

Snapshot Disjointness in Temporal Graphs

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Abstract

In the study of temporal graphs, only paths respecting the flow of time are relevant. In this context, many concepts of walks disjointness have been proposed over the years, and the validity of Menger's Theorem, as well as the complexity of related problems, has been investigated. Menger's Theorem states that the maximum number of disjoint paths between two vertices is equal to the minimum number of vertices required to disconnect them. In this paper, we introduce and investigate a type of disjointness that is only time dependent. Two walks are said to be *snapshot disjoint* if they are not active in a same snapshot (also called timestep). The related paths and cut problems are then defined and proved to be $W[1]$ -hard and XP-time solvable when parameterized by the size of the solution. Additionally, in the light of the definition of Mengerian graphs given by Kempe, Kleinberg and Kumar in their seminal paper (STOC'2000), we define a *Mengerian graph for time* as a graph G for which there is no time labeling for its edges where Menger's Theorem does not hold in the context of snapshot disjointness. We then give a characterization of Mengerian graphs in terms of forbidden structures and provide a polynomial-time recognition algorithm. Finally, we also prove that, given a temporal graph (G, λ) and a pair of vertices $s, z \in V(G)$, deciding whether at most h multiedges can separate s from z is NP-complete, where one multiedge uv is the set of all edges with endpoints u and v .

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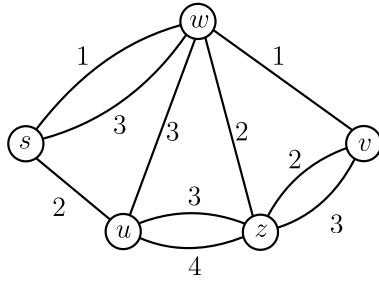
1 Introduction

A temporal graph can be described as a graph that varies in time. Such objects can be modeled in different ways, usually according to the application being considered, and have appeared in the literature under many names as for instance dynamic networks [19], temporal networks [9], time-varying graphs [4], etc. For surveys we refer the reader to [9, 14]. In this paper, we consider a *temporal graph* to be a pair (G, λ) , where G is a multigraph (henceforward called just graph) and λ is a function, called *timefunction*, that relates each edge to a discrete label telling when such edge is *active*. The value $\max_{e \in E(G)} \lambda(e)$ is called *lifetime* and is denoted by τ . Also, graph G is called the *base graph*. See Figure 1a for an example.

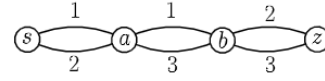
Many practical problems are modeled as temporal graphs (see [9] for a nice collection of practical examples), and among the most common ones are those related to temporal walks and connectivity. A *temporal walk* is a walk that respects the flow of time; for simplicity, we

¹ This work was partially developed during this author's stay as visiting professor at the University of Florence.





(a) Temporal graph with lifetime 4.



(b) Pair s, z whose cut must be bigger than the number of snapshot disjoint paths.

■ **Figure 1** Examples used throughout the text to clarify some concepts.

represent temporal walks as sequences of vertices and timesteps. For instance, in Figure 1a, the sequence $(w, 2, z, 2, v, 3, z, 4, u)$ is a temporal walk between w and u (also called temporal w, u -walk). A *temporal path* is then defined as a temporal walk whose vertices are not repeated. Hence, the previously mentioned temporal w, u -walk is not a path, while the walk $(w, 2, z, 4, u)$ is a temporal path. Additionally, some authors deal only with walks and paths whose edges are active in strictly increasing times; in such case, $(s, 1, w, 2, z, 3, v)$ is a valid temporal s, v -path, while $(s, 2, u, 3, z, 3, v)$ is not. To distinguish from these, we say that a walk/path is *strict* if the edges are active in strictly increasing times, and that it is *non-strict* if they are active in non-decreasing times.

In contrast with classic graph theory, when dealing with walks and paths in temporal graphs, sometimes the connectivity problems defined on walks differ from those defined on paths. See for instance [1, 5, 7, 10]. This is not the case for the problems investigated here, and this is why we interchangeably use the terms walks and paths.

