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Improved formulations of the joint order batching and picker routing problem

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ABSTRACT
Order picking is the process of retrieving ordered products from storage locations in warehouses. In picker-to-parts order picking systems, two or more customer orders may be grouped and assigned to a single picker. Then routing decision regarding the visiting sequence of items during a picking tour must be made. (J.Won and S.Olafsson 2005) found that solving the integrated problem of batching and routing enables warehouse managers to organize order picking operations more efficiently compared with solving the two problems separately and sequentially. We therefore investigate the mathematical programming formulation of this integrated problem.

We present several improved formulations for the problem based on the findings of (Valle, Beasley, and da Cunha 2017), that can significantly improve computational results. More specifically, we reconstruct the connectivity constraints and generate new cutting planes in our branch-and-cut framework. We also discuss some problem properties by studying the structure of the graphical representation, and we present two types of additional constraints. We also consider the no-reversal case of this problem. We present efficient formulations by building different auxiliary graphs. Finally, we present computational results for publicly available test problems for single-block and multiple-block warehouse configurations.

KEYWORDS
Integer programming; inventory management; order batching; order picking; picker routing

1. Introduction

In modern business environments, warehousing and relative order picking processes are essential components of any supply chain (Manzini 2012). Order picking is the process of retrieving ordered products from storage locations in warehouses, and it typically accounts for 55% of the total warehouse operating expense and has long been recognized as the most labor-intensive and costly activity for warehouses. Therefore, the order picking process should be robustly designed and optimally controlled to handle requirements efficiently (de Koster, Le-Duc, and Roodbergen 2007; Tompkins et al. 2010).

The order picking process should be investigated within a system context. In picker-to-parts systems, pickers walk or ride through the picking area to collect the requested items. In parts-to-picker systems, automated cranes move along the aisle, retrieve unit

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loads, and bring them to a pick position. This study focuses on order picking operations in picker-to-parts systems, which still account for the large majority of all order picking systems (de Koster, Le-Duc, and Roodbergen 2007; Marchet, Melacini, and Perotti 2015; van Gils et al. 2018).

According to (van Gils et al. 2018), decisions to manage order picking in picker-to-parts systems can be classified into strategic, tactical or operational decisions. In this paper we focus on the operational planning problems that typically concern daily operations. Operational planning problems include (1) how batches of orders are generated (batching), (2) how each picker is routed (routing), (3) how the daily required number of order pickers is determined (workforce level), (4) how a given workforce is allocated to the picking and sorting operations (workforce allocation), and (5) how the picking orders are sequenced (job assignment). Some researchers focus on individual planning problems, have developed many efficient algorithms, and have successfully applied theses algorithms to suboptimal problems. However, simultaneously optimizing multiple order picking planning problems can result in even more efficient picking operations. (van Gils et al. 2018) presented a comprehensive review of research on combination problems in order picking systems. According to their study, the joint order batching and picker routing problem (JOBPRP), which is also the focus of attention in this study, has received more attention than other combination problems (for example, the integrated problem of order batching and job assignment). One of the first works that considers this integrated problem was (J.Won and S.Olafsson 2005). The crucial observation from their simulation experiment is that a simultaneous solution yields significantly better performance benefits than a sequential solution. More recently, (Scholz and Wäscher 2017) integrated different routing algorithms into a heuristic approach for the batching problem. Their numerical experiments demonstrated the benefits from solving the joint problem.

We now introduce the JOBPRP from practical application viewpoints. In picker-to-parts systems, an order picker is guided by a pick list that comprises one or more customer orders on his/her picking tour. A customer order typically requires a list of distinct products, and customer orders can be converted into a pick list until the capacity of the picking device is exhausted. To minimize the total travel distance, order pickers have to decide how the orders should be assigned to picking tours, which give rise to the so-called order batching problem. For each picking tour, the shortest path to visit a set of picking locations should be determined, which give rise to the so-called picker routing problem. Those two problems are thought to be strongly linked, because solving the routing problem is dependent on the solution of the batching problem. Obviously, solving the integrated problem, the JOBPRP, can improve the efficiency of order pickers.

Although there are many heuristic-solving frameworks, very few exact algorithms for the JOBPRP have been proposed in the literature. (Valle, Beasley, and da Cunha 2017) proposed a novel exact algorithm that incorporates a non-compact integer programming formulation and a branch-and-cut procedure. The original model proposed by (Valle, Beasley, and Cunha 2016) can be significantly improved by adding valid inequalities based on the standard layout of warehouses. Inspired by (Valle, Beasley, and da Cunha 2017), we analyze existing models for the JOBPRP and propose modified solution approaches to improve computational performance. The following are the main contributions of our study:

(1) We reconstruct the connectivity constraints by making full use of the properties of a rectangular warehouse with multiple blocks. The improved formulations achieve computational efficiency.

(2) We also discuss some problem properties by studying the structure of a graphical representation, and we present two types of additional constraints.

(3) We also consider the no-reversal JOBPRP. We propose a traveling salesman problem (TSP) formulation for single-block and 2-block warehouses by introducing two auxiliary graphs.

(4) We conduct a series of numerical experiments to evaluate our formulations. The test instances used here are generated using the method provided by (Valle, Beasley, and Cunha 2016).

The remainder of this paper is organized as follows. Section 2 comprises a literature review regarding the JOBPRP and some closely related problems. Section 3 briefly introduces the JOBPRP and presents a graph-based formulation for this problem. Section 4 presents two improved formulations, and Section 5 introduces two types of additional constraints to improve solution quality. Section 6 discusses the no-reversal JOBPRP in detail. Section 7 reports on some computational results and observations, and Section 8 concludes the study.

2. Literature review

Picker routing and order batching problems have received considerable interest since the 1980s (Elsayed 1981; Vannelli and Kumar 1986; Ratliff and Rosenthal 1983; Cornuéjols, Fonlupt, and Naddef 1985). In this section, we provide a summary of some previous studies on order batching and picker routing problems.

2.1. *Picker routing problem*

This problem can be solved to optimality using any exact approach to the TSP. However, more efficient solution approaches can be obtained using a particular warehouse layout. The first attempt to provide a problem-specific exact approach to picker routing problems was proposed by (Ratliff and Rosenthal 1983). They constructed a sparse graph representation for a rectangular warehouse containing a single block and presented a polynomial-time dynamic programming algorithm. Researchers then developed two generalized algorithms based on the work of (Ratliff and Rosenthal 1983). (Cornuéjols, Fonlupt, and Naddef 1985) interpreted the routing problem as a Steiner travelling salesman problem (STSP) and extended the Ratliff-Rosenthal algorithm to all series-parallel graphs; (Roodbergen and Koster 2001) modified the Ratliff-Rosenthal algorithm and introduced routing heuristics for 2-block and more complex layouts. Recently, (Scholz et al. 2016) proposed an exact approach regarding the unique structure of a single-block warehouse. They introduced integer programming whose size is independent of the number of picking locations and demonstrated that this formulation can significantly improve computational performance.

Because the TSP is NP-hard, optimal routing is often regarded as difficult to determine. Many heuristics have been proposed for the problem from a practical standpoint based on different routing strategies, including S-shape (Goetschalckx and Ratliff 1988), midpoint(Hall 1993), largest gap (Hall 1993), combined (Petersen 1997) and aisle-by-aisle (Vaughan 1999). TSP heuristics can also be used to solve the picker routing problem. (Theys et al. 2010) discovered that the Lin–Kernighan–Helsgaun heuristic (Helsgaun 2000) outperforms the S-shape heuristic when there are two or more blocks in the warehouse. (Cambazard and Catusse 2018) developed a dynamic programming approach for a rectilinear TSP, and the algorithm is also applicable to the picker routing problem. However, the complexity grows exponentially with the number of blocks. Other heuristics are created from metaheuristics. For example, (Ho and Tseng 2006b) proposed a simulating annealing heuristic which is integrated with the largest gap routing strategy. The reader interested in the picker routing problem may refer to (Masae, Glock, and Grosse 2020) for a comprehensive review.

2.2. *Order batching problem*

The order batching problem can be formally defined as follows: How, given the capacity of the picking device and the adopted routing strategy, can a given set of customer orders with known storage locations be grouped into picking orders such that the total lengths of all picker tours is minimized? (Wäscher 2004; Manzini 2012)

Because the order batching problem is known to be NP-hard (Gademann, Berg, and Hoff 2001), exact approaches are typically impractical for instances with relatively large sizes. As a result, many scholars consider heuristics and metaheuristics. According to (Manzini 2012), batching heuristics can be distinguished into savings, seed, or priority rule-based algorithms as well as other algorithms. Savings algorithms are based on the algorithm of (Clarke and Wright 1964) for the vehicle routing problem. The initial version of the savings algorithm for the batching problem can be described as follows: savings are computed in terms of reducing the travel distance by collecting items for two customer orders on a single picking tour instead of collecting them separately, and then, orders are sequentially assigned to batches based on the savings (e.g., see (Elsayed and Unal 1989; Bozer and Kile 2008)). Meanwhile, the seed algorithm introduced by (Elsayed 1981) generates batches by means of a two-phase procedure: a seed order is first selected and added to a new batch according to a seed selection rule, and then, unassigned orders are added to this batch according to an order addition rule (e.g., see (Gibson and Sharp 1992; ROSENWEIN 1996; Ho and Tseng 2006a)). The priority rule-based algorithm also consists of a two-step procedure: first, priorities are assigned to the customer orders, and then, customer orders are assigned successively to batches in the sequence given by the priorities (e.g., see (Pan and Liu 1995; Ruben and Jacobs 1999)). There are also some metaheuristics for the order batching problem. The interested reader may refer to (Manzini 2012; van Gils et al. 2018; Cergibozan and Taşan 2019) for further details.

2.3. *Joint order batching and picker routing problem*

Considering the strong relationship between batching and routing, solving these planning problems in a detailed manner would be beneficial. In recent decades, many efficient heuristic and metaheuristic methods have been proposed to solve the JOBPRP.

(J.Won and S.Olafsson 2005) was one of the first to formulate the batching and routing problem jointly as a combinatorial optimization problem. Their proposed two-step heuristic first constructs batches sequentially and then solves the subsequent routing problem. (Hong, Johnson, and Peters 2012) presented a route-selection based formulation that enumerates all possible routes and compared their heuristic solution with a lower bound developed by a relaxation model. (Kulak, Şahin, and Taner 2012) proposed a tabu search algorithm integrated with a clustering algorithm that generates an initial solution. They also proposed two constructive heuristics to solve the picker routing problem. (Grosse, Glock, and Ballester-Ripoll 2014) developed a simulated annealing algorithm to determine order batches and picker routes and applied four different heuristics to form initial order batches. (Cheng et al. 2015) proposed a hybrid approach consisting of a particle swarm optimization for batching, whereas (Li, Huang, and Dai 2017) proposed a constructive heuristic based on similarity coefficient for batching; both used an ant colony optimization algorithm in the routing procedure. (Scholz and Wäscher 2017) introduced an iterated local search algorithm, which allows for integrating different routing algorithms. (Arbex Valle and Beasley 2020) presented an approximate formulation for this problem. They also proposed a partial integer optimization heuristic based on their formulation. (Briant et al. 2020) proposed a heuristic based on column generation to deal with an exponential linear programming formulation of the JOBPRP. (Aerts, Cornelissens, and Sörensen 2021) modeled the JOBPRP as a clustered vehicle routing problem and applied a two-level variable neighborhood search algorithm developed by (Defryn and Sörensen 2017). (Attari et al. 2021) presented a model of JOBRPR under uncertainty; metaheuristics, such as genetic, particle swarm optimization, and artificial honeybee colony algorithms are used as approaches to solve the formulated model.

To enhance efficiency and customer service, some researchers also take the due dates of the customer orders into account, which initiates the picking sequencing problem. (Tsai, Liou, and Huang 2008) considered earliness and tardiness penalties and suggested a genetic algorithm under the assumption that splitting customer orders is allowed. (Chen et al. 2015) developed a genetic algorithm for the order batching and sequencing processes. For the routing decision for each batch, they adopted an ant colony algorithm. (Scholz, Schubert, and Wäscher 2017) introduced a mixed-integer linear formulation whose size increases polynomially with the number of orders. They also proposed a variable neighborhood descent algorithm that could work with very large problems. Meanwhile, (van Gils et al. 2019) proposed an iterated local search algorithm to solve the problem effectively and efficiently. They also showed the substantial performance benefits gained from integrating planning problems using a real-life case study.

Apart from these heuristic and metaheuristic methods, only a few exact approaches have been proposed in the literature. (Valle, Beasley, and Cunha 2016) presented three basic formulations of the JOBPRP; one of them involves exponentially many constraints and the remaining two are based on network flows. They used the branch-and-cut algorithm presented by (Padberg and Rinaldi 1991) for the first formulation. A JOBPRP-test instance generator based on publicly available real-world data was also introduced. The non-compact formulations proposed by (Valle, Beasley, and Cunha 2016) was improved by (Valle, Beasley, and da Cunha 2017). They introduced a significant number of valid inequalities to strengthen the linear relaxation of their formulation.

