i\xi t} f(t) dt,$$

to the function $\varphi_{x,c,\varepsilon}$.

Let $\Delta = \varphi_{x,c,\varepsilon} - f_x$ and assume $x > 2/|\log(\varepsilon)|$. It then suffices to prove the following identities:

$$(3.2) \quad w_s(\hat{\varphi}_{x,c,\varepsilon}) = \sum_{\rho} a_{c,\varepsilon}(\rho) \frac{x^{\rho} - 1}{\rho} - x - \log x + 1$$

$$(3.3) \quad w_f(\varphi_{x,c,\varepsilon}) = -\psi_{c,\varepsilon}(x)$$

$$(3.4) \quad w_{\infty}(f_x) = -\log x - \frac{\gamma}{2} - \frac{1}{2} \log \pi - \frac{1}{2} \log(1 - x^{-2})$$

$$(3.5) \quad w_{\infty}(\Delta) = \Theta(8\varepsilon|\log \varepsilon|).$$

The identities (3.2) and (3.3) follow directly from the definitions of the functionals. So we begin with the proof of (3.4). We have

$$w_{\infty}(f_x) = \frac{1}{2} \frac{\Gamma'}{\Gamma}(1/4) - \frac{1}{2} \log \pi - \int_0^{\log x} \frac{1 - e^{-t/2}}{1 - e^{-2t}} dt + \int_{\log x}^{\infty} \frac{e^{-t/2}}{1 - e^{-2t}} dt.$$

Using

$$\frac{1}{2} \frac{\Gamma'}{\Gamma}(1/4) = \int_0^{\infty} \frac{e^{-2t}}{t} - \frac{e^{-t/2}}{1 - e^{-2t}} dt$$

and

$$- \int_0^{\log x} \frac{1 - e^{-t/2}}{1 - e^{-2t}} dt = -\log x + \int_0^{\log x} \frac{e^{-t/2} - e^{-2t}}{1 - e^{-2t}} dt,$$

we get

$$\begin{aligned} w_{\infty}(f_x) &= -\log x - \frac{1}{2} \log \pi + \int_0^{\log x} \frac{e^{-2t}}{2t} - \frac{e^{-2t}}{1 - e^{-2t}} dt + \int_{\log x}^{\infty} \frac{e^{-2t}}{2t} dt \\ &= -\log x - \frac{1}{2} \log \pi - \frac{1}{2} \log(1 - x^{-2}) + \frac{1}{2} \lim_{\delta \searrow 0} (E_1(2\delta) - \log(1 - e^{-2\delta})), \end{aligned}$$

where

$$E_1(y) = \int_y^{\infty} \frac{e^{-t}}{t} dt$$

denotes the first exponential integral. Since

$$E_1(y) = -\gamma - \log(y) + O(y)$$

holds for $y \searrow 0$ [Olv97, p. 40], we get

$$\lim_{\delta \searrow 0} (E_1(2\delta) - \log(1 - e^{-2\delta})) = -\gamma + \log\left(\frac{1 - e^{-2\delta}}{2\delta}\right) = -\gamma,$$

which concludes the proof of (3.4).

It remains to show (3.5), and we start by bounding $\Delta(t)$:

Lemma 1. *Let ε and x be as in the proposition. Then $\Delta(t)$ vanishes for $t \notin B_{\varepsilon}(0) \cup B_{\varepsilon}(\log x)$. Moreover, we have*

$$(3.6) \quad \Delta(t) + \Delta(-t) = 2\Delta(0) + \Theta(2t) \quad \text{for } 0 \leq t \leq \varepsilon,$$

$$(3.7) \quad |\Delta(t)| \leq \frac{1}{2} e^{\varepsilon} \sqrt{x} \quad \text{for } t \in B_{\varepsilon}(\log x),$$

and

$$(3.8) \quad |\Delta(0)| \leq \varepsilon.$$

Proof. Under the conditions imposed on x and ε , we have $B_\varepsilon(0) \cap B_\varepsilon(\log x) = \emptyset$ and

$$(3.9) \quad e^{t+\tau} = e^t + \Theta(2|\tau|)$$

for $\max\{|t|, |\tau|\} \leq \varepsilon$.

Since $\exp(\cdot/2) * \eta_{c,\varepsilon}(t) = \lambda_{c,\varepsilon} \exp(t/2)$ this gives

$$\Delta(0) = \frac{1}{2\lambda_{c,\varepsilon}} \int_0^\varepsilon \eta_{c,\varepsilon}(\tau) (e^{\tau/2} - e^{-\tau/2}) d\tau = \Theta(\varepsilon),$$

so we get (3.8). Moreover, we have

$$\begin{aligned} \Delta(t) + \Delta(-t) &= \frac{1}{\lambda_{c,\varepsilon}} \int_t^\varepsilon \eta_{c,\varepsilon}(\tau) (e^{\frac{\tau-t}{2}} - e^{\frac{t-\tau}{2}}) dt \\ &= \frac{1}{\lambda_{c,\varepsilon}} \int_t^\varepsilon \eta_{c,\varepsilon}(\tau) (e^{\tau/2} - e^{-\tau/2}) dt + \Theta(t) \\ &= \frac{1}{\lambda_{c,\varepsilon}} \int_0^\varepsilon \eta_{c,\varepsilon}(\tau) (e^{\tau/2} - e^{-\tau/2}) dt + \Theta(2t), \end{aligned}$$

