A simplified ordinal analysis of first-order reflection

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Abstract

In this note we give a simplified ordinal analysis of first-order reflection. An ordinal notation system OT is introduced based on ψ-functions. Provable Σ_1-sentences on $L_{\omega_1^{CK}}$ are bounded through cut-elimination on operator controlled derivations.

1 Introduction

Let ORD denote the class of all ordinals, $A \subset ORD$ and α a limit ordinal. α is said to be Π_n*-reflecting* on A iff for any Π_n-formula $\phi(x)$ and any $b \in L_\alpha$, if $\langle L_\alpha, \in\rangle \models \phi(b)$, then there exists a $\beta \in A \cap \alpha$ such that $b \in L_\beta$ and $\langle L_\beta, \in\rangle \models \phi(b)$. Let us write $\alpha \in rM_n(A) :\Leftrightarrow \alpha$ is Π_n-reflecting on A. Also α is said to be Π_n*-reflecting* iff α is Π_n-reflecting on ORD.

It is not hard for us to show that the assumption that the universe is Π_n-reflecting is proof-theoretically reducible to iterabilities of the lower operation rM_{n-1} (and Mostowski collapsings), cf. [3].

In this paper we aim an ordinal analysis of Π_n-reflection. Such an analysis was done by Pohlers and Stegert [7] using reflection configurations introduced in M. Rathjen [9], and an alternative analysis in [1, 2, 4] with the complicated combinatorial arguments of ordinal diagrams and finite proof figures. Our approach is simpler in view of combinatorial arguments. In [1], a Π_n-reflecting universe is resolved in ramified hierarchies of lower Mahlo operations, and ultimately in iterations of recursively Mahlo operations. Our ramification process is akin to a tower, i.e., has an exponential structure. It is natural that an exponential structure emerges in lowering and eliminating first-order formulas (in reflections), cf. ordinal analysis for the fragments $I\Sigma_{n-3}$ of the first-order arithmetic. Mahlo classes $Mh_k(\xi)$ defined in Definition 2.5 to resolve or approximate

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Π_n-reflection are based on similar structure. As in Rathjen's analysis for Π_3-reflection in [8], thinning operations are applied on the Mahlo classes $Mh_k(\xi)$, and this yields an exponential structure similar to the one in [1] as follows.

Let us consider the simplest case $N = 4$. Let $\Lambda := \varepsilon_{\mathbb{K}+1}$, the next epsilon number above the lease Π_4-reflecting ordinal $\mathbb{K}$. Roughly $\pi \in Mh_3(\xi)$ designates the fact that an ordinal π is Π_3-reflecting on $Mh_3(\nu)$ for any $\nu < \xi < \Lambda$. Suppose a Π_3-sentence θ on L_π is derived from the assumption $\pi \in Mh_3(\xi)$. We need to find an ordinal $\kappa < \pi$ for which $L_\kappa \models \theta$ holds. It turns out that $\kappa \in Mh_2(\Lambda^\xi a)$ suffices for an ordinal $a < \Lambda$, where the ordinal κ in the class $Mh_2(\Lambda^\xi a)$ is Π_2-reflecting on classes $Mh_2(\Lambda^\xi b) \cap Mh_3(\nu)$ for any $b < a$ and any $\nu < \xi$. Note that the class $Mh_2(\Lambda^\xi a)$ is not obtained through iterations of recursively Mahlo operations since it involves Π_4-definable classes $Mh_3(\nu)$. The classes $Mh_3(\nu)\,(\nu < \xi)$ for the assumption $\pi \in Mh_3(\xi)$ are thinned out with the new classes $Mh_2(\Lambda^\xi b)\,(b < \Lambda)$, cf. Lemma 5.1.

Our theorem runs as follows. Let $\mathsf{KP}\Pi_N$ denote the set theory for Π_N-reflecting universes, and $\mathsf{KP}\omega$ the Kripke-Platek set theory with the axiom of infinity. OT is a computable notation system of ordinals defined in section 3, $\Omega = \omega_1^{CK}$ and ψ_Ω is a collapsing function such that $\psi_\Omega(\alpha) < \Omega$. $\mathbb{K}$ is an ordinal term denoting the least Π_N-reflecting ordinal in the theorems.

Theorem 1.1 *Suppose* $\mathsf{KP}\Pi_N \vdash \theta$ *for a* $\Sigma_1(\Omega)$*-sentence* θ*. Then we can find an* $n < \omega$ *such that for* $\alpha = \psi_\Omega(\omega_n(\mathbb{K}+1))$, $L_\alpha \models \theta$.

Actually the bound is seen to be tight, cf. [5].

Theorem 1.2 $\mathsf{KP}\Pi_N$ *proves that* each *initial segment* $\{\alpha \in OT : \alpha < \psi_\Omega(\omega_n(\mathbb{K}+1))\}\ (n = 1, 2, \ldots)$ *is well-founded.*

Thus the ordinal $\psi_\Omega(\varepsilon_{\mathbb{K}+1})$ is seen to be the proof-theoretic ordinal of $\mathsf{KP}\Pi_N$.

Theorem 1.3

$$\psi_\Omega(\varepsilon_{\mathbb{K}+1}) = |\mathsf{KP}\Pi_N|_{\Sigma_1^\Omega} := \min\{\alpha \leq \omega_1^{CK} : \forall\theta \in \Sigma_1(\mathsf{KP}\Pi_N \vdash \theta^{L_\Omega} \Rightarrow L_\alpha \models \theta)\}.$$

$A \subset ORD$ is Π^1_n*-indescribable* in α iff for any Π^1_n-formula $\phi(X)$ and any $B \subset ORD$, if $\langle L_\alpha, \in; B \cap \alpha\rangle \models \phi(B \cap \alpha)$, then there exists a $\beta \in A \cap \alpha$ such that $\langle L_\beta, \in; B \cap \beta\rangle \models \phi(B \cap \beta)$. A regular cardinal π is Π^1_n*-indescribable* ifff ORD is Π^1_n-indescribable in π.

Let us mention the contents of this paper. In the next section 2 we define simultaneously iterated Skolem hulls $\mathcal{H}_\alpha(X)$ of sets X of ordinals, ordinals $\psi_\kappa^{\vec{\xi}}(\alpha)$ for regular cardinals κ, $\alpha < \varepsilon_{\mathbb{K}+1}$ and sequences $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ of ordinals $\xi_i < \varepsilon_{\mathbb{K}+2}$, and classes $Mh_k^\alpha(\xi)$ under the *assumption* that a Π^1_{N-2}-indescribable cardinal $\mathbb{K}$ exists. It is shown that for $2 \leq k < N$, $\alpha < \varepsilon_{\mathbb{K}+1}$ and each $\xi < \varepsilon_{\mathbb{K}+2}$, ($\mathbb{K}$ is a Π^1_{N-2}-indescribable cardinal) $\to \mathbb{K} \in Mh_k^\alpha(\xi)$ in $\mathsf{ZF} + (V = L)$.

In section 3 a computable notation system OT of ordinals is extracted. Following W. Buchholz [6], operator controlled derivations for $\mathsf{KP}\Pi_N$ is introduced in section 4, and inference rules for Π_N-reflection are eliminated from derivations in section 5. This completes a proof of Theorem 1.1 for an upper bound.

IH denotes the Induction Hypothesis, MIH the Main IH and SIH the Subsidiary IH. We are assuming tacitly the axiom of constructibility $V = L$. Throughout of this paper $N \geq 3$ is a fixed integer.

2 Ordinals for Π_N-reflection

In this section we work in the set theory ZFLK_N obtained from $\mathsf{ZFL} = \mathsf{ZF}+(V = L)$ by adding the axiom $\exists\mathbb{K}(\mathbb{K}$ is Π^1_{N-2}-indescribable) for a fixed integer $N \geq 3$. For ordinals α, $\varepsilon(\alpha)$ denotes the least epsilon number above α.

Let $ORD \subset V$ denote the class of ordinals, $\mathbb{K}$ the least Π^1_{N-2}-indescribable cardinal, and Reg the set of regular ordinals below $\mathbb{K}$. Θ denotes finite sets of ordinals$\leq \mathbb{K}$. $u, v, w, x, y, z, \ldots$ range over sets in the universe, $a, b, c, \alpha, \beta, \gamma, \ldots$ range over ordinals$< \Lambda$, $\xi, \zeta, \nu, \mu, \iota, \ldots$ range over ordinals$< \varepsilon(\Lambda) = \varepsilon_{\mathbb{K}+2}$, $\vec{\xi}, \vec{\zeta}, \vec{\nu}, \vec{\mu}, \vec{\iota}, \ldots$ range over finite sequences over ordinals$< \varepsilon(\Lambda)$, and $\pi, \kappa, \rho, \sigma, \tau, \lambda, \ldots$ range over regular ordinals. θ denotes formulas.

Let $\vec{\xi} = (\xi_0, \ldots, \xi_{m-1})$ be a sequence of ordinals. The *length* $lh(\vec{\xi}) := m$. Sequences consisting of a single element (ξ) is identified with the ordinal ξ, and $\emptyset$ denotes the *empty sequence*. $\vec{0}$ denotes ambiguously a zero-sequence $(0, \ldots, 0)$ with its length $0 \leq lh(\vec{0}) \leq N-1$. $\vec{\xi} * \vec{\mu} = (\xi_0, \ldots, \xi_{m-1}) * (\mu_0, \ldots, \nu_{n-1}) = (\xi_0, \ldots, \xi_{m-1}, \mu_0, \ldots, \mu_{n-1})$ denotes the *concatenated* sequence of $\vec{\xi}$ and $\vec{\mu}$.

$\Lambda = \varepsilon(\mathbb{K}) = \varepsilon_{\mathbb{K}+1}$ denotes the next epsilon number above the least Π_{N-2}-indescribable cardinal $\mathbb{K}$, and $\varepsilon(\Lambda) = \varepsilon_{\mathbb{K}+2}$ the next epsilon number above Λ.

Definition 2.1 For a non-zero ordinal $\xi < \varepsilon(\Lambda)$, its Cantor normal form with base Λ is uniquely determined as

$$\xi =_{NF} \sum_{i \leq m} \Lambda^{\xi_i} a_i = \Lambda^{\xi_m} a_m + \cdots + \Lambda^{\xi_0} a_0 \tag{1}$$

where $\xi_m > \cdots > \xi_0$, $0 < a_i < \Lambda$.

1. $K(\xi) = \{a_i : i \leq m\} \cup \bigcup\{K(\xi_i) : i \leq m\}$ is the set of *components* of ξ with $K(0) = \emptyset$. For a sequence $\vec{\xi} = (\xi_0, \ldots, \xi_{n-1})$ of ordinals $\xi_i < \varepsilon(\Lambda)$, $K(\vec{\xi}) := \bigcup\{K(\xi_i) : i < n\}$.

2. For $\xi > 1$, $te(\xi) = \xi_0$ in (1) is the *tail exponent*, and $he(\xi) = \xi_m$ is the *head exponent* of ξ, resp. The *head* $Hd(\xi) := \Lambda^{\xi_m} a_m$, and the *tail* $Tl(\xi) := \Lambda^{\xi_0} a_0$ of ξ.

3. $he^{(i)}(\xi)$ is the *i-th head exponent* of ξ, defined recursively by $he^{(0)}(\xi) = \xi$, $he^{(i+1)}(\xi) = he(he^{(i)}(\xi))$.

   The *i-th tail exponent* $te^{(i)}(\xi)$ is defined similarly.

4. ζ is a *part* of ξ, denoted by $\zeta \leq_{pt} \xi$ iff
$\zeta =_{NF} \sum_{i\geq n} \Lambda^{\xi_i} a_i = \Lambda^{\xi_m} a_m + \cdots + \Lambda^{\xi_n} a_n$ for an $n\,(0 \leq n \leq m+1)$.
$\zeta <_{pt} \xi :\Leftrightarrow \zeta \leq_{pt} \xi \,\&\, \zeta \neq \xi$.

5. A sequence $\vec{\mu} = (\mu_0, \ldots, \mu_n)$ is an *iterated tail parts* of ξ, denoted by $\vec{\mu} \subset_{pt} \xi$ iff $\mu_0 \leq_{pt} \xi \,\&\, \forall i < n(\mu_{i+1} \leq_{pt} te(\mu_i))$.

6. $\vec{\nu} = (\nu_0, \ldots, \nu_n) * \vec{0} < \xi$ iff there exists a sequence $\vec{\mu} = (\mu_0, \ldots, \mu_n)$ such that $\vec{\mu} \subset_{pt} \xi$ and $\nu_i < \mu_i$ for every $i \leq n$.

7. Let $\vec{\nu} = (\nu_0, \ldots, \nu_n)$ and $\vec{\xi} = (\xi_0, \ldots, \xi_n)$ be sequences of ordinals in the same length, and $0 \leq k \leq n$.
$\vec{\nu} <_k \vec{\xi} :\Leftrightarrow \forall i < k(\nu_i \leq \xi_i) \wedge (\nu_k, \ldots, \nu_n) < \xi_k$.

8. ζ is a *step-down* of ξ, denoted by $\zeta <_{sd} \xi$ iff
$\zeta = \Lambda^{\xi_m} a_m + \cdots + \Lambda^{\xi_1} a_1 + \Lambda^{\xi_0} b + \nu$ for some ordinals $b < a_0$ and $\nu < \Lambda^{\xi_0}$.

9. $\vec{\nu} = (\nu_0, \ldots, \nu_n) * \vec{0} <_{sd} \xi$ iff $\nu_i <_{sd} te^{(i)}(\xi)$ for every $i \leq n$.

10. $\zeta \leq_{sp} \xi :\Leftrightarrow \exists \mu \leq_{pt} \xi(\zeta \leq_{sd} \mu)$, and $\zeta <_{sp} \xi :\Leftrightarrow \exists \mu \leq_{pt} \xi(\zeta <_{sd} \mu)$.

11. $\vec{\nu} <_{sp} \xi$ iff $\vec{\nu} <_{sd} \mu$ for a $\mu \leq_{pt} \xi$.
Let $p(\vec{\nu}, \xi)$ denote the number $p\,(0 \leq p < m)$ such that $\xi =_{NF} \mu + \sum_{i<p} \Lambda^{\xi_i} a_i$ for $\mu = \Lambda^{\xi_m} a_m + \cdots + \Lambda^{\xi_p} a_p$ and $\vec{\nu} <_{sd} \mu$.

Note that $(\nu) * \vec{0} < \xi \Leftrightarrow \nu < \xi$, and $(\xi, te(\xi), te^{(2)}(\xi), \ldots) \subset_{pt} \xi$. Also $\zeta <_{sd} \xi \Leftrightarrow \zeta < \xi$ if $\xi < \Lambda$.

Proposition 2.2 $\xi < \mu < \varepsilon(\Lambda) \Rightarrow te(\xi) \leq he(\xi) \leq he(\mu)$.

Proposition 2.3 $\vec{\nu} < \xi \leq \zeta \Rightarrow \vec{\nu} < \zeta$.

Proof by induction on the lengths $n = lh(\vec{\nu})$. Let $\vec{\mu} = (\mu_0, \ldots, \mu_{n-1})$ be a sequence for $\vec{\nu} = (\nu_0, \ldots, \nu_{n-1})$ such that $\vec{\mu} \subset_{pt} \xi$ and $\forall i \leq n-1(\nu_i < \mu_i)$, cf. Definition 2.1.6.

If $n = 1$, then $\nu_0 < \mu_0 \leq_{pt} \xi \leq \zeta$. $\nu_0 < \zeta \leq_{pt} \zeta$ yields $\vec{\nu} = (\nu_0) < \zeta$.

Let $n > 1$. We have $(\nu_1, \ldots, \nu_{n-1}) < te(\mu_0)$ with $(\mu_1, \ldots, \mu_{n-1}) \subset_{pt} te(\mu_0)$. We show the existence of a λ such that $\mu_0 \leq \lambda \leq_{pt} \zeta$ and $te(\mu_0) \leq te(\lambda)$. Then IH yields $(\nu_1, \ldots, \nu_{n-1}) < te(\lambda)$, and $\vec{\nu} < \zeta$ follows.

If $\mu_0 \leq_{pt} \zeta$, then $\lambda = \mu_0$ works. Suppose $\mu_0 \not\leq_{pt} \zeta$. On the other hand we have $\mu_0 \leq_{pt} \xi \leq \zeta$. This means that $\xi < \zeta$ and there exists a $\lambda \leq_{pt} \zeta$ such that $\mu_0 < \lambda$ and $te(\mu_0) \leq te(\lambda)$. □

2.1 Ordinals

Definition 2.4 1. For $i < \omega$ and $\xi < \varepsilon(\Lambda)$, $\Lambda_i(\xi)$ is defined recursively by $\Lambda_0(\xi) = \xi$ and $\Lambda_{i+1}(\xi) = \Lambda^{\Lambda_i(\xi)}$.

2. For $A \subset ORD$, limit ordinals α and $i \geq 0$, let $\alpha \in M_{2+i}(A)$ iff $A \cap \alpha$ is Π^1_i-indescribable in α.

3. κ^+ denotes the next regular ordinal above κ.

4. $\Omega_\alpha := \omega_\alpha$ for $\alpha > 0$, $\Omega_0 := 0$, and $\Omega = \Omega_1$.

Define simultaneously classes $\mathcal{H}_\alpha(X)$, $Mh^\alpha_k(\xi)$, and ordinals $\psi^{\vec{\xi}}_\kappa(\alpha)$ as follows. We see that these are Σ_1-definable as a fixed point in ZFL, cf. Proposition 2.7.