Connectivity problems concern the *robustness* of a network, which translates into knowing how many independent (or disjoint) ways there are to go from one vertex to another, and how easy it is to break such connections. In this paper, we introduce a new robustness concept that relies on the time aspect of a network. To better understand these concepts, consider the following scenario. Suppose a temporal graph (G, λ) models a communication network. Such network might be prone to interruptions of all communications at a given timestep due to attacks, blackouts, maintenance, etc. A good measure of robustness of such networks could then be the minimum number of timesteps in which the communications must get interrupted in order to break all possible connections between a pair of vertices. A network with higher measure means that it is less susceptible to failing under such interruptions and hence is considered more robust. In Figure 1a, for instance, if there is an interruption on timesteps 2 and 3, then vertex s cannot relay a message to z anymore, while it still can relay messages to v through the path $(z, 1, w, 1, v)$.

To model such scenario, we say that two temporal s, z -paths P and Q are *snapshot disjoint* if, at any given timestep, at most one between P and Q is traversing any edges. For example, in Figure 1a, paths $(s, 1, w, 1, v, 2, z)$ and $(s, 3, w, 3, u, 3, z)$ are two snapshot disjoint temporal s, z -paths. We also say that a set S of timesteps is a *snapshot s, z -cut* if every temporal s, z -path uses an edge active in timestep i for some $i \in S$. For example, in Figure 1a, $S = \{2, 3\}$ is a snapshot s, z -cut. The following problems are then defined.

$\leq h$ -SNAPSHOT s, z -CUT

Input. A temporal graph (G, λ) , vertices $s, z \in V(G)$, and an integer h .

Question. Is there a snapshot s, z -cut in (G, λ) of size at most h ?

$\geq k$ -SNAPSHOT DISJOINT TEMPORAL s, z -PATHS

Input. A temporal graph (G, λ) , a pair of vertices $s, z \in V(G)$, and a positive integer k .

Question. Is there a set of snapshot disjoint temporal s, z -paths in (G, λ) of size at least k ?

We prove that, when parameterized by h and k respectively, both problems are $W[1]$ -hard, and that this is best possible, i.e., that they are also XP. While the XP algorithm for SNAPSHOT s, z -CUT follows easily from the definition and the fact that we can test all possible cuts in XP time (namely, $O(\tau^h)$ time), the algorithm for SNAPSHOT DISJOINT TEMPORAL s, z -PATHS is much more involved and uses a technique similar to the one applied to find k vertex disjoint paths between k given pairs of vertices (also known as the k -LINKAGE PROBLEM) in a DAG [17]. As we will see in the related works, this is the first result of this kind, with all previously defined disjointness either having the related paths problem polynomial-time solvable or para-NP-complete (i.e., NP-complete for fixed values of k).

A celebrated result in classic graph theory tells us that, in a graph G and for every pair $s, z \in V(G)$, the maximum number of internally vertex disjoint s, z -paths is equal to the minimum size of an s, z -cut (vertices whose removal breaks all s, z -paths). This is the well known Menger's Theorem, and it holds on both undirected and directed graphs, as well as for edge-disjoint paths and edge cuts. When translating these concepts to temporal graphs, it is natural to ask whether a version of Menger's Theorem holds. The answer in our context is no, as can be witnessed by the example in Figure 1b. Note that any two temporal s, z -paths intersect in some timestep, while there is no snapshot s, z -cut of size 1. Indeed, $(s, 1, a, 1, b, 2, z)$ does not use edges active in timestep 3, $(s, 2, a, 3, b, 3, z)$ does not use 1, and $(s, 1, a, 1, b, 3, z)$ does not use 2.