which gives (3.6). The remaining inequality (3.7) follows easily from

$$\Delta(t) = \frac{\chi(\log x, \infty)(t)}{\lambda_{c,\varepsilon}} \int_{t-\log x}^\varepsilon \eta_{c,\varepsilon}(\tau) e^{\frac{t-\tau}{2}} d\tau - \frac{\chi(0, \log x)(t)}{\lambda_{c,\varepsilon}} \int_{-\varepsilon}^{t-\log x} \eta_{c,\varepsilon}(\tau) e^{\frac{t-\tau}{2}} d\tau,$$

which holds for $t \in B_\varepsilon(\log x)$. \square

Now, we divide the integral in $w_\infty(\Delta)$ as follows

$$(3.10) \quad \int_0^\infty \frac{\Delta(t) + \Delta(-t) - 2\Delta(0)}{1 - e^{-2t}} e^{-t/2} dt = \int_0^\varepsilon \frac{\Delta(t) + \Delta(-t) - 2\Delta(0)}{1 - e^{-2t}} e^{-t/2} dt \\ - 2 \int_\varepsilon^\infty \frac{\Delta(0)}{1 - e^{-2t}} e^{-t/2} dt + \int_{B_\varepsilon(\log x)} \frac{\Delta(t)}{1 - e^{-2t}} e^{-t/2} dt.$$

Since the mapping $t \mapsto \frac{1 - \exp(-2t)}{t}$ is monotonously decreasing in $(0, \infty)$, we have

$$(3.11) \quad 1 - e^{-2t} \geq 1.8t$$

for $0 \leq t \leq \varepsilon \leq 0.1$. So, using (3.6), we obtain the bound

$$\int_0^\varepsilon \frac{|\Delta(t) + \Delta(-t) - 2\Delta(0)|}{1 - e^{-2t}} e^{-t/2} dt \leq \int_0^\varepsilon \frac{2t}{1.8t} dt \leq 1.2\varepsilon$$

for the first integral on the right hand side of (3.10).

For the second integral we use (3.8) and the bound $|\log \varepsilon| \geq 2.3$, which gives

$$2|\Delta(0)| \int_\varepsilon^\infty \frac{e^{-t/2}}{1 - e^{-t}} dt \leq 2|\Delta(0)| \left| \log \frac{e^{\varepsilon/2} - 1}{e^{\varepsilon/2} + 1} \right| \leq 2\varepsilon \left| \log \frac{\varepsilon}{2 \cdot 2.1} \right| \leq 3.4\varepsilon |\log \varepsilon|.$$

It remains to bound the third integral on the right hand side of (3.10). From (3.11) we get

$$1 - e^{-2t} \geq 1 - \exp(2\varepsilon - 2\log x) \geq 1 - \exp\left(-\frac{4}{|\log \varepsilon|}\right) \geq \frac{2}{|\log \varepsilon|}$$

for $t \in B_\varepsilon(\log x)$ which, together with (3.7), implies

$$\int_{B_\varepsilon(\log x)} \frac{|\Delta(t)|}{1 - e^{-2t}} e^{-t/2} dt \leq \frac{1}{2} e^{\varepsilon/2} \sqrt{x} \frac{e^{\varepsilon/2}}{\sqrt{x}} \int_{B_\varepsilon(\log x)} \frac{dt}{1 - e^{-2t}} \leq \varepsilon |\log \varepsilon|.$$

By the Gauß-Digamma theorem [AAR99, Theorem 1.2.7], we have

$$\frac{\Gamma'}{\Gamma}(1/4) = -\gamma - \frac{\pi}{2} - 3 \log 2,$$

so (3.8) gives the bound

$$\left| \left(\frac{\Gamma'}{\Gamma}(1/4) - \log \pi \right) \Delta(0) \right| \leq 5.4 \varepsilon$$

for the remaining summand in $w_\infty(\Delta)$. Therefore, we arrive at

$$|w_\infty(\Delta)| \leq \varepsilon(5.4 + 1.2 + (3.4 + 1)|\log \varepsilon|) \leq 8 \varepsilon |\log \varepsilon|,$$

which concludes the proof of the proposition. \square

4. BOUNDING THE SUM OVER ZEROS

We provide several bounds for parts of the sum over zeros in the explicit formula for $\psi_{c,\varepsilon}(x)$. First we truncate the sum, making use of the sharp cutoff property of the Logan function.

Proposition 3. *Let $x > 1$, $\varepsilon \leq 10^{-3}$ and $c \geq 3$. Then we have*

$$(4.1) \quad \sum_{|\Im(\rho)| > \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^\rho}{\rho} \right| \leq 0.16 \frac{x+1}{\sinh(c)} e^{0.71\sqrt{c\varepsilon}} \log(3c) \log\left(\frac{c}{\varepsilon}\right).$$

Furthermore, if $a \in (0, 1)$ such that $a \frac{c}{\varepsilon} \geq 10^3$ holds, and if the Riemann hypothesis holds for all zeros with imaginary part in $(0, \frac{c}{\varepsilon}]$, then we have

$$(4.2) \quad \sum_{\frac{ac}{\varepsilon} < |\Im(\rho)| \leq \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^\rho}{\rho} \right| \leq \frac{1 + 11c\varepsilon}{\pi c a^2} \log\left(\frac{c}{\varepsilon}\right) \frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)} \sqrt{x}.$$

Proof. Since $\exp(t/2)$ is convex and $\eta_{c,\varepsilon}$ is non-negative and even, we have

$$\lambda_{c,\varepsilon} \exp(t/2) = \exp(\cdot/2) * \eta_{c,\varepsilon}(t) \geq \exp(t/2),$$

and therefore $\lambda_{c,\varepsilon} \geq 1$. Thus

$$\left| a_{c,\varepsilon}(\rho) \frac{x^\rho}{\rho} \right| \leq x^{\Re(\rho)} \frac{|\ell_{c,\varepsilon}(\frac{\rho}{i} - \frac{1}{2i})|}{|\Im(\rho)|}$$

holds for every non-trivial zero ρ . From this one obtains (4.2) from [Büt, Lemma 5.5], pairing ρ and $1 - \bar{\rho}$ for every zero off the critical line, and (4.1) follows from the following lemma. \square