3. Problem description and basic formulation

In this section, we provide more background information regarding the warehouse layout. We then introduce a basic formulation of the JOBPRP.

3.1. *Background information*

We consider a rectangular warehouse with a manual picker-to-parts order-picking system. The warehouse is composed of one origin, several vertical picking aisles, and several horizontal cross-aisles. We call the part between two adjacent cross-aisles a

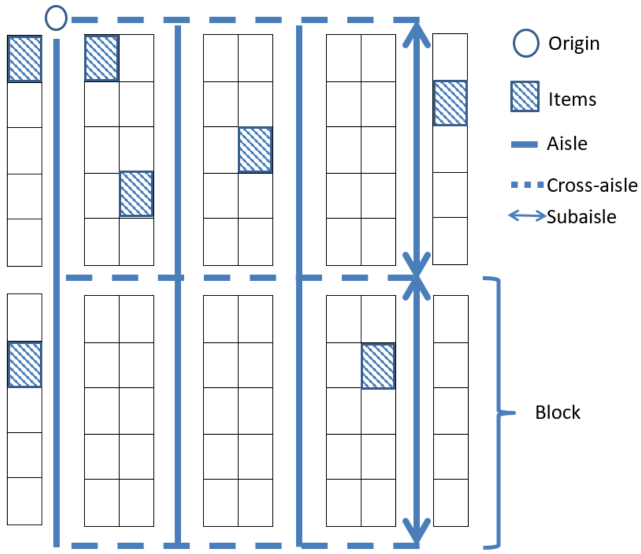


Figure 1. Example of a 2-block warehouse layout

block, and the section of a picking aisle within a block a subaisle. If a warehouse has q blocks, any picking aisle in the warehouse can be partitioned into q subaisles. We illustrate these concepts in Figure 1.

Each vertical aisle contains a set of picking locations on both sides and a picking location contains several storage slots. We assume that each slot holds one type of product and each product type is assigned to only one slot. In addition, products are divided into several classes, and products belonging to the same class are placed in consecutive slots. As a result, most items in a subaisle belong to the same class. The adopted storage assignment policy is actually a variant of the class-based storage policy (Manzini 2012).

Before a shift starts, the number of order pickers available for carrying out the picking operations has been determined. Pickers start at the origin, visit a set of picking locations to retrieve the order products and return to the origin. During a picking tour, a picker is equipped with a trolley that accommodates a limited number of baskets. The necessary number of baskets to carry each customer order is assumed to be known. A customer order typically requires a list of distinct products, and we assume that a customer order cannot be split over various batches. The reason is that mixing and dividing orders can result in an unacceptable consolidation effort (Valle, Beasley, and Cunha 2016; Scholz and Wäscher 2017).

3.2. *Model formulation*

The basic formulation of the JOBPRP that we discuss in this section is based on the STSP. Thus we first introduce a formulation for the STSP in the context of the single-picker routing problem. Then, we provide the basic formulation for the JOBPRP.

Let V_L denote a set of picking locations, each of which is on a subaisle and contains one or more requested products. Let V_I denote a set of endpoints of each subaisle, and we call these points 'artificial locations'. For simplicity, we assume that the origin s is located at the top left corner of the warehouse, and it just overlaps the first artificial

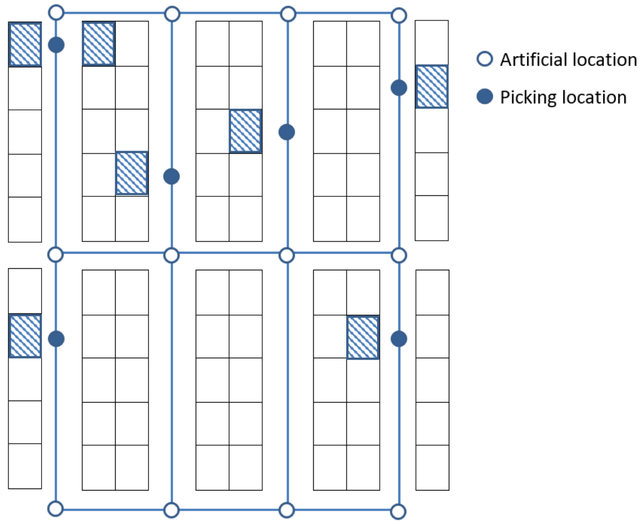


Figure 2. The related graph optimization problem

location. We define a sparse graph using the vertex set V and the edge set E. Edges in E connect the following pair of vertices: (1) two neighboring locations within a picking aisle and (2) two neighboring artificial locations within a cross-aisle. The length d_e of any edge $e \in E$ is equal to the direct distance between two locations. The graphical representation of a warehouse is shown in Figure 2.

Now we present a non-compact formulation, inspired by the work of (Letchford, Nasiri, and Theis 2013), for the single-picker routing problem. Before a picker enters the warehouse, the graphical representation $G = (V, E)$ is already known. The task is to find a closed walk by which every $v \in V_L$ is visited. Note that the walk need not be Hamiltonian or Euler circuits, that is, a vertex or an edge can be visited more than once by the walk. Let $[u, v]$ denote the unordered pair of location u and location v, i.e., $[u, v]$ is the edge connecting u and v. For any node set $S \subset V$, $\delta(S)$ denotes the set of edges with exactly one end-node inside S. For a single vertex $v \in V$, let $\delta(v) = \delta(\{v\})$. We introduce a nonnegative decision variable $x_e \in \mathbb{Z}$ to represent the number of times edge e is traversed. We also use a binary decision variable y_v to indicate whether vertex $v \in V \backslash \{s\}$ is visited by the walk. The single-picker routing problem can be easily described by the following program.

$$
\begin{aligned}
\min \quad & \sum_{e \in E} d_e x_e && && (1)\\
\text{s.t.} \quad & \sum_{e \in \delta(v)} x_e \geq 1, && \forall v \in \{s\} \cup V_L && (2)\\
& y_v \geq \min\{x_e, 1\}, && \forall v \in V \backslash \{s\}, e \in \delta(v) && (3)\\
& \sum_{e \in \delta(S)} x_e \geq y_v, && \forall v \in S, S \subset V \backslash \{s\}, |S| \geq 2 && (4)\\
& \sum_{e \in \delta(v)} x_e \text{ is an even integer}, && \forall v \in V && (5)\\
& x_e \in \{0, 1, 2, ...\}, && \forall e \in E && (6)\\
& y_v \in \{0, 1\}, && \forall v \in V && (7)
\end{aligned}
$$

Constraints (2) ensure that each picking location is visited by the picker. Constraints (3) define the y variables for each vertex. Constraints (4) guarantee that the multigraph induced by the walk is connected. In the TSP, constraints (4) are also known as subtour elimination constraints. Constraints (3), (4), and (5) ensure that Euler circuit exists in the multigraph induced by the walk.

The abovementioned formulation is treated as a starting point for the JOBPRP. The basic formulation for the JOBPRP is constructed by including the assignment of orders to batches as additional constraints. We start by creating a directed graph $\tilde{G} = (V, \tilde{E})$ from graph $G = (V, E)$: any edge $e = [u, v] \in E$ is replaced with two directed arcs $e_1 = (u, v)$ and $e_2 = (v, u)$. A formulation based on graph $\tilde{G}$ for the single-picker routing problem can be formulated similarly. Constraints (5) will then be replaced by flow constraints, which are known to be totally unimodular constraints. Furthermore, a very useful theorem proposed in (Valle, Beasley, and Cunha 2016) is as follows.

Theorem 3.1. *Each directed arc in $\tilde{E}$ can only be traversed once by any optimal walk.*

When multiple pickers and pick capacity are considered, we should assign orders to pickers. Assume that a picker has a capacity of B units, the set of all orders is denoted by O, and any order $o \in O$ has a capacity of b_o units. Two or more customer orders can be batched together if the total capacity of customer orders assigned to a picker does not exceed its available capacity. After the order batching process, O will be partitioned into several subsets, and each subset of orders will be assigned to one picker. Each order o contains a subset of picking locations $L_o \subset V$. Let $O_t \subset O$ denote the subset of orders assigned to picker t, then picker t must visit all nodes in $\cup_{o \in O_t} L_o$ to collect a set of items for orders in O_t.

Now we present a basic formulation of the JOBPRP based on formulation (1)-(7) and Theorem 3.1. For any node set $S \subset V$, let $\delta^+(S) = \{(u, v) \in \tilde{E} : u \in S, v \notin S\}$ and $\delta^-(S) = \{(u, v) \in \tilde{E} : u \notin S, v \in S\}$. For a single vertex $v \in V$, let $\delta^+(v) = \delta^+(\{v\})$ and $\delta^-(v) = \delta^-(\{v\})$. Let T be the number of available pickers, and let $\mathcal{T} = \{1, 2, ..., T\}$. We introduce the binary variables x_{tuv} to indicate whether arc (u, v) is traversed by picker t, y_{tv} to indicate whether vertex v is visited by picker t and z_{ot} to indicate whether picker t picks order o. The basic formulation is formally given as follows.

$$\min \quad \sum_{t=1}^{T} \sum_{(u,v)\in \tilde{E}} d_{uv} x_{tuv} \tag{8}$$

$$\text{s.t.} \quad \sum_{(s,v)\in\delta^+(s)} x_{tsv} \geq 1, \qquad \forall t \in \mathcal{T} \tag{9}$$

$$\sum_{(u,v)\in\delta^+(u)} x_{tuv} \geq z_{ot}, \qquad \forall t \in \mathcal{T}, o \in O, u \in L_o \tag{10}$$

$$y_{tu} \geq x_{tuv}, \qquad \forall t \in \mathcal{T}, u \in V\backslash\{s\}, (u,v) \in \delta^+(u) \tag{11}$$

$$\sum_{(u,v)\in\delta^+(S)} x_{tuv} \geq y_{tu_0}, \qquad \forall t \in \mathcal{T}, S \subset V\backslash\{s\}, |S| \geq 2, u_0 \in S \tag{12}$$

$$\sum_{(v,u)\in\delta^+(v)} x_{tvu} = \sum_{(u,v)\in\delta^-(v)} x_{tuv}, \qquad \forall t \in \mathcal{T}, v \in V \tag{13}$$

$$\sum_{t\in\mathcal{T}} z_{ot} = 1, \qquad \forall o \in O \tag{14}$$

$$\sum_{o\in O} b_o z_{ot} \leq B, \qquad \forall t \in \mathcal{T} \tag{15}$$

$$x_{tuv} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, (u,v) \in \tilde{E} \tag{16}$$

$$y_{tv} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, v \in V \tag{17}$$

$$z_{ot} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, o \in O \tag{18}$$

Constraints (9)-(13) are similar to constraints (2)-(5); each of them describes a picker routing process. Constraints (14) ensure that each order is assigned to precisely one picker. Constraints (15) are capacity constraints. Constraints (14)-(15) describe the order batching process. The feasible region of the basic formulation is denoted by P_{basic}, i.e., $P_{basic} = \{(x,y,z) : \textit{constraints } (9)-(18)\}$. This formulations is non-compact because it involves exponentially many constraints (12) to enforce connectivity. A branch-and-cut algorithm that separates these constraints should be implemented when this formulation is adopted.

The basic formulation is a slightly different version of the original formulation (Valle, Beasley, and Cunha 2016). The original formulation assumes that it is unnecessary for a picker to depart from the origin when no order is assigned to it, but constraints (9) force all pickers to depart from the origin. The main reasons for this assumption are the following:

(1) We do not tackle the workforce level planning problem in this paper, and this assumption helps in simplifying the formulation.
(2) This is not a critical assumption because constraints (9) can easily be modified to deal with the previous assumption.

In the rest of the paper, we simply let $|\mathcal{T}|$ be the necessary number of pickers to carry all products, which can be obtained by solving a bin-packing problem.

4. Improved formulations

In this section, we propose two improved formulations P_G and P_F for the JOBPRP by reformulating the connectivity constraints (12). The key idea of reformulation is to enforce connectivity using the subaisle cuts (Valle, Beasley, and da Cunha 2017). To improve readability, we present two tables that summarize the most important notations and formulations in the appendix.