Let $a < \Lambda$, and φ denote the binary Veblen function. Let us define a Skolem hull $\mathcal{H}_a(X)$ of $\{0, \mathbb{K}\} \cup X$ under the functions $+, \alpha \mapsto \omega^\alpha, (\alpha, \beta) \mapsto \varphi\alpha\beta\, (\alpha, \beta < \mathbb{K}), \alpha \mapsto \Omega_\alpha\, (\alpha < \mathbb{K})$ and ψ-functions. *Reg* denotes the set of regular ordinals$\leq \mathbb{K}$.

Definition 2.5 $\mathcal{H}_a[Y](X) := \mathcal{H}_a(Y \cup X)$ for sets $Y \subset \mathbb{K}$.

1. (Inductive definition of $\mathcal{H}_a(X)$).

   (a) $\{0, \mathbb{K}\} \cup X \subset \mathcal{H}_a(X)$.

   (b) $x, y \in \mathcal{H}_a(X) \Rightarrow x + y \in \mathcal{H}_a(X)$, $x \in \mathcal{H}_a(X) \Rightarrow \omega^x \in \mathcal{H}_a(X)$, and $x, y \in \mathcal{H}_a(X) \cap \mathbb{K} \Rightarrow \varphi xy \in \mathcal{H}_a(X)$.

   (c) $\mathbb{K} > \alpha \in \mathcal{H}_a(X) \Rightarrow \Omega_\alpha \in \mathcal{H}_a(X)$.

   (d) If $\pi \in \mathcal{H}_a(X) \cap Reg$ and $b \in \mathcal{H}_a(X) \cap a$, then $\psi_\pi(b) \in \mathcal{H}_a(X)$.

   (e) If $\{b, \xi\} \subset \mathcal{H}_a(X)$ with $\xi \leq b < a$, then $\kappa = \psi^{\vec{0} * (\xi)}_{\mathbb{K}}(b) \in \mathcal{H}_a(X)$, where $lh(\vec{0}) = N - 3$.

   (f) Let $\{\pi, b, c\} \subset \mathcal{H}_a(X)$ with $\pi < \mathbb{K}$, $2 \leq k < N - 1$ an integer, and $\vec{\xi} = (\xi_2, \ldots, \xi_k, \xi_{k+1}) * \vec{0}$ a sequence of ordinals $\xi_i < \varepsilon(\Lambda)$ with $lh(\vec{0}) = N - 2 - k$ such that $\xi_{k+1} \neq 0$ and $K(\vec{\xi}) \subset \mathcal{H}_a(X)$. Assume $\max(K(\vec{\xi}) \cup \{c\}) \leq b < a$, and $\pi \in Mh^b_2(\vec{\xi})$.
   Then $\kappa = \psi^{\vec{\nu}}_\pi(b) \in \mathcal{H}_a(X)$ for the sequence $\vec{\nu} = (\xi_2, \ldots, \xi_k + \Lambda^{\xi_{k+1}} c) * \vec{0}$ with $lh(\vec{0}) = N - 1 - k$.

   (g) Let $\{\pi, b\} \subset \mathcal{H}_a(X)$ with $\pi < \mathbb{K}$, and $0 \neq \xi < \varepsilon(\Lambda)$ an ordinal with $K(\xi) \subset \mathcal{H}_a(X)$. Let $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$ be a sequence of ordinals$< \varepsilon(\Lambda)$ such that $K(\vec{\nu}) \subset \mathcal{H}_a(X)$. Assume $\max K(\vec{\nu}) \leq b < a$, $K(\vec{\nu}) \subset \mathcal{H}_b(\pi)$, $\pi \in Mh^b_2(\xi)$, and $\vec{\nu} < \xi$, cf. Definition 2.1.6.
   Then $\kappa = \psi^{\vec{\nu}}_\pi(b) \in \mathcal{H}_a(X)$.

2. (Definitions of $Mh^a_k(\xi)$ and $Mh^a_k(\vec{\xi})$)
   First let $\mathbb{K} \in Mh^a_N(0) :\Leftrightarrow \mathbb{K} \in M_N \Leftrightarrow \mathbb{K}$ is Π^1_{N-2}-indescribable.

The classes $Mh_k^a(\xi)$ are defined for $2 \leq k < N$, and ordinals $a < \Lambda$, $\xi < \varepsilon(\Lambda)$. Let π be a regular ordinal$\leq \mathbb{K}$. Then for $\xi > 0$

$$\begin{aligned}\pi \in Mh_k^a(\xi) :\Leftrightarrow\ & \{\pi, a\} \cup K(\xi) \subset \mathcal{H}_a(\pi)\,\& \\ & \forall \vec{\nu} < \xi\,(K(\vec{\nu}) \subset \mathcal{H}_a(\pi) \Rightarrow \pi \in M_k(Mh_k^a(\vec{\nu})))\end{aligned} \tag{2}$$

where $\vec{\nu} = (\nu_k, \ldots, \nu_n)\,(2 \leq k \leq n \leq N-1)$ varies through non-empty sequences of ordinals$< \varepsilon(\Lambda)$ and

$$\pi \in Mh_k^a(\vec{\nu}) :\Leftrightarrow \pi \in \bigcap_{k \leq i \leq n} Mh_i^a(\nu_i).$$

By convention, let for $2 \leq k < N$, $\pi \in Mh_k^a(0) :\Leftrightarrow \pi \in Mh_2^a(\emptyset) :\Leftrightarrow$ π is a limit ordinal. Note that by letting $\vec{\nu} = (0)$, $\pi \in Mh_k^a(\xi) \Rightarrow \pi \in M_k$ for $\xi > 0$. Also $\vec{0} < 1$, and $Mh_k^a(1) = M_k$.

3. (Definition of $\psi_\pi^{\vec{\xi}}(a)$)
   Let $a < \Lambda$ be an ordinal, $\pi \leq \mathbb{K}$ a regular ordinal and $\vec{\xi}$ a sequence of ordinals$< \varepsilon(\Lambda)$ such that $lh(\vec{\xi}) = N-2$. Then let

$$\psi_\pi^{\vec{\xi}}(a) := \min(\{\pi\} \cup \{\kappa \in Mh_2^a(\vec{\xi}) \cap \pi : \mathcal{H}_a(\kappa) \cap \pi \subset \kappa, K(\vec{\xi}) \cup \{\pi, a\} \subset \mathcal{H}_a(\kappa)\}) \tag{3}$$

   Let $\psi_\pi a := \psi_\pi^{\vec{0}} a$, where $lh(\vec{0}) = N-2$, $Mh_2^a(\vec{0}) = Lim$, and $\pi \in M_2$, i.e., π is a regular ordinal.

Note that $\pi \in Mh_k^a(\xi) \Rightarrow \forall \nu < \xi\,(\pi \in M_k(Mh_k^a(\nu)))$, since $(\nu) < \xi$ holds with $(\xi) \subset_{pt} \xi$ for $\nu < \xi$.

Proposition 2.6 *$b + c \in \mathcal{H}_a[\Theta](d) \Rightarrow c \in \mathcal{H}_a[\Theta](d)$, and $\omega^c \in \mathcal{H}_a[\Theta](d) \Rightarrow c \in \mathcal{H}_a[\Theta](d)$.*

The following Proposition 2.7 is easy to see.

Proposition 2.7 *Each of $x = \mathcal{H}_a(y)\,(a < \Lambda, y < \mathbb{K})$, $x = \psi_\kappa a$, $x \in Mh_k^a(\xi)$ and $x = \psi_\kappa^{\vec{\xi}}(a)$, is a Σ_1-predicate as fixed points in* ZFL.

Proof. This is seen from the facts that there exists a universal Π_n^1-formula, and by using it, $\alpha \in M_n(x)$ iff $\langle L_\alpha, \in\rangle \models m_n(x \cap L_\alpha)$ for some Π_{n+1}^1-formula $m_n(R)$ with a unary predicate R. □

Let $A(a)$ denote the conjunction of $\forall u < \mathbb{K} \exists! x[x = \mathcal{H}_a(u)]$, and $\forall \vec{\xi} \forall x(\max K(\vec{\xi}) \leq a\,\&\,K(\vec{\xi}) \cup \{\kappa, a\} \subset x = \mathcal{H}_a(\kappa) \to \exists! b \leq \kappa(b = \psi_\kappa^{\vec{\xi}}(a)))$, where $lh(\vec{\xi}) = N-2$.

Since the cardinality of the set $\mathcal{H}_{\varepsilon_{\mathbb{K}+1}}(\pi)$ is π for any infinite cardinal $\pi \leq \mathbb{K}$, pick an injection $f : \mathcal{H}_\Lambda(\mathbb{K}) \to \mathbb{K}$ so that $f\text{"}\mathcal{H}_\Lambda(\pi) \subset \pi$ for any weakly inaccessibles $\pi \leq \mathbb{K}$.

Lemma 2.8 *1.* $\forall a < \Lambda\, A(a)$.

2. $\pi \in Mh^a_k(\xi)$ *is a* Π^1_{k-1}*-class on* L_π *uniformly for weakly inaccessible cardinals* $\pi \leq \mathbb{K}$ *and* a, ξ. *This means that for each* k *there exists a* Π^1_{k-1}*-formula* $mh^a_k(x)$ *such that* $\pi \in Mh^a_k(\xi)$ *iff* $L_\pi \models mh^a_k(\xi)$ *for any weakly inaccessible cardinals* $\pi \leq \mathbb{K}$ *with* $f\text{"}(\{a\} \cup K(\xi)) \subset L_\pi$.

3. $\mathbb{K} \in Mh^\alpha_{N-1}(\Lambda) \cap M_{N-1}(Mh^\alpha_{N-1}(\Lambda))$.

Proof.
2.8.1. We show that $A(a)$ is progressive, i.e., $\forall a < \Lambda[\forall c < a\, A(c) \to A(a)]$.

Assume $\forall c < a\, A(c)$ and $a < \Lambda$. $\forall b < \mathbb{K}\exists! x[x = \mathcal{H}_a(b)]$ follows from IH in ZFL. $\exists! b \leq \kappa(b = \psi^{\vec{\xi}}_\kappa a)$ follows from this.

2.8.2. Let π be a weakly inaccessible cardinal with $f\text{"}(\{a\} \cup K(\xi)) \subset L_\pi$. Let f be an injection such that $f\text{"}\mathcal{H}_\Lambda(\pi) \subset L_\pi$. Then for $\forall \alpha \in K(\xi)(f(\alpha) \in f\text{"}\mathcal{H}_\alpha(\pi))$, $\pi \in Mh^a_k(\xi)$ iff for any $f(\vec{\nu}) = (f(\nu_k), \ldots, f(\nu_{N-1}))$, each of $f(\nu_i) \in L_\pi$, if $\forall \alpha \in K(\vec{\nu})(f(\alpha) \in f\text{"}\mathcal{H}_a(\pi))$ and $\vec{\nu} < \xi$, then $\pi \in M_k(Mh^a_k(\vec{\nu}))$, where $f\text{"}\mathcal{H}_a(\pi) \subset L_\pi$ is a class in L_π.

2.8.3. We show the following $B(a)$ is progressive in $a < \Lambda$:

$$B(a) :\Leftrightarrow \mathbb{K} \in Mh^\alpha_{N-1}(a) \cap M_{N-1}(Mh^\alpha_{N-1}(a))$$

Note that $a \in \mathcal{H}_a(\mathbb{K})$ holds for any $a < \Lambda$.

Suppose $\forall b < a\, B(b)$. We have to show that $Mh^\alpha_{N-1}(a)$ is Π^1_{N-3}-indescribable in $\mathbb{K}$. It is easy to see that if $\pi \in M_{N-1}(Mh^\alpha_{N-1}(a))$, then $\pi \in Mh^\alpha_{N-1}(a)$ by induction on π. Let $\theta(u)$ be a Π^1_{N-3}-formula such that $L_\mathbb{K} \models \theta(u)$.

By IH we have $\forall b < a[\mathbb{K} \in M_{N-1}(Mh^\alpha_{N-1}(b))]$. In other words, $\mathbb{K} \in Mh^\alpha_{N-1}(a)$, i.e., $L_\mathbb{K} \models mh^\alpha_{N-1}(a)$, where $mh^\alpha_{N-1}(a)$ is a Π^1_{N-2}-sentence in Proposition 2.8.2. Since the universe $L_\mathbb{K}$ is Π^1_{N-2}-indescribable, pick a $\pi < \mathbb{K}$ such that L_π enjoys the Π^1_{N-2}-sentence $\theta(u) \wedge mh^\alpha_{N-1}(a)$, and $\{f(\alpha), f(a)\} \subset L_\pi$. Therefore $\pi \in Mh^\alpha_{N-1}(a)$ and $L_\pi \models \theta(u)$. Thus $\mathbb{K} \in M_{N-1}(Mh^\alpha_{N-1}(a))$. □

2.2 Normal forms in ordinal notations

In this subsection we introduce an *irreducibility* of sequences, which is needed to define a normal form in ordinal notations.

Proposition 2.9 $\pi \in Mh^a_k(\zeta)\, \&\, \xi \leq \zeta \Rightarrow \pi \in Mh^a_k(\xi)$.

Proof. (2) for $\pi \in Mh^a_k(\xi)$ in Definition 2.5.2 follows from $\pi \in Mh^a_k(\zeta)$ and Proposition 2.3. □

Lemma 2.10 (Cf. Lemma 3 in [1].) *Assume* $\mathbb{K} \geq \pi \in Mh^a_k(\xi) \cap Mh^a_{k+1}(\xi_0)$ *with* $2 \leq k \leq N-1$, $he(\mu) \leq \xi_0$ *and* $\{a\} \cup K(\mu) \subset \mathcal{H}_a(\pi)$. *Then* $\pi \in Mh^a_k(\xi + \mu)$ *holds. Moreover if* $\pi \in M_{k+1}$, *then* $\pi \in M_{k+1}(Mh^a_k(\xi + \mu))$ *holds.*

Proof. Suppose $\pi \in Mh^a_k(\xi) \cap Mh^a_{k+1}(\xi_0)$ and $K(\mu) \subset \mathcal{H}_a(\pi)$ with $he(\mu) \leq \xi_0$. We show $\pi \in Mh^a_k(\xi+\mu)$ by induction on ordinals μ. First note that if $b \in \mathcal{H}_a(\pi)$, then $f(b) \in f\text{"}\mathcal{H}_\Lambda(\pi) \subset L_\pi$. We have $K(\xi+\mu) \subset \mathcal{H}_a(\pi)$. $\pi \in M_{k+1}(Mh^a_k(\xi+\mu))$ follows from $\pi \in Mh^a_k(\xi+\mu)$ and $\pi \in M_{k+1}$.

Let $(\zeta)*\vec{\nu} < \xi+\mu$ and $K(\zeta) \cup K(\vec{\nu}) \subset \mathcal{H}_a(\pi)$ for $\vec{\nu} = (\nu_0, \ldots, \nu_{n-1})$. We need to show that $\pi \in M_k(Mh^a_k((\zeta)*\vec{\nu}))$. By Definition 2.1.6, let $(\zeta_0)*(\mu_0, \ldots, \mu_{n-1})$ be a sequence such that $\zeta < \zeta_0 \leq_{pt} \xi+\mu$, $\mu_0 \leq_{pt} te(\zeta_0)$, $\forall i \leq n-1(\nu_i < \mu_i)$, and $\forall i < n-1(\mu_{i+1} \leq_{pt} te(\mu_i))$.

If $\zeta_0 \leq_{pt} \xi$, then $(\zeta)*\vec{\nu} < \xi$, and $\pi \in M_k(Mh^a_k((\zeta)*\vec{\nu}))$ by $\pi \in Mh^a_k(\xi)$.

Let $\zeta_0 = \xi + \zeta_1$ with $0 < \zeta_1 \leq_{pt} \mu$. If $\zeta_1 <_{pt} \mu$, then by IH with $he(\zeta_1) = he(\mu)$ we have $\pi \in Mh^a_k(\zeta_0)$. On the other hand we have $(\zeta)*\vec{\nu} < \zeta_0$. Hence $\pi \in M_k(Mh^a_k((\zeta)*\vec{\nu}))$.

Finally consider the case when $0 < \zeta_1 = \mu$. Then we obtain $\vec{\nu} < te(\xi+\mu) = te(\mu) \leq he(\mu) \leq \xi_0$. $\pi \in Mh^a_{k+1}(\xi_0)$ with Proposition 2.9 yields $\pi \in M_{k+1}(Mh^a_{k+1}(\vec{\nu}))$.

On the other side we see $\pi \in Mh^a_k(\zeta)$ as follows. We have $\zeta < \xi+\mu$. If $\zeta \leq \xi$, then this follows from $\pi \in Mh^a_k(\xi)$ and Proposition 2.9, and if $\zeta = \xi+\lambda < \xi+\mu$, then IH yields $\pi \in Mh^a_k(\zeta)$.