Lemma 2. *Let $0 < \varepsilon < 10^{-3}$ and let $c \geq 3$. Then we have*

$$\sum_{|\Im(\rho)| > \frac{c}{\varepsilon}} \frac{|\ell_{c,\varepsilon}(\frac{\rho}{i} - \frac{1}{2i})|}{|\Im(\rho)|} \leq 0.32 \frac{e^{0.71\sqrt{c\varepsilon}}}{\sinh(c)} \log(3c) \log\left(\frac{c}{\varepsilon}\right).$$

Proof. This is a more flexible version of [FKBJar, Lemma 2.4], which is proven in detail in [Büt15]. We give a brief outline of the proof: We may weaken the condition $T > 10^6$ to $T \geq 100$ by replacing the constant 0.4 by 0.82 in Corollary 2.2 and by replacing $M + 6$ by $M + 18$ in Corollary 2.3. In the proof of Lemma 2.4 we replace the definition of $f(z)$ by $\frac{\sinh(c)}{c} e^{-0.71\sqrt{c\varepsilon}} \ell_{c,1}(z)$. It is then straightforward to show that (2.7) and (2.8) and the final inequality remain true, which gives the desired result. \square

For the remaining part of the zeros, we will also be needing the following lemma.

Lemma 3. *Let $t_2 > t_1 \geq 14$. Then we have*

$$(4.3) \quad \sum_{t_1 \leq \Im(\rho) < t_2} \frac{1}{\Im(\rho)} \leq \frac{1}{4\pi} \left[\log\left(\frac{t_2}{2\pi}\right)^2 - \log\left(\frac{t_1}{2\pi}\right)^2 \right] + \Theta\left(5 \frac{\log t_1}{t_1}\right),$$

and for $t_2 \geq 5000$ we have

$$\sum_{0 < \Im(\rho) < t_2} \frac{1}{\Im(\rho)} \leq \frac{1}{4\pi} \log\left(\frac{t_2}{2\pi}\right)^2.$$

Proof. Let $N(t)$ denote the zero-counting function. Using the notation $N(t) = g(t) + R(t)$, where $g(t) = \frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{7}{8}$, we get

$$\sum_{t_1 \leq \Im(\rho) < t_2} \frac{1}{\Im(\rho)} = \int_{t_1}^{t_2} \frac{g'(t)}{t} dt + \int_{t_1}^{t_2} \frac{dR(t)}{t}.$$

Here the first integral gives the main term in (4.3). Furthermore, Rosser's estimate [Ros41, p. 223] implies $|R(t)| \leq \log t$ for $t \geq 14$. Consequently, we get

$$\begin{aligned} \int_{t_1}^{t_2} \frac{dR(t)}{t} &= \left[\frac{R(t)}{t} \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{R(t)}{t^2} dt \\ &\leq 2 \frac{\log t_1}{t_1} + \int_{t_1}^{t_2} \frac{\log t}{t^2} dt \\ &\leq 4 \frac{\log t_1}{t_1} + \frac{1}{t_1} \leq 5 \frac{\log t_1}{t_1}. \end{aligned}$$

In particular, we have

$$\begin{aligned} \sum_{0 < \Im(\rho) < t_2} \frac{1}{\Im(\rho)} &\leq \frac{1}{4\pi} \log\left(\frac{t_2}{2\pi}\right)^2 + \sum_{0 < \Im(\rho) < 5000} \frac{1}{\Im(\rho)} - \frac{1}{4\pi} \log\left(\frac{5000}{2\pi}\right)^2 + 5 \frac{\log(5000)}{5000} \\ &\leq \frac{1}{4\pi} \log\left(\frac{t_2}{2\pi}\right)^2 + 3.54 - 3.55 + 0.0086 < \frac{1}{4\pi} \log\left(\frac{t_2}{2\pi}\right)^2 \end{aligned}$$

for $t_1 \geq 5000$. \square

5. BOUNDING THE SUM OVER PRIME POWERS

The modified Chebyshev function $\psi_{c,\varepsilon}$ can be used to trivially bound $\psi(x)$, choosing $\alpha = 1$ in Proposition 1, but one obtains considerably better results choosing α close to zero and bounding the sum over prime powers.

We introduce the auxiliary functions

$$\mu_{c,\varepsilon}(t) = \begin{cases} -\int_{-\infty}^t \eta_{c,\varepsilon}(\tau) d\tau & t < 0, \\ -\mu_{c,\varepsilon}(-t) & t > 0, \\ 0 & t = 0 \end{cases}$$

and

$$\nu_{c,\varepsilon}(t) = \int_{-\infty}^t \mu_{c,\varepsilon}(\tau) d\tau.$$

Proposition 4. *Let $0 \leq \alpha < 1$, $x > 100$, and $\varepsilon < 10^{-2}$, such that*

$$B = \frac{\varepsilon x e^{-\varepsilon} |\nu_c(\alpha)|}{2(\mu_c)_+(\alpha)} > 1$$

holds. We define

$$A(x, c, \varepsilon, \alpha) = e^{2\varepsilon} \log(e^\varepsilon x) \left[\frac{2\varepsilon x |\nu_c(\alpha)|}{\log B} + 2.01\varepsilon\sqrt{x} + \frac{1}{2} \log \log(2x^2) \right].$$

Then we have

$$\psi(e^{-\alpha\varepsilon} x) \leq \psi_{c,\varepsilon}(x) + A(x, c, \varepsilon, \alpha)$$

and

$$\psi(e^{\alpha\varepsilon} x) \geq \psi_{c,\varepsilon}(x) - A(x, c, \varepsilon, \alpha).$$

We will use the following two Lemmas from [Büt].