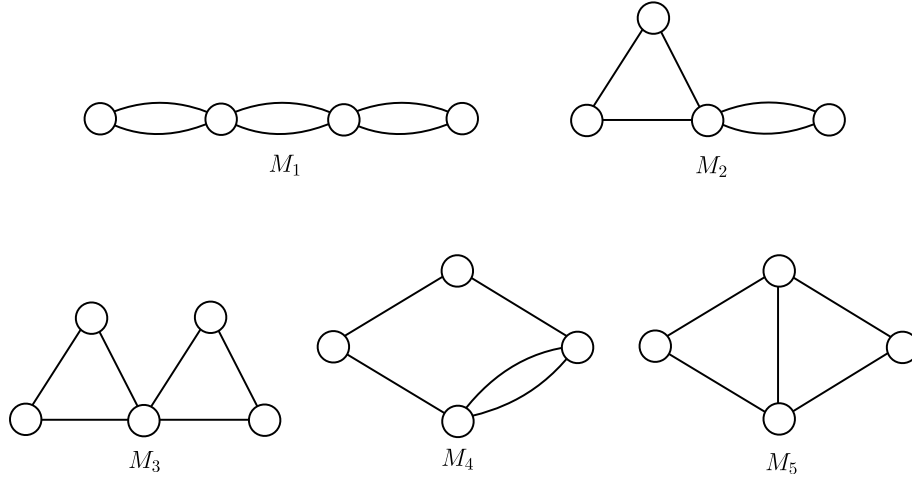
In their seminal paper, Kempe, Kleinberg and Kumar [13], in the context of vertex disjoint temporal paths, defined a Mengerian graph as being a graph where Menger's Theorem would hold for every choice of timefunction. They then characterize these graphs when constrained to simple graphs (every multiedge has multiplicity 1), and more recently their result was generalized to allow for multigraphs [11]. Here, we say that G is *Mengerian for time* if, for every timefunction λ and every $s, z \in V(G)$, the maximum number of snapshot disjoint temporal s, z -paths in (G, λ) is equal to the minimum size of a snapshot s, z -cut. In other words, the snapshot disjoint version of Menger's Theorem always holds on temporal graphs whose base graph is G . We then give the following characterization, which will be proved in Section 5. The formal definition of an m -topological minor is given in Section 2, but for now it suffices to say that it is a generalization of topological minors that preserves the multiplicity of the multiedges.

► **Theorem 1.** *Let G be a graph. Then G is Mengerian for time if and only if G does not have any of the graphs in Figure 2 as m -topological minor. Moreover, we can recognize whether G is Mengerian for time in polynomial time.*

Finally, in order to fill an open entry related to multiedge disjoint temporal paths (the definition is presented shortly), we prove in Section 6 that the related cut problem is NP-complete even if the temporal graph has lifetime equal to 2.

Related problems

As snapshot disjointness is a newly introduced concept, no previous results exist. We then refer the reader to the many results about Menger's related concepts in temporal graphs. In this context, the vertex disjoint version of Menger's Theorem was proved not to hold by Berman [3]. Since then a number of papers have investigated the complexity of



■ **Figure 2** Graphs in the set \mathcal{M} .

related problems [13, 20], as well as new structural concepts like the definition of Mengerian graphs [13, 11], and adaptations to temporal vertex disjoint versions [16, 10]. Because our problem is more closely related to edge connectivity, we refrain from commenting in detail the results on vertex connectivity, but refer the reader to [10] for an overview of such results. In what follows, we present the edge-related concepts and existing results. These are summarized in Table 1.

■ **Table 1** On the leftmost column, we specify the type of disjointness. Above, τ denotes the lifetime of the temporal graph, k denotes the number of paths, h denotes the size of the cut, NPc stands for NP-completeness, and W[1] or XP stands for W[1]-hardness or XP results when parameterizing by the size of the solution. Gray cells are proved in this paper.