Since $\pi \in Mh^a_k(\zeta)$ is a Π^1_{k-1}-sentence holding on L_π by Lemma 2.8.2 and $\{a\} \cup K(\zeta) \subset \mathcal{H}_a(\pi)$, we obtain $\pi \in M_{k+1}(Mh^a_k((\zeta)*\vec{\nu}))$, a fortiori $\pi \in M_k(Mh^a_k((\zeta)*\vec{\nu}))$. $\Box$

Definition 2.11 For sequences of ordinals $\vec{\xi} = (\xi_k, \ldots, \xi_{N-1})$ and $\vec{\nu} = (\nu_k, \ldots, \nu_{N-1})$ and $2 \leq k, m, n \leq N-1$,

$$Mh^a_m(\vec{\nu}) \prec_k Mh^a_n(\vec{\xi}) :\Leftrightarrow \forall \pi \in Mh^a_n(\vec{\xi})(\{a, \pi\} \cup K(\vec{\nu}) \subset \mathcal{H}_a(\pi) \Rightarrow \pi \in M_k(Mh^a_m(\vec{\nu}))).$$

Corollary 2.12 *Let $\vec{\nu}$ be a sequence defined from a sequence $\vec{\xi}$ as follows. $\forall i < k(\nu_i = \xi_i)$, $\forall i > k(\nu_i = 0)$, and $\nu_k = \xi_k + \Lambda^{\xi_{k+1}} b$, where $2 \leq k < N$, $b < \Lambda$ and $\xi_{k+1} \neq 0$. Then $Mh^a_2(\vec{\nu}) \prec_{k+1} Mh^a_2(\vec{\xi})$ holds. In particular if $\pi \in Mh^a_2(\vec{\xi})$ and $K(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{H}_a(\pi)$, then $\psi^{\vec{\nu}}_\pi(a) < \pi$.*

Proof. This is seen from Lemma 2.10. $\Box$

Proposition 2.13 *Let $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$, $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ be sequences of ordinals$< \varepsilon(\Lambda)$ such that $\vec{\nu} <_k \vec{\xi}$ for an integer k with $2 \leq k \leq N-1$. Then $Mh^a_2(\vec{\nu}) \prec_k Mh^a_2(\vec{\xi})$. In particular if $\pi \in Mh^a_2(\vec{\xi})$ and $K(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{H}_a(\pi)$, then $\psi^{\vec{\nu}}_\pi(a) < \pi$.*

Proof. Assume $\pi \in Mh^a_2(\vec{\xi})$ and $K(\vec{\nu}) \subset \mathcal{H}_a(\pi)$. We have $\pi \in Mh^a_k(\xi_k)$. By the definition (2) and $(\nu_k, \ldots, \nu_{N-1}) < \xi_k$, we obtain $\pi \in M_k(\bigcap_{k \leq i \leq N-1} Mh^a_i(\nu_i))$.

On the other hand we have $\pi \in \bigcap_{i<k} Mh^a_i(\xi_i)$, and hence $\pi \in \bigcap_{i<k} Mh^a_i(\nu_i)$ by $\forall i < k(\nu_i \leq \xi_i)$ and Proposition 2.9. Since $\pi \in \bigcap_{i<k} Mh^a_i(\nu_i)$ is a Π^1_{k-2}-sentence holding in L_π, we obtain $\pi \in M_k(\bigcap_{i \leq N-1} Mh^a_i(\nu_i)) = M_k(Mh^a_2(\vec{\nu}))$, a fortiori $\pi \in M_2(Mh^a_2(\vec{\nu}))$.

Suppose $\{\pi, a\} \subset \mathcal{H}_a(\pi)$. The set $C = \{\kappa < \pi : \mathcal{H}_a(\kappa) \cap \pi \subset \kappa, K(\vec{\nu}) \cup \{\pi, a\} \subset \mathcal{H}_a(\kappa)\}$ is a club subset of the regular cardinal π. This shows the existence of a $\kappa \in Mh_2^a(\vec{\nu}) \cap C \cap \pi$, and hence $\psi_\pi^{\vec{\nu}}(a) < \pi$ by the definition (3). □

Proposition 2.14 *Let $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ be a sequence of ordinals $< \varepsilon(\Lambda)$ such that $\{\pi, a\} \cup K(\vec{\xi}) \subset \mathcal{H}_a(\pi)$. Assume $Tl(\xi_i) < \Lambda_k(\xi_{i+k}+1)$ for some $i < N-1$ and $k > 0$. Then $\pi \in Mh_2^a(\vec{\xi}) \Leftrightarrow \pi \in Mh_2^a(\vec{\mu})$, where $\vec{\mu} = (\mu_2, \ldots, \mu_{N-1})$ with $\mu_i = \xi_i - Tl(\xi_i)$ and $\mu_j = \xi_j$ for $j \neq i$.*

Proof. When $0 < \xi_i = \Lambda^{\gamma_m} a_m + \cdots + \Lambda^{\gamma_1} a_1 + \Lambda^{\gamma_0} a_0$ with $\gamma_m > \cdots > \gamma_1 > \gamma_0$, $0 < a_i < \Lambda$, $\mu_i = \Lambda^{\gamma_m} a_m + \cdots + \Lambda^{\gamma_1} a_1$ for $Tl(\xi_i) = \Lambda^{\gamma_0} a_0$. If $\xi_i = 0$, then so is $\mu_i = 0$.

Let $\pi \in Mh_2^a(\vec{\mu})$ and $Tl(\xi_i) < \Lambda_k(\xi_{i+k}+1)$. We obtain $\forall j \leq k(he^{(j)}(Tl(\xi_i)) < \Lambda_{k-j}(\xi_{i+k}+1))$, and $he^{(k)}(Tl(\xi_i)) \leq \xi_{i+k}$. On the other hand we have $\pi \in Mh_{i+k}^a(\xi_{i+k})$. From Lemma 2.10 we see inductively that for any $j < k$, $\pi \in Mh_{i+j}^a(he^{(j)}(Tl(\xi_i)))$. In particular $\pi \in Mh_{i+1}^a(he(Tl(\xi_i)))$, and once again by Lemma 2.10 and $\pi \in Mh_i^a(\mu_i)$ we obtain $\pi \in Mh_i^a(\xi_i)$. Hence $\pi \in Mh_2^a(\vec{\xi})$. □

Definition 2.15 A sequence of ordinals $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ is said to be *irreducible* iff $\forall i < N-1 \forall k > 0(\xi_i > 0 \Rightarrow Tl(\xi_i) \geq \Lambda_k(\xi_{i+k}+1))$.

Proposition 2.16 *Let $\vec{\nu} = (\nu_k, \ldots, \nu_{N-1}) \neq \vec{0}$ be an irreducible sequence, and $k_0 \geq k$ be the least number such that $\nu_{k_0} \neq 0$. Assume $\nu_{k_0} < he^{(k_0-k)}(\xi)$. Then $\vec{\nu} < \xi$ holds in the sense of Definition 2.1.6.*

Proof. Let $\ell < N-k$ be the largest number such that $\nu_{k+\ell} \neq 0$. We show $(\nu_k, \ldots, \nu_{k+\ell}) < \xi$. Since $\vec{\nu}$ is irreducible, we have $\Lambda_i(\nu_{k_0+i}+1) \leq Tl(\nu_{k_0})$. From $\nu_{k_0} < he^{(k_0-k)}(\xi)$ and $te(\mu) \leq he(\mu)$ we obtain $\nu_{k_0+i} < \nu_{k_0+i}+1 \leq he^{(i)}(\nu_{k_0}) \leq he^{(k_0-k+i)}(\xi)$. Let $(\mu_k, \ldots, \mu_{N-1}) \subset_{pt} \xi$ such that $\mu_k = Hd(\xi)$ and $\mu_{i+1} = he(\mu_i) = te(Hd(\mu_i))$. Then $te(\mu_{k+i}) = he(\mu_{k+i})$ and $\mu_{k_0+i} = he(\mu_{k_0+i-1}) = he^{(k_0-k+i)}(\xi)$ for $k_0-k+i > 0$. Therefore $(\mu_k, \ldots, \mu_{k+\ell}) \subset_{pt} \xi$ witnesses $(\nu_k, \ldots, \nu_{k+\ell}) < \xi$. □

Definition 2.17 Let $\vec{\xi} = (\xi_k, \ldots, \xi_{N-1})$, $\vec{\nu} = (\nu_k, \ldots, \nu_{N-1})$ and $\vec{\nu} \neq \vec{\xi}$. Let $i \geq k$ be the minimal number such that $\nu_i \neq \xi_i$. Suppose $(\xi_i, \ldots, \xi_{N-1}) \neq \vec{0}$, and let $k_1 \geq i$ be the minimal number such that $\xi_{k_1} \neq 0$. Then $\vec{\nu} <_{lx,k} \vec{\xi}$ iff one of the followings holds:

1. $(\nu_i, \ldots, \nu_{N-1}) = \vec{0}$.
2. In what follows assume $(\nu_i, \ldots, \nu_{N-1}) \neq \vec{0}$, and let $k_0 \geq i$ be the minimal number such that $\nu_{k_0} \neq 0$ $(i = \min\{k_0, k_1\})$. Then $\vec{\nu} <_{lx,k} \vec{\xi}$ iff one of the followings holds:
   (a) $i = k_0 < k_1$ and $he^{(k_1-k_0)}(\nu_{k_0}) \leq \xi_{k_1}$.
   (b) $k_0 \geq k_1 = i$ and $\nu_{k_0} < he^{(k_0-k_1)}(\xi_{k_1})$.

Proposition 2.18 *Suppose that both of $\vec{\nu}$ and $\vec{\xi}$ are irreducible. Then $\vec{\nu} <_{lx,k} \vec{\xi} \Rightarrow Mh_k^a(\vec{\nu}) \prec_k Mh_k^a(\vec{\xi})$.*

Proof. Let $\pi \in Mh_k^a(\vec{\xi})$, $K(\vec{\nu}) \subset \mathcal{H}_a(\pi)$, and $i \geq k$ be the minimal number such that $\nu_i \neq \xi_i$. We have $\pi \in \bigcap_{k \leq j < i} Mh_j^a(\nu_j)$, which is a Π^1_{i-2}-sentence holding on L_π. In the case $\xi_i \neq 0$, it suffices to show that $\pi \in M_i(\bigcap_{j \geq i} Mh_j^a(\nu_j))$, since then we obtain $\pi \in M_i(Mh_k^a(\vec{\nu}))$ by $\pi \in Mh_i^a(\xi_i) \subset M_i$, a fortiori $\pi \in M_k(Mh_k^a(\vec{\nu}))$.

If $(\nu_i, \ldots, \nu_{N-1}) = \vec{0}$, then $\xi_i \neq 0$ and $\bigcap_{j \geq i} Mh_j^a(\nu_j)$ denotes the class of limit ordinals. Obviously $\pi \in M_i(\bigcap_{j \geq i} Mh_j^a(\nu_j))$.

In what follows assume $(\nu_i, \ldots, \nu_{N-1}) \neq \vec{0}$, and let $k_0 \geq i$ be the minimal number such that $\nu_{k_0} \neq 0$, and $k_1 \geq i$ be the minimal number such that $\xi_{k_1} \neq 0$.
Case 1. $k_0 \geq k_1 = i$: Then we have $\nu_{k_0} < he^{(k_0-k_1)}(\xi_{k_1})$. Proposition 2.16 yields $(\nu_{k_0}, \ldots, \nu_{N-1}) < \xi_{k_1} = \xi_i$, which in turn yields $\pi \in M_i(\bigcap_{j \geq i} Mh_j^a(\nu_j))$ by the definition (2) of $\pi \in Mh_i^a(\xi_i)$.

Case 2. $i = k_0 < k_1$: Then we have $he^{(k_1-i)}(\nu_i) \leq \xi_{k_1}$. Also $\nu_{i+p} < he^{(p)}(\nu_i)$ for any $p > 0$ since $\vec{\nu}$ is irreducible and $\nu_i \neq 0$. Let $j \geq k_1$. Then $\nu_j < he^{(j-i)}(\nu_i) \leq he^{(j-k_1)}(\xi_{k_1})$. In particular $\nu_{k_1} < \xi_{k_1}$. Proposition 2.16 yields $(\nu_{k_1}, \ldots, \nu_{N-1}) < \xi_{k_1}$. $\pi \in Mh_{k_1}^a(\xi_{k_1})$ yields $\pi \in M_{k_1}(\bigcap_{j \geq k_1} Mh_j^a(\nu_j))$. Moreover for any $p < k_1 - i$, $he^{(k_1-i-p)}(\nu_{i+p}) \leq \xi_{k_1}$ by Proposition 2.2. Lemma 2.10 yields $\pi \in \bigcap_{k_1 > j \geq i} Mh_j^a(\nu_j)$. Therefore $\pi \in M_{k_1}(Mh_k^a(\vec{\nu}))$, a fortiori $\pi \in M_k(Mh_k^a(\vec{\nu}))$. □

Proposition 2.19 (Cf. Proposition 4.20 in [8])
Let $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$, $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ be irreducible sequences of ordinals$<\varepsilon(\Lambda)$, and assume that $\psi_\pi^{\vec{\nu}}(b) < \pi$ and $\psi_\kappa^{\vec{\xi}}(a) < \kappa$.

Then $\beta_1 = \psi_\pi^{\vec{\nu}}(b) < \psi_\kappa^{\vec{\xi}}(a) = \alpha_1$ iff one of the following cases holds:

1. *$\pi \leq \psi_\kappa^{\vec{\xi}}(a)$.*
2. *$b < a$, $\psi_\pi^{\vec{\nu}}(b) < \kappa$ and $K(\vec{\nu}) \cup \{\pi, b\} \subset \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$.*
3. *$b > a$ and $K(\vec{\xi}) \cup \{\kappa, a\} \not\subset \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))$.*
4. *$b = a$, $\kappa < \pi$ and $\kappa \notin \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))$.*
5. *$b = a$, $\pi = \kappa$, $K(\vec{\nu}) \subset \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$, and $\vec{\nu} <_{lx,2} \vec{\xi}$.*
6. *$b = a$, $\pi = \kappa$, $K(\vec{\xi}) \not\subset \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))$.*

Proof. If the case (2) holds, then $\psi_\pi^{\vec{\nu}}(b) \in \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a)) \cap \kappa \subset \psi_\kappa^{\vec{\xi}}(a)$.

If one of the cases (3) and (4) holds, then $K(\vec{\xi}) \cup \{\kappa, a\} \not\subset \mathcal{H}_a(\psi_\pi^{\vec{\nu}}(b))$. On the other hand we have $K(\vec{\xi}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$. Hence $\psi_\pi^{\vec{\nu}}(b) < \psi_\kappa^{\vec{\xi}}(a)$.

If the case (5) holds, then Proposition 2.18 yields $Mh_2^a(\vec{\nu}) \prec_2 Mh_2^a(\vec{\xi}) \ni \psi_\kappa^{\vec{\xi}}(a)$. Hence $\psi_\kappa^{\vec{\xi}}(a) \in M_2(Mh_2^a(\vec{\nu}))$. Since $K(\vec{\nu}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$, the set $\{\rho < \psi_\kappa^{\vec{\xi}}(a) : \mathcal{H}_a(\rho) \cap \kappa \subset \rho, K(\vec{\nu}) \cup \{\kappa, a\} \subset \mathcal{H}_a(\rho)\}$ is club in $\psi_\kappa^{\vec{\xi}}(a)$. Therefore $\psi_\pi^{\vec{\nu}}(b) = \psi_\kappa^{\vec{\nu}}(a) < \psi_\kappa^{\vec{\xi}}(a)$ by (3) in Definition 2.5.3.

Finally assume that the case (6) holds. Since $K(\vec{\xi}) \subset \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$, $\psi_\pi^{\vec{\nu}}(b) < \psi_\kappa^{\vec{\xi}}(a)$ holds.

Conversely assume that $\psi_\pi^{\vec{\nu}}(b) < \psi_\kappa^{\vec{\xi}}(a)$ and $\psi_\kappa^{\vec{\xi}}(a) < \pi$.

First consider the case $b < a$. Then we have $K(\vec{\nu}) \cup \{\pi, b\} \subset \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b)) \subset \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$. Hence (2) holds.

Next consider the case $b > a$. $K(\vec{\xi}) \cup \{\kappa, a\} \subset \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))$ would yield $\psi_\kappa^{\vec{\xi}}(a) \in \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b)) \cap \pi \subset \psi_\pi^{\vec{\nu}}(b)$, a contradiction $\psi_\kappa^{\vec{\xi}}(a) < \psi_\pi^{\vec{\nu}}(b)$. Hence (3) holds.

Finally assume $b = a$. Consider the case $\kappa < \pi$. $\kappa \in \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b)) \cap \pi$ would yield $\psi_\kappa^{\vec{\xi}}(a) < \kappa < \psi_\pi^{\vec{\nu}}(b)$, a contradiction. Hence $\kappa \notin \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))$, and (4) holds. If $\pi < \kappa$, then $\pi \in \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b)) \cap \kappa \subset \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a)) \cap \kappa$, and $\pi < \psi_\kappa^{\vec{\xi}}(a)$, a contradiction, or we should say that (1) holds. Finally let $\pi = \kappa$. We can assume that $K(\vec{\xi}) \subset \mathcal{H}_b(\psi_\pi^{\vec{\nu}}(b))$, otherwise (6) holds. If $\vec{\xi} <_{lx,2} \vec{\nu}$, then by (5) $\psi_\kappa^{\vec{\xi}}(a) < \psi_\pi^{\vec{\nu}}(b)$ would follow. If $K(\vec{\nu}) \not\subset \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$, then by (6) again $\psi_\kappa^{\vec{\xi}}(a) < \psi_\pi^{\vec{\nu}}(b)$ would follow. Hence $K(\vec{\nu}) \subset \mathcal{H}_a(\psi_\kappa^{\vec{\xi}}(a))$ and $\vec{\nu} \leq_{lx} \vec{\xi}$. If $\vec{\nu} = \vec{\xi}$, then $\psi_\kappa^{\vec{\xi}}(a) = \psi_\pi^{\vec{\nu}}(b)$. Therefore (5) must be the case. □

Definition 2.20 is utilized to define a computable notation system in the next section 3.

Definition 2.20 A set SD of sequences $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1})$ of ordinals $\xi_i < \varepsilon(\Lambda)$ is defined recursively as follows.