Lemma 4 ([Büt, Lemma 3.5]). *Let $x \geq 100$, $\varepsilon \leq \frac{1}{100}$ and let $I = [e^{-\varepsilon} x, e^\varepsilon x]$. Then we have*

$$\sum_{\substack{p^m \in I \\ m \geq 2}} \frac{1}{m} \leq 4.01\varepsilon\sqrt{x} + \log \log(2x^2).$$

Lemma 5 ([Büt, Lemma 5.8]). *Let $x > 1$, $\varepsilon < 1$ and $\alpha \in (0, 1)$, such that*

$$B := \frac{\varepsilon x e^{-\varepsilon} |\nu_c(\alpha)|}{2\mu_c(\alpha)} > 1$$

holds. Furthermore, let $I_\alpha^+ = [e^{\alpha\varepsilon} x, e^\varepsilon x]$ and $I_\alpha^- = [e^{-\varepsilon} x, e^{-\varepsilon\alpha} x]$. Then we have

$$\sum_{p \in I_\alpha^\pm} \left| \mu_{c,\varepsilon} \left(\log \frac{p}{x} \right) \right| \leq 2 \frac{\varepsilon x e^\varepsilon |\nu_c(\alpha)|}{\log B}.$$

Proof of Proposition 4. By Proposition 1, it suffices to show that

$$\left| \sum_{p^m \in I_\alpha^\pm} \frac{1}{m} M_{x,c,\varepsilon}(p^m) \right| \leq A(x, c, \varepsilon, \alpha).$$

From (3.9) and (2.3) one easily obtains the bound

$$M_{x,c,\varepsilon}(t) = \frac{\log t}{\lambda_{c,\varepsilon}} \mu_{c,\varepsilon} \left(\log \frac{t}{x} \right) (1 + \Theta(\varepsilon)) \leq \frac{e^\varepsilon}{2} \log(e^\varepsilon x).$$

Then Lemma 5 gives the bound

$$2e^{2\varepsilon} \log(e^\varepsilon x) \frac{\varepsilon x |\nu_c(\alpha)|}{\log B}$$

for the contribution of the prime numbers in I_α^\pm , and Lemma 4 gives the bound

$$\frac{e^\varepsilon}{2} \log(e^\varepsilon x) [4.01\varepsilon\sqrt{x} + \log \log(2x)^2]$$

for the contribution of the remaining prime powers in I_α^\pm . \square

Analyzing the asymptotic behavior of $\mu_c(\alpha)$ and $\nu_c(\alpha)$ as functions of c for arbitrary α seems difficult. However, we can do this for the case $\alpha = 0$, which is usually not too far from the optimal choice. To this end, we introduce the modified Bessel function of the first kind for real parameters $\gamma \geq 0$ by

$$(5.1) \quad I_\gamma(x) = \left(\frac{x}{2}\right)^\gamma \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n! \Gamma(\gamma + n + 1)}.$$

Then we have the following proposition.

Proposition 5. *For $c_0 > 0$ let*

$$D(c_0) = \sqrt{\frac{\pi c_0}{2}} \frac{I_1(c_0)}{\sinh(c_0)}.$$

Then the inequalities

$$\frac{D(c_0)}{\sqrt{2\pi c}} \leq |\nu_c(0)| \leq \frac{1}{\sqrt{2\pi c}}$$

hold for all $c \geq c_0$. Furthermore, we have $D(c_0) \nearrow 1$ for $c_0 \rightarrow \infty$.

Proof. Since

$$|\nu_c(0)| = \frac{I_1(c)}{2 \sinh(c)}$$

[FKBJar, p. 15] and since $I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh(x)$ the assertion follows directly from the following lemma. \square

Lemma 6. *Let $\alpha, \beta \in [0, \infty)$ such that $\alpha < \beta$ holds. Then the function*

$$\frac{I_\beta(x)}{I_\alpha(x)}$$

is positive and monotonously increasing in $(0, \infty)$ and converges to 1 for $x \rightarrow \infty$.

Proof. The proof is based on the Sturm Monotony Principle [Stu36], [Wat44, p. 518]. We define the auxiliary function

$$f_\gamma(x) = \sqrt{x} I_\gamma(x).$$

The Bessel differential equation

$$\frac{d^2}{dx^2} I_\gamma + \frac{1}{x} \frac{d}{dx} I_\gamma - \left(1 + \frac{\gamma^2}{x^2}\right) I_\gamma = 0$$

then implies

$$\frac{d^2}{dx^2} f_\gamma - \left(1 - \frac{1}{4x} + \frac{\gamma^2}{x^2}\right) f_\gamma = 0.$$

Consequently, we have

$$f_\beta f_\alpha'' - f_\beta'' f_\alpha = \frac{\beta^2 - \alpha^2}{x^2} f_\alpha f_\beta > 0$$

in $(0, \infty)$ and thus

$$\left[f_\beta f'_\alpha - f'_\beta f_\alpha \right]_\varepsilon^x > 0$$

for $x > \varepsilon$ and every $\varepsilon > 0$. Since

$$f_\beta f'_\alpha - f'_\beta f_\alpha = I_\beta(xI'_\alpha + I_\alpha) - I_\alpha(xI'_\beta + I_\beta)$$

vanishes for $x \rightarrow 0$ we thus get

$$f_\beta f'_\alpha - f'_\beta f_\alpha \geq 0.$$

Consequently, the function $f_\beta/f_\alpha = I_\beta/I_\alpha$ increases monotonously in $(0, \infty)$, and since

$$I_\gamma(x) \sim \frac{e^x}{\sqrt{2\pi x}}$$

holds for every $\gamma \geq 0$, it converges to 1 for $x \rightarrow \infty$. \square

6. BOUNDS OF CHEBYSHOV TYPE

The previous results give rise to a simple method to calculate bounds of the form

$$|\psi(x) - x| \leq \delta_0 x \quad \text{for } x \geq x_0,$$

which will be needed in the proof of the main result.