We first introduce the subaisle cuts and discuss the relationships between these cuts and the basic formulation. Let the number of subaisles be W_{sub}. Subaisles are indexed by the elements in set $[W_{sub}] = \{1, 2, ..., W_{sub}\}$. Let the set of picking locations within subaisle i be $V_{sub}(i)$. The northern artificial location is denoted by $f(i)$ and the southern artificial location is denoted by $l(i)$. For any picking location $v \in V_{sub}(i)$, the adjacent northern location is denoted by $n(v)$ and the adjacent southern location is denoted by $s(v)$. $s(f(i))$ and $n(l(i))$ are defined similarly. By using auxiliary binary variables α and β, we consider the following feasible region P_{sub} containing only subaisle cuts:

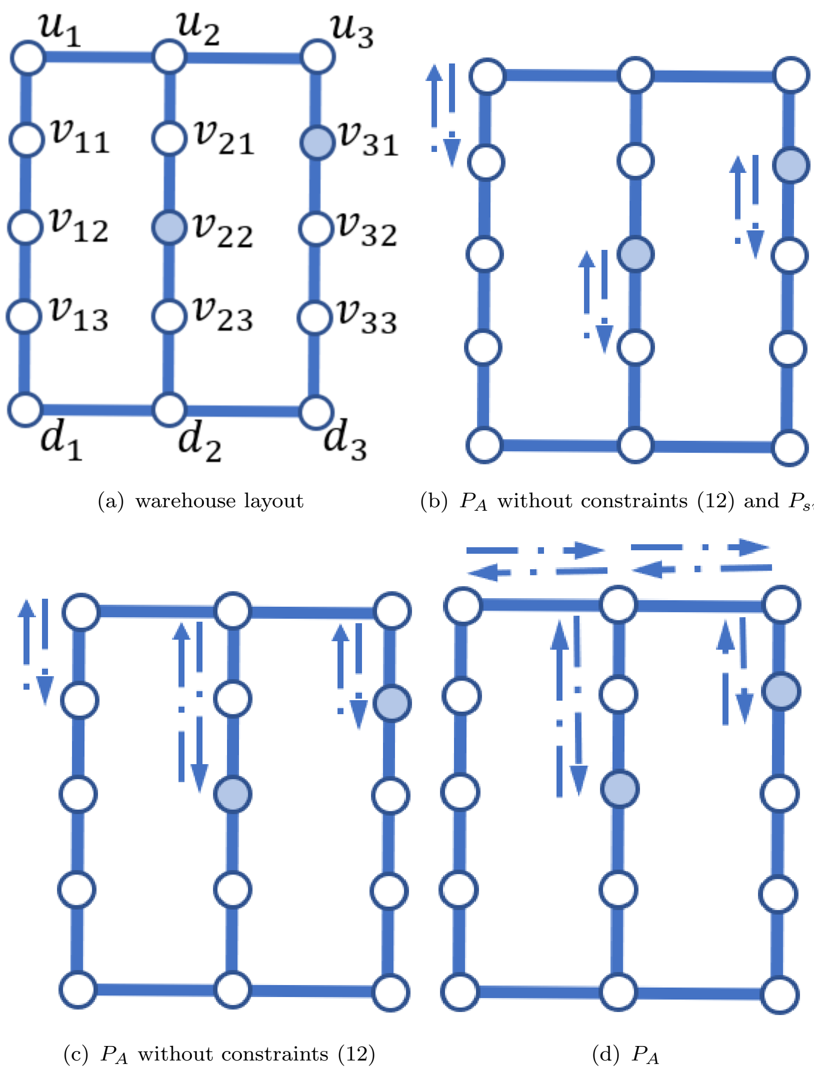
$$\alpha_{tv} \geq \alpha_{ts(v)}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], v \in V_{sub}(i) \backslash \{n(l(i))\} \tag{19}$$
$$x_{tn(v)v} \geq \alpha_{tv}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], v \in V_{sub}(i) \tag{20}$$
$$\beta_{tv} \geq \beta_{tn(v)}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], v \in V_{sub}(i) \backslash \{s(f(i))\} \tag{21}$$
$$x_{ts(v)v} \geq \beta_{tv}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], v \in V_{sub}(i) \tag{22}$$
$$\alpha_{tv} + \beta_{tv} \geq z_{ot}, \qquad \forall t \in \mathcal{T}, o \in O, v \in L_o \tag{23}$$
$$\alpha_{tv} \in \{0, 1\}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], v \in V_{sub}(i) \tag{24}$$
$$\beta_{tv} \in \{0, 1\}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], v \in V_{sub}(i) \tag{25}$$

Those additional constraints work as follows: we suppose that there exists a feasible solution $(x^*, \alpha^*, \beta^*) \in P_{sub}$. For any picking location v in subaisle i, if $\alpha^*_{tv} = 1$ then there exists a straight path connecting $f(i)$ and v in walk t, and the same holds when $\beta^*_{tv} = 1$. If walk t pass through picking location v, then there must exist a path connecting v and the northern or southern artificial location. Therefore, constraints (23) are valid constraints. Remark that $\alpha_{tv} + \beta_{tv}$ may not be 2 when there exist two paths in walk t, one of them connects picking location v and the northern artificial location, and others connect v and the southern artificial location. In other words, there is no surjection from P_{basic} to $P_A = \{(x, y, z, \alpha, \beta) : (x, y, z) \in P_{basic}, (x, z, \alpha, \beta) \in P_{sub}\}$.

Subaisle cuts can be regarded as a class of connectivity constraints Therefore, those cuts can partially replace constraints (12). To illustrate the relationship between P_{sub} and constraints (12), consider the following example.

Example 4.1. Consider the warehouse illustrated in Figure 3. Suppose that there is an order o with $L_o = \{v_{22}, v_{31}\}$. We provide feasible solutions for various relaxations of P_A in Figure 3:

(1) When we remove all connectivity constraints, a feasible solution could only consist of several cycles.
(2) When we remove constraints (12), v_{22} and v_{31} are forced to be connected with neighboring artificial locations.
(3) When all connectivity constraints are used, we are able to generate a Eulerian tour from a feasible solution.



(a) warehouse layout

(b) P_A without constraints (12) and P_{sub}

(c) P_A without constraints (12)

(d) P_A

Figure 3. Warehouse layout and feasible solutions for different formulations

It is possible to observe that we only need to focus on the connectivity of the graph induced by artificial locations when subaisle cuts have been added to the formulation (Figure 3(c)). In the remainder of this section, we reconstruct constraints (12) using this observation. For any artificial location v, we let $Q_W(v)$ be the adjacent artificial location u lying to the west of v; we let $Q_E(v)$ be the adjacent artificial location u lying to the east of v. For any subaisle i, we let $Q_N(l(i)) = f(i)$ and $Q_S(f(i)) = l(i)$. We illustrate all of these concepts in the reduced graph in Figure 4. Let $\tilde{E}'$ be the arc set of the reduced graph, that is, $\tilde{E}$ is a set of edges connecting neighboring artificial locations while ignoring picking locations within subaisles. For each arc $(u,v) \in \tilde{E}'$, we introduce an auxiliary binary variable γ_{tuv}. For any node set $S \subset V_I$, let $\eta^+(S) = \{(u,v) \in \tilde{E}' : u \in S, v \notin S\}$ and $\eta^-(S) = \{(u,v) \in \tilde{E}' : u \notin S, v \in S\}$. We have the following feasible region P_g:

$$\sum_{(s,v)\in\delta^+(s)} x_{tsv} \geq 1, \qquad \forall t \in \mathcal{T} \tag{26}$$

$$\sum_{(u,v)\in\delta^+(u)} x_{tuv} \geq z_{ot}, \qquad \forall t \in \mathcal{T}, o \in O, u \in L_o \tag{27}$$

$$y_{tu} \geq x_{tuv}, \qquad \forall t \in \mathcal{T}, u \in V_I \backslash \{s\}, (u,v) \in \delta^+(u) \tag{28}$$

$$x_{tvQ_W(v)} = \gamma_{tvQ_W(v)}, \qquad \forall t \in \mathcal{T}, v \in V_I, (v, Q_W(v)) \in \tilde{E}' \tag{29}$$

$$x_{tvQ_E(v)} = \gamma_{tvQ_E(v)}, \qquad \forall t \in \mathcal{T}, v \in V_I, (v, Q_E(v)) \in \tilde{E}' \tag{30}$$

$$\alpha_{tn(l(i))} \geq \gamma_{tf(i)l(i)}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}] \tag{31}$$

$$x_{tn(l(i))l(i)} \geq \gamma_{tf(i)l(i)}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}] \tag{32}$$

$$\beta_{ts(f(i))} \geq \gamma_{tl(i)f(i)}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}] \tag{33}$$

$$x_{ts(f(i))f(i)} \geq \gamma_{tl(i)f(i)}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}] \tag{34}$$

$$\sum_{(u,v)\in\eta^+(S)} \gamma_{tuv} \geq y_{tu_0}, \qquad \forall t \in \mathcal{T}, S \subset V_I \backslash \{s\}, |S| \geq 2, u_0 \in S \tag{35}$$

$$\sum_{(v,u)\in\delta^+(v)} x_{tvu} = \sum_{(u,v)\in\delta^-(v)} x_{tuv}, \qquad \forall t \in \mathcal{T}, v \in V \tag{36}$$

$$\sum_{t\in\mathcal{T}} z_{ot} = 1, \qquad \forall o \in O \tag{37}$$

$$\sum_{o\in O} b_o z_{ot} \leq B, \qquad \forall t \in \mathcal{T} \tag{38}$$

$$x_{tuv} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, (u,v) \in \tilde{E} \tag{39}$$

$$y_{tv} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, v \in V \tag{40}$$

$$z_{ot} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, o \in O \tag{41}$$

$$\gamma_{tuv} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, (u,v) \in \tilde{E}' \tag{42}$$

The auxiliary binary variable γ_{tuv} tries to indicate whether $(u,v) \in \tilde{E}'$ is traversed by walk t. Consider a feasible solution $(x^*, y^*, z^*, \alpha^*, \beta^*, \gamma^*) \in P_g$. If $\gamma^*_{tuv} = 1$, artificial locations u,v are in the same connected component in walk t according to constraints (29)-(34). If at most the subaisle between u and v is partially traversed in the direction $u \to v$ by walk t, then we have $\gamma^*_{tuv} = 0$. Thus, it suffices to add the connectivity constraints (35) rather than constraints (12) to ensure that all artificial

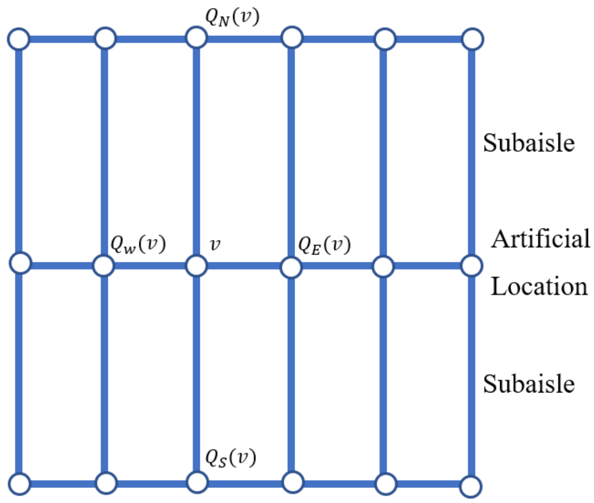


Figure 4. The related reduced graph

locations are in the same connected component. Let $f(x) = \sum_{t=1}^{T} \sum_{(u,v)\in\tilde{E}} d_{uv} x_{tuv}$ and $P_G = \{(x, y, z, \alpha, \beta, \gamma) : (x, y, z, \gamma) \in P_g, (x, z, \alpha, \beta) \in P_{sub}\}$. As shown in the following theorem, it suffices to compute an optimal solution that satisfies P_G.

Theorem 4.2. $min\{f(x) : (x, y, z, \alpha, \beta, \gamma) \in P_G\} = min\{f(x) : (x, y, z) \in P_{basic}\}$

Proof. For some $(x^*, y^*, z^*) \in P_{basic}$, $(\alpha^*, \beta^*, \gamma^*)$ are defined as follows:

(1) For any picking location v within subaisle i, we set $\alpha^*_{tv} = 1$ if $x^*_{tus(u)} = 1$ for $u \in \{f(i), s(f(i)), s(s(f(i))), ..., n(v)\}$; otherwise, we set $\alpha^*_{tv} = 0$.
(2) For any picking location v within subaisle i, we set $\beta^*_{tv} = 1$ if $x^*_{tun(u)} = 1$ for $u \in \{l(i), n(l(i)), n(n(l(i))), ..., s(v)\}$; otherwise, we set $\beta^*_{tv} = 0$.
(3) For any aritificial location v, we set $\gamma^*_{tvQ_W(v)} = x^*_{tvQ_W(v)}$ and $\gamma^*_{tvQ_E(v)} = x^*_{tvQ_E(v)}$. We set $\gamma^*_{tvQ_N(v)} = 1$ if $v = l(i)$ and $x^*_{tun(u)} = 1$ for $u \in \{l(i), n(l(i)), ..., s(f(i))\}$; otherwise, we set $\gamma^*_{tvQ_N(v)} = 0$. We define $\gamma^*_{tvQ_S(v)}$ similarly.

It is easy to check that $(x^*, y^*, z^*, \alpha^*, \beta^*, \gamma^*) \in P_G$, therefore the following holds:

$$min\{f(x) : (x, y, z, \alpha, \beta, \gamma) \in P_G\} \leq min\{f(x) : (x, y, z) \in P_{basic}\}$$

Let us now consider an optimal solution $(x^*, y^*, z^*, \alpha^*, \beta^*, \gamma^*) \in P_G$. For picker $t \in \mathcal{T}$, the graph indicated by x^* is denoted by H. The connected component of H that contains the origin is denoted by H_0. We are able to generate a feasible picking tour from H_0 because any picking location v, which belongs to a location set L_o with $z^*_{ot} = 1$, also belongs to the vertex set of H_0. This gives us the following result.

$$min\{f(x) : (x, y, z, \alpha, \beta, \gamma) \in P_G\} \geq min\{f(x) : (x, y, z) \in P_{basic}\}$$

$\square$

Similar to P_{basic}, the improved formulation P_G involves exponentially many constraints to enforce connectivity. The connectivity constraints should be added to the model via a separation procedure when using a branch-and-cut algorithm. For any candidate integral solution in the branch-and-cut tree, we verify via a depth-first search whether the graph is connected and we add connectivity constraints for every connected component except the one containing the origin when the graph is not connected. Note that because a subset of constraints (12) is actually dominated by a single constraint (35) according to constraints (29)-(34), constraints (35) seem to be more suitable for a branch-and-cut procedure.

