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MODELLING INTEGER LINEAR PROGRAMS WITH PETRI NETS (*, **)

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Abstract. – *We show in this paper that timed Petri nets, with one resource shared by all the transitions, are directly connected to the modelling of integer linear programs (ILP). To an ILP can be automatically associated an equivalent Petri net. The optimal reachability delay is an optimal solution of the ILP. We show next that a net can model any ILP. I order to do this, we give a new sufficient reachability condition for the marking equation, which also holds for general Petri nets without timed transitions.*

Keywords: Petri nets, integer linear programming.

1. INTRODUCTION

Linear programming is a general and useful approach to solve operational research problems. Analytical equations define the objective function and the linear constraints of the problem. In particular cases, the problem can be formulated in different ways. For instance some problems can be both represented by a graph and a linear program. Besides models based on graph lead to a better understanding of the studied problems.

Petri nets give a graphical view of systems. When the net is extended with a time feature, performance evaluation of the modelled system can then be made. Timed Petri nets give a good representation of scheduling problems. They allow multiple resolution approaches like branch and bound, linear programming and many others like constraint logic programming. Furthermore, the study of subnet classes defines fundamental problems. For

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instance, computing the average cycle time in a strongly connected marked graph defines the basic cyclic scheduling problem. We assume hereafter the reader is familiar with basic concepts of Petri nets. If it is not the case, one can refer to [2].

In this paper, we show the link between scheduling a special class of timed Petri nets and solving Integer Linear Programs (ILP). We illustrate the modelling technique with the examples of the Simple Transportation Problem and the Knapsack Problem. We prove that any Integer Linear Program can be modelled by a Petri net.

2. MODELLING INTEGER LINEAR PROGRAMS WITH PETRI NETS

We define a special class of timed Petri nets for modelling Integer Linear Program (ILP). We define a sub-class of timed Petri nets, called after STPN for short. A timed Petri net is a Petri net with temporisation associated to transitions. STPN is a timed Petri net with a special place, which leads to the mutual exclusion of all transition fires.

DEFINITION 1: *A sequential timed Petri net is:*
- *A Petri net* $< P, T, Pre, Post >$. $C = Post\text{-}Pre$. $|P| = n$, $|T| = m$;
- $d : T \rightarrow N$, *is the delay mapping to transitions.* $D = (d_t)$;
- $p_{me} \in P$ *is such that* $\forall t \in T$, $p_{me} \in \{^\bullet t\}$ *and* $p_{me} \in \{t^\bullet\}$;
- $M_0(p_{me}) = 1$.

It directly follows from the previous definition that a marking is reachable in a STPN if, and only if, it is reachable on the underlying untimed net since no parallelism is allowed in the transition firings. Thus the reachable markings follow the classical marking equation:

$$M = M_0 + CX$$

where M is the reached marking, M_0 is the initial marking, C is the incidence matrix, and X is the characteristic vector (number of transition fires).

An ILP, in its standard form (slack variables have been added) is defined by:

$$Min\, z = D^T X$$
$$\text{Subject to}\, CX = b.$$

In order to show the link between Petri nets and Integer Linear Programming, let us assume that delays are the costs of the vector D. Then $D^T X$ models the duration in order to fire the transitions x_t times. Let C be an incidence matrix and $b = M_f - M_0$, then the constraints of the ILP model the classical marking equation of Petri nets. For instance the net modelling the constraint $x_1 - x_2 + 2x_3 - 4x_4 = 3$ is given in Figure 1. Every firing of a transition increases by one the value of the corresponding variable in the modelled ILP. The behaviour of the net must ensure that $M_f(p) - M_0(p) = 3$, in order to respect the linear program constraint modelled by a place in the STPN. The mutual exclusion place ensures that the reachability delay will be expressed as a sum of delays (d_t) weighted by the characteristic vector (x_t). The reachability time is the linear objective function: $z = \sum_{t \in T} d_t x_t$. Since the characteristic vector is integer, the linear program associated to a sequential Petri net is in integer numbers.

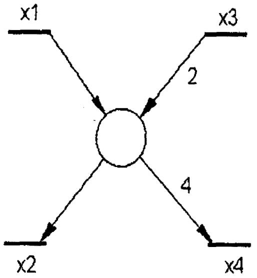

Figure 1. – Modelling of a linear program constraint (the mutual exclusion place is not drawn).