Theorem 1. *Let $0 < \varepsilon < 10^{-3}$, $c \geq 3$, $x_0 \geq 100$ and $\alpha \in [0, 1)$ such that the inequality*

$$B_0 := \frac{\varepsilon e^{-\varepsilon} x_0 |\nu_c(\alpha)|}{2(\mu_c)_+(\alpha)} > 1$$

holds. We denote the zeros of the Riemann zeta function by $\rho = \beta + i\gamma$ with $\beta, \gamma \in \mathbb{R}$. Then, if $\beta = 1/2$ holds for $0 < \gamma \leq c/\varepsilon$, the inequality

$$|\psi(x) - x| \leq x \cdot e^{\alpha\varepsilon} (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$$

holds for all $x \geq e^{\alpha\varepsilon} x_0$, where

$$\mathcal{E}_1 = e^{2\varepsilon} \log(e^\varepsilon x_0) \left[\frac{2\varepsilon |\nu_c(\alpha)|}{\log B_0} + \frac{2.01\varepsilon}{\sqrt{x_0}} + \frac{\log \log(2x_0^2)}{2x_0} \right] + (e^{\alpha\varepsilon} - 1),$$

$$\mathcal{E}_2 = 0.16 \frac{1 + x_0^{-1}}{\sinh(c)} e^{0.71\sqrt{c\varepsilon}} \log\left(\frac{c}{\varepsilon}\right),$$

and

$$(6.1) \quad \mathcal{E}_3 = \frac{2}{\sqrt{x_0}} \sum_{0 < \gamma \leq c/\varepsilon} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} + \frac{2}{x_0}.$$

It is noteworthy that Theorem 1 gives better estimates than the more sophisticated method in [FK15] in a large range, as can be seen from Tables 1 and 2.

Proof. Under the conditions of the theorem we get

$$\psi(e^{-\alpha\varepsilon} x) - e^{-\alpha\varepsilon} x \leq \psi_{c,\varepsilon}(x) - e^{-\alpha\varepsilon} x + \frac{A(x_0, c, \varepsilon, \alpha)}{x_0} x \leq \psi_{c,\varepsilon}(x) - x + \mathcal{E}_1 x$$

from Proposition 4, since $A(x, c, \varepsilon, \alpha)/x$ decreases monotonously. A similar calculation for the lower bound then gives

$$|\psi(e^{\pm\alpha\varepsilon} x) - e^{\pm\alpha\varepsilon} x| \leq |\psi_{c,\varepsilon}(x) - x| + \mathcal{E}_1 x.$$

TABLE 1. Bounds for $T \leq 3.061 \times 10^{10}$. The value δ_0 is an upper bound for $e^{\alpha\varepsilon}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$ in Theorem 1, applied with $\varepsilon = c/T$.

$e^{\alpha\varepsilon}x_0$	c	T	α	δ_0
e^{45}	25	3.5×10^9	0.11	1.11742×10^{-8}
e^{50}	30	3.061×10^{10}	0.11	1.16465×10^{-9}
e^{55}	30	3.061×10^{10}	0.1	2.88434×10^{-10}
e^{60}	28	3.061×10^{10}	0.09	2.08162×10^{-10}
e^{65}	28	3.061×10^{10}	0.09	1.96865×10^{-10}
e^{70}	28	3.061×10^{10}	0.08	1.91910×10^{-10}
e^{80}	28	3.061×10^{10}	0.07	1.84848×10^{-10}
e^{90}	29	3.061×10^{10}	0.06	1.79330×10^{-10}
e^{100}	29	3.061×10^{10}	0.05	1.75185×10^{-10}
e^{500}	29	3.061×10^{10}	0.01	1.47067×10^{-10}
e^{1000}	29	3.061×10^{10}	0.005	1.43770×10^{-10}
e^{3000}	29	3.061×10^{10}	0.001	1.41594×10^{-10}

Furthermore, we get

$$|\psi_{c,\varepsilon}(x) - x| \leq \sum_{|\Im(\rho)| \leq c/\varepsilon} \left| a_{c,\varepsilon}(\rho) \frac{x^\rho}{\rho} \right| + 2 + \mathcal{E}_2 x \leq (\mathcal{E}_2 + \mathcal{E}_3)x$$

from Propositions 2 and 3, so the assertion follows. \square

6.1. Numerical estimates for \mathcal{E}_1 and \mathcal{E}_3 . The sum over zeros in (6.1) can either be evaluated, which is recommended if c/ε is small, or the sum can be estimated piecewise, using the following lemma.

Lemma 7. *Let $c, \varepsilon > 0$ and let $14 \leq T_0 < T_1 < c/\varepsilon$. Then we have*

$$\sum_{T_0 \leq \gamma < T_1} \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} \leq \frac{\ell_{c,\varepsilon}(T_0)}{4\pi} \left[\log\left(\frac{T_1}{2\pi}\right)^2 - \log\left(\frac{T_0}{2\pi}\right)^2 + 20\pi \frac{\log(T_1)}{T_1} \right].$$

Proof. This follows directly from $\ell_{c,\varepsilon}$ being monotonously decreasing in $[0, c/\varepsilon]$ and Lemma 3. \square

The values $\mu_c(\alpha)$ and $\nu_c(\alpha)$ can be evaluated by power series representations, as shown in [FKBJar]. Alternatively, these values can be bounded by Riemann sums.