We also provide a compact formulation of this problem, i.e., a formulation with a polynomial number of variables and constraints. The main trick is introducing an auxiliary multicommodity flow problem. Assume that there is a salesman in each selected artificial location. The salesmen must determine feasible paths to the origin. Once all salesmen can get to the origin, the origin and all selected artificial locations are in the same connected component. We introduce flow variables $\sigma_{tuv}^{v_0}$ to indicate the amount of commodity from artificial location v_0 passing through arc $(u,v) \in \tilde{E}'$ in walk t. P_f is given by constraints (26)-(34), (36)-(42) and the following:

$$\sum_{(v_0,v)\in\eta^+(v_0)} \sigma_{tv_0v}^{v_0} - \sum_{(v,v_0)\in\eta^-(v_0)} \sigma_{tvv_0}^{v_0} = y_{tv_0}, \qquad \forall t \in \mathcal{T}, v_0 \in V_I \tag{43}$$

$$\sum_{(u,v)\in\eta^+(u)} \sigma_{tuv}^{v_0} - \sum_{(v,u)\in\eta^-(u)} \sigma_{tvu}^{v_0} = 0, \qquad \forall t \in \mathcal{T}, v_0 \in V_I, u \in V_I\backslash\{s, v_0\} \tag{44}$$

$$\sum_{(s,v)\in\eta^+(u)} \sigma_{tsv}^{v_0} - \sum_{(v,s)\in\eta^-(s)} \sigma_{tvs}^{v_0} = -y_{tv_0}, \qquad \forall t \in \mathcal{T}, v_0 \in V_I \tag{45}$$

$$0 \le \sigma_{tuv}^{v_0} \le \gamma_{tuv}, \qquad \forall t \in \mathcal{T}, v_0 \in V_I, (u,v) \in \tilde{E}' \tag{46}$$

Let $P_F = \{(x,y,z,\alpha,\beta,\gamma,\sigma) : (x,y,z,\gamma,\sigma) \in P_f, (x,z,\alpha,\beta) \in P_{sub}\}$. We immediately obtain the following.

Theorem 4.3. $min\{f(x) : (x,y,z,\alpha,\beta,\gamma,\sigma) \in P_F\} = min\{f(x) : (x,y,z) \in P_{basic}\}$

Proof. For some $(x^*,y^*,z^*) \in P_{basic}$, $(\alpha^*,\beta^*,\gamma^*)$ are defined as in the proof of Theorem 4.2 and σ^* is defined as follows.

For picker $t \in \mathcal{T}$, the graph indicated by x^* is denoted by H. For any artificial location v_0 with $y_{tv_0}^* = 1$, we consider an arbitrary path in H that goes from v_0 to the origin (because $(x^*,y^*,z^*) \in P_{basic}$, such a path exists). We set $\sigma_{tuv}^{v_0*} = 1$ if (u,v) is traversed by this path; otherwise, we set $\sigma_{tuv}^{v_0*} = 0$.

It is easy to check that $(x^*,y^*,z^*,\alpha^*,\beta^*,\gamma^*,\sigma^*) \in P_F$, and therefore the following holds:

$$min\{f(x) : (x,y,z,\alpha,\beta,\gamma,\sigma) \in P_F\} \le min\{f(x) : (x,y,z) \in P_{basic}\}$$

The rest of the proof is similar to that of Theorem 4.2. □

We finish this section by showing the relationship between the LP relaxations of P_G and P_F. Formally, let P_{LP} denote the LP relaxation of P, let $proj_x(P)$ denote the projection of P onto the $x-space$ and we prove the following theorem:

Theorem 4.4. $proj_{(x,y,z,\alpha,\beta,\gamma)}((P_F)_{LP}) = (P_G)_{LP}$

Proof. According to [Theorem 2. (Letchford, Nasiri, and Theis 2013)], there exists a feasible flow $\sigma_t^{u_0}$ satisfying constraints (43)-(46) if and only if

$$\sum_{(u,v)\in\eta^+(S)} \gamma_{tuv} \geq y_{tu_0} \qquad \forall S \in \{S : u_0 \in S, S \subset V_I \backslash \{s\}\}$$

Because there should be a feasible flow $\sigma_t^{u_0}$ for any $u_0 \in V_I \backslash \{s\}$ according to our construction, the proof is ended. □

5. Additional constraints

In this section, we introduce two types of additional constraints, called strengthened connectivity constraints and single traversing constraints. Strengthened connectivity constraints link the routing decisions and the batching decisions where classical connectivity constraints cannot take batching decisions into account. Single traversing constraints mainly focus on the problem property of a rectangular warehouse and can cut off some non-optimal solutions.

5.1. *Strengthened connectivity constraints and basic cuts*

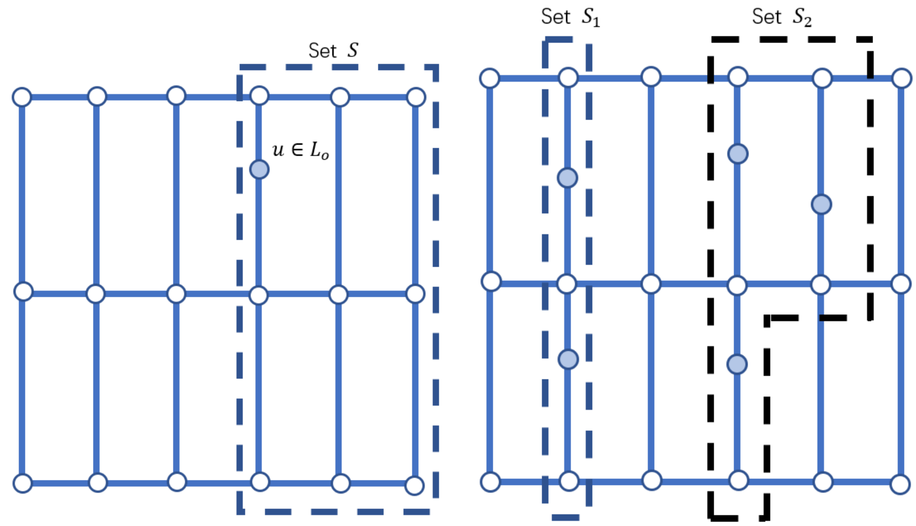
Considering the relationship between routing decisions and batching decisions, it would be interesting to investigate the constraints that jointly deal with both decisions.

We first introduce the concept of the strengthened connectivity constraints. Let us have a look at connectivity constraints (12): a constraint of type (12) can be generated by fixing a proper vertex set S. For any picking location $u_0 \in S$, we know that y_{tu_0} is actually dominated by any z_{ot} satisfying $u_0 \in L_o$ according to constraints (10)-(11). Therefore, we have the following strengthened connectivity constraints:

$$\sum_{(u,v)\in\delta^+(S)} x_{tuv} \geq z_{ot}, \qquad \forall t \in \mathcal{T}, S \subset V\backslash\{s\}, |S| \geq 2, o \in \{o : S \cap L_o \neq \emptyset\} \tag{47}$$

Notice that there are an exponential number of connectivity constraints (12). Thus, constraints (47) cannot be added directly to a formulation. To make use of these constraints, we intend to find a family $\mathcal{S}$ of a polynomial number of vertex sets S. All constraints induced by $\mathcal{S}$ will be added directly to a formulation. A carefully selected $\mathcal{S}$ leads to powerful constraints. For instance, if $S \in \mathcal{S}$ is composed of locations on the right side of some aisle (as shown in Figure 5), constraints (47) are reduced to aisle cuts proposed by (Valle, Beasley, and da Cunha 2017).

Here, we present another type of strengthened connectivity constraints, which we call basic cuts. $\mathcal{S}$ can be obtained by the following operations. For any order o, let $\tilde{E}_0$ denote a set of $e \in \tilde{E}'$ (which represents a subaisle) that contains at least one picking location $u \in L_o$. Let $\tilde{V}_0$ be a set of vertices associated with $\tilde{E}_0$. Then, any $e \in \tilde{E}'$ with both ends in $\tilde{V}_0$ will be added to $\tilde{E}_0$. By executing a depth-first search algorithm, we can discover all connected components of the graph $(\tilde{V}_0, \tilde{E}_0)$. Figure 5 shows an example of the two connected components induced by an order. Let S be a vertex set of a connected component that does not contain the origin. All picking locations within the subaisle that has both ends in S will then be added to S. Then S will be added to $\mathcal{S}$. We call the strengthened connectivity constraints induced by $\mathcal{S}$ basic cuts.



(a) Aisle cut (b) Basic cut

Figure 5. Strengthened connectivity constraints

5.2. *Single traversing constraints*

In this subsection, we derive additional constraints by investigating the graph property of the warehouse. We first restrict our attention to a single-block warehouse. By 'traversing', we mean going from the north artificial vertex to the south artificial vertex or vice-versa.

Theorem 5.1. *For a single-block warehouse, each subaisle will be traversed at most once by any optimal walk.*

Corollary 5.2. *(Single traversing constraints) For a single-block warehouse, no optimal solution will be cut off by the following constraints:*

$$\alpha_{tv} + \beta_{tv} \leq 1, \qquad \forall t \in \mathcal{T}, o \in O, v \in L_o \tag{48}$$

We remark that a subset of feasible solutions may not satisfy Theorem 5.1; therefore, the inequalities it yields might not be strictly valid inequalities. Furthermore, Theorem 5.1 can induce different constraints other than (48) and it is still unclear whether (48) is the most proper formulation.

Corollary 5.2 can be obtained as a direct consequence of Theorem 5.1, and thus, it suffices to prove Theorem 5.1. For a single-block warehouse, let the artificial locations in the first cross-aisle be $u_1, u_2, ..., u_n$, and the artificial locations in the second cross-aisle be $d_1, d_2, ..., d_n$. Let picking locations in the path north to south in aisle i be $v_{i1}, v_{i2}, ..., v_{is_i}$. We illustrate all of these concepts in Figure 6. A Eulerian tour can be specified using a sequence of vertices, such as

$$r = (u_1 v_{11} v_{12} ... d_1 d_2 ... u_2 u_1)$$

Let $V(r)$ denotes the set of the vertices in r. We say that Eulerian tour r' is better than Eulerian tour r if r, r' satisfy

(1) $\{V(r) \cap V_L\} \subseteq \{V(r') \cap V_L\}$

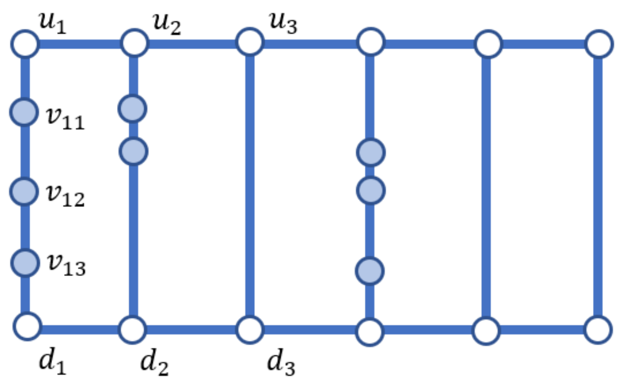


Figure 6. notations that we used in subsection 5.2

(2) r' is shorter than r

Now, we introduce three lemmas to prove Theorem 5.1.

Lemma 5.3. *For any tour r, if there exists an aisle i such that tour r contains subpath $(u_i v_{i1} v_{i2} ... v_{is_i} d_i v_{is_i} v_{i(s_i-1)} ... v_{i1} u_i)$, then there exists a tour r' better than r.*

Proof. We simply assume that $r = (r_1 r_2 ... r_p u_i v_{i1} v_{i2} ... v_{is_i} d_i v_{is_i} v_{i(s_i-1)} ... v_{i1} u_i r_q ... r_R)$. Then $r' = (r_1 r_2 ... r_p u_i v_{i1} v_{i2} ... v_{i(s_i-1)} v_{is_i} v_{i(s_i-1)} ... v_{i1} u_i r_q ... r_R)$ is a better tour. □

The reader may find a different application of Lemma 5.3 in (Valle, Beasley, and da Cunha 2017), in which it also induced the artificial vertex reversal cuts.