1. $\vec{0} * (a) \in SD$ for each $a < \Lambda$.

2. (Cf. Definition 2.1.9.) Let $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1}) \in SD$, $1 \leq k < N-1$, $\zeta < \varepsilon(\Lambda)$ be an ordinal such that $(\xi_{k+1}, \ldots, \xi_{N-1}) <_{sd} \zeta$, and $(\xi_2, \ldots, \xi_{k-1}, \xi_k, \zeta) * \vec{0} \in SD$. Then for $\zeta_k = \xi_k + \Lambda^\zeta a$ with an ordinal $a < \Lambda$, $(\xi_2, \ldots, \xi_{k-1}) * (\zeta_k) * (\xi_{k+1}, \ldots, \xi_{N-1}) \in SD$ and $(\xi_2, \ldots, \xi_{k-1}) * (\zeta_k) * \vec{0} \in SD$.

Proposition 2.21 *Let $\vec{\xi} = (\xi_2, \ldots, \xi_{N-1}) \in SD$.*

1. *$(\xi_2, \ldots, \xi_i) * \vec{0} \in SD$ for each i with $1 \leq i < N$.*

2. *For $2 \leq i < j < k < N$, if $\xi_i \neq 0$ and $\xi_k \neq 0$, then $\xi_j \neq 0$.*

3. *Let $\xi_i \neq 0$. Then $(\xi_{i+1}, \ldots, \xi_{N-1}) <_{sd} te(\xi_i)$.*

4. *$\vec{\xi}$ is irreducible.*

Proof. Let $1 \leq k < N-1$, $\zeta < \varepsilon(\Lambda)$ be an ordinal such that $(\xi_{k+1}, \ldots, \xi_{N-1}) <_{sd} \zeta$, and $(\xi_2, \ldots, \xi_{k-1}, \xi_k, \zeta) * \vec{0} \in SD$. Also let $\zeta_k = \xi_k + \Lambda^\zeta a$ with an ordinal $a < \Lambda$.
2.21.1 is seen by induction on the recursive definition of $\vec{\xi} \in SD$.
2.21.2 is seen by induction on the recursive definition of $\vec{\xi} \in SD$. Suppose $\xi_i \neq 0$ for an $i < k$. From $(\xi_2, \ldots, \xi_{k-1}, \xi_k, \zeta) * \vec{0} \in SD$ and $\zeta \neq 0$, IH yields $\xi_k \neq 0$.
2.21.3 and 2.21.4. We show these by simultaneous induction on the recursive definition of $\vec{\xi} \in SD$.
2.21.3. We show Proposition 2.21.3 for the sequence $(\xi_2, \ldots, \xi_{k-1}) * (\zeta_k) * (\xi_{k+1}, \ldots, \xi_{N-1}) \in SD$. The proposition holds for the sequence $\vec{\xi}$, and we can

assume $a \neq 0$. We obtain $(\xi_{i+1}, \ldots, \xi_{N-1}) <_{sd} te(\xi_i)$ for $i > k$ if $\xi_i \neq 0$, and $(\xi_{k+1}, \ldots, \xi_{N-1}) <_{sd} te(\zeta_k) = \zeta$ by the assumption. Let $2 \leq i < k$ and $\xi_i \neq 0$. We show $(\xi_{i+1}, \ldots, \xi_{k-1}) * (\zeta_k) * (\xi_{k+1}, \ldots, \xi_{N-1}) <_{sd} te(\xi_i)$. It suffices to show that $\zeta_k <_{sd} te^{(k-i)}(\xi_i)$. By IH we have $\xi_k <_{sd} te^{(k-i)}(\xi_i)$. On the other hand we have $\xi_k \neq 0$ by $(\xi_2, \ldots, \xi_{k-1}, \xi_k, \zeta) * \vec{0} \in SD$, $\zeta \neq 0$, and Proposition 2.21.2. Moreover $(\xi_2, \ldots, \xi_{k-1}, \xi_k, \zeta) * \vec{0}$ is irreducible by Proposition 2.21.4, and hence $Tl(\xi_k) \geq \Lambda^{\zeta+1}$. Therefore $te(\xi_k) > \zeta$. This means that $\zeta_k =_{NF} \xi_k + \Lambda^{\zeta} a$, and $\xi_k <_{sd} te^{(k-i)}(\xi_i)$ yields $\zeta_k <_{sd} te^{(k-i)}(\xi_i)$ by Definition 2.1.8.
2.21.4. If $(\xi_{i+1}, \ldots, \xi_{N-1}) <_{sd} te(\xi_i)$ for $\xi_i \neq 0$, then $\xi_{i+k} <_{sd} te^{(k)}(\xi_i)$ for $k > 0$, and $\xi_{i+k} + 1 \leq te^{(k)}(\xi_i)$. Hence $\Lambda_k(\xi_{i+k} + 1) \leq \Lambda^{te(\xi_i)} \leq Tl(\xi_i)$, and $\vec{\xi}$ is irreducible. □

3 Computable notation system OT

In this section (except Propositions 3.3) we work in a weak fragment of arithmetic, e.g., in the fragment $I\Sigma_1$ or even in the bounded arithmetic S^1_2. Referring Proposition 2.19 the sets of ordinal terms $OT \subset \Lambda = \varepsilon_{\mathbb{K}+1}$ and $E \subset \varepsilon(\Lambda) = \varepsilon_{\mathbb{K}+2}$ over symbols $\{0, \mathbb{K}, \Lambda, +, \omega, \varphi, \Omega, \psi\}$ are defined recursively. OT is isomorphic to a subset of $\mathcal{H}_\Lambda(0)$. Simultaneously we define finite sets $K_\delta(\alpha) \subset OT$ for $\delta, \alpha \in OT$, and sequences $(m_k(\alpha))_{2 \leq k \leq N-1}$ for $\alpha \in OT \cap \mathbb{K}$, where in $\alpha = \psi^{\vec{\nu}}_\pi(a)$, $m_k(\alpha) = \nu_k$, i.e., $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1}) = (m_2(\alpha), \ldots, m_{N-1}(\alpha)) = (m_k(\alpha))_k = \vec{m}(\alpha)$. For $\{\alpha_0, \ldots, \alpha_m, \beta\} \subset OT$, $K_\delta(\alpha_0, \ldots, \alpha_m) := \bigcup_{i \leq m} K_\delta(\alpha_i)$, $K_\delta(\alpha_0, \ldots, \alpha_m) < \beta :\Leftrightarrow \forall \gamma \in K_\delta(\alpha_0, \ldots, \alpha_m)(\gamma < \beta)$, and $\beta \leq K_\delta(\alpha_0, \ldots, \alpha_m) :\Leftrightarrow \exists \gamma \in K_\delta(\alpha_0, \ldots, \alpha_m)(\beta \leq \gamma)$.

An ordinal term in OT is said to be a *regular* term if it is one of the form $\mathbb{K}$, $\Omega_{\beta+1}$ or $\psi^{\vec{\nu}}_\pi(a)$ with the non-zero sequences $\vec{\nu} \neq \vec{0}$. $\mathbb{K}$ and the latter terms $\psi^{\vec{\nu}}_\pi(a)$ are *Mahlo* terms.

$\alpha =_{NF} \alpha_m + \cdots + \alpha_0$ means that $\alpha = \alpha_m + \cdots + \alpha_0$ and $\alpha_m \geq \cdots \geq \alpha_0$ and each α_i is a non-zero additive principal number. $\alpha =_{NF} \varphi\beta\gamma$ means that $\alpha = \varphi\beta\gamma$ and $\beta, \gamma < \alpha$. $\alpha =_{NF} \omega^\beta$ means that $\alpha = \omega^\beta > \beta$. $\alpha =_{NF} \Omega_\beta$ means that $\alpha = \Omega_\beta > \beta$.

Let $pd(\psi^{\vec{\nu}}_\pi(a)) = \pi$ (even if $\vec{\nu} = \vec{0}$). Moreover for n, $pd^{(n)}(\alpha)$ is defined recursively by $pd^{(0)}(\alpha) = \alpha$ and $pd^{(n+1)}(\alpha) \simeq pd(pd^{(n)}(\alpha))$.

For terms $\pi, \kappa \in OT$, $\pi \prec \kappa$ denotes the transitive closure of the relation $\{(\pi, \kappa) : \exists \vec{\xi} \exists b[\pi = \psi^{\vec{\xi}}_\kappa(b)]\}$, and its reflexive closure $\pi \preceq \kappa :\Leftrightarrow \pi \prec \kappa \lor \pi = \kappa \Leftrightarrow \exists n(\kappa = pd^{(n)}(\pi))$.

For each ordinal term $\alpha = \psi^{\vec{\nu}}_\pi(a)$, a series $(\pi_i)_{i \leq L}$ of ordinal terms is uniquely determined as follows: $\pi_L = \alpha$, $\pi_i = pd(\pi_{i+1})$ and $\pi_0 = \mathbb{K}$. Let us call the series $(\pi_i)_{i \leq L}$ the *collapsing series* of $\alpha = \pi_L$.

Then we see that an ordinal term $\alpha = \psi^{\vec{\nu}}_\pi(a)$ with $\vec{\nu} \neq \vec{0}$ is constructed by Definition 3.1.2g below iff $L = 1$. α is constructed by Definition 3.1.2i iff $L \equiv 1 \pmod{(N-2)}$. Otherwise α is constructed by Definition 3.1.2h.

Definition 3.1 $\ell\alpha$ denotes the number of occurrences of symbols $\{0, \mathbb{K}, \Lambda, +, \omega, \varphi, \Omega, \psi\}$ in terms $\alpha \in OT \cup E$.

1. (a) $0 \in E$.

   (b) If $0 < a \in OT$, then $a \in E$. $K(a) = \{a\}$.

   (c) If $\{\xi_i : i \leq m\} \subset E$, $\xi_m > \cdots > \xi_0 > 0$ and $0 < b_i \in OT$, then $\sum_{i \leq m} \Lambda^{\xi_i} b_i = \Lambda^{\xi_m} b_m + \cdots + \Lambda^{\xi_0} b_0 \in E$. $K(\sum_{i \leq m} \Lambda^{\xi_i} b_i) = \{b_i : i \leq m\} \cup \bigcup\{K(\xi_i) : i \leq m\}$.

   (d) For sequences $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$, let $K(\vec{\nu}) = \bigcup_{2 \leq i \leq N-1} K(\nu_i)$.

2. (a) $0, \mathbb{K} \in OT$. $m_k(0) = 0$ for any k, and $K_\delta(0) = K_\delta(\mathbb{K}) = \emptyset$.

   (b) If $\alpha =_{NF} \alpha_m + \cdots + \alpha_0\, (m > 0)$ with $\{\alpha_i : i \leq m\} \subset OT$, then $\alpha \in OT$, and $m_k(\alpha) = 0$ for any k. $K_\delta(\alpha) = K_\delta(\alpha_0, \ldots, \alpha_m)$.

   (c) If $\alpha =_{NF} \varphi\beta\gamma$ with $\{\beta, \gamma\} \subset OT \cap \mathbb{K}$, then $\alpha \in OT$, and $m_k(\alpha) = 0$ for any k. $K_\delta(\alpha) = K_\delta(\beta, \gamma)$.

   (d) If $\alpha =_{NF} \omega^\beta$ with $\mathbb{K} < \beta \in OT$, then $\alpha \in OT$, and $m_k(\alpha) = 0$ for any k. $K_\delta(\alpha) = K_\delta(\beta)$.

   (e) If $\alpha =_{NF} \Omega_\beta$ with $\beta \in OT \cap \mathbb{K}$, then $\alpha \in OT$. $m_2(\alpha) = 1, m_k(\alpha) = 0$ for any $k > 2$ if β is a successor ordinal. Otherwise $m_k(\alpha) = 0$ for any k. In each case $K_\delta(\alpha) = K_\delta(\beta)$.

   (f) Let $\alpha = \psi_\pi(a) := \psi_\pi^{\vec{0}}(a)$ where π is a regular term , i.e., either $\pi = \mathbb{K}$ or $\vec{m}(\pi) \neq \vec{0}$, and $K_\alpha(\pi, a) < a$.

   Then $\alpha = \psi_\pi(a) \in OT$. Let $m_k(\alpha) = 0$ for any k. $K_\delta(\psi_\pi(a)) = \emptyset$ if $\alpha < \delta$. $K_\delta(\psi_\pi(a)) = \{a\} \cup K_\delta(a, \pi)$ otherwise.

   (g) Let $\alpha = \psi_{\mathbb{K}}^{\vec{\nu}}(a)$ with $\vec{\nu} = \vec{0} * (b)\, (lh(\vec{\nu}) = N - 2)$ and $b, a \in OT$ such that $0 < b \leq a$ and $K_\alpha(b, a) < a$.

   Then $\alpha = \psi_{\mathbb{K}}^{\vec{\nu}}(a) \in OT$. Let $m_{N-1}(\alpha) = b$, $m_k(\alpha) = 0$ for $k < N - 1$. $K_\delta(\psi_{\mathbb{K}}^{\vec{\nu}}(a)) = \emptyset$ if $\alpha < \delta$. $K_\delta(\psi_{\mathbb{K}}^{\vec{\nu}}(a)) = \{a\} \cup \bigcup\{K_\delta(\gamma) : \gamma \in K(\nu)\}$ otherwise.

   (h) Let $\pi \in OT \cap \mathbb{K}$ be such that $m_{k+1}(\pi) \neq 0$ and $\forall i > k+1 (m_i(\pi) = 0)$ for a $k\, (2 \leq k \leq N - 2)$, and $b, a \in OT$ such that $0 < b \leq a$. Let $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1})$ be a sequence defined by $\forall i < k(\nu_i = m_i(\pi))$, $\nu_k = m_k(\pi) + \Lambda^{m_{k+1}(\pi)} b$, and $\forall i > k(\nu_i = 0)$.

   Then $\alpha = \psi_\pi^{\vec{\nu}}(a) \in OT$ if $K_\alpha(\pi, a, b) \cup K_\alpha(K(\vec{m}(\pi))) < a$. Let $m_i(\alpha) = \nu_i$ for each i. $K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \emptyset$ if $\alpha < \delta$. Otherwise $K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \{a\} \cup K_\delta(a, \pi) \cup \bigcup\{K_\delta(b) : b \in K(\vec{\nu})\}$.

   (i) Let $\pi \in OT \cap \mathbb{K}$ be such that $m_2(\pi) \neq 0$ and $\forall i > 2(m_i(\pi) = 0)$, and $a \in OT$. Let $\vec{0} \neq \vec{\nu} = (\nu_2, \ldots, \nu_{N-1}) \in SD$ be a sequence of ordinal terms $\nu_i \in E$ such that $\vec{\nu} <_{sp} m_2(\pi)$.

   Then $\alpha = \psi_\pi^{\vec{\nu}}(a)$ if $K_\alpha(\pi, a) < a$, and

   $$\forall k(K_\alpha(\nu_k) < \max K(\nu_k)) \tag{4}$$

   Let $m_i(\alpha) = \nu_i$ for each i.

   $K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \emptyset$ if $\alpha < \delta$. Otherwise $K_\delta(\psi_\pi^{\vec{\nu}}(a)) = \{a\} \cup K_\delta(a, \pi) \cup \bigcup\{K_\delta(b) : b \in K(\vec{\nu})\}$.

Let $\{\pi, a, \xi\} \subset \mathcal{H}_a(\pi)$. Then $\xi = m_k(\pi)$ is intended to be equivalent to $\pi \in Mh^a_k(\xi)$. For Definition 3.1.2h, see Corollary 2.12, and for Definition 3.1.2i, see Proposition 2.13.

Proposition 3.2 *For each Mahlo term $\alpha = \psi_\pi^{\vec{\nu}}(a) \in OT$, $\vec{m}(\alpha) = \vec{\nu} \in SD$ for the class SD in Definition 2.20.*

Proposition 3.3 *For any $\alpha \in OT$ and any δ such that $\delta = 0, \mathbb{K}$ or $\delta = \psi_\pi^{\vec{\nu}}(b)$ for some $\pi, b, \vec{\nu}$, $\alpha \in \mathcal{H}_\gamma(\delta) \Leftrightarrow K_\delta(\alpha) < \gamma$.*

Proof. By induction on $\ell\alpha$. □

Lemma 3.4 *$(OT, <)$ is a computable notation system of ordinals. In particular the order type of the initial segment $\{\alpha \in OT : \alpha < \Omega_1\}$ is less than ω_1^{CK}.*

Specifically each of $\alpha < \beta$ and $\alpha = \beta$ is decidable for $\alpha, \beta \in OT$, and $\alpha \in OT$ is decidable for terms α over symbols $\{0, \mathbb{K}, \Lambda, +, \omega, \varphi, \Omega, \psi\}$.

Proof. These are shown simultaneously referring Propositions 2.19 and 3.3. Let us give recursive definitions only for terms $\Omega_\alpha, \psi_\kappa^{\vec{\nu}}(a) \in OT$.

First $\Omega_{\psi_\kappa^{\vec{\nu}}(a)} = \psi_\kappa^{\vec{\nu}}(a)$, i.e., $\Omega_\alpha < \psi_\kappa^{\vec{\nu}}(a) \Leftrightarrow \alpha < \psi_\kappa^{\vec{\nu}}(a)$, $\psi_\kappa^{\vec{\nu}}(a) < \Omega_\alpha \Leftrightarrow \psi_\kappa^{\vec{\nu}}(a) < \alpha$. Next $\Omega_\alpha < \psi_{\Omega_{\alpha+1}}(a) < \Omega_{\alpha+1}$.