Lemma 8. *Let $\alpha \in (0, 1)$, $K \in \mathbb{N}$ and let $h = \frac{1-\alpha}{K}$. Then we have*

$$hc \sum_{k=0}^{K-1} \frac{I_0(c\sqrt{2kh - k^2h^2})}{2 \sinh(c)} \leq \mu_c(\alpha) \leq hc \sum_{k=1}^K \frac{I_0(c\sqrt{2kh - k^2h^2})}{2 \sinh(c)}$$

TABLE 2. Bounds for $T \leq 2.445 \times 10^{12}$. The value δ_0 is an upper bound for $e^{\alpha\varepsilon}(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)$ in Theorem 1, applied with $\varepsilon = c/T$.

$e^{\alpha\varepsilon}x_0$	c	T	α	δ_0
e^{55}	39	8.5×10^{11}	0.1	1.12494×10^{-10}
e^{60}	33	2.445×10^{12}	0.11	1.22147×10^{-11}
e^{65}	33	2.445×10^{12}	0.1	3.57125×10^{-12}
e^{70}	33	2.445×10^{12}	0.09	2.79233×10^{-12}
e^{75}	32	2.445×10^{12}	0.08	2.70358×10^{-12}
e^{80}	33	2.445×10^{12}	0.08	2.61079×10^{-12}
e^{90}	33	2.445×10^{12}	0.07	2.52129×10^{-12}
e^{100}	33	2.445×10^{12}	0.06	2.45229×10^{-12}
e^{500}	33	2.445×10^{12}	0.012	1.99986×10^{-12}
e^{1000}	33	2.445×10^{12}	0.005	1.94751×10^{-12}
e^{2000}	33	2.445×10^{12}	0.003	1.92155×10^{-12}
e^{3000}	33	2.445×10^{12}	0.001	1.91298×10^{-12}
e^{4000}	33	2.445×10^{12}	0.001	1.90866×10^{-12}

and

$$h^2 c \sum_{k=0}^{K-1} \sum_{j=0}^k \frac{I_0(c\sqrt{2jh - j^2h^2})}{2 \sinh(c)} \leq |\nu_c(\alpha)| \leq h^2 c \sum_{k=1}^K \sum_{j=1}^k \frac{I_0(c\sqrt{2jh - j^2h^2})}{2 \sinh(c)}.$$

Proof. This follows from $\mu'_c = -\eta_{c,1}$ in $(0, 1)$ and $\nu'_c = \mu_c$, since both $\eta_{c,1}$ and μ_c are monotonously decreasing and non-negative in this region. \square

7. A PARTIAL PRIME NUMBER THEOREM

We now come to the main result of this paper, the proof of Schoenfeld's bounds [Sch76] for the functions $\psi(x)$,

$$\pi(x) = \sum_p \chi_{[0,x]}^*(p), \quad \vartheta(x) = \sum_p \chi_{[0,x]}^*(p) \log(p), \quad \text{and} \quad \pi^*(x) = \sum_{p^m} \frac{1}{m} \chi_{[0,x]}^*(p^m),$$

in limited ranges under partial RH assumptions. This is a slight improvement of [Büt15, Theorem 6.1].

Theorem 2. *Let $T > 0$ such that the Riemann hypothesis holds for $0 < \Im(\rho) \leq T$. Then, under the condition $4.92\sqrt{\frac{x}{\log x}} \leq T$, the following estimates hold:*

$$(7.1) \quad \begin{aligned} |\psi(x) - x| &\leq \frac{\sqrt{x}}{8\pi} \log(x)^2 && \text{for } x > 59, \\ |\vartheta(x) - x| &\leq \frac{\sqrt{x}}{8\pi} \log(x)^2 && \text{for } x > 599, \\ |\pi^*(x) - \text{li}(x)| &\leq \frac{\sqrt{x}}{8\pi} \log(x) && \text{for } x > 59, \end{aligned}$$

and

$$(7.2) \quad |\pi(x) - \text{li}(x)| \leq \frac{\sqrt{x}}{8\pi} \log(x) \quad \text{for } x > 2657.$$

In particular the numerical verification in [Pla15] ($T \approx 3.061 \times 10^{10}$) gives these bounds for $x \leq 1.89 \times 10^{21}$, the result in [FKBJar] ($T = 10^{11}$) gives them for $x \leq 2.1 \times 10^{22}$ and the result in [Gou04] ($T \approx 2.445 \times 10^{12}$) gives them for $x \leq 1.4 \times 10^{25}$.

Proof. We will first prove the stronger bounds

$$(7.3) \quad |\psi(x) - x| \leq \frac{\sqrt{x}}{8\pi} \log(x)(\log(x) - 3) \quad \text{for } x \geq 5000,$$

and

$$(7.4) \quad |\vartheta(x) - x| \leq \frac{\sqrt{x}}{8\pi} \log(x)(\log(x) - 2) \quad \text{for } x \geq 5000.$$