	Non-strict		Strict	
	$\geq k$ -Walks	$\leq h$ -Cut	$\geq k$ -Walks	$\leq h$ -Cut
Multiedge	NPc [3], if G dir., even for $k = \tau = 2$	NPc for $\tau = 2$ (Theorem 13)	NPc for $\tau = 5$ [12] and $k = 2$ [15]	NPc for $\tau = 4$ [2] and W[1] for h [8]
Edge	Polynomial [3]		Polynomial [16]	
Snapshot	W[1] for k (Th. 6) XP for k (Th. 5)	W[1] for h (Th. 7) XP for h (Th. 4)	Open	
Node dep.	Open		Polynomial [16]	

A set of temporal s, z -walks are *edge disjoint* if they share no edges, and are *multiedge disjoint* if they share no multiedges, i.e, if no two walks in the set use consecutively a same pair of vertices. For example, in Figure 1a, the paths $(s, 1, w, 2, z)$ and $(s, 3, w, 3, u, 3, z)$ are edge disjoint, but are not multiedge disjoint, since they share the multiedge with endpoints sw . A set of (multi)edges is a *temporal (multi)edge s, z -cut* if they intersect every temporal s, z -walk. For example, in Figure 1a, sw and su form a multiedge s, z -cut, but if we want an edge s, z -cut, we have to pick both edges whose endpoints are sw . Some works define a temporal graph *with lifetime τ* as a pair (G, λ) where $\lambda : E(G) \rightarrow 2^{[\tau]}$. Observe that there is equivalence with our model.

In [3], Berman showed that the edge problems for non-strict temporal paths are polynomial-time solvable, and that deciding the existence of at least k multiedge disjoint temporal paths is NP-complete, G directed or undirected, and if G is directed, then the same holds even if $k = \tau = 2$. Up to our knowledge, no result concerning the cut problem related to multiedges is known. By a simple modification of a proof in [20], we present in Section 6 a proof of NP-completeness of the multiedge cut problem. Our proof also works for the case where G is

a directed multigraph. Concerning strict paths, the complexities of these problems follow directly from results about problems on bounded length paths [2, 8]. Additionally, the strict problems related to edge disjoint paths was shown to be polynomial-time solvable in [16]. We mention that, among all the problems appearing in Table 1, Menger’s Theorem holds only for edge disjoint paths in both the strict and non-strict contexts [3, 16], and node departure disjoint strict paths, defined below.

Another related concept is that of *node departure disjoint*, introduced in [16]. Given a temporal graph (G, λ) with lifetime τ , a set of strict temporal s, z -walks is *node departure disjoint* if no two of these paths leave a vertex in the same timestep. For example, in Figure 1a, $(s, 1, w, 2, z)$ and $(s, 3, w, 3, u, 3, z)$ are node departure disjoint. Additionally, a set $S \subseteq V(G) \times [\tau]$ is a *node departure s, z -cut* if each strict temporal s, z -walks contains an edge departing from u in time t , for some $(u, t) \in S$. For example, in Figure 1a, the set $S = \{(s, 1), (s, 2), (s, 3)\}$ is a node departure s, z -cut. In [16], the authors prove that the maximum number of node departure disjoint s, z -walks is equal to the minimum size of a node departure s, z -cut. Even though the authors do not comment on the complexity of the related problems, their proof leads to a polynomial time algorithm as it consists of building a flow network and proving that the searched values are equivalent to applying the famous Maxflow-Mincut Theorem. Up to our knowledge, their results have not been investigated for the non-strict context.

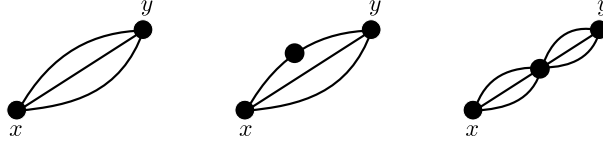
The text is organized as follows. In Section 2, we present definitions, terminology and some basic results. In Section 3, we present our XP algorithms. In Section 4, we prove that $\leq h$ -SNAPSHOT s, z -CUT and $\geq k$ -SNAPSHOT DISJOINT TEMPORAL s, z -PATHS are $W[1]$ -hard when parameterized by h and k , respectively. In Section 5, we characterize Mengerian graphs. Finally, in Section 6, we prove that $\leq h$ -Multiedge cut is NP-complete, and in Section 7 we present our concluding remarks.