Finally for $\psi_\pi^{\vec{\nu}}(b), \psi_\kappa^{\vec{\xi}}(a) \in OT$, $\psi_\pi^{\vec{\nu}}(b) < \psi_\kappa^{\vec{\xi}}(a)$ iff one of the following cases holds:

1. $\pi \leq \psi_\kappa^{\vec{\xi}}(a)$.
2. $b < a$, $\psi_\pi^{\vec{\nu}}(b) < \kappa$, and $K_{\psi_\kappa^{\vec{\xi}}(a)}(\{\pi, b\} \cup K(\vec{\nu})) < a$.
3. $b \geq a$, and $b \leq K_{\psi_\pi^{\vec{\nu}}(b)}(\{\kappa, a\} \cup K(\vec{\xi}))$.
4. $b = a$, $\pi = \kappa$, $K_{\psi_\kappa^{\vec{\xi}}(a)}(K(\vec{\nu})) < a$, and $\vec{\nu} <_{lx,2} \vec{\xi}$.

□

Proposition 3.5 *1. Let $\beta = \psi_\pi^{\vec{\nu}}(b)$ with $\pi = \psi_\kappa^{\vec{\xi}}(a)$. Then $a < b$.*

2. For $\alpha = \psi_\pi^{\vec{\nu}}(a) \in OT$, $\max K(\vec{\nu}) \leq a$ holds.

Proof. 3.5.1. Let $\beta = \psi_\pi^{\vec{\nu}}(b)$ with $\pi = \psi_\kappa^{\vec{\xi}}(a)$. Then $K_\beta(\{\pi, b\} \cup K(\vec{\nu})) < b$. On the other hand we have $\beta < \pi$. Hence $a \in K_\beta(\pi) < b$.

3.5.2. This is seen by induction on $\ell\alpha$. Ww have $c < a$ by Proposition 3.5.1 when $\pi = \psi_\sigma^{\vec{\mu}}(c)$

When α is constructed by Definition 3.1.2h, $\nu_k = m_k(\pi) + \Lambda^{m_{k+1}(\pi)} b$ holds for $b \leq a$. By IH we have $\max K(\vec{m}(\pi)) \leq c < a$ when $\pi = \psi_\sigma^{\vec{\mu}}(c)$.

Suppose α is constructed by Definition 3.1.2i. We obtain $\vec{\nu} <_{sp} m_2(\pi)$, and hence $\max K(\vec{\nu}) \leq \max K(m_2(\pi)) \leq c < a$ by IH. □

4 Operator controlled derivations

In this section, operator controlled derivations are defined, which are introduced by W. Buchholz [6].

In this and the next sections except otherwise stated $\alpha, \beta, \gamma, \ldots, a, b, c, d, \ldots$ range over ordinal terms in $OT \subset \mathcal{H}_\Lambda(0)$, $\xi, \zeta, \nu, \mu, \iota, \ldots$ range over ordinal terms in E, $\vec{\xi}, \vec{\zeta}, \vec{\nu}, \vec{\mu}, \vec{\iota}, \ldots$ range over finite sequences over ordinal terms in E, and $\pi, \kappa, \rho, \sigma, \tau, \lambda, \ldots$ range over regular ordinal terms $\mathbb{K}$, $\Omega_{\beta+1}$, $\psi_\pi^{\vec{\nu}}(a)$ with $\vec{\nu} \neq \vec{0}$. *Reg* denotes the set of regular ordinal terms. We write $\alpha \in \mathcal{H}_a(\beta)$ for $K_\beta(\alpha) < a$.

4.1 Classes of sentences

Following Buchholz [6] let us introduce a language for ramified set theory RS.

Definition 4.1 *RS-terms* and their *levels* are inductively defined.

1. For each $\alpha \in OT \cap \mathbb{K}$, L_α is an RS-term of level α.

2. If $\phi(x, y_1, \ldots, y_n)$ is a set-theoretic formula in the language $\{\in\}$, and $a_1, \ldots, a_n$ are RS-terms of levels$< \alpha$, then $[x \in L_\alpha : \phi^{L_\alpha}(x, a_1, \ldots, a_n)]$ is an RS-term of level α.

Each ordinal term α is denoted by the ordinal term $[x \in L_\alpha : x$ is an ordinal$]$, whose level is α.

Definition 4.2 1. $|a|$ denotes the level of RS-terms a, and $Tm(\alpha)$ the set of RS-terms of level$< \alpha$. $Tm = Tm(\mathbb{K})$ is then the set of RS-terms, which are denoted by $a, b, c, d, \ldots$

2. *RS-formulas* are constructed from *literals* $a \in b, a \notin b$ by propositional connectives $\vee, \wedge$, bounded quantifiers $\exists x \in a, \forall x \in a$ and unbounded quantifiers $\exists x, \forall x$. Unbounded quantifiers $\exists x, \forall x$ are denoted by $\exists x \in L_\mathbb{K}, \forall x \in L_\mathbb{K}$, resp.

3. For RS-terms and RS-formulas ι, $\mathsf{k}(\iota)$ denotes the set of ordinal terms α such that the constant L_α occurs in ι.

4. For a set-theoretic Σ_n-formula $\psi(x_1, \ldots, x_m)$ in $\{\in\}$ and $a_1, \ldots, a_m \in Tm(\kappa)$, $\psi^{L_\kappa}(a_1, \ldots, a_m)$ is a $\Sigma_n(\kappa)$*-formula*, where $n = 0, 1, 2, \ldots$ and $\kappa \leq \mathbb{K}$. $\Pi_n(\kappa)$-formulas are defined dually.

5. For $\theta \equiv \psi^{L_\kappa}(a_1, \ldots, a_m) \in \Sigma_n(\kappa)$ and $\lambda < \kappa$, $\theta^{(\lambda,\kappa)} :\equiv \psi^{L_\lambda}(a_1, \ldots, a_m)$.

Note that the level $|t| = \max(\{0\} \cup \mathsf{k}(t))$ for RS-terms t. In what follows we need to consider *sentences*. Sentences are denoted A, C possibly with indices.

The assignment of disjunctions and conjunctions to sentences is defined as in [6].

Definition 4.3 *1.* For $b, a \in Tm(\mathbb{K})$ with $|b| < |a|$,

$$(b\varepsilon a) :\equiv \begin{cases} A(b) & \text{if } a \equiv [x \in L_\alpha : A(x)] \\ b \notin L_0 & \text{if } a \equiv L_\alpha \end{cases}$$

and $(a = b) :\equiv (\forall x \in a(x \in b) \land \forall x \in b(x \in a))$.

2. For $b, a \in Tm(\mathbb{K})$ and $J := Tm(|a|)$

$$(b \in a) :\simeq \bigvee(c\varepsilon a \land c = b)_{c\in J} \text{ and } (b \notin a) :\simeq \bigwedge(c \not\varepsilon a \lor c \neq b)_{c\in J}$$

3. $(A_0 \lor A_1) :\simeq \bigvee(A_\iota)_{\iota\in J}$ and $(A_0 \land A_1) :\simeq \bigwedge(A_\iota)_{\iota\in J}$ for $J := 2$.

4. For $a \in Tm(\mathbb{K}) \cup \{L_\mathbb{K}\}$ and $J := Tm(|a|)$

$$\exists x \in a\, A(x) :\simeq \bigvee(b\varepsilon a \land A(b))_{b\in J} \text{ and } \forall x \in a\, A(x) :\simeq \bigwedge(b \not\varepsilon a \lor A(b))_{b\in J}.$$

The rank $\text{rk}(\iota)$ of sentences or terms ι is defined as in [6].

Definition 4.4 *1.* $\text{rk}(\neg A) := \text{rk}(A)$.

2. $\text{rk}(L_\alpha) = \omega\alpha$.

3. $\text{rk}([x \in L_\alpha : A(x)]) = \max\{\omega\alpha + 1, \text{rk}(A(L_0)) + 2\}$.

4. $\text{rk}(a \in b) = \max\{\text{rk}(a) + 6, \text{rk}(b) + 1\}$.

5. $\text{rk}(A_0 \lor A_1) := \max\{\text{rk}(A_0), \text{rk}(A_1)\} + 1$.

6. $\text{rk}(\exists x \in a\, A(x)) := \max\{\omega\text{rk}(a), \text{rk}(A(L_0)) + 2\}$ for $a \in Tm(\mathbb{K}) \cup \{L_\mathbb{K}\}$.

Proposition 4.5 *Let A be a sentence with $A \simeq \bigvee(A_\iota)_{\iota\in J}$ or $A \simeq \bigwedge(A_\iota)_{\iota\in J}$.*

1. $\text{rk}(A) < \mathbb{K} + \omega$.

2. $|A| \leq \text{rk}(A) \in \{\omega|A| + i : i \in \omega\}$.

3. $\forall \iota \in J(\text{rk}(A_\iota) < \text{rk}(A))$.

4. $\text{rk}(A) < \lambda \Rightarrow A \in \Sigma_0(\lambda)$

4.2 Operator controlled derivations

By an *operator* we mean a map $\mathcal{H}$, $\mathcal{H} : \mathcal{P}(OT) \to \mathcal{P}(OT)$, such that

1. $\forall X \subset OT[X \subset \mathcal{H}(X)]$.
2. $\forall X, Y \subset OT[Y \subset \mathcal{H}(X) \Rightarrow \mathcal{H}(Y) \subset \mathcal{H}(X)]$.

For an operator $\mathcal{H}$ and $\Theta, \Theta_1 \subset OT$, $\mathcal{H}[\Theta](X) := \mathcal{H}(X \cup \Theta)$, and $\mathcal{H}[\Theta][\Theta_1] := (\mathcal{H}[\Theta])[\Theta_1]$, i.e., $\mathcal{H}[\Theta][\Theta_1](X) = \mathcal{H}(X \cup \Theta \cup \Theta_1)$.

Obviously $\mathcal{H}_\alpha$ is an operator for any α, and if $\mathcal{H}$ is an operator, then so is $\mathcal{H}[\Theta]$.

Sequents are finite sets of sentences, and inference rules are formulated in one-sided sequent calculus. Let $\mathcal{H} = \mathcal{H}_\gamma\,(\gamma \in OT)$ be an operator, Θ a finite set of $\mathbb{K}$, Γ a sequent, $a \in OT$ and $b \in OT \cap (\mathbb{K} + \omega)$.

We define a relation $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$, which is read 'there exists an infinitary derivation of Γ which is Θ-*controlled* by $\mathcal{H}_\gamma$, and whose height is at most a and its cut rank is less than b'.

$\kappa, \lambda, \sigma, \tau, \pi$ ranges over regular ordinal terms.

Definition 4.6 $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$ holds if

$$\mathsf{k}(\Gamma) \cup \{a\} \subset \mathcal{H}_\gamma[\Theta] \tag{5}$$

and one of the following cases holds:

($\bigvee$) $A \simeq \bigvee\{A_\iota : \iota \in J\}$, $A \in \Gamma$ and there exist $\iota \in J$ and $a(\iota) < a$ such that

$$|\iota| < a \tag{6}$$

and $(\mathcal{H}_\gamma, \Theta) \vdash^{a(\iota)}_b \Gamma, A_\iota$.

($\bigwedge$) $A \simeq \bigwedge\{A_\iota : \iota \in J\}$, $A \in \Gamma$ and for every $\iota \in J$ there exists an $a(\iota) < a$ such that $(\mathcal{H}_\gamma, \Theta \cup \{\mathsf{k}(\iota)\}) \vdash^{a(\iota)}_b \Gamma, A_\iota$.

(*cut*) There exist $a_0 < a$ and C such that $\mathrm{rk}(C) < b$ and $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_b \Gamma, \neg C$ and $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_b C, \Gamma$.

($\Omega \in M_2$) There exist ordinals a_ℓ, $a_r(\alpha)$ and a sentence $C \in \Pi_2(\Omega)$ such that $\sup\{a_\ell + 1, a_r(\alpha) + 1 : \alpha < \Omega\} \leq a$, $b \geq \Omega$, $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_b \Gamma, C$ and $(\mathcal{H}_\gamma, \Theta \cup \{\omega\alpha\}) \vdash^{a_r(\alpha)}_b \neg C^{(\alpha,\Omega)}, \Gamma$ for any $\alpha < \Omega$.

($\mathrm{rfl}(\pi, k, \vec{\xi}, \vec{\nu})$) There exist a Mahlo ordinal $\mathbb{K} \geq \pi \in \mathcal{H}_\gamma[\Theta] \cap (b+1)$, an integer $2 \leq k \leq N$ and sequences $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1}), \vec{\xi} = (\xi_2, \ldots, \xi_{N-1}) \in SD$ of ordinals $\nu_i, \xi_i \in E$, ordinals $a_\ell, a_r(\rho), a_0$, and a finite set Δ of $\Sigma_k(\pi)$-sentences enjoying the following conditions: When $\pi = \mathbb{K}$, $k = N$ and $\vec{\nu} = \vec{0}$ with $lh(\vec{\nu}) = N - 1$ hold. Also let $\vec{\xi} = \vec{0}$ in this case. When $\pi < \mathbb{K}$, $\xi_k \neq 0$ with $k < N$, $\vec{0} \neq \vec{\xi}$, and $\forall i(\xi_i \leq_{sp} m_i(\pi))$.

1. When $\pi < \mathbb{K}$, cf. Definitions 2.1.9,

$$\forall i < k(\nu_i = \xi_i)\,\&\,(\nu_k, \ldots, \nu_{N-1}) <_{sd} \xi_k\,\&\,K(\vec{\nu}) \cup K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta] \tag{7}$$

and

$$\forall \mu \in \vec{\nu} \cup \vec{\xi} \cup \vec{m}(\pi)(K(\mu) \subset \mathcal{H}_{\max K(\mu)}[\Theta]) \tag{8}$$

cf. (4).

2. For each $\delta \in \Delta$, $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_{b} \Gamma, \neg\delta$.
3. $H(\vec{\nu}, \pi, \gamma, \Theta)$ denotes the *resolvent class* for $\vec{\nu}$, π, γ and Θ defined as follows:

$$\begin{aligned} C(\pi, \gamma, \Theta) &:= \{\rho < \pi : \mathcal{H}_\gamma(\rho) \cap \pi \subset \rho \,\&\, \Theta \cap \pi \subset \rho\} \quad (9) \\ \rho \in H(\vec{\nu}, \pi, \gamma, \Theta) &:\Leftrightarrow \forall i(\nu_i \leq_{sp} m_i(\rho) \land K(m_i(\rho)) \subset \mathcal{H}_{\max K(m_i(\rho))}(\rho)) \end{aligned}$$

   for $\rho \in Reg \cap C(\pi, \gamma, \Theta)$.

   Then for each $\rho \in H(\vec{\nu}, \pi, \gamma, \Theta)$, $(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash^{a_r(\rho)}_{b} \Gamma, \Delta^{(\rho,\pi)}$.
4. $$\sup\{a_\ell, a_r(\rho) : \rho \in H(\vec{\nu}, \pi, \gamma, \Theta)\} \leq a_0 \in \mathcal{H}_\gamma[\Theta] \cap a \quad (10)$$

In the inference rule $(\mathrm{rfl}(\pi, k, \vec{\xi}, \vec{\nu}))$ for $\pi = \psi^{\vec{\xi}}_{\sigma}(c) < \mathbb{K}$, we have $\pi \in Mh^c_2(\vec{\xi})$. In particular, $\pi \in \bigcap_{i<k} Mh^c_i(\xi_i) \cap Mh^c_k(\xi_k)$. Also we are assuming $(\nu_k, \ldots, \nu_{N-1}) <_{sd} \xi_k$, a fortiori $(\nu_k, \ldots, \nu_{N-1}) < \xi_k$. Since $\pi \in \bigcap_{i<k} Mh^c_i(\nu_i)$ is a Π_k-sentence holding on L_π, we obtain $\pi \in M_k(Mh^c_2(\vec{\nu}))$. Thus the reflection rule $(\mathrm{rfl}(\pi, k, \vec{\nu}))$ says that π is Π_k-reflecting on the class $H(\vec{\nu}, \pi, \gamma, \gamma_0, \Theta)$ for the club subset $C(\pi, \gamma, \Theta)$ of π, cf. Proposition 2.13. On the other side we see $\rho \in Mh^a_2(\vec{\nu})$ from Proposition 2.9 if $\forall i(\nu_i \leq m_i(\rho))$ for $\rho \in Mh^a_2(\vec{m}(\rho))$.

We will state some lemmas for the operator controlled derivations. These can be shown as in [6]. In what follows by an operator $\mathcal{H}$ we mean an $\mathcal{H}_\gamma$ for an ordinal γ.

Lemma 4.7 *Let* $(\mathcal{H}_\gamma, \Theta) \vdash^a_b \Gamma$.

1. $(\mathcal{H}_{\gamma'}, \Theta \cup \Theta_0) \vdash^{a'}_{b'} \Gamma, \Delta$ *for any* $\gamma' \geq \gamma$, *any* Θ_0, *and any* $a' \geq a$, $b' \geq b$ *such that* $\mathsf{k}(\Delta) \cup \{a'\} \subset \mathcal{H}_{\gamma'}[\Theta \cup \Theta_0]$.
2. *Assume* $\Theta_1 \cup \{c\} = \Theta$, $c \in \mathcal{H}_\gamma[\Theta_1]$. *Then* $(\mathcal{H}_\gamma, \Theta_1) \vdash^a_b \Gamma$.

Lemma 4.8 (Tautology) $(\mathcal{H}, \mathsf{k}(\Gamma \cup \{A\})) \vdash^{2\mathrm{rk}(A)}_{0} \Gamma, \neg A, A$.