Lemma 5.4. *For any tour r, if there exists an aisle i such that tour r contains subpath $(u_{i-1} u_i v_{i1} v_{i2} ... v_{is_i} d_i d_{i+1})$ and $(d_i v_{is_i} v_{i(s_i-1)} ... v_{i_1} u_i)$, then there exists a tour r' better than r.*

Proof. We simply assume that $r = (r_1 r_2 ... r_p u_t u_{t+1} ... u_{i-1} u_i v_{i1} v_{i2} ... v_{is_i} d_i d_{i+1} ... d_k v_{ks_k} r_q ... r_R)$. Then $r' = (r_1 r_2 ... r_p u_t u_{t+1} ... u_{i-1} u_i ... u_k v_{k1} v_{k2} ... v_{ks_k} r_q ... r_R)$ is a better tour. □

Lemma 5.5. *For any tour r, if there exists an aisle i such that tour r contains subpath $(u_{i-1} u_i v_{i1} v_{i2} ... v_{is_i} d_i d_{i-1})$ and $(d_i v_{is_i} v_{i(s_i-1)} ... v_{i_1} u_i)$, then there exists a tour r' better than r.*

Proof. We simply assume that $r = (r_1 r_2 ... r_p u_t u_{t+1} ... u_{i-1} u_i v_{i1} v_{i2} ... v_{is_i} d_i d_{i-1} ... d_k v_{ks_k} r_q ... r_R)$ with $k \geq t$. Then $r' = (r_1 r_2 ... r_p u_t u_{t+1} ... u_{i-1} u_i ... u_k v_{k1} v_{k2} ... v_{ks_k} r_q ... r_R)$ is a better tour. □

Now, we can give the proof for Theorem 5.1.

Proof. We assume by contradiction that there exists an optimal tour $r = (r_1 r_2 ... r_p u_{i-1} u_i v_{i1} ... v_{is_i} d_i r_q ... r_R)$ and r contains subpath $(d_i v_{is_i} v_{i(s_i-1)} ... v_{i_1} u_i)$. Then

(1) If $(r_q r_{q+1} ... r_{q+s_i}) = (v_{is_i} v_{i(s_i-1)} ... v_{i_1} u_i)$, lemma 5.3 provides a better tour.
(2) If $r_q = d_{i+1}$, lemma 5.4 provides a better tour.
(3) If $r_q = d_{i-1}$, lemma 5.5 provides a better tour.

Thus, no such optimal tour exists. □

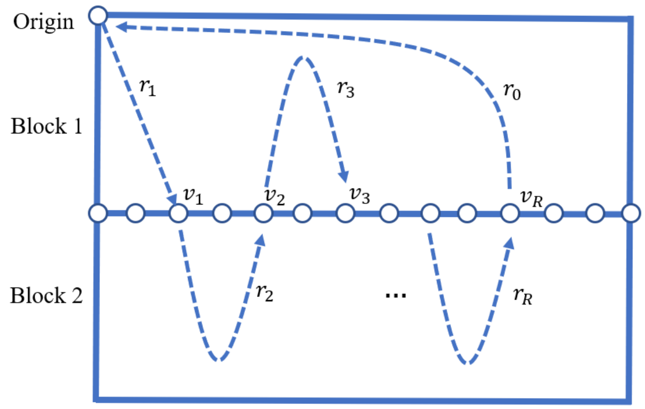


Figure 7. The partition of the walk r

In the remainder of this subsection, we extend the result for a 2-block warehouse. Let 1 be the index of the topmost left subaisle, and we have the following theorem.

Theorem 5.6. *For a 2-block warehouse, there exists an optimal walk that traverses any subaisle $i \in \{2, 3, ..., W_{sub}\}$ at most once.*

Proof. Let r be an optimal walk. We can partition this tour into several parts $r_1, r_2, ..., r_R, r_0$ and each part is located in the first block or the second block (see Figure 7). As shown in the proof of Theorem 5.1, a subaisle cannot be traversed twice by any subtour $r_k, k \in \{0, 1, ..., R\}$. Furthermore, there always exists an optimal solution such that a subaisle will not be traversed by different subtours r_k, r_t where $k, t \in \{1, 2, ..., R\}$. The construction of such an optimal solution is similar to the proofs of Lemma 5.4-5.5. Thus, there exists an optimal walk r such that any subaisle is traversed at most once by $r' = (r_1, r_2, ..., r_R)$. If a picker needs to pick some products in r_0, then any subaisle is traversed at most once by r; otherwise, the picker needs to return to the origin immediately, and we can assume that the picker first goes to the south artificial vertex of subaisle 1 and then goes to the origin. This completes the proof.

□

Corollary 5.7. *(Single traversing constraints) For a 2-block warehouse, there exists an optimal solution satisfying the following constraints:*

$$\alpha_{tv} + \beta_{tv} \leq 1, \qquad \forall t \in \mathcal{T}, o \in O, v \in L_o \backslash V_{sub}(1) \tag{49}$$

Proof. This follows immediately from Theorem 5.6. □

6. No-reversal case

In this section, we introduce the no-reversal case of the JOBPRP and present new formulations that are less influenced by symmetric solutions.

We note that it would be impractical to search for an optimal solution when there are a vast number of orders. By assuming that routing is conducted in a no-reversal fashion, (Valle, Beasley, and da Cunha 2017) obtained an efficiently solvable formulation.

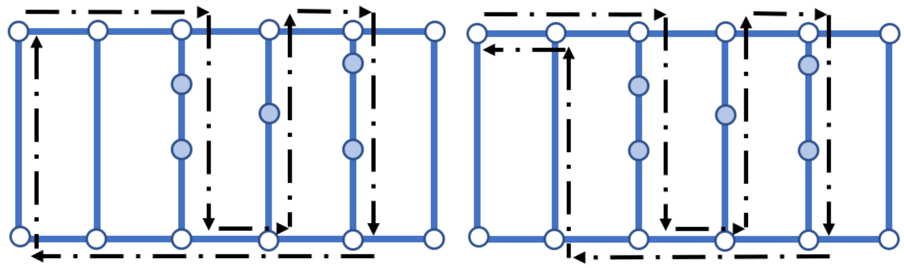


Figure 8. Symmetry routes

The no-reversal constraints are described as follows.

$$x_{tn(v)v} = x_{tv}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], v \in V_{sub}(i) \cup \{l(i)\} \tag{50}$$

$$x_{ts(v)v} = x_{tv}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], v \in V_{sub}(i) \cup \{f(i)\} \tag{51}$$

In the case of no-reversal constraints, a picker is not allowed to reverse in a subaisle. Many feasible solutions, including some or all optimal solutions, will be cut off by these constraints. Note that the no-reversal JOBPRP can also be regarded as an approximate problem, and by solving this problem, we can quickly make order batching decisions. Then each picker can be rerouted by applying other methods such as a classical TSP model.

Although these constraints can greatly simplify the decision problem, this problem still suffers severely from the presence of symmetry. A large amount of equivalence of solutions induced by symmetry routes might confound the branch-and-bound (or branch-and-cut) process, as illustrated in Figure 8. In other words, there is also a substantial room for the further improvement of the no-reversal formulation.

6.1. *TSP formulation for a single-block warehouse*

A single-block warehouse is first considered. We start with an undirected graph $G^+ = (V_I, E^+)$, which is constructed as follows: Recall that $e = [u, v]$ denote the unordered pair of location u and location v. E^+ is defined as $E_1^+ \cup E_2^+$ where $E_1^+ = \{[u, v] : (u, v) \in \tilde{E}'\}$ and $E_2^+ = \{[s, v] : v \in V_I \backslash \{s\}\}$. For any subaisle i, let $e(i)$ denote the edge $[f(i), l(i)]$. The resulting feasible region P_U^1 then consists of the following

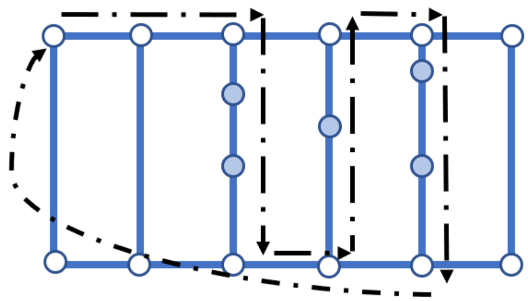


Figure 9. The only path back to the origin

constraints.

$$x_{t[s,l(1)]} + x_{t[s,f(2)]} \geq 1, \qquad \forall t \in \mathcal{T} \tag{52}$$

$$\sum_{[s,v]\in\delta(s)} x_{t[s,v]} + \tilde{x}_t = 2, \qquad \forall t \in \mathcal{T} \tag{53}$$

$$x_{t[u,v]} \geq z_{ot}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], [u,v] = e(i), o \in \{o : V_{sub}(i) \cap L_o \neq \emptyset\} \tag{54}$$

$$\sum_{[l(1),v]\in\delta(l(1))} x_{t[l(1),v]} + \tilde{x}_t = 2y_{tu}, \qquad \forall t \in \mathcal{T} \tag{55}$$

$$\sum_{[u,v]\in\delta(u)} x_{t[u,v]} = 2y_{tu}, \qquad \forall t \in \mathcal{T}, u \in V_I \backslash \{s, l(1)\} \tag{56}$$

$$\sum_{[u,v]\in\delta(S)} x_{t[u,v]} \geq 2y_{tu_0}, \qquad \forall t \in \mathcal{T}, S \subset V_I \backslash \{s\}, |S| \geq 2, u_0 \in S \tag{57}$$

$$\sum_{t\in\mathcal{T}} z_{ot} = 1, \qquad \forall o \in O \tag{58}$$

$$\sum_{o\in O} b_o z_{ot} \leq B, \qquad \forall t \in \mathcal{T} \tag{59}$$

$$x_{t[u,v]} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, [u,v] \in E^+ \tag{60}$$

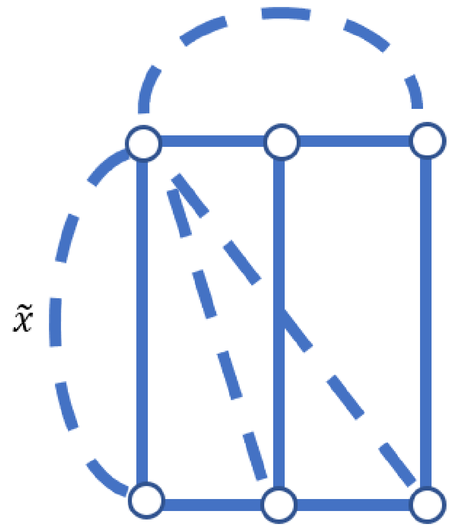
$$\tilde{x}_t \in \{0,1\}, \qquad \forall t \in \mathcal{T} \tag{61}$$

$$y_{tv} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, v \in V_I \tag{62}$$

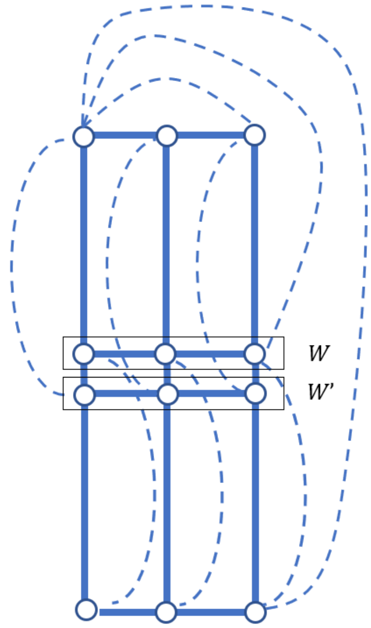
$$z_{ot} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, o \in O \tag{63}$$

Here, $x_{t[u,v]}$ is a binary variable, which takes the value 1 if and only if edge $[u,v]$ is traversed by walk t. We let binary variable $\tilde{x}$ indicate whether a parallel edge between $f(1)$ and $s(1)$ should be added (note that the origin $s = f(1)$). We actually formulate the standard TSP on E^+ (possibly with a parallel edge of $e(1)$) using constraints (52)-(57). We illustrate how our formulation works in Figure 9.

The TSP-based formulation is highly dependent on the construction of the auxiliary graph. Once we define the undirected graphs for other warehouses, we are able to propose extended formulations. In the next subsection, we discuss the auxiliary graph for a 2-block warehouse.



(a) E^+ for a single-block warehouse



(b) $\cup_{i=1}^{3} E_i^+$ for a 2-block warehouse

Figure 10. Auxiliary graphs

6.2. *TSP formulation for a 2-block warehouse*

We should propose a more sophisticated auxiliary graph which can induce a TSP formulation that contains an optimal solution. An optimal route in a 2-block warehouse may be different from an optimal route in a single-block warehouse such that:

(1) The picker may require to traverse a subaisle twice to enter the other block.
(2) The picker may require to traverse some edge in the second cross-aisle twice before retrieving all products from storage (after which the picker should go back to the origin).

From the above observations, we should add more parallel edges to the graph to preserve optimal solutions. In fact, we are describing the problem property when adding parallel edges. Once the auxiliary graph maintains a subset of optimal solutions, we obtain a TSP-based formulation.

Now, we show how to construct the auxiliary graph. Let W be a set of artificial locations in the second cross-aisle. We construct a copy v' for any $v \in W$ and let the set of copies be W'. Let E_1^+ be the set of edges connecting neighboring vertex. E_2^+ and E_3^+ are defined as follows.