2 Definitions and Terminology

Given positive integers $i, j \in \mathbb{N}$ such that $j \geq i$, we denote by $[i, j]$ the set $\{i, i + 1, \dots, j\}$ and by $[j]$ the set $\{1, \dots, j\}$.

A *graph* is a triple (V, E, f) where V and E are finite sets that we call *vertex set* and *edge set* respectively, and f is a function that, for each $e \in E$ associates a pair xy of elements in V , where $x \neq y$. We say that edge e is *incident* to x and y , that x, y are the *endpoints* of e , and that e *connects* x and y . We omit f in the rest of the paper and refer simply to the endpoints of e instead. We also call the pair xy a *multiedge*, and the number of edges with endpoints xy is the *multiplicity* of the multiedge xy . If the multiplicity of each edge is 1, we say that G is a *simple graph*. We denote by $U(G)$ the simple graph obtained from G by decreasing the multiplicity of all multiedges to 1. See [18] for further basic definitions of graph theory.

Given a graph G and a set of vertices $Z \subseteq V(G)$, the *identification of Z* is the graph obtained from $G - Z$ by adding a new vertex z and, for every edge e with endpoints $z'u$ where $z' \in Z$ and $u \notin Z$, add an edge e' with endpoints zu . The graph G' obtained from G by a *subdivision* of an edge e with endpoints uv is the graph having $V(G) \cup \{z_e\}$ as vertex set, and $E(G - e) \cup \{e', e''\}$ as edge set, where e' has endpoints uz_e and e'' has endpoints $z_e v$. Finally, the graph obtained from G by an *m-subdivision* of a multiedge xy is the graph obtained by subdividing all the edges with endpoints xy and then identifying the new vertices. Observe Figure 3 for an illustration of these definitions. The definition of m-subdivision has been introduced in [11]. Given a graph H , if G has a subgraph that can be obtained from m-subdivisions of H , then we say that H is an *m-topological minor* of G .



■ **Figure 3** From left to right: the multiedge xy , the subdivision of an edge with endpoints xy , and the m -subdivision of xy .

A *temporal graph* is a pair (G, λ) where G is a graph and $\lambda: E(G) \rightarrow \mathbb{N} \setminus \{0\}$. We refer to the elements of $\mathbb{N} \setminus \{0\}$ as *timesteps*. If an edge e is such that $\lambda(e) = \alpha$ we say that e is *active* or *appears* at timestep α . A *temporal x_1, x_q -walk* in (G, λ) is a sequence P that alternates vertices and edges $(x_0, e_1, x_1, \dots, e_q, x_q)$ such that for every $i \in [q]$, e_i is an edge between x_i and x_{i-1} and $\lambda(e_1) \leq \dots \leq \lambda(e_q)$. If $x_i \neq x_j$ for every $i, j \in [q]$ with $i \neq j$, we say that such temporal walk is a *temporal path*. Moreover, we define $V(P) = \{x_1, \dots, x_q\}$ and $E(P) = \{e_1, \dots, e_q\}$. For our purposes, we can assume that the subgraph active at a given timestep is simple, i.e., that if e and e' have both endpoints xy , then $\lambda(e) \neq \lambda(e')$. Such assumption allows us to define a path as a sequence of vertices and timesteps $(x_0, t_1, x_1, \dots, t_{q-1}, x_q)$ such that, for each $i \in [q]$, there is an edge connecting $x_i x_{i-1}$ active at timestep t_i . The *lifetime* of (G, λ) is denoted by $\tau(\lambda)$ and is the maximum integer such that there is an edge of G active at such timestep. For each timestep $i \in \mathbb{N} \setminus \{0\}$, the *i -th snapshot* of (G, λ) is the subgraph of G defined as $H = (V(G), \lambda^{-1}(i))$.