Lemma 4.9 (Inversion) *Let* $A \simeq \bigwedge (A_\iota)_{\iota \in J}$, *and* $(\mathcal{H}, \Theta) \vdash^a_b \Gamma$ *with* $A \in \Gamma$. *Then for any* $\iota \in J$, $(\mathcal{H}, \Theta \cup \mathsf{k}(\iota)) \vdash^a_b \Gamma, A_\iota$ *holds.*

Lemma 4.10 (Boundedness) *Suppose* $(\mathcal{H}, \Theta) \vdash^a_c \Gamma, C$ *for a* $C \in \Sigma_1(\lambda)$, and $a \leq b \in \mathcal{H} \cap \lambda$. *Then* $(\mathcal{H}, \Theta) \vdash^a_c \Gamma, C^{(b,\lambda)}$.

Lemma 4.11 (Persistency) *Suppose* $(\mathcal{H}, \Theta) \vdash^a_c \Gamma, C^{(b,\lambda)}$ *for a* $C \in \Sigma_1(\lambda)$ *and a* $b < \lambda \in \mathcal{H}[\Theta]$. *Then* $(\mathcal{H}, \Theta) \vdash^a_c \Gamma, C$.

Lemma 4.12 (Predicative Cut-elimination) *Suppose* $(\mathcal{H}, \Theta) \vdash^b_{c+\omega^a} \Gamma$, $a \in \mathcal{H}[\Theta]$ *and* $]c, c+\omega^a] \cap Reg = \emptyset$. *Then* $(\mathcal{H}, \Theta) \vdash^{\varphi ab}_{c} \Gamma$.

Lemma 4.13 (Embedding of Axioms)
For each axiom A *in* $\mathsf{KP\Pi}_N$, *there is an* $m < \omega$ *such that for any operator* $\mathcal{H} = \mathcal{H}_\gamma$, $(\mathcal{H}, \emptyset) \vdash^{\mathbb{K}\cdot 2}_{\mathbb{K}+m} A$ *holds.*

Proof. The axiom $\neg A, \exists z\, A^{(z)}$ for Π_N-reflection follows from $A, \neg A$ and $\exists z\, A^{(z)}, \neg A^{(\rho)}$ for regular ordinals $\rho < \mathbb{K}$ by an inference $(\mathrm{rfl}(\mathbb{K}, N, \vec{0}, \vec{0}))$. □

Lemma 4.14 (Embedding) *If* $\mathsf{KP}\Pi_N \vdash \Gamma$ *for sets* Γ *of sentences, there are* $m, k < \omega$ *such that for any operator* $\mathcal{H} = \mathcal{H}_\gamma$, $(\mathcal{H}, \emptyset) \vdash^{\mathbb{K}\cdot 2+k}_{\mathbb{K}+m} \Gamma$ *holds*

5 Lowering and eliminating higher Mahlo operations

In the section inferences $(\mathrm{rfl}(\mathbb{K}, N, \vec{0}, \vec{0}))$ for Π_N-reflecting ordinals $\mathbb{K}$ are eliminated from operator controlled derivations of Σ_1-sentences φ^{L_Ω} over Ω.

$\alpha \# \beta$ denotes the natural (commutative) sum of ordinal terms α, β.

Lemma 5.1 *For a Mahlo term* $\pi \in OT$, $\vec{\xi} \in SD$ *denotes a sequence with* $lh(\vec{\xi}) = N-2$, *and* $2 \leq k \leq N-1$ *an integer for which the following hold: When* $\pi = \mathbb{K}$, *let* $\vec{\xi} = \vec{0}$ *and* $k = N-1$. *Otherwise* $\vec{\xi} = (\xi_2, \ldots, \xi_{k+1}) * \vec{0}$ *with* $\xi_{k+1} \neq 0$ *such that* $\forall i \leq k+1 (\xi_i \leq_{sp} m_i(\pi))$.

For ordinal terms $\gamma, a \in OT$ *let us define a sequence* $\vec{\zeta}(a) := (\zeta_2(a), \ldots, \zeta_k(a)) * \vec{0}$ *with* $lh(\vec{\zeta}(a)) = N-2$ *as follows.* $\vec{\zeta}(a) = \vec{0} * (\gamma + a)$ *when* $\pi = \mathbb{K}$. *Otherwise* $\zeta_k(a) = \xi_k + \Lambda^{\xi_{k+1}}(\gamma + a)$ *and* $\zeta_i(a) = \xi_i$ *for* $i < k$.

Let $\kappa \in H(\vec{\zeta}(a), \pi, \gamma, \Theta)$ *for a finite set* $\Theta \subset OT$.

Now suppose $(\mathcal{H}_\gamma, \Theta) \vdash^a_\pi \Gamma$ *where* $\{\gamma, \pi\} \cup K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta]$, $\Theta \subset \pi$, $\forall i (K(\xi_i) \subset \mathcal{H}_{\max K(\xi_i)}[\Theta])$, *and* $\Gamma \subset \Pi_{k+1}(\pi)$.

Let $\gamma(a, b) = \gamma \# a \# b$, $\beta(a, b) = \psi_\pi(\gamma(a, b))$, *and* $c > \gamma(a, \kappa)$. *Then the following holds:*

$$(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a,\kappa)}_\kappa \Gamma^{(\kappa,\pi)} \tag{11}$$

Proof by induction on a. Let $\kappa \in H(\vec{\zeta}(a), \pi, \gamma, \Theta)$. We see $\vec{\zeta}(a) \in SD$, and from (5) and $\Theta \subset \kappa$ that

$$\mathsf{k}(\Gamma) \cap \pi \subset \mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa \tag{12}$$

For any $a \in \mathcal{H}_\gamma[\Theta]$, we obtain $\{\gamma, \pi, a, \kappa\} \subset \mathcal{H}_\gamma(\pi)$ by $\Theta \cup \{\kappa\} \subset \pi$. Hence for $\gamma(a, \kappa) = \gamma \# a \# \kappa$, $\{\gamma(a, \kappa), \pi\} \subset \mathcal{H}_\gamma(\pi)$, and $\{\gamma(a, \kappa), \pi\} \subset \mathcal{H}_{\gamma(a,\kappa)}(\beta(a, \kappa))$ by the definition (3). Therefore $\kappa \in \mathcal{H}_{\gamma(a,\kappa)}(\beta(a, \kappa)) \cap \pi \subset \beta(a, \kappa)$ by Proposition 2.6, and $\Theta \subset \beta(a, \kappa) < \pi$. Thus we obtain

$$\{a_0, a_1\} \subset \mathcal{H}_\gamma[\Theta \cup \Theta_0] \,\&\, a_0 < a_1 \,\&\, \Theta_0 \subset \kappa \Rightarrow \beta(a_0, \kappa) < \beta(a_1, \kappa).$$

Case 1. First consider the case when the last inference is a $(\mathrm{rfl}(\pi, k+1, \vec{\xi}, \vec{\nu}))$.

We have $a_\ell \in \mathcal{H}_\gamma[\Theta] \cap a$, $a_r(\rho) \in \mathcal{H}_\gamma[\Theta \cup \{\rho\}] \cap a$, and a finite set Δ of $\Sigma_{k+1}(\pi)$-sentences. We have for each $\delta \in \Delta$

$$(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_\pi \Gamma, \neg\delta \tag{13}$$

and for each $\rho \in H(\vec{\nu}, \pi, \gamma, \Theta)$

$$(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash_\pi^{a_r(\rho)} \Gamma, \Delta^{(\rho,\pi)} \tag{14}$$

When $\pi < \mathbb{K}$, $\vec{\nu} = (\nu_2, \ldots, \nu_{N-1}) \in SD$ is a sequence such that $\forall i < k+1(\nu_i = \xi_i)$, $(\nu_{k+1}, \ldots, \nu_{N-1}) <_{sd} \xi_{k+1}$, $K(\vec{\nu}) \cup K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta]$, and $\forall i(K(\nu_i) \subset \mathcal{H}_{\max K(\nu_i)}[\Theta])$, cf. (7) and (8).

Let $\Gamma_0 = \Gamma \cap \Sigma_k(\pi)$ and $\{\forall x \in L_\pi\, \theta_i(x) : i = 1, \ldots, n\}\, (n \geq 0) = \Gamma \setminus \Gamma_0$ for $\Sigma_k(\pi)$-formulas $\theta_i(x)$. Let us fix $\vec{d} = \{d_1, \ldots, d_n\} \subset Tm(\kappa)$ arbitrarily. Put $\mathsf{k}(\vec{d}) = \bigcup\{\mathsf{k}(d_i) : i = 1, \ldots, n\}$ and $\Gamma(\vec{d}) = \Gamma_0 \cup \{\theta_i(d_i) : i = 1, \ldots, n\}$.

By Inversion lemma 4.9 from (13) we obtain for each $\delta \in \Delta$

$$(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\vec{d})) \vdash_\pi^{a_\ell} \Gamma(\vec{d}), \neg\delta \tag{15}$$

Let $\rho \in C(\kappa, c, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$. We see $\rho < \kappa$, and $\mathsf{k}(\vec{d}) < \rho$ from $\mathsf{k}(\vec{d}) < \kappa$. By $\Theta \cap \pi \subset \mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa$ and $\gamma \leq c$ we obtain $C(\kappa, c, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \subset C(\pi, \gamma, \Theta)$. Namely, cf. (9)

$$\rho \in H(\vec{\nu}, \kappa, c, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \Rightarrow \rho \in H(\vec{\nu}, \pi, \gamma, \Theta) \tag{16}$$

For each $\rho \in H(\vec{\nu}, \kappa, c, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$, IH with (14) and (16) yields for $c > \gamma(a_r(\rho), \kappa)$ and $\kappa \in H(\vec{\zeta}(a_r(\rho)), \pi, \gamma, \Theta \cup \{\rho\})$

$$(\mathcal{H}_c, \Theta \cup \{\rho, \kappa\}) \vdash_\kappa^{\beta(a_r(\rho),\kappa)} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\pi)} \tag{17}$$

Let $\rho \in M_\ell := \{\rho \in Reg : \forall i(\zeta_i(a_\ell) \leq_{sp} m_i(\rho))\} \cap H(\vec{\nu}, \kappa, c, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$. Then $M_\ell \subset H(\vec{\zeta}(a_\ell), \pi, \gamma, \Theta \cup \mathsf{k}(\vec{d}))$ and $\Theta \cup \mathsf{k}(\vec{d}) \subset \rho$. For each $\delta \in \Delta$, IH with (15) yields for $c > \gamma(a_\ell, \rho)$

$$(\mathcal{H}_c, \Theta \cup \mathsf{k}(\vec{d}) \cup \{\rho\}) \vdash_\rho^{\beta(a_\ell,\rho)} \Gamma(\vec{d})^{(\rho,\pi)}, \neg\delta^{(\rho,\pi)} \tag{18}$$

From (17) and (18) by several (cut)'s of $\delta^{(\rho,\pi)}$ with $\mathrm{rk}(\delta^{(\rho,\pi)}) < \kappa$ we obtain for $a(\rho) = \max\{a_\ell, a_r(\rho)\}$ and some $p < \omega$

$$\{(\mathcal{H}_c, \Theta \cup \mathsf{k}(\vec{d}) \cup \{\kappa, \rho\}) \vdash_\kappa^{\beta(a(\rho),\kappa)+p} \Gamma(\vec{d})^{(\rho,\pi)}, \Gamma^{(\kappa,\pi)} : \rho \in M_\ell\} \tag{19}$$

On the other hand we have by Tautology lemma 4.8 for each $\theta(\vec{d})^{(\kappa,\pi)} \in \Gamma(\vec{d})^{(\kappa,\pi)}$

$$(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(\vec{d}) \cup \{\kappa\}) \vdash_0^{2\mathrm{rk}(\theta(\vec{d})^{(\kappa,\pi)})} \Gamma(\vec{d})^{(\kappa,\pi)}, \neg\theta(\vec{d})^{(\kappa,\pi)} \tag{20}$$

where $2\mathrm{rk}(\theta(\vec{d})^{(\kappa,\pi)}) \leq \kappa + p$ for some $p < \omega$.

Moreover we have $\sup\{2\mathrm{rk}(\theta(\vec{d})^{(\kappa,\pi)}), \beta(a(\rho), \kappa) + p : \rho \in M_\ell\} \leq \beta(a_0, \kappa) + p \in \mathcal{H}_\gamma[\Theta \cup \{\kappa\}]$, where $\sup\{a_\ell, a_r(\rho) : \rho \in H(\vec{\nu}, \pi, \gamma, \Theta)\} \leq a_0 < a$ by (10).

Now let $\vec{\mu} = (\mu_2, \ldots, \mu_{N-1}) = \max\{\vec{\zeta}(a_\ell), \vec{\nu}\}$ with $\mu_i = \max\{\zeta_i(a_\ell), \nu_i\}$. Since $\nu_i = \xi_i \leq_{pt} \zeta_i(a_\ell)$ for $i < k+1$, we obtain $\mu_i = \begin{cases} \zeta_i(a_\ell) & i \leq k \\ \nu_i & i > k \end{cases}$. We see that $M_\ell = H(\vec{\mu}, \kappa, c, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d}))$. Moreover we have $\forall i < k(\mu_i = \xi_i = \zeta_i(a))$ and $(\mu_k, \ldots, \mu_{N-1}) = (\zeta_k(a_\ell)) * (\nu_{k+1}, \ldots, \nu_{N-1}) <_{sd} \zeta_k(a)$. Also $\forall i(K(\zeta_i(a)) \subset \mathcal{H}_{\max K(\zeta_i(a))}[\Theta])$ and $\forall i(K(\mu_i) \subset \mathcal{H}_{\max K(\mu_i)}[\Theta])$. For $\neg\Gamma(\vec{d})^{(\kappa,\pi)} \subset \Pi_k(\kappa)$, by an inference rule $(\mathrm{rfl}(\kappa, k, \vec{\zeta}(a), \vec{\mu}))$ with its resolvent class M_ℓ, we conclude from (20) and (19) that $(\mathcal{H}_c, \Theta \cup \{\kappa\} \cup \mathsf{k}(\vec{d})) \vdash_\kappa^{\beta(a_0,\kappa)+p+1} \Gamma(\vec{d})^{(\kappa,\pi)}, \Gamma^{(\kappa,\pi)}$. Since $\vec{d} \subset Tm(\kappa)$ is arbitrary, several $(\bigwedge)$'s yield (11).

Case 2. Second consider the case when the last inference is a $(\mathrm{rfl}(\pi, j, \vec{\xi}, \vec{\nu}))$ for a $j < k+1$. We have $(\mathcal{H}_\gamma, \Theta) \vdash_\pi^{a_\ell} \Gamma, \neg\delta$ for each $\delta \in \Delta \subset \Sigma_j(\pi)$ with $a_\ell \in \mathcal{H}_\gamma[\Theta] \cap a$, and $(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash_\pi^{a_r(\rho)} \Gamma, \Delta^{(\rho,\pi)}$ for each $\rho \in H(\vec{\nu}, \pi, \gamma, \Theta)$ with $a_r(\rho) \in \mathcal{H}_\gamma[\Theta \cup \{\rho\}] \cap a$. $\vec{\nu} \in SD$ is a sequence such that $\forall i < j(\nu_i = \xi_i)$ and $(\nu_j, \ldots, \nu_{N-1}) <_{sd} \xi_j$.

We see that the resolvent class $H(\vec{\nu}, \kappa, c_1, \Theta \cup \{\kappa\})$ is a subclass of $H(\vec{\nu}, \pi, \gamma, \Theta)$. By IH we have $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash_\kappa^{\beta(a_\ell,\kappa)} \Gamma^{(\kappa,\pi)}, \neg\delta^{(\kappa,\pi)}$ for each $\delta \in \Delta$, and $(\mathcal{H}_c, \Theta \cup \{\kappa, \rho\}) \vdash_\kappa^{\beta(a_r(\rho),\kappa)} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\pi)}$ for each $\rho \in H(\vec{\nu}, \kappa, c, \Theta \cup \{\kappa\})$ with $\Delta^{(\rho,\pi)} = (\Delta^{(\kappa,\pi)})^{(\rho,\kappa)}$. We claim that $\forall i \leq j(\xi_j \leq_{sp} m_i(\kappa))$. Consider the case when $i = j = k$. Then we have $\xi_k \leq_{sp} m_k(\pi)$ and $\zeta_k(a) \leq_{sp} m_k(\kappa)$ with $\xi_k <_{pt} \zeta_k(a)$. We obtain $\xi_k \leq_{sp} m_k(\kappa)$. Hence by an inference rule $(\mathrm{rfl}(\kappa, j, \vec{\xi}(j), \vec{\nu}))$ for the sequence $\vec{\xi}(j) = (\xi_2, \ldots, \xi_j) * \vec{0} \in SD$, cf. Proposition 2.21.1, we obtain (11).

Case 3. Third consider the case when the last inference is a $(\mathrm{rfl}(\sigma, j, \vec{\mu}, \vec{\nu}))$ for a $\sigma < \pi$. We have $(\mathcal{H}_\gamma, \Theta) \vdash_\pi^{a_\ell} \Gamma, \neg\delta$ for each $\delta \in \Delta \subset \Sigma_j(\sigma)$, and $(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash_\pi^{a_r(\rho)} \Gamma, \Delta^{(\rho,\sigma)}$ for each $\rho \in H(\vec{\nu}, \sigma, \gamma, \Theta)$. We obtain $\sigma < \kappa$ by (12) for $\sigma \in \mathcal{H}_\gamma[\Theta]$. Hence $\Delta \subset \Sigma^1_0(\sigma) \subset \Sigma_0(\kappa)$ and $\delta^{(\kappa,\pi)} \equiv \delta$ for any $\delta \in \Delta$. Let $H(\vec{\nu}, \sigma, c, \Theta \cup \{\kappa\})$ be the resolvent class for σ, $\vec{\nu}$, c and $\Theta \cup \{\kappa\}$. Then $H(\vec{\nu}, \sigma, c, \Theta \cup \{\kappa\}) \subset H(\vec{\nu}, \sigma, \gamma, \Theta)$.