These imply the bounds in (7.1) and (7.2) for $x \geq 5000$, since if (f, g) is one of the tuples (ψ, π^*) or (ϑ, π) , we have

$$g(x) - g(a) = \text{li}(x) - \text{li}(a) - \frac{x - f(x)}{\log(x)} + \frac{a - f(a)}{\log a} - \int_a^x \frac{t - f(t)}{t \log(t)^2} dt$$

by partial summation, and so we get

$$\begin{aligned} |\pi^*(x) - \text{li}(x)| &\leq \frac{\sqrt{x}}{8\pi} (\log(x) - 3) + \left| \pi^*(5000) - \text{li}(5000) - \frac{\psi(5000) - 5000}{\log(5000)} \right| \\ &\quad + \frac{\sqrt{x}}{4\pi} - \frac{\sqrt{5000}}{4\pi} < \frac{\sqrt{x}}{8\pi} \log(x) \end{aligned}$$

and

$$\begin{aligned} |\pi(x) - \text{li}(x)| &\leq \frac{\sqrt{x}}{8\pi} (\log(x) - 2) + \left| \pi(5000) - \text{li}(5000) - \frac{\vartheta(5000) - 5000}{\log(5000)} \right| \\ &\quad + \frac{\sqrt{x}}{4\pi} - \frac{\sqrt{5000}}{4\pi} < \frac{\sqrt{x}}{8\pi} \log(x). \end{aligned}$$

For the remaining values of x the validity of the claimed inequalities is easily checked by a short computer calculation (the author did this with the pari/gp calculator).

We will prove (7.3) for $x \geq 10^{19}$ first, choosing

$$c = \frac{1}{2} \log(x) + 5$$

and

$$\varepsilon = \frac{\log(x)^{3/2}}{8\sqrt{x}}$$

in Proposition 2. In particular, we then have $c > 26$ and $\varepsilon < 1.2 \times 10^{-8}$. If we take into account that

$$\left| \sum_{\rho} \frac{a_{c,\varepsilon}(\rho)}{\rho} \right| = \left| \sum_{\Im(\rho) > 0} \frac{a_{c,\varepsilon}(\rho)}{\rho(1-\rho)} \right| \leq \frac{e^{\varepsilon/2} |1 + i100|}{100} \sum_{\rho}^* \frac{1}{\rho} \leq 0.024$$

holds under these conditions, (3.1) can be simplified to

$$(7.5) \quad x - \psi_{c,\varepsilon}(x) = \sum_{\rho}^* \frac{a_{c,\varepsilon}(\rho)}{\rho} x^{\rho} + \Theta(2).$$

Furthermore, we have

$$\frac{c}{\varepsilon} \leq 4.92 \sqrt{\frac{x}{\log x}} \leq T,$$

so we may assume $\Re(\rho) = 1/2$ for all zeros ρ with imaginary part up to c/ε .

We divide the sum in (7.5) into three parts. For $|\Im(\rho)| > c/\varepsilon$ we get

$$(7.6) \quad \sum_{|\Im(\rho)| > \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^{\rho}}{\rho} \right| \leq 0.16 \frac{x+1}{\sinh(c)} e^{0.71\sqrt{c\varepsilon}} \log(3c) \log\left(\frac{c}{\varepsilon}\right) \\ \leq 0.0013 \sqrt{x} \log(x) \log \log(x) =: \mathcal{E}_1(x)$$

from Proposition 3. Furthermore, choosing $a = \sqrt{\frac{2}{c}}$ in Proposition 3 gives

$$(7.7) \quad \sum_{\frac{\sqrt{2c}}{\varepsilon} < |\Im(\rho)| \leq \frac{c}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^{\rho}}{\rho} \right| \leq \frac{1+11c\varepsilon}{2\pi} \log\left(\frac{c}{\varepsilon}\right) \frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)} \sqrt{x} \\ \leq \frac{1.001}{4\pi e} \log(x) \sqrt{x} \leq 0.03 \log(x) \sqrt{x} =: \mathcal{E}_2(x).$$

For the remaining part of the sum we bound $|a_{c,\varepsilon}(\rho)/\rho|$ trivially by $1/|\Im(\rho)|$ and use Lemma 3, which gives

$$(7.8) \quad \sum_{0 < |\Im(\rho)| \leq \frac{\sqrt{2c}}{\varepsilon}} \left| a_{c,\varepsilon}(\rho) \frac{x^{\rho}}{\rho} \right| \leq \frac{\sqrt{x}}{2\pi} \log\left(\frac{\sqrt{2c}}{2\pi\varepsilon}\right)^2 \\ \leq \frac{\sqrt{x}}{2\pi} \left(\frac{1}{2} \log(x) + \log(1.45) - \log \log(x) \right)^2 \\ \leq \frac{\sqrt{x}}{8\pi} \log(x)^2 + \mathcal{E}_3(x),$$

where

$$\mathcal{E}_3(x) = \sqrt{x} \left(0.061 \log(x) + 0.16 \log \log(x)^2 \right. \\ \left. + 0.024 - 0.15 \log(x) \log \log(x) - 0.114 \log \log(x) \right).$$

Next, we treat the difference $\psi(x) - \psi_{c,\varepsilon}(x)$. Lemma 6 implies

$$\frac{0.98}{\sqrt{2\pi c}} \leq |\nu_c(0)| = \frac{I_1(c)}{2 \sinh(c)} \leq \frac{1}{\sqrt{2\pi c}}$$

for $c > 26$, so that we get

$$(7.9) \quad |\psi(x) - \psi_{c,\varepsilon}(x)| \leq \frac{2.001\sqrt{x} \log(x)^{5/2}}{8\sqrt{\pi(\log(x)+10)}} \log\left(\frac{0.97\sqrt{x} \log(x)^{3/2}}{8\sqrt{\pi(\log(x)+10)}}\right)^{-1} \\ + \frac{2.02}{8} \log(x)^{5/2} + 0.51 \log \log(2x^2) \log(x)$$

from Proposition 4. Since we have $\sqrt{\frac{\log(x)}{\log(x)+10}} \geq 0.9$, the first summand on the right hand side is bounded by