(1) $E_2^+ = \{[v', Q_N(v)] : v \in W\} \cup \{[v, Q_S(v')] : v \in W\}$
(2) $E_3^+ = \{[s, v] : v \in \{V \cup W'\} \backslash \{s\}\}$

E_1^+ enables the picker to reverse direction in the second cross-aisle; E_2^+ enables the picker to traverse a subaisle twice; E_3^+ enables the picker to return to the origin after picking is completed. We compare the two auxiliary graphs in this section, as shown in Figure 10.

For each picker $t \in \mathcal{T}$, let $x_{t[u,v]} \in \{0, 1\}$ be an indicator variable equal to 1 if $[u, v] \in E_1^+ \cup E_2^+$ is traversed, and $\tilde{x}_{t[u,v]} \in \{0, 1\}$ be an indicator variable equal to

1 if $[u,v] \in E_3^+$ is traversed. The TSP-based formulation P_U^2 is similar to P_U^1, and is described by the following constraints.

$$\sum_{[s,v]\in\delta(s)} x_{t[s,v]} \geq 1, \qquad \forall t \in \mathcal{T} \tag{64}$$

$$\sum_{[s,v]\in\delta(s)} (x_{t[s,v]} + \tilde{x}_{t[s,v]}) = 2, \qquad \forall t \in \mathcal{T} \tag{65}$$

$$x_{t[u,v]} \geq z_{ot}, \qquad \forall t \in \mathcal{T}, i \in [W_{sub}], [u,v] = e(i), o \in \{o : V_{sub}(i) \cap L_o \neq \emptyset\} \tag{66}$$

$$\sum_{[u,v]\in\delta(u)} (x_{t[u,v]} + \tilde{x}_{t[u,v]}) = 2y_{tu}, \qquad \forall t \in \mathcal{T}, u \in V_I \backslash \{s\} \tag{67}$$

$$\sum_{[u,v]\in\delta(S)} (x_{t[u,v]} + \tilde{x}_{t[u,v]}) \geq 2y_{tu_0}, \qquad \forall t \in \mathcal{T}, S \subset V_I \backslash \{s\}, |S| \geq 2, u_0 \in S \tag{68}$$

$$\sum_{t\in\mathcal{T}} z_{ot} = 1, \qquad \forall o \in O \tag{69}$$

$$\sum_{o\in O} b_o z_{ot} \leq B, \qquad \forall t \in \mathcal{T} \tag{70}$$

$$x_{t[u,v]} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, [u,v] \in E_1^+ \cup E_2^+ \tag{71}$$

$$\tilde{x}_{t[u,v]} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, [u,v] \in E_3^+ \tag{72}$$

$$y_{tv} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, v \in V_I \tag{73}$$

$$z_{ot} \in \{0,1\}, \qquad \forall t \in \mathcal{T}, o \in O \tag{74}$$

A question immediately arises: is there always an optimal picking tour that can be induced by a feasible solution of P_U^2? In the remainder of this section, we reveal the existence of such a tour.

Assume that we have known the set $K_1(K_2)$ of subaisles in the first (second) block, which contains at least one product to be picked. For simplicity, we assume that $K_1 \neq \emptyset$ and $K_2 \neq \emptyset$. Let i_0 be the first subaisle in K_1. The following two routes, which we call S-shape routes, are considered.

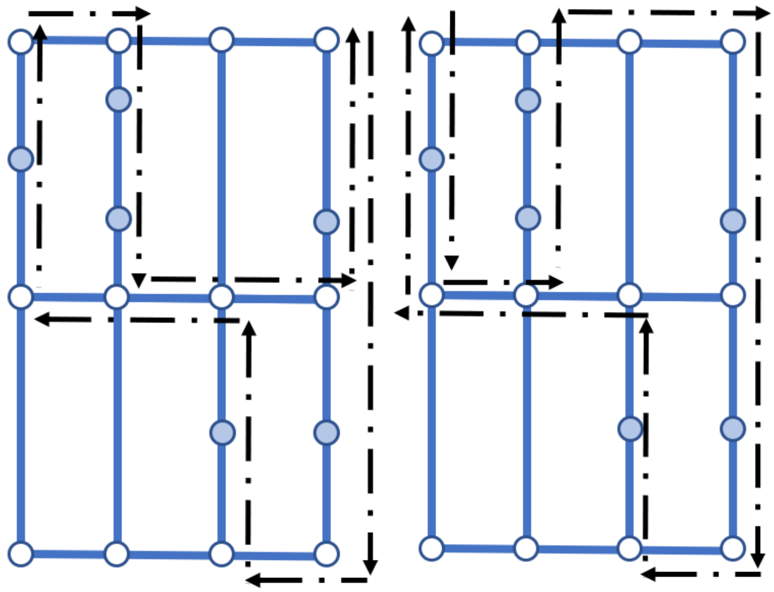
(1) r_S^1 will first visit all subaisles in $K_1 \backslash \{i_0\}$, then visit all subaisles in K_2 and finally visit subaisle i_0 (as shown in Figure 11(a)).
(2) r_S^2 will first visit all subaisles in K_1, then visit all subaisles in K_2 (as shown in Figure 11(b)).

Obviously, routes of type r_S^1 or r_S^2 can always be represented by a feasible solution of P_U^2. Furthermore, the following theorem guarantees the existence of an optimal tour.

Theorem 6.1. *There always exists an optimal picking tour which is done in an S-shape fashion.*

Proof. We first consider the case where subaisle $1 \in K_1$. Let the total vertical distance of route r be l_r and the optimal route r^*. Let d denotes the length of a subaisle. It can be seen that $l_{r^*} \geq |K_1 \cup K_2|d$. We also have the following results for r_S^1:

(1) If $|K_1|$ and $|K_2|$ are odd, we have that $l_{r_S^1} - |K_1 \cup K_2|d = 2d$ when $1 \in K_1$.
(2) If $|K_1|$ and $|K_2|$ are even, we have that $l_{r_S^1} - |K_1 \cup K_2|d = 0$ when $1 \in K_1$.
(3) If $|K_1|$ is odd and $|K_2|$ is even, we have that $l_{r_S^1} - |K_1 \cup K_2|d = d$ when $1 \in K_1$.



(a) r_S^1 (b) r_S^2

Figure 11. The S-shape route

(4) If $|K_1|$ is even and $|K_2|$ is odd, we have that $l_{r_S^1} - |K_1 \cup K_2|d = d$ when $1 \in K_1$.

We now show that r_S^1 is an optimal picking tour. We only deal with the case when $|K_1|$ is odd and $|K_2|$ is even. Assume by contradiction that $l_{r^*} = |K_1 \cup K_2|d$, which means r^* will traverse each selected subaisle exactly once. r^* can be described as a sequence of subaisles $i_1, i_2, ...i_n$. If $i_1, i_k \in K_1$ and $i_2, ..., i_{k-1} \in K_2$, we can find that k is even. $i_1, i_2, ...i_n$ can be reformulated as $Q_1 = \{i_1, i_2, ..., i_{k_1}\}$,$Q_2 = \{i_{k_1+1}, i_2, ..., i_{k_2}\}$,...,$Q_{2n+1} = \{i_{k_{2n}+1}, i_{k_{2n}+2}, ..., i_{k_{2n+1}}\}$ where $Q_{2t+1} \subset K_1$ and $Q_{2t} \subset K_2$. Note that $|Q_2|, ..., |Q_{2n}|$ are even and $|K_1|$ is odd, we can assume that $|Q_1|$ is even. This implies that r^* must traverse i_{k_1} twice, which is a contradiction.

Similarly, we can prove that there exist an optimal picking tour which is of type r_S^1 or r_S^2 when $1 \notin K_1$. This completes the proof. □

Note that the second cross-aisle is passed through by r_S^1 or r_S^2 exactly twice, and thus, we can further tighten the feasible region. Let V_S be the set of vertices in the second block, i.e., $V_S = \{W' \cup \{Q_S(v) : v \in W'\}\}$. We use the following constraint to ensure that the second cross-aisle can be passed through at most twice by the picker. This constraint is very effective at reducing solution times.

$$\sum_{[u,v] \in \delta(V_S)} (x_{t[u,v]} + \tilde{x}_{t[u,v]}) \leq 2, \qquad \forall t \in \mathcal{T} \tag{75}$$

We finish this section with the following corollary.

Corollary 6.2. *There always exists an optimal solution satisfying constraints (75).*

Proof. This follows immediately from Theorem 6.1. □

7. Computational results

The experiments were performed on an AMD Ryzen 7 4800H @2.90 GHz processor and 32 GB of RAM. The code was written in Python and GUROBI 9.1.1 was used as the mixed-integer solver. The performances of our MIP formulations and additional constraints are tested by comparing the computational difficulties to find optimal solutions.

7.1. *Test problems*

Our formulations are tested over the publicly available benchmark instances at `http://www.dcc.ufmg.br/arbex/orderpicking.html`. It comes from a database of anonymized customer purchases over two years for a chain of supermarkets. A single order is generated by combing the purchases of a customer over the first Δ days with $\Delta \in \{5, 10, 20\}$. The warehouse layouts considered in our experiments are similar to that of (Valle, Beasley, and da Cunha 2017); the slight difference is that we assume the origin is right from the first artificial location while they set the distance from the origin to the first artificial vertex is 4 meters. Our formulations for their warehouse layout are essentially the same. We set the available capacity of the picking vehicle $B = 8$. For every test instance, we define T by solving a bin-packing problem.

7.2. *Computational results*

In this section, we compare our formulations with that of (Valle, Beasley, and da Cunha 2017). The formulation for the JOBPRP (resp., for the no-reversal JOBPRP) presented by (Valle, Beasley, and da Cunha 2017) is denoted as P_O (resp., P_U). To further improve our formulation, we take into account existing constraints including aisles cuts, artificial vertex reversal constraints (Valle, Beasley, and da Cunha 2017) and column inequalities (Kaibel and Pfetsch 2006). To have a better comparison, we also use column inequalities to improved P_O. We do not present experimental results to verify the effectiveness of these constraints that existing studies have already illustrated. Details regarding test formulations are shown in Table 1.

Table 1.: Details about test formulations

Notation	Explanation
P_O	P_{basic} with the valid inequalities defined in (Valle, Beasley, and da Cunha 2017) and column inequalities
P_G^+	P_G with aisles cuts, artificial vertex reversal constraints and column inequalities
P_F^+	P_F with artificial vertex reversal constraints and column inequalities
P_U	the no-reversal formulation in (Valle, Beasley, and da Cunha 2017) with column inequalities
P_U^{1+}	P_U^1 with column inequalities
P_U^{2+}	P_U^2 with column inequalities

Remark that all formulations except P_F^+ are non-compact. The exponentially many constraints are generated sequentially in a branch-and-cut framework. For each candidate integral solution, we add constraints when the graph for some picker is disconnected. The connectivity condition is verified by a depth-first search. In addition, P_F seems unable to benefit from aisles cuts and therefore P_F^+ does not include aisles cuts.

In Table 2, we compare formulation P_O, P_G^+ and P_F^+ on the selected instances by setting a time limit of 2400 s. Column O corresponds to the number of orders. Column T corresponds to the total time. Columns UB and LB represent the best upper and lower bounds obtained at the end of the search, respectively, when either the instance was solved to prove optimality or the time limit has hit. GAP is defined as $100\% \times \frac{UB-LB}{UB}$.

The branch-and-cut algorithm based on P_G^+ managed to solve most instances to proven optimality. Furthermore, it obtained the lowest gap or had the shortest computing time. Thus, we can state that P_G^+ outperforms the existing formulation P_O. However, P_F^+ is not as strong as P_G^+. Although it provided a better gap (than P_O) for some instances (for example, instance with $\Delta = 5, O \in [19, 30]$), it performs poorly when $\Delta = 10$. Furthermore, an increasing number of orders cause a fast increase in the solution time, even for P_G^+. One possible reason is that as the number of order pickers increases, the number of symmetry branches in the search tree grows exponentially. In fact, even the relaxation $min\{f(x) : (x, z, \alpha, \beta) \in P_{sub}, constraints\ (9) - (10), (13) - (16), (18)\}$ (see Figure 3(c)) is very difficult to solve when there are a large number of orders (and it cannot figure out the route for each picker).

We also compare our formulations with two commonly used heuristics for order batching problem: the seed algorithm and the Clarke and Wright algorithm(II) (Koster, der Poort, and Wolters 1999). For each batch, we find an S-shape route to estimate the traveling distance (see section 6.2). Typically, the Seed and CWII can provide feasible solutions within several seconds. However, the solutions seem to be far from optimal. Numerical results are given in Table 3 where the last column shows the currently best known solution.

We analyze the efficiency of the basic cuts and the single traversing constraints by adding them to P_O, P_G^+ and P_F^+. Note that the basic cuts are generated by performing a depth-first search algorithm, and the total running time is typically less than 0.5 seconds (0.03s-0.5s). Therefore we do not need to take into account the processing time of constructing the basic cuts. The original model and the strengthened model are compared by counting winning instances. An instance is a winner for model A compared with model B, if

(1) A finished within the time limit and B did not finish or required a longer CPU time or
(2) A obtained a lower gap than B.