From IH we have $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash_\kappa^{\beta(a_\ell,\kappa)} \Gamma^{(\kappa,\pi)}, \neg\delta$ for each $\delta \in \Delta$, and $(\mathcal{H}_c, \Theta \cup \{\kappa, \rho\}) \vdash_\kappa^{\beta(a_r(\rho),\kappa)} \Gamma^{(\kappa,\pi)}, \Delta^{(\rho,\sigma)}$ for each $\rho \in H(\vec{\nu}, \sigma, c, \Theta \cup \{\kappa\})$. We obtain (11) by an inference rule $(\mathrm{rfl}(\sigma, j, \vec{\mu}, \vec{\nu}))$ with the resolvent class $H(\vec{\nu}, \sigma, c, \Theta \cup \{\kappa\})$.

Case 4. Fourth consider the case when the last inference $(\bigwedge)$ introduces a $\Pi_{k+1}(\pi)$-sentence $(\forall x \in L_\pi\, \theta(x)) \in \Gamma$. We have $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(d)) \vdash_\pi^{a(d)} \Gamma, \theta(d)$ for each $d \in Tm(\pi)$. For each $d \in Tm(\kappa)$, IH with $\mathsf{k}(d) < \kappa$ yields $(\mathcal{H}_c, \Theta \cup \{\kappa\} \cup \mathsf{k}(d)) \vdash_\kappa^{\beta(a(d),\kappa)} \Gamma^{(\kappa,\pi)}, \theta(d)^{(\kappa,\pi)}$. $(\bigwedge)$ yields (11) for $\forall x \in L_\kappa\, \theta(x)^{(\kappa,\pi)} \equiv (\forall x \in L_\pi\, \theta(x))^{(\kappa,\pi)} \in \Gamma^{(\kappa,\pi)}$.

Case 5. Fifth consider the case when the last inference $(\bigwedge)$ introduces a $\Sigma_0(\pi)$-sentence $(\forall x \in c\, \theta(x)) \in \Gamma$ for a $c \in Tm(\pi)$. We have $(\mathcal{H}_\gamma, \Theta \cup \mathsf{k}(d)) \vdash_\pi^{a(d)} \Gamma, \theta(d)$ for each $d \in Tm(|c|)$. Then we have $|d| < |c| < \kappa$ by (12). IH yields $(\mathcal{H}_c, \Theta \cup \{\kappa\} \cup \mathsf{k}(d) \vdash_\kappa^{\beta(a(d),\kappa)} \Gamma^{(\kappa,\pi)}, \theta(d)$, and we obtain (11) by an inference $(\bigwedge)$.

Case 6. Sixth consider the case when the last inference $(\bigvee)$ introduces a $\Sigma_k(\pi)$-sentence $(\exists x \in L_\pi\, \theta(x)) \in \Gamma$. We have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_\pi \Gamma, \theta(d)$ for a $d \in Tm(\pi)$. Without loss of generality we can assume that $\mathsf{k}(d) \subset \mathsf{k}(\theta(d))$. Then we see that $|d| < \kappa$ from (12), and $d \in Tm(\kappa)$. Also $|d| < \kappa < \beta(a, \kappa)$ for (6). IH yields with $(\exists x \in L_\pi\, \theta(x))^{(\kappa,\pi)} \equiv (\exists x \in L_\kappa\, \theta(x)^{(\kappa,\pi)}) \in \Gamma^{(\kappa,\pi)}$, $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a_0,\kappa)}_\kappa \Gamma^{(\kappa,\pi)}, \theta(d)^{(\kappa,\pi)}$, and we obtain (11) by an inference $(\bigvee)$.

Case 7. Seventh consider the case when the last inference is a (cut). We have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_\pi \Gamma, \neg C$ and $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_\pi C, \Gamma$ for $a_0 < a$ with $\mathrm{rk}(C) < \pi$. Then $C \in \Sigma_0(\pi)$ by Proposition 4.5.4. On the other side $\mathsf{k}(C) \subset \pi$ holds by Proposition 4.5.2. Then $\mathsf{k}(C) \subset \kappa$ by (12). Hence $C^{(\kappa,\pi)} \equiv C$ and $\mathrm{rk}(C^{(\kappa,\pi)}) < \kappa$ again by Proposition 4.5.2. IH yields $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a_0,\kappa)}_\kappa \Gamma^{(\kappa,\pi)}, \neg C^{(\kappa,\pi)}$ and $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a_0,\kappa)}_\kappa C^{(\kappa,\pi)}, \Gamma^{(\kappa,\pi)}$. Hence by a (cut) we obtain (11).

Case 8. Eighth consider the case when the last inference is an $(\Omega \in M_2)$. We have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_\pi \Gamma, C$ and $(\mathcal{H}_\gamma, \Theta \cup \{\omega\alpha\}) \vdash^{a_r(\alpha)}_\pi \neg C^{(\alpha,\Omega)}, \Gamma$ for each $\alpha < \Omega$ with $\sup\{a_\ell + 1, a_r(\alpha) + 1 : \alpha < \Omega\} \leq a$ and $C \in \Pi_2(\Omega)$.

We obtain $\omega\alpha < \kappa$ for $\alpha < \Omega$. IH with $C^{(\kappa,\pi)} \equiv C$ yields for each $\alpha < \Omega$, $(\mathcal{H}_c, \Theta \cup \{\kappa, \omega\alpha\}) \vdash^{\beta(a_r(\alpha),\kappa)}_\kappa \neg C^{(\alpha,\Omega)}, \Gamma^{(\kappa,\pi)}$, and $(\mathcal{H}_c, \Theta \cup \{\kappa\}) \vdash^{\beta(a_\ell,\kappa)}_\kappa \Gamma^{(\kappa,\pi)}, C$. An $(\Omega \in M_2)$ yields (11)

All other cases are seen easily from IH. □

Lemma 5.2 *Let $\lambda \leq \pi$ be a regular ordinal term such that $\forall i(K(m_i(\pi)) \subset \mathcal{H}_{\max K(m_i(\pi))}[\Theta])$, and $\Gamma \subset \Sigma_1(\lambda)$.*

Suppose for an ordinal term $a \in OT$

$$(\mathcal{H}_\gamma, \Theta) \vdash^a_\pi \Gamma$$

where $\{\gamma, \lambda, \pi\} \subset \mathcal{H}_\gamma[\Theta]$.

Assume

$$\forall \rho \in [\lambda, \pi] \forall d [\Theta \subset \psi_\rho(\gamma \# d)] \tag{21}$$

Let $\hat{a} = \gamma \# \omega^{\pi + a + 1}$ and $\beta = \psi_\lambda(\hat{a})$. Then the following holds

$$(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^\beta_\beta \Gamma \tag{22}$$

Proof by main induction on π with subsidiary induction on a. We can assume $a > 0$.

We see that $\Theta \subset \beta = \psi_\lambda(\hat{a})$ from (21). Hence

$$a_0 \in \mathcal{H}_\gamma[\Theta] \cap a \Rightarrow \psi_\lambda(\widehat{a_0}) < \psi_\lambda(\hat{a})$$

Let $\vec{\xi} \in SD$ be a sequence of ordinals and k a number for which the following hold: If $\pi = \mathbb{K}$, then let $\vec{\xi} = \vec{0}$ with $lh(\vec{\xi}) = N - 1$ and $k = N - 1$. Let $\pi < \mathbb{K}$. If $\vec{m}(\pi) \neq \vec{0}$, then $K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta]$, $\vec{\xi} \leq \vec{m}(\pi)$ and $k = \max\{k \leq N - 2 : \xi_{k+1} > 0\}$. Otherwise let $\vec{\xi} = \vec{0}$ and $k = 1$. By the assumption (21), and (5) we obtain

$$\forall \rho \in [\lambda, \pi] \forall b \in K(\vec{\xi}) \forall d [\mathsf{k}(\Gamma) \cup \{\gamma, \lambda, a, \pi, b\} \subset \mathcal{H}_\gamma(\psi_\rho(\gamma \# d))] \tag{23}$$

Case 1. First consider the case when $k \geq 2$.

Let $\vec{\xi} = \vec{m}(\pi)$, and $\vec{\zeta}(a) := (\zeta_2(a), \ldots, \zeta_k(a)) * \vec{0}$ be the sequence defined as in Lemma 5.1 from γ, a: $\vec{\zeta}(a) = \vec{0} * (\gamma + a)$ when $\pi = \mathbb{K}$, otherwise $\zeta_k(a) = \xi_k + \Lambda^{\xi_{k+1}}(\gamma + a)$ and $\zeta_i(a) = \xi_i$ for $i < k$. Also let $\gamma(a, b) = \gamma \# a \# b$ and $\beta(a, b) = \psi_\pi \gamma(a, b)$.

Let $\kappa := \psi_\pi^{\vec{\zeta}(a)}(\gamma(a, 0))$. By the assumption (21) we have $\Theta \subset \psi_\pi(\gamma \# a)$. On the other hand we have $\psi_\pi(\gamma \# a) = \psi_\pi(\gamma(a, 0)) \leq \kappa$, and $\Theta \subset \kappa$. $\pi \in \mathcal{H}_\gamma[\Theta]$ with $\Theta \subset \pi$ yields $K(\vec{\xi}) = K(\vec{m}(\pi)) \subset \mathcal{H}_\gamma[\Theta] \subset \mathcal{H}_{\gamma(a,0)}(\kappa)$. Hence $K(\vec{\xi}) \cup \{\pi, \gamma(a, 0)\} \subset \mathcal{H}_{\gamma(a,0)}(\kappa)$, and $\kappa \in OT$ by $\gamma(a, 0) = \gamma \# a > 0$ and Definition 3.1.2h such that $\kappa < \pi$ and $\mathcal{H}_\gamma(\kappa) \cap \pi \subset \kappa$. Moreover we have $\forall i (K(\zeta_i(a)) \subset \mathcal{H}_{\max K(\zeta_i(a))}[\Theta])$ by $\forall i (K(m_i(\pi)) \subset \mathcal{H}_{\max K(m_i(\pi))}[\Theta])$ and $\{\gamma, a\} \subset \mathcal{H}_\gamma[\Theta]$ with $\Theta \subset \kappa$. In other words, $\kappa \in H(\vec{\zeta}(a), \pi, \gamma, \Theta)$.

By Lemma 5.1 we obtain $(\mathcal{H}_{\gamma(a,\kappa)+1}, \Theta \cup \{\kappa\}) \vdash_\kappa^{\beta(a,\kappa)} \Gamma^{(\kappa,\pi)}$, and Lemma 4.7.2 with $\kappa \in \mathcal{H}_{\gamma(a,0)+1}[\Theta]$

$$(\mathcal{H}_{\gamma(a,\kappa)+1}, \Theta) \vdash_\kappa^{\beta(a,\kappa)} \Gamma^{(\kappa,\pi)} \tag{24}$$

If $\lambda = \pi$, then $\Gamma^{(\kappa,\pi)} \subset \Sigma_1(\kappa) \subset \Sigma_0(\lambda)$. We have $\Theta \subset \psi_\pi(\hat{a}) = \beta$, and $\kappa \in \mathcal{H}_{\hat{a}}(\beta)$. Hence $\{\gamma, \pi, a, \kappa\} \subset \mathcal{H}_{\hat{a}}(\beta)$, and $\gamma(a, \kappa) = \gamma \# a \# \kappa < \gamma \# \omega^{\pi+a+1} = \hat{a}$. Therefore $\kappa < \beta(a, \kappa) \leq \psi_\pi(\hat{a}) = \beta$. We obtain (22) by Persistency lemma 4.11.

Next consider the case when $\lambda < \pi$. Then $\lambda < \kappa$ and $\Gamma^{(\kappa,\pi)} = \Gamma$. We have for (21), $\forall d \forall \rho \in [\lambda, \kappa](\Theta \subset \psi_\rho(\gamma(a, \kappa) + 1 \# d))$. By MIH on (24) we obtain $(\mathcal{H}_{b_0+1}, \Theta) \vdash_{\beta_0}^{\beta_0} \Gamma$ for $\beta_0 = \psi_\lambda(b_0)$ with $b_0 = (\gamma(a, \kappa) + 1) \# \omega^{\kappa+\beta(a,\kappa)+1}$. We have $b_0 = \gamma \# a \# \kappa \# 1 \# \omega^{\beta(a,\kappa)+1} < \gamma \# \omega^{\pi+a+1} = \hat{a}$ by $\beta(a, \kappa) < \pi$. This yields $\psi_\lambda(b_0) = \beta_0 < \beta = \psi_\lambda(\hat{a})$ by $\Theta \subset \beta$ and $\{\gamma, \kappa, \pi, a\} \subset \mathcal{H}_{\hat{a}}(\beta)$. Hence (22) follows.

In what follows suppose $k = 1$.

Case 2. Consider the case when the last inference rule is a $(\mathrm{rfl}(\pi, 2, \vec{\xi}, \vec{\nu}))$.

We have an ordinal term $a_\ell \in \mathcal{H}_\gamma[\Theta] \cap a$, and a finite set Δ of $\Sigma_2(\pi)$-sentences for which $(\mathcal{H}_\gamma, \Theta) \vdash_\pi^{a_\ell} \Gamma, \neg\delta$ holds for each $\delta \in \Delta$. On the other hand we have sequences $\vec{\nu}, (\xi_2) * \vec{0} \in SD$ such that $\vec{\nu} <_{sd} \xi_2$ and $K(\vec{\nu}) \cup K(\vec{\xi}) \subset \mathcal{H}_\gamma[\Theta]$ by (7), and an ordinal term $a_r(\rho) \in \mathcal{H}_\gamma[\Theta \cup \{\rho\}] \cap a$ for which $(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash_\pi^{a_r(\rho)} \Gamma, \Delta^{(\rho,\pi)}$ holds for each $\rho \in H(\vec{\nu}, \pi, \gamma, \Theta)$, where $\xi_2 \leq_{sp} m_2(\pi)$.

Let $\rho := \psi_\pi^{\vec{\nu}}(\widehat{a_\ell} \# \pi)$ for $\widehat{a_\ell} = \gamma \# \omega^{\pi+a_\ell+1}$. By the assumption (21) we have $\Theta \subset \psi_\pi(\widehat{a_\ell}) \subset \rho$. $K(\vec{\nu}) \cup \{\pi, \gamma, a\} \subset \mathcal{H}_\gamma[\Theta]$ yields $K(\vec{\nu}) \cup \{\pi, \widehat{a_\ell}\} \subset \mathcal{H}_{\widehat{a_\ell} \# \pi}(\rho)$. Next consider the condition (4). We have $\forall i (K(\nu_i) \subset \mathcal{H}_{\max K(\nu_i)}[\Theta])$ by (8), and hence $\forall i (K(\nu_i) \subset \mathcal{H}_{\max K(\nu_i)}(\rho))$ by $\Theta \subset \rho$. Therefore $\rho \in OT$ by Definition 3.1.2i. Moreover $\rho \in C(\pi, \gamma, \Theta)$, i.e., $\mathcal{H}_\gamma(\rho) \cap \pi \subset \rho \,\&\, \Theta \cap \pi \subset \rho$. Hence $\rho \in H(\vec{\nu}, \pi, \gamma, \Theta)$.

By Inversion lemma 4.9 we obtain for each $\delta \equiv (\exists x \in L_\pi \delta_1(x)) \in \Delta$ and each $d \in Tm(\rho)$ with $|d| = \max(\{0\} \cup \mathsf{k}(d))$, $(\mathcal{H}_{\gamma \# |d|}, \Theta \cup \mathsf{k}(d)) \vdash_\pi^{a_\ell} \Gamma, \neg\delta_1(d)$.