$$\mathcal{E}_4(x) := 0.283\sqrt{x} \frac{\log(x)^{3/2}}{\sqrt{\log(x)+10}}.$$

So if we define

$$\mathcal{E}_5(x) := 0.26 \log(x)^{5/2} + 0.51 \log(x) \log \log(2x)^2 + 2,$$

we get

$$|\psi(x) - x| \leq \frac{\sqrt{x}}{8\pi} \log(x)^2 + \mathcal{E}_1(x) + \mathcal{E}_2(x) + \mathcal{E}_3(x) + \mathcal{E}_4(x) + \mathcal{E}_5(x)$$

from (7.5), (7.6), (7.7), (7.8), and (7.9). Differentiating with respect to the variable $y = \log(x)$ shows that

$$\frac{1}{\sqrt{x} \log(x)} (\mathcal{E}_1(x) + \mathcal{E}_2(x) + \mathcal{E}_3(x) + \mathcal{E}_4(x) + \mathcal{E}_5(x))$$

is monotonously decreasing for $x \geq 10^{19}$ and smaller than $-\frac{3}{8\pi}$, so (7.3) holds in this region.

For $\exp(18) \leq x \leq \exp(44)$ (7.3) can be proven by calculating a sufficient amount of Chebyshev bounds with the method from the previous section. To this end, it suffices to verify

$$(7.10) \quad |\psi(x) - x| \leq \delta_n x$$

for $x \geq y_n = \exp(n/4)$, with a δ_n satisfying

$$(7.11) \quad \delta_n y_n \leq e^{-1/8} \frac{\sqrt{y_n}}{8\pi} \log(y_n) (\log(y_n) - 3),$$

since then (7.10) implies (7.3) for $x \in [y_n, y_{n+1}]$ by concavity of the right hand side. This has been carried out with the choice $x_0 = \exp(-\alpha\varepsilon)y_n$, $c = n/8 + 5$, $T = 2\sqrt{y_n}$, $\varepsilon = c/T$ and $\alpha = 0.2$ in Theorem 1 for $72 \leq n \leq 129$, and with the altered choice $T = 4\sqrt{y_n/\log(y_n)}$ and $\alpha = 0.1$ for $129 \leq n \leq 175$. In all cases (7.11) turned out to hold.

For the remaining $x \in [5000, \exp(18)]$ the validity of (7.3) is easily checked numerically by evaluating $\psi(x)$ at all prime powers in this interval.

Since we have

$$\psi(x) - \psi(\sqrt{x}) \leq \vartheta(x) \leq \psi(x),$$

(7.3) implies (7.4) for $x \geq 10^{11}$. For the remaining x (7.4) follows from the bound

$$0 \leq x - \vartheta(x) \leq 1.938\sqrt{x} \quad \text{for } 5000 \leq x \leq 10^{11},$$

which the author obtained numerically. □

ACKNOWLEDGMENT

The author wishes to thank Jens Franke for pointing out the Sturm Monotony Principle, which was essential for the proof of Lemma 6. He also wishes to thank the referee for his or her comments. Furthermore, he wishes to thank Allysa Lumley for pointing out a mistaken sign in Lemma 8.

REFERENCES

- [AAR99] George E. Andrews, Richard Askey, and Ranjan Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.
- [Bar81] Klaus Barner, *On A. Weil's explicit formula*, J. Reine Angew. Math. **323** (1981), 139–152.
- [Bre79] Richard P. Brent, *On the zeros of the Riemann zeta function in the critical strip*, Math. Comp. **33** (1979), no. 148, 1361–1372.
- [Büt] J. Büthe, *An improved analytic method for calculating $\pi(x)$* , arXiv:1410.7008.
- [Büt15] ———, *Untersuchung der Primzahlzählfunktion und verwandter Funktionen*, Ph.D. thesis, Bonn University, March 2015.
- [FK15] Laura Faber and Habiba Kadiri, *New bounds for $\psi(x)$* , Math. Comp. **84** (2015), no. 293, 1339–1357.
- [FKBJar] J. Franke, Th. Kleinjung, J. Büthe, and A. Jost, *A practical analytic method for calculating $\pi(x)$* , Math. Comp. (to appear).
- [Gou04] Xavier Gourdon, *The 10^{13} first zeros of the Riemann Zeta function and zeros computation at very large height*, URL = <http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf>, October 2004.
- [Log88] B. F. Logan, *Bounds for the tails of sharp-cutoff filter kernels*, SIAM J. Math. Anal. **19** (1988), no. 2, 372–376.
- [Olv97] Frank W. J. Olver, *Asymptotics and special functions*, AKP Classics, A K Peters Ltd., Wellesley, MA, 1997.
- [OS88] A. M. Odlyzko and A. Schönhage, *Fast algorithms for multiple evaluations of the Riemann zeta function*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 797–809.
- [Pla15] David J. Platt, *Computing $\pi(x)$ analytically*, Math. Comp. **84** (2015), no. 293, 1521–1535.
- [Ros41] Barkley Rosser, *Explicit bounds for some functions of prime numbers*, Amer. J. Math. **63** (1941), 211–232.
- [Sch76] Lowell Schoenfeld, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II*, Math. Comp. **30** (1976), no. 134, 337–360.
- [Stu36] C. Sturm, *Memoire sur les équation différentielles linéaire du second ordre*, J. Math. Pure Appl. (1) **1** (1836), pp. 106–186.
- [vM95] H. von Mangoldt, *Zu Riemanns Abhandlungen "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"*, J. Reine Angew. Math. **114** (1895), pp. 255–305.
- [Wat44] G. N. Watson, *A treatise on the theory of bessel functions*, 2nd ed., Cambridge University Press, 1944.

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