Let (G, λ) be a temporal graph with lifetime τ . Also, let $s, z \in V(G)$ be vertices in G and Q, J temporal s, z -paths. We say that Q and J are *snapshot disjoint* if $\lambda(E(Q)) \cap \lambda(E(J)) = \emptyset$. A subset $S \subseteq [\tau]$ is a *snapshot s, z -cut* if every temporal s, z -path uses an edge active at some timestep in S . We denote by $sp_{G, \lambda}(s, z)$ the maximum number of snapshot disjoint temporal s, z -paths and by $sc_{G, \lambda}(s, z)$ the minimum size of a snapshot s, z -cut. Observe that if the above definitions are made in terms of temporal paths, then these parameters would not change. A graph G is *Mengerian (for time)* if, for every timefunction λ on $E(G)$, and every $s, z \in V(G)$, $s \neq z$, we have that $sp_{G, \lambda}(s, z) = sc_{G, \lambda}(s, z)$. The following will be useful later.

► **Proposition 2.** *If G is non-Mengerian, then an m -subdivision of G is also non-Mengerian.*

Proof. Let G be a non-Mengerian graph and consider $\lambda \in E(G) \rightarrow \mathbb{N} \setminus \{0\}$ and $s, z \in V(G)$ to be such that $sp_{G, \lambda}(s, z) < sc_{G, \lambda}(s, z)$. Also, suppose that H is obtained from G by m -subdividing a multiedge, say xy . We construct a function λ' from λ that proves that H is also non-Mengerian.

Let $D \subseteq E(G)$ be the set of edges of G with endpoints xy , and denote by v_{xy} the vertex of H created by the m -subdivision of xy . Moreover, denote by D_x and D_y the sets of edges of H with endpoints xv_{xy} and $v_{xy}y$, respectively. Finally, define λ' to be such that $\lambda'(e) = \lambda(e)$, for every $e \in E(G) \setminus D$, and $\lambda'(D_x) = \lambda'(D_y) = \lambda(D)$. We show that $sp_{G, \lambda}(s, z) = sp_{H, \lambda'}(s, z)$ and $sc_{G, \lambda}(s, z) = sc_{H, \lambda'}(s, z)$, which finishes our proof.

Given a set of snapshot disjoint temporal s, t -paths in (G, λ) , if some of these paths, say P , uses the edge xy , then in (H, λ') we can substitute such edge by an edge in D_x and another in D_y active at the same time to obtain a temporal path P' such that $V(P') = V(P) \cup \{v_{xy}\}$. This gives us a set of snapshot disjoint temporal s, t -paths in (H, λ') . In the other direction, if it is given a set of snapshot disjoint temporal s, t -paths in (H, λ') , if some of them uses the vertex v_{xy} , let f_j be the edge used in D_j for $j \in \{x, y\}$. Suppose without loss of generality that $\lambda(f_x) \geq \lambda(f_y)$. Then we substitute both edges incident to v_{xy} by an edge in D appearing at time $\lambda(f_x)$. This implies that $sp_{G, \lambda}(s, z) = sp_{G, \lambda'}(s, z)$. To see that $sc_{G, \lambda}(s, z) = sc_{H, \lambda'}(s, z)$ one can need to recall that $\lambda'(D_x) = \lambda'(D_y) = \lambda(D)$. ◀

► **Proposition 3.** *G is Mengerian if and only if H is Mengerian, for every subgraph H of G .*

Proof. To prove necessity, suppose that $H \subseteq G$ is non-Mengerian, and let s, z, λ be such that $sp_{H,\lambda}(s, z) < sc_{H,\lambda}(s, z)$. Consider the timefunction λ' in $E(G)$ defined as follows.

$$\lambda'(e) = \begin{cases} \lambda(e) + 1 & , \text{ for every } e \in E(H), \\ 1 & , \text{ for every } e \in E(G) \setminus