We have $\{\pi, \gamma, |d|\} \subset \mathcal{H}_{\gamma \# |d|}(\pi)$ by $|d| < \rho < \pi$, and this yields $|d| \in \mathcal{H}_{\gamma \# |d|}(\psi_\pi(\gamma \# |d|)) \cap \pi \subset \psi_\pi(\gamma \# |d|)$. Hence $|d| < \psi_\pi(\gamma \# |d|)$, and $\forall e (\Theta \cup \mathsf{k}(d) \subset$

$\psi_\pi(\gamma\#|d|\#e))$, i.e., (21) holds for $\lambda=\pi$ and $\gamma\#|d|$. Let $\beta_d=\psi_\pi(\widehat{a_d})$ for $\widehat{a_d}=\gamma\#|d|\#\omega^{\pi+a_\ell+1}=\widehat{a_\ell}\#|d|$. SIH yields $(\mathcal{H}_{\widehat{a_d}+1},\Theta\cup\mathsf{k}(d))\vdash^{\beta_d}_{\beta_d}\Gamma,\neg\delta_1(d)$, which in turn Boundedness lemma 4.10 yields $(\mathcal{H}_{\widehat{a_\pi}+1},\Theta\cup\mathsf{k}(d))\vdash^{\beta_d}_{\beta_d}\Gamma,\neg\delta_1^{(\beta_d,\pi)}(d)$ for $\widehat{a_\pi}=\gamma\#\pi\#\omega^{\pi+a_\ell+1}=\widehat{a_\ell}\#\pi$. By persistency we obtain $(\mathcal{H}_{\widehat{a_\pi}+1},\Theta\cup\mathsf{k}(d))\vdash^{\beta_d}_{\rho}\Gamma,\neg\delta_1^{(\rho,\pi)}(d)$ for $\beta_d<\psi_\pi(\widehat{a_\pi})=\rho\in\mathcal{H}_\gamma[\Theta]$. Since $d\in Tm(\rho)$ is arbitrary, $(\bigwedge)$ yields

$$(\mathcal{H}_{\widehat{a_\pi}+1},\Theta)\vdash^{\rho}_{\rho}\Gamma,\neg\delta^{(\rho,\pi)} \tag{25}$$

Now pick the ρ-th branch from the right upper sequents

$$(\mathcal{H}_{\widehat{a_\pi}+1},\Theta\cup\{\rho\}\vdash^{a_r(\rho)}_{\pi}\Gamma,\Delta^{(\rho,\pi)}$$

By $\rho\in\mathcal{H}_{\widehat{a_\pi}+1}[\Theta]$ and Lemma 4.7.2 we obtain

$$(\mathcal{H}_{\widehat{a_\pi}+1},\Theta)\vdash^{a_r(\rho)}_{\pi}\Gamma,\Delta^{(\rho,\pi)} \tag{26}$$

Case 2.1. First consider the case $\lambda=\pi$. Then $\Delta^{(\rho,\pi)}\subset\Sigma_0(\lambda)$. Let $\beta_\rho=\psi_\pi(b_\rho)$ with $b_\rho=\widehat{a_\pi}\#1\#\omega^{\pi+a_r(\rho)+1}=\gamma\#\omega^{\pi+a_\ell+1}\#\omega^{\pi+a_r(\rho)+1}\#\pi\#1$. Then $\beta_\rho>\rho$ and $\forall d[\Theta\cup\{\rho\}\subset\psi_\pi(\widehat{a_\pi}+1\#d)]$. SIH yields for (26)

$$(\mathcal{H}_{b_\rho+1},\Theta)\vdash^{\beta_\rho}_{\beta_\rho}\Gamma,\Delta^{(\rho,\pi)} \tag{27}$$

Several (cut)'s with (27), (25) yield $(\mathcal{H}_{\hat{a}+1},\Theta)\vdash^{\beta_\rho+p}_{\beta_\rho}\Gamma$ for $\beta_\rho\geq\rho$, $\widehat{a_\pi}<b_\rho<\hat{a}$ and some $p<\omega$, where $\beta_\rho<\beta=\psi_\pi(\hat{a})$ by $b_\rho<\hat{a}$. (22) follows.

Case 2.2. Next consider the case when $\lambda<\pi$. Then $\lambda<\rho$ and $\Delta^{(\rho,\pi)}\subset\Sigma_1(\rho^+)$ with $\rho^+=\Omega_{\rho+1}$. SIH with (26) yields $(\mathcal{H}_{b_\rho+1},\Theta\cup\{\rho\})\vdash^{\beta_{\rho^+}}_{\beta_{\rho^+}}\Gamma,\Delta^{(\rho,\pi)}$ for $\beta_{\rho^+}=\psi_{\rho^+}(b_\rho)>\rho$, and by Lemma 4.7.2 we obtain

$$(\mathcal{H}_{b_\rho+1},\Theta)\vdash^{\beta_{\rho^+}}_{\beta_{\rho^+}}\Gamma,\Delta^{(\rho,\pi)} \tag{28}$$

Several (cut)'s with (25), (28) yield $(\mathcal{H}_{b_0+1},\Theta)\vdash^{\beta_{\rho^+}+p}_{\beta_{\rho^+}}\Gamma$ for $\beta_{\rho^+}>\rho$ and $b_0=\gamma\#(\omega^{\pi+a_\ell+1}\cdot 2)\#\omega^{\pi+a_r(\rho)+1}\#1\geq\max\{b_\ell,b_\rho\}$. Predicative cut-elimination lemma 4.12 yields for $\beta_1=\varphi(\beta_{\rho^+})(\beta_{\rho^+}+p)<\rho^+$

$$(\mathcal{H}_{b_0+1},\Theta)\vdash^{\beta_1}_{\rho}\Gamma \tag{29}$$

We obtain $\lambda<\rho\in\mathcal{H}_{b_0+1}[\Theta]$ by $\gamma<\widehat{a_\ell}<b_0$. MIH with (29) yields $(\mathcal{H}_{c+1},\Theta)\vdash^{\psi_\lambda c}_{\psi_\lambda c}\Gamma$ for $c=b_0\#1\#\omega^{\rho+\beta_1+1}$. We obtain $c=b_0\#\omega^{\rho+\beta_1+1}\#1=\gamma\#(\omega^{\pi+a_\ell+1}\cdot 2)\#\omega^{\pi+a_r(\rho)+1}\#\omega^{\rho+\beta_1+1}\#2<\gamma\#\omega^{\pi+a+1}=\hat{a}$ since $a_\ell,a_r(\rho)<a$ and $\rho,\beta_1<\rho^+<\pi$. Hence $\psi_\lambda c<\psi_\lambda(\hat{a})=\beta$, and (22) follows.

Case 3. Third consider the case when the last inference introduces a $\Sigma_1(\lambda)$-sentence $(\forall x\in c\,\theta(x))\in\Gamma$ for $c\in Tm(\lambda)$. We have $(\mathcal{H}_\gamma,\Theta\cup\mathsf{k}(d))\vdash^{a(d)}_{\pi}\Gamma,\theta(d)$

for each $d \in Tm(|c|)$. Then we see from (23) that $|d| < |c| \in \mathcal{H}_\gamma(\psi_\rho(\gamma\# e)) \cap \rho \subset \psi_\rho(\gamma\# e)$ for any $\rho \in [\lambda, \pi]$ and any e. Hence $|d| \in \psi_\rho(\gamma\# e)$. (21) is enjoyed for $\Theta \cup \mathsf{k}(d)$. SIH yields $(\mathcal{H}_{\hat{a}+1}, \Theta \cup \mathsf{k}(d)) \vdash^{\beta_d}_{\beta_d} \Gamma, \theta(d)$ for $\beta_d = \psi_\lambda(\widehat{a(d)})$. $(\bigwedge)$ yields (22) for $\beta = \psi_\lambda(\hat{a}) > \beta_d$.

Case 4. Fourth consider the case when the last inference introduces a $\Sigma_1(\lambda)$-sentence $(\exists x \in L_\lambda\, \theta(x)) \in \Gamma$. We have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_{\pi} \Gamma, \theta(d)$ for a $d \in Tm(\lambda)$. SIH yields $(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta_0}_{\beta_0} \Gamma, \theta(d)$ for $\beta = \psi_\lambda(\hat{a}) > \psi_\lambda(\widehat{a_0}) = \beta_0$. Without loss of generality we can assume that $\mathsf{k}(d) \subset \mathsf{k}(\theta(d))$. Then we see from (23) that $|d| \in \mathcal{H}_\gamma(\psi_\lambda(\gamma+1)) \cap \lambda \subset \psi_\lambda(\gamma+1) < \beta$. Thus is enjoyed in the following inference rule $(\bigvee)$. We obtain $(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta}_{\beta} \Gamma$ by a $(\bigvee)$, which enjoys (6).

Case 5. Fifth consider the case when the last inference is a $(\mathrm{rfl}(\tau, j, \vec{\mu}, \vec{\nu}))$ for a $\tau \in \mathcal{H}_\gamma[\Theta] \cap \pi$. We have an $a_\ell < a$ and a finite set Δ of $\Sigma_j(\tau)$-sentences such that $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_{\pi} \Gamma, \neg\delta$ for each $\delta \in \Delta$. On the other hand we have a sequence $\vec{\nu}$ and an ordinal term $a_r(\rho) < a$ for each $\rho \in H(\vec{\nu}, \tau, \gamma, \Theta)$ such that $(\mathcal{H}_\gamma, \Theta \cup \{\rho\}) \vdash^{a_r(\rho)}_{\pi} \Gamma, \Delta^{(\rho,\tau)}$. By (23), for any $\rho \in H(\vec{\nu}, \tau, \gamma, \Theta)$ we obtain

$$\forall e \forall \kappa[\max\{\tau+1, \lambda\} \leq \kappa \leq \pi \Rightarrow \rho < \tau \in \mathcal{H}_\gamma(\psi_\kappa(\gamma\# e)) \cap \kappa \subset \psi_\kappa(\gamma\# e)] \quad (30)$$

Case 5.1. First consider the case when $\tau < \lambda$. Then $\rho < \psi_\kappa(\gamma\# e)$ for any $\kappa \in [\lambda, \pi]$ and e. From SIH with (30) we obtain $(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta_\ell}_{\beta_\ell} \Gamma, \neg\delta$ for each $\delta \in \Delta$ with $\beta_\ell = \psi_\lambda(\widehat{a_\ell})$, and $(\mathcal{H}_{\hat{a}+1}, \Theta \cup \{\rho\}) \vdash^{\beta_r(\rho)}_{\beta_r(\rho)} \Gamma, \Delta^{(\rho,\tau)}$ for each $\rho \in H(\vec{\nu}, \tau, \gamma, \Theta)$ with $\beta_r(\rho) = \psi_\lambda(\widehat{a_r(\rho)})$. We see $\max\{\beta_\ell, \beta_r(\rho), \tau\} < \beta = \psi_\lambda(\hat{a})$, and an inference rule $(\mathrm{rfl}(\tau, j, \vec{\mu}, \vec{\nu}))$ yields $(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\beta}_{\beta} \Gamma$.

Case 5.2. Second consider the case when $\lambda \leq \tau$. Then $\Delta \cup \Delta^{(\rho,\tau)} \subset \Sigma_1(\tau^+)$, and $\rho < \psi_\kappa(\gamma\# e)$ for $\tau < \kappa \leq \pi$ and e by (30). SIH yields $(\mathcal{H}_{\widehat{a_\ell}+1}, \Theta) \vdash^{\beta_2}_{\beta_2} \Gamma, \neg\delta$ for each $\delta \in \Delta$, where $\beta_2 = \psi_{\tau^+}(\widehat{a_\ell})$. On the other side SIH yields $(\mathcal{H}_{\widehat{a_r(\rho)}+1}, \Theta \cup \{\rho\}) \vdash^{\beta_\rho}_{\beta_\rho} \Gamma, \Delta^{(\rho,\tau)}$ for each $\rho \in H(\vec{\nu}, \tau, \gamma, \Theta)$, where $\beta_\rho = \psi_{\tau^+}\left(\widehat{a_r(\rho)}\right)$. Predicative cut-elimination lemma 4.12 yields $(\mathcal{H}_{\widehat{a_\ell}+1}, \Theta) \vdash^{\delta_2}_{\tau} \Gamma, \neg\delta$ and $(\mathcal{H}_{\widehat{a_r(\rho)}+1}, \Theta \cup \{\rho\}) \vdash^{\delta_\rho}_{\tau} \Gamma, \Delta^{(\rho,\tau)}$ for $\delta_2 = \varphi(\beta_2)(\beta_2)$ and $\delta_\rho = \varphi(\beta_\rho)(\beta_\rho)$. From these with the inference rule $(\mathrm{rfl}(\tau, j, \vec{\mu}, \vec{\nu}))$ we obtain

$$(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\delta_0+1}_{\tau} \Gamma \quad (31)$$

where $\sup\{\delta_2, \delta_\rho : \rho \in H(\vec{\nu}, \tau, \widehat{a_0}+1, \Theta)\} \leq \delta_0 := \varphi(\beta_0)(\beta_0) \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$ with $\sup\{\beta_2, \beta_\rho : \rho \in H(\vec{\nu}, \tau, \gamma, \Theta)\} \leq \beta_0 := \psi_{\tau^+}(\widehat{a_0})$, and $\sup\{a_\ell, a_r(\rho) : \rho \in H(\vec{\nu}, \tau, \gamma, \Theta)\} \leq a_0 \in \mathcal{H}_\gamma[\Theta] \cap a$, cf. (10).

MIH with (31) yields $(\mathcal{H}_{\hat{a}+1}, \Theta) \vdash^{\delta}_{\delta} \Gamma$ for $\delta = \psi_\lambda((\widehat{a_0}+1)\#\omega^{\tau+\delta_0+2})$ and $(\widehat{a_0}+1)\#\omega^{\tau+\delta_0+2} < \hat{a}$. We have $\delta = \psi_\lambda(\widehat{a_0}\#1\#\omega^{\tau+\delta_0+2}) < \psi_\lambda(\hat{a}) = \beta$ by $\widehat{a_0} < \hat{a}$ and $\tau, \delta_0 < \tau^+ < \pi$ and $\tau \in \mathcal{H}_\gamma[\Theta]$. (22) follows.

Case 6. Sixth consider the case when the last inference is a (cut). For an $a_0 < a$ and a C with $\mathrm{rk}(C) < \pi$, we have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_\pi \Gamma, \neg C$ and $(\mathcal{H}_\gamma, \Theta) \vdash^{a_0}_\pi C, \Gamma$.

Case 6.1. First consider the case when $\mathrm{rk}(C) < \lambda$. Then $C \in \Sigma_0(\lambda)$. SIH yields the lemma.

Case 6.2. Second consider the case when $\lambda \leq \mathrm{rk}(C) < \pi$. Let $\rho^+ = (\mathrm{rk}(C))^+ = \min\{\kappa \in Reg : \mathrm{rk}(C) < \kappa\}$. Then $C \in \Sigma_0(\rho^+)$ and $\lambda \leq \rho \in \mathcal{H}_\gamma[\Theta] \cap \pi$. SIH yields $(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\beta_0}_{\beta_0} \Gamma, \neg C$ and $(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\beta_0}_{\beta_0} C, \Gamma$ for $\beta_0 = \psi_{\rho^+}(\widehat{a_0}) \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$. By a (cut) we obtain $(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\beta_1}_{\beta_1} \Gamma$ for $\beta_1 = \max\{\beta_0, \mathrm{rk}(C)\} + 1$ with $\rho < \beta_1 < \rho^+$. Predicative cut-elimination lemma 4.12 yields $(\mathcal{H}_{\widehat{a_0}+1}, \Theta) \vdash^{\delta_1}_\rho \Gamma$ for $\delta_1 = \varphi(\beta_1)(\beta_1)$, where $\widehat{a_0} \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$, and $\forall e \forall \tau \in [\lambda, \rho][\Theta \subset \psi_\tau(\widehat{a_0}\# e)]$ hold. Hence MIH with $\rho \in \mathcal{H}_{\widehat{a_0}+1}[\Theta]$ yields $(\mathcal{H}_{b+1}, \Theta) \vdash^{\psi_\lambda(b)}_{\psi_\lambda(b)} \Gamma$ for $b = \widehat{a_0}\#1\#\omega^{\rho+\delta_1+1}$. We see $b < \hat{a}$ and $\psi_\lambda(b) < \psi_\lambda(\hat{a}) = \beta$, and (22) follows.

Case 7. Seventh consider the case when the last inference is an $(\Omega \in M_2)$. We have $(\mathcal{H}_\gamma, \Theta) \vdash^{a_\ell}_\pi \Gamma, C$ for an $a_\ell < a$, and $(\mathcal{H}_\gamma, \Theta \cup \{\alpha\}) \vdash^{a_r(\alpha)}_\pi \neg C^{(\alpha,\Omega)}, \Gamma$ for an $a_r(\alpha) < a$ for each $\alpha < \Omega$, where $C \in \Pi_2(\Omega)$.

The case $\lambda > \Omega$ is seen as in **Case 5.1**. The case $\lambda = \Omega$ is seen as in **Case 5.2**. □

Let us conclude Theorem 1.1. Let $\Omega = \Omega_1$.

Proof of Theorem 1.1. Let $\mathsf{KP}\Pi_N \vdash \theta$. By Embedding lemma 4.14 pick an m so that $(\mathcal{H}_0, \emptyset) \vdash^{\mathbb{K}\cdot 2+m}_{\mathbb{K}+m} \theta$. Predicative cut-elimination lemma 4.12 yields $(\mathcal{H}_0, \emptyset) \vdash^{\omega_{m+1}(\mathbb{K}+1)}_{\mathbb{K}} \theta$ for $\omega_m(\mathbb{K}\cdot 2 + m) < \omega_{m+1}(\mathbb{K}+1)$. Lemma 5.2 yields $(\mathcal{H}_{a+1}, \emptyset) \vdash^\beta_\beta \theta$ for $a = \omega^{\mathbb{K}+\omega_{m+1}(\mathbb{K}+1)+1}$ and $\beta = \psi_\Omega(a)$. Predicative cut-elimination lemma 4.12 yields $(\mathcal{H}_{a+1}, \emptyset) \vdash^{\varphi(\beta)(\beta)}_0 \theta$. We obtain $\varphi(\beta)(\beta) < \alpha := \psi_\Omega(\omega_n(\mathbb{K}+1))$ for $n = m+3$, and hence $(\mathcal{H}_{\omega_n(\mathbb{K}+1)}, \emptyset) \vdash^\alpha_0 \theta$. Boundedness lemma 4.10 yields $(\mathcal{H}_{\omega_n(\mathbb{K}+1)}, \emptyset) \vdash^\alpha_0 \theta^{(\alpha,\Omega)}$. Since each inference rule other than reflection rules $(\mathrm{rfl}(\pi, k, \vec{\xi}, \vec{\nu}))$ and $(\Omega \in M_2)$ is sound, we see by induction up to $\alpha = \psi_\Omega(\omega_n(\mathbb{K}+1))$ that $L_\alpha \models \theta$.

This completes a proof of Theorem 1.1.

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