DEFINITION 2: *The integer linear program associated to a sequential Petri net N, noted* ILP(N), is:

$$[Opt]z = X^T D$$

Subject to

$$CX = M_f - M_0$$
$$X \geq 0$$

where OPT is the optimisation sense (Min or Max). C is the incidence matrix of the net. M_f is the final marking and M_0 is the initial one. X is the characteristic vector, and D is the vector of delays associated to the transitions.

Since the marking equation is a necessary condition for the reachability criterion, every firing sequence in the net N is associated to a feasible solution in ILP(N). If it is also sufficient, then to every feasible solution X of ILP(N) can be associated a feasible schedule of the transitions that leads from the initial marking to the final one.

We now consider the example of the Simple Transportation Problem (STP). Materials are available in a set of plants (P_i). A set of clients required these materials in some quantities (D_j). For each couple (i, j) we know the unitary cost of transportation from i to $j(c_{ij})$. The linear program of this problem is

$$[Min]z = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}.x_{ij}$$

$$s.t. - \sum_{j=1}^{m} x_{ij} = -P_i \,\forall\, i = 1.n$$

$$\sum_{i=1}^{n} x_{ij} = D_j \,\forall\, j = 1.m.$$

We assume that the sums of demands and availabilities are equal. If it is not the case, one can introduce new fictitious plants or demands. For this problem it is well known that the matrix of constraints is totally unimodular. Thus if the values of the problems are integers, then the solutions will be integers too. The generic model of the Simple Transportation Problem based on Petri net is given in Figure 2.

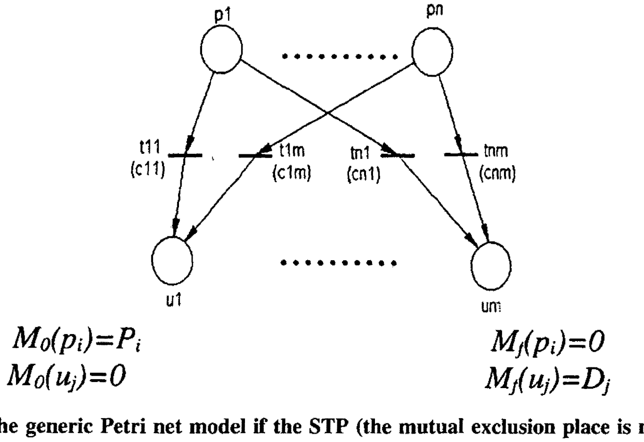

Figure 2. – The generic Petri net model if the STP (the mutual exclusion place is not drawn).

It is easy to see that the incidence matrix of this model is exactly the matrix of constraints of the STP. The two representations (Petri net model and linear program) are equivalent. The net is an acyclic state machine. It is well known that the incidence matrix of a state machine is totally unimodular and since the net is acyclic, the marking equation is a necessary and sufficient reachability condition [2]. But in the general case, the marking equation is only a necessary condition for the reachability problem. In this latter case, solving the ILP through its Petri net model leads to a lower bound of the optimal solution.

Let us give a second example, which deals with another classical operational research problem: the Knapsack problem. Unlike the STP problem, the Knapsack problem is weakly NP-Complete. We only recall the well-known analytical formulation in integer numbers:

$$Max\, z = \sum_{i=1}^{n} c_i x_i$$

$$s.t.$$

$$\sum_{i=1}^{n} p_i x_i + s = P.$$

In the Petri net model (Fig. 3), we consider initially that the place is marked with P tokens. The objective is to find the optimal schedule so that the place becomes empty.

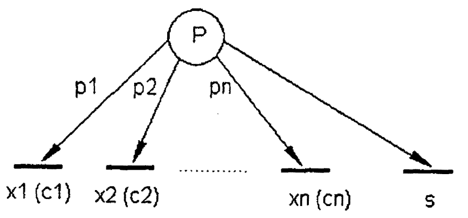

Figure 3. – Petri net model of the Knapsack problem (the mutual exclusion place is not drawn).

As in the STP case, the Petri net model of the Knapsack problem is defined by an acyclic graph. So solving the problem while scheduling the transitions of net leads to the optimal solution for the Knapsack problem.

3. GENERALITY OF THE MODELLING APPROACH

An important problem is the generality of our approach: can we model every ILP by means of STPN? Precisely, can we build a STPN so that the optimal solution X^* of the ILP is firable on the net (*i.e.* there is a firable sequence associated to the characteristic vector which is a solution of the integer program). We prove hereafter that it is always possible to define such an equivalent net.