If the difference between the times or gaps is below 1 s or 0.1%, respectively, the instance is not counted. For example, we can compare P_O and P_F^+ in Table 2 by only considering the instances with $\Delta = 5$. Then we can observe that P_O has 3 winners and P_F^+ has 10 winners. Table 4,5 show the impacts of adding these additional constraints. Except the aggregated results, we also provide more detailed results in the appendix.

Table 4 shows the efficiency of the basic cuts. For formulation P_O, the basic cuts can improve at most 69.2% instances (when $\Delta = 20$). Similarly, the basic cuts can improve at most 75% instances for P_G^+ (when $\Delta = 5$). However, due to being compact and not requiring an explicit branch-and-cut implementation, P_F^+ seems to benefit less

Table 2. Comparison of the branch-and-cut algorithm based on formulations P_O,P_G^+ and P_F^+.

		P_O				P_G^+				P_F^+			
Δ	O	T(seconds)	UB	LB	GAP(%)	T(seconds)	UB	LB	GAP(%)	T(seconds)	UB	LB	GAP(%)
5	5	0.2	346	346	0	0.24	346	346	0	0.51	346	346	0
	10	2.91	578	578	0	1.5	578	578	0	92	578	578	0
	15	15	650	650	0	7.8	650	650	0	41	650	650	0
	16	390	766	766	0	37	766	766	0	236	766	766	0
	17	90	802	802	0	30	802	802	0	230	802	802	0
	18	1821	840	840	0	81	840	840	0	691	840	840	0
	19	2400	856	851	0.6	135	856	856	0	690	856	856	0
	20	2400	906	758	16.3	86	864	864	0	770	864	864	0
	21	2400	892	884	0.9	136	892	892	0	1585	892	892	0
	22	2400	902	868	3.8	171	892	892	0	1386	892	892	0
	23	2400	912	877	3.8	290	908	908	0	2400	908	901	0.8
	24	2400	1118	723	35.3	2400	1059	925	12.7	2400	1056	862	18.3
	25	2400	1104	815	26.2	2400	1102	954	13.4	2400	1104	869	21.3
	30	2400	1200	843	29.8	2400	1206	961	20.3	2400	1206	864	28.4
10	5	0.06	368	368	0	0.06	368	368	0	0.12	368	368	0
	10	40	656	656	0	6.31	656	656	0	34	656	656	0
	15	195	874	874	0	59	874	874	0	263	874	874	0
	16	178	926	926	0	65	926	926	0	311	926	926	0
	17	1188	960	960	0	123	960	960	0	996	960	960	0
	18	892	970	970	0	106	970	970	0	1112	970	970	0
	19	375	978	978	0	166	978	978	0	607	978	978	0
	20	454	984	984	0	209	984	984	0	1145	984	984	0
	21	320	990	990	0	143	990	990	0	1462	990	990	0
	22	1400	1000	1000	0	180	1000	1000	0	1880	1000	1000	0
	23	2400	1162	959	17.5	2400	1140	978	14.2	2400	1132	881	22.2
	24	2400	1218	824	32.3	2400	1162	1012	12.9	2400	1188	907	23.7
	25	2400	1192	935	21.6	2400	1220	1006	17.5	2400	1182	966	18.3
	30	2400	1326	955	28	2400	1320	1083	18	2400	1284	969	24.5
20	5	19	570	570	0	1.03	570	570	0	7.5	570	570	0
	10	68	912	912	0	25	912	912	0	119	912	912	0
	15	2400	1026	1009	1.7	58	1022	1022	0	255	1022	1022	0
	16	2400	1206	1085	10	745	1200	1200	0	1409	1200	1200	0
	17	2400	1292	1029	20.4	2059	1250	1250	0	2400	1250	1161	7.1
	18	2400	1324	1070	19.2	1166	1288	1288	0	2400	1310	1145	12.6
	19	2400	1356	1093	19.4	2400	1326	1146	13.6	2400	1346	1108	17.7
	20	2400	1334	1087	18.5	2400	1340	1292	3.6	2400	1340	1189	11.3
	21	2400	1554	960	38.2	2400	1542	1145	25.7	2400	1534	1078	29.7
	22	2400	1702	909	46.6	2400	1578	1162	26.4	2400	1558	1135	27.2
	23	2400	1638	1030	37.1	2400	1624	1130	30.4	2400	1620	1115	31.2
	24	2400	1672	968	42.1	2400	1640	1193	27.3	2400	1652	1040	37
	25	2400	-*	930	-	2400	1648	1141	30.8	2400	1644	1083	34.1
	30	2400	-	987	-	2400	1900	1125	40.8	2400	1944	988	49.2

*The symbol '-' shows that GUROBI failed to find a feasible solution.

Table 3. Experimental results for heuristic solution approaches

	Δ = 5			Δ = 10			Δ = 20		
O	Seed	CWII	Best	Seed	CWII	Best	Seed	CWII	Best
5	382	382	346	382	382	368	620	620	570
10	636	636	578	726	726	656	982	982	912
15	724	764	650	922	922	874	1146	1146	1022
16	910	882	766	1058	1058	926	1372	1372	1200
17	902	930	802	1058	1058	960	1488	1450	1250
18	942	980	840	1058	1058	970	1490	1490	1288
19	980	1030	856	1096	1096	978	1490	1490	1326
20	980	1018	864	1096	1096	984	1488	1488	1334
21	1018	1018	892	1146	1146	990	1716	1728	1534
22	1018	1018	892	1146	1146	1000	1754	1766	1558
23	1058	1058	908	1284	1284	1132	1812	1832	1620
24	1186	1158	1056	1332	1332	1162	1872	1872	1640
25	1246	1312	1102	1372	1362	1182	1872	1872	1644
30	1440	1360	1200	1452	1490	1284	2128	2166	1900

Table 4. Behavior of the basic cuts

	P_O	P_O+ basic cuts
Δ	Win Rate(%)	Win Rate(%)
5	69.2	30.8
10	46.2	53.8
20	30.8	69.2
	P_G^+	P_G^++ basic cuts
Δ	Win Rate(%)	Win Rate(%)
5	25	75
10	41.7	58.3
20	46.2	53.8
	P_F^+	P_F^++ basic cuts
Δ	Win Rate(%)	Win Rate(%)
5	58.3	41.7
10	53.8	46.2
20	71.4	28.6

Table 5. Behavior of the single traversing constraints

	P_O	P_O+ single traversing constraints
Δ	Win Rate(%)	Win Rate(%)
5	61.5	38.5
10	53.8	46.2
20	58.3	41.7
	P_G^+	P_G^++ single traversing constraints
Δ	Win Rate(%)	Win Rate(%)
5	50	50
10	25	75
20	30.8	69.2
	P_F^+	P_F^++ single traversing constraints
Δ	Win Rate(%)	Win Rate(%)
5	23.1	76.9
10	41.7	58.3
20	58.3	41.7

from the basic cuts. Table 5 shows the efficiency of the single traversing constraints. Although both P_G^+ (at most 75% instances) and P_F^+ (at most 76.9 instances) are able to benefit from the single traversing constraints, more than half of instances P_O cannot be improved by these constraints. We also note that the basic cuts and the single traversing constraints sometimes increase the solution time, which could be due to the interaction of these constraints and some built-in general-purpose cuts. Furthermore, we believe that the single traversing constraints should be given more consideration; This can lead to a much smaller feasible region and may induce other constraints or formulations.

In Table 6, we compare formulation P_U and P_U^{1+} for a single-block warehouse setting a time limit of 300 s. We reduce the time limit, mainly because the no-reversal JOBPRP is much simpler than the JOBPRP. The branch-and-cut algorithm based on P_U^{1+} can solve all instances within several seconds. The main reason could be that we successfully cut many symmetric solutions by designing an auxiliary graph. Similarly, we compare formulation P_U and P_U^{2+} for a 2-block warehouse in Table 7, and P_U^{2+} still outperforms P_U for all instances.

8. Conclusions

In this article, we investigate the JOBPRP, which is pivotal for the efficiency of order picking operations. To fully utilize the structure of the warehouse, we reconstruct the connectivity constraints. The obtained formulations, which consider separately the graph properties of picking locations and artificial locations, can significantly improve computational performance. We also provide two types of relevant additional constraints: one aims at dealing with batching decisions and routing decisions in an integrated way; the other aims at cutting off a subset of feasible solutions by the property of an optimal routing. Additionally, we consider the optimal routing for the no-reversal special case of this problem and propose TSP-based formulations. Our experimental results also show that the TSP-based formulations are very powerful and can significantly improve solution quality.

There are several potential topics for future research. First, graph-based mathematical formulations should consider the warehouse structure, which implies a need for polyhedral studies of different warehouses. For example, one might investigate the graph representation and the associated polytope for the HappyChic warehouse considered by (Briant et al. 2020), which is slightly different from the rectangular warehouse considered in this paper. Second, one might improve traditional heuristic algorithms by analyzing the property of optimal solutions. Third, both the routing and batching problems suffer severely from the presence of symmetry. If we treat the batching problem as a partitioning problem, we can find many symmetry breaking methods (for example, column inequalities (Kaibel and Pfetsch 2006)). One might make use of these symmetry breaking methods to improve different heuristics or exact methods. Fourth, as no-reversal routes are easy to implement in practice, it might be worthwhile to pay more attention to this special case. (Arbex Valle and Beasley 2020) demonstrated the feasibility of using easy-to-solve approximation programs to obtain high-quality no-reversal solutions. One might build up an approximation model that only considers some features of a feasible solution, and might study the accuracy of the estimation.

Table 6. Comparison of the branch-and-cut algorithm based on formulations P_U and P_U^{1+} for a single-block warehouse.

		P_U				P_U^{1+}			
Δ	O	T(seconds)	UB	LB	GAP(%)	T(seconds)	UB	LB	GAP(%)
5	5	0.02	358	358	0	0.01	358	358	0
	10	0.13	634	634	0	0.05	634	634	0
	15	0.05	716	716	0	0.01	716	716	0
	20	1.72	982	982	0	0.16	982	982	0
	21	26	1064	1064	0	0.21	1064	1064	0
	22	41	1064	1064	0	0.25	1064	1064	0
	23	20	1064	1064	0	0.27	1064	1064	0
	24	300	1248	1140	8.7	1.64	1248	1248	0
	25	98	1258	1258	0	2.21	1258	1258	0
	26	64	1268	1268	0	1.3	1268	1268	0
	27	215	1278	1278	0	1.17	1278	1278	0
	28	235	1330	1330	0	2.62	1330	1330	0
	29	300	1350	1340	0.7	1.2	1350	1350	0
	30	300	1350	1304	0.3	1.66	1350	1350	0
10	5	0.01	358	358	0	0.01	358	358	0
	10	0.07	716	716	0	0.01	716	716	0
	15	1.26	972	972	0	0.19	972	972	0
	20	1.27	992	992	0	0.23	992	992	0
	21	10	1064	1064	0	0.18	1064	1064	0
	22	2.56	1064	1064	0	0.25	1064	1064	0
	23	73	1248	1248	0	1.22	1248	1248	0
	24	70	1248	1248	0	0.72	1248	1248	0
	25	4.53	1268	1268	0	0.65	1268	1268	0
	26	35	1268	1268	0	0.66	1268	1268	0
	27	214	1330	1330	0	0.63	1330	1330	0
	28	119	1340	1340	0	0.52	1340	1340	0
	29	77	1340	1340	0	0.7	1340	1340	0
	30	82	1340	1340	0	0.6	1340	1340	0
20	5	0.09	706	706	0	0.03	706	706	0
	10	0.88	992	992	0	0.1	992	992	0
	15	0.88	1074	1074	0	0.12	1074	1074	0
	20	31	1422	1422	0	0.34	1422	1422	0
	21	300	1698	1675	1.4	1.92	1698	1698	0
	22	300	1698	1672	1.5	1.04	1698	1698	0
	23	300	1770	1699	4	1.21	1770	1770	0
	24	300	1770	1693	4.4	1.13	1770	1770	0
	25	300	1780	1736	2.5	0.88	1780	1780	0
	26	300	1780	1774	0.3	0.66	1780	1780	0
	27	33	1780	1780	0	1.85	1780	1780	0
	28	300	2056	1903	7.4	4.66	2056	2056	0
	29	300	2056	1652	19.6	5.19	2056	2056	0
	30	300	2056	1780	13.4	4.47	2056	2056	0

Table 7. Comparison of the branch-and-cut algorithm based on formulations P_U and P_U^{2+} for a 2-block warehouse.