The result is based on a special class of firing sequences having a regular pattern of interleaving transitions. These kinds of sequences have also been used in other fields than Petri nets theory. But the main interesting result for our study is the works of Karp *et al.* on the scheduling of the computations of uniform recurrence equations [1]. Let us first define the concept of regular sequence that will be used hereafter.

DEFINITION 1: *A regular sequence is a firing sequence σ of length l that is defined by l subsequences: $\sigma = \sigma_0\sigma_1\ldots\sigma_{l-1}$ and in every subsequence k, $0 \leq k < l$ is verified:*

$$t_i \in \sigma_k \quad \text{if} \ \left\lceil \frac{k+1}{l}\vec{\sigma}(t_i) \right\rceil - \left\lceil \frac{k}{l}\vec{\sigma}(t_i) \right\rceil.$$

We must note that the sequence of transitions in each subsequence $\sigma_0, \sigma_1, \ldots, \sigma_{l-1}$ is not fixed through this definition. The proof technique is based on a well-known result of Karp *et al.* (1967). Their result leads in fact to a new result in the Petri net theory: a necessary and sufficient reachability condition for the marking equation. The proof uses a technical lemma that is based on the bounded distance between integer markings and continuous markings belonging to the segment $[M_0, M_f]$. The proof of Theorem 1 uses iteratively this technical lemma and shows that if the initial marking M_0, and the marking M_f to be reached are in the domain defined by Karp's bound, then there is a regular sequence σ so that $M_0 \xrightarrow{\sigma} M_f$.

Karp's bound B is defined by the sum of the columns of the incidence matrix, where only absolute values of components are considered. The Karp's domain K_B is the domain defined by all the markings greater than or equal to the Karp's bound.

$$B = \sum_{i=1}^{m} |C_i| \qquad K_B = \{M \in \mathbb{N}^n, M \geq B\}.$$

THEOREM 1: *Let N be a Petri net, M_0, M_f be the initial and final markings belonging to Karp's domain and so that $M = M_0 + Cx$, $x \geq 0$, then there is a regular sequence σ, $M_0 \xrightarrow{\sigma} M_f$ and its length is $|\sigma| = \sum_{i=1}^{m} x_i$.*

(The original proof can be found in [1] (pp. 372-373), or in [3] for its Petri nets version.)

To a given ILP, with the constraints expressed as $CX = b$, one can always modify the constraints so that b becomes non-negative, by multiplying the constraints by -1. Assuming that $b_i \geq 0$ for each $i = 1 \ldots n$, the STPN can be built as follows:

– the incidence matrix is C,

– $M_0 = \sum_{i=1}^{m} |C_i|$ and $M_f(p) = M_0(p) + b(p)$ for $p = 1...n$.

Thus solutions X of $M_f = M_0 + CX$ are feasible, *i.e.* there is an associated firable regular sequence. Now we shall show that the above reduction is polynomial. Size (x) denotes the memory space required to store x.

THEOREM 2: *The reduction is polynomial.*

Proof: First, from the time complexity point of view, the main computation is due to Karp's bound. This can be performed in O(m.n), thus in polynomial time. Secondly, we must verify that the size of the obtained net is polynomially bounded in the size of the ILP. The size of the net is defined by:

$$size(N, M_0, M_f) = size(C) + size(M_0) + size(M_f).$$

We verify that:

$$size(C) = size(A) \text{ and } size(M_0) = size(B) \leq size(A).$$

Furthermore: $size(M_f) = size(M_0 + b) \leq size(M_0) + size(b)$ since for every integers x, $y > 1$ it holds that $\log(x + y) \leq \log(x) + \log(y)$. Thus $size(N) \leq 3size(A) + size(b)$, which is a polynomial space bound according to the size of the ILP.

4. CONCLUSION

We have presented the ability of Petri nets to model integer linear programs. To any ILP can be associated an STPN so that there is a firable sequence of transitions to each feasible solution of the ILP. This result is based on a new necessary and sufficient reachability condition for Petri nets. This condition is only based on the values of the initial marking, the final marking, and the incidence matrix of the net. The optimisation step of the ILP can then be viewed as a scheduling problem in the STPN. In [3], we have shown that the results can be easily extended to linear program while considering continuous Petri nets. Those principles have been used to design computer-aided software for production planning in the bottle-glass industry.

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