		P_U				P_U^{2+}			
Δ	O	T(seconds)	UB	LB	GAP(%)	T(seconds)	UB	LB	GAP(%)
5	5	0.05	382	382	0	0.02	382	382	0
	10	0.51	608	608	0	0.2	608	608	0
	15	1.47	696	696	0	0.22	696	696	0
	20	20	940	940	0	1.91	940	940	0
	21	5.34	940	940	0	1.73	940	940	0
	22	7.18	940	940	0	1.2	940	940	0
	23	8.2	950	950	0	1.54	950	950	0
	24	273	1108	1108	0	31	1108	1108	0
	25	152	1146	1146	0	41	1146	1146	0
	26	300	1194	1118	6.4	45	1176	1176	0
	27	300	1206	1185	1.7	52	1206	1206	0
	28	300	1234	1151	6.7	67	1206	1206	0
	29	300	1254	1184	5.6	67	1254	1254	0
	30	300	1254	1151	8.2	48	1254	1254	0
10	5	0.02	382	382	0	0.02	382	382	0
	10	1.29	724	724	0	0.29	724	724	0
	15	4.74	922	922	0	3.08	922	922	0
	20	8	1020	1020	0	0.97	1020	1020	0
	21	132	1058	1058	0	1.42	1058	1058	0
	22	26	1058	1058	0	1.13	1058	1058	0
	23	237	1214	1214	0	22	1214	1214	0
	24	266	1254	1254	0	19	1254	1254	0
	25	213	1254	1254	0	14	1254	1254	0
	26	300	1302	1283	1.5	17	1302	1302	0
	27	300	1342	1302	3	35	1342	1342	0
	28	300	1352	1231	8.9	23	1352	1352	0
	29	300	1352	1264	6.5	20	1352	1352	0
	30	300	1352	1255	7.2	22	1352	1352	0
20	5	0.2	620	620	0	0.09	620	620	0
	10	2.68	982	982	0	0.69	982	982	0
	15	59	1108	1108	0	0.67	1108	1108	0
	20	300	1432	1411	1.5	11	1430	1430	0
	21	300	1638	1513	7.6	154	1626	1626	0
	22	300	1646	1526	7.3	114	1626	1626	0
	23	300	1684	1534	8.9	162	1648	1648	0
	24	300	1744	1600	8.2	234	1708	1708	0
	25	300	1718	1565	8.9	125	1714	1714	0
	26	300	1746	1562	10.5	148	1716	1716	0
	27	300	1756	1589	9.5	179	1736	1736	0
	28	300	1932	1430	26	300	1894	1538	18.8
	29	300	1990	1444	27.4	300	1932	1629	15.7
	30	300	2066	1476	28.6	300	1990	1734	12.9

Data availability statement

The data that support the findings of this study are available from the corresponding author, C.H. Gao, upon reasonable request.

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Appendix

Notation	Explanation
Sets	
$\mathcal{T}$	set of available trolleys
O	set of orders
L_o	set of picking locations of order o
V	set of all locations
V_L	set of picking locations
V_I	set of artificial locations
$V_{sub}(i)$	set of picking locations within subaisle i
$\tilde{E}$	set of directed edges connecting neighboring locations
$\tilde{E}'$	set of directed edges connecting neighboring artificial locations while ignoring picking locations
$\delta(S)$	set of undirected edges with one end in set S
$\delta^+(S)/\delta^-(S)$	set of directed edges in $\tilde{E}$ that leave/enter set S
$\eta^+(S)/\eta^-(S)$	set of directed edges in $\tilde{E}'$ that leave/enter set S
W_{sub}	the number of subaisles
Constants	
s	the origin of the warehouse
(u, v)	the ordered pair of location u and location v, which represents a directed edge
$[u, v]$	the unordered pair of location u and location v, which represents an undirected edge
$f(i)/l(i)$	the northern/southern artificial location of subaisle i
$n(v)/s(v)$	the adjacent northern/southern location of v
$Q_N(v)/Q_S(v)/Q_E(v)/Q_W(v)$	the adjacent northern/southern/eastern/western artificial location of artificial location v
b_o	size of order o
B	available capacity of a trolley
Variables	
x_{tuv}	Binary variable that takes value 1 if and only if (u, v) $(\in \tilde{E})$ is traversed by walk t
y_{tv}	Binary variable that takes value 1 if and only if trolley t visits location v
z_{ot}	Binary variable that takes value 1 if and only if trolley t picks order o
α_{tv}/β_{tv}	Binary variable that takes value 1 only if there exists a straight path connecting the northern/southern artificial location and v in walk t
γ_{tuv}	Binary variable that takes value 1 only if $[u, v]$ $(\in \tilde{E}')$ is traversed by walk t
$\sigma_{tuv}^{v_0}$	Continuous variable that indicate the volume of flow from artificial location v_0 passing through arc (u, v) $(\in \tilde{E}')$ in walk t

Formulation	Constraints	Explanation
For Analysis		
P_{sub}	(19)-(25)	feasible region of subaisle cuts
P_{basic}	(9)-(18)	the basic formulation for the JOBPRP
P_A	(9)-(25)	the basic formulation with subaisle cuts
P_g	(26)-(42)	a formulation which only force artificial locations to be in the same connected component
P_f	(26)-(34), (36)-(46)	a flow-based formulation which only force artificial locations to be in the same connected component
P_G	(19)-(42)	a non-compact improved formulation for the JOBPRP
P_F	(19)-(34), (36)-(46)	a flow-based improved formulation for the JOBPRP
P_U^1	(52)-(63)	a TSP-based no-reversal formulation for a single-block warehouse
P_U^2	(64)-(74)	a TSP-based no-reversal formulation for a 2-block warehouse
For Experiment		
P_O	-	P_{basic} with the valid inequalities defined in (Valle, Beasley, and da Cunha 2017) and column inequalities
P_G^+	-	P_G with aisles cuts, artificial vertex reversal constraints and column inequalities
P_F^+	-	P_F with artificial vertex reversal constraints and column inequalities
P_U	-	the no-reversal formulation in (Valle, Beasley, and da Cunha 2017) with column inequalities
P_U^{1+}	-	P_U^1 with column inequalities
P_U^{2+}	-	P_U^2 with column inequalities

Table 9. Detailed results for the basic cuts

		P_O+ basic cuts				P_G^++ basic cuts				P_F^++ basic cuts			
Δ	O	T(seconds)	UB	LB	GAP(%)	T(seconds)	UB	LB	GAP(%)	T(seconds)	UB	LB	GAP(%)
5	5	0.34	346	346	0	0.21	346	346	0	0.4	346	346	0
	10	6.96	578	578	0	3.23	578	578	0	66	578	578	0
	15	26	650	650	0	9.4	650	650	0	46	650	650	0
	16	406	766	766	0	37	766	766	0	292	766	766	0
	17	239	802	802	0	46	802	802	0	281	802	802	0
	18	2400	870	806	7.4	95	840	840	0	830	840	840	0
	19	2020	856	856	0	72	856	856	0	605	856	856	0
	20	2067	864	864	0	75	864	864	0	745	864	864	0
	21	2400	902	846	6.2	270	892	892	0	2400	892	850	4.7
	22	2400	892	886	1.1	190	892	892	0	1045	892	892	0
	23	2400	918	868	5.5	416	908	908	0	1465	908	908	0
	24	2400	-	733	-	2400	1064	934	12.2	2400	1076	820	23.8
	25	2400	1112	890	20	2400	1108	944	14.8	2400	1120	839	25.1
	30	2400	1232	844	31.5	2400	1212	942	22.3	2400	1202	862	28.3
10	5	0.08	368	368	0	0.12	368	368	0	0.09	368	368	0
	10	45	656	656	0	8.67	656	656	0	27	656	656	0
	15	122	874	874	0	36	874	874	0	224	874	874	0
	16	138	926	926	0	65	926	926	0	430	926	926	0
	17	408	960	960	0	97	960	960	0	501	960	960	0
	18	410	970	970	0	151	970	970	0	1055	970	970	0
	19	300	978	978	0	156	978	978	0	1421	978	978	0
	20	524	984	984	0	166	984	984	0	1640	984	984	0
	21	1410	990	990	0	281	990	990	0	2192	990	990	0
	22	2203	1000	1000	0	125	1000	1000	0	2400	1014	938	7.5
	23	2400	1344	743	44.7	2400	1132	1011	10.7	2400	1152	926	19.6
	24	2400	1184	911	23.1	2400	1168	1058	9.4	2400	1194	937	21.5
	25	2400	1230	945	23.2	2400	1220	991	18.8	2400	1196	962	19.6
	30	2400	1306	958	26.6	2400	1286	1052	18.2	2400	1288	954	25.9
20	5	13	570	570	0	1.29	570	570	0	8.7	570	570	0
	10	57	912	912	0	47	912	912	0	113	912	912	0
	15	2400	1022	1012	1	63	1022	1022	0	385	1022	1022	0
	16	2400	1200	1088	9.3	377	1200	1200	0	1607	1200	1200	0
	17	2400	1282	1047	18.3	1602	1250	1250	0	2400	1250	1183	5.4
	18	2400	1330	1093	17.8	2400	1296	1216	6.2	2400	1302	1160	10.9
	19	2400	1342	1188	11.5	1799	1304	1304	0	2400	1322	1187	10.2
	20	2400	1352	1115	17.5	2400	1332	1266	5	2400	1352	1121	17.1
	21	2400	1620	903	44.3	2400	1520	1193	21.5	2400	1518	1036	31.8
	22	2400	1820	966	46.9	2400	1532	1180	23	2400	1536	1107	27.9
	23	2400	-	989	-	2400	1598	1175	26.5	2400	1618	1044	35.5
	24	2400	-	916	-	2400	1640	1185	27.7	2400	1652	1097	33.6
	25	2400	1736	987	43.1	2400	1674	1164	30.5	2400	1654	1086	34.3
	30	2400	-	979	-	2400	1934	1104	42.9	2400	1940	970	50

Table 10. Detailed results for the single traversing constraints

Δ	O	P_O+ single traversing				P_G^++ single traversing				P_F^++ single traversing			
		T(seconds)	UB	LB	GAP(%)	T(seconds)	UB	LB	GAP(%)	T(seconds)	UB	LB	GAP(%)
5	5	0.17	346	346	0	0.17	346	346	0	0.38	346	346	0
	10	5.3	578	578	0	2.16	578	578	0	27	578	578	0
	15	18	650	650	0	17	650	650	0	36	650	650	0
	16	223	766	766	0	33	766	766	0	128	766	766	0
	17	1460	802	802	0	40	802	802	0	208	802	802	0
	18	1681	840	840	0	68	840	840	0	781	840	840	0
	19	2400	872	831	4.7	78	856	856	0	363	856	856	0
	20	2400	888	755	15	139	864	864	0	701	864	864	0
	21	2400	898	871	3	161	892	892	0	1880	892	892	0
	22	2400	898	857	4.6	180	892	892	0	735	892	892	0
	23	2400	908	887	2.3	251	908	908	0	1385	908	908	0
	24	2400	1112	819	26.3	2400	1062	933	12.1	2400	1078	831	22.9
	25	2400	1136	764	32.7	2400	1112	941	15.4	2400	1098	894	18.6
	30	2400	1234	847	31.4	2400	1194	962	19.4	2400	1198	919	23.3
20	5	0.07	368	368	0	0.13	368	368	0	0.2	368	368	0
	10	33	656	656	0	6.44	656	656	0	30	656	656	0
	15	162	874	874	0	50	874	874	0	313	874	874	0
	16	296	926	926	0	45	926	926	0	253	926	926	0
	17	398	960	960	0	168	960	960	0	447	960	960	0
	18	1519	970	970	0	91	970	970	0	895	970	970	0
	19	742	978	978	0	195	978	978	0	2400	978	958	2
	20	297	984	984	0	155	984	984	0	1502	984	984	0
	21	1015	990	990	0	125	990	990	0	1515	990	990	0
	22	325	1000	1000	0	200	1000	1000	0	1449	1000	1000	0
	23	2400	1178	923	21.6	2400	1128	1006	10.8	2400	1150	939	18.3
	24	2400	1188	904	23.9	2400	1162	1021	12.1	2400	1182	904	23.5
	25	2400	1192	875	26.6	2400	1210	1039	14.1	2400	1196	952	20.4
	30	2400	1320	862	34.7	2400	1296	1074	17.1	2400	1268	962	24.1
5	5	8.9	570	570	0	1.1	570	570	0	4.4	570	570	0
	10	125	912	912	0	21	912	912	0	157	912	912	0
	15	2400	1022	1008	1.4	68	1022	1022	0	315	1022	1022	0
	16	2400	1238	1045	15.6	530	1200	1200	0	2394	1200	1200	0
	17	2400	1282	938	26.8	1266	1250	1250	0	2400	1250	1190	4.8
	18	2400	1318	1187	9.9	1568	1288	1288	0	2400	1308	1076	17.7
	19	2400	1350	1114	17.5	2400	1316	1251	4.9	2400	1308	1168	10.7
	20	2400	1360	1072	21.2	2400	1332	1264	5.1	2400	1358	1109	18.3
	21	2400	1744	1014	41.9	2400	1510	1187	21.4	2400	1542	1094	29.1
	22	2400	1590	999	37.2	2400	1534	1206	21.4	2400	1566	1116	28.7
	23	2400	1742	1020	41.4	2400	1606	1165	27.5	2400	1618	1117	31
	24	2400	1672	940	43.8	2400	1660	1143	31.1	2400	1664	1133	31.9
	25	2400	-	917	-	2400	1680	1191	29.1	2400	1672	1060	36.6
	30	2400	-	967	-	2400	1896	1134	40.2	2400	1944	987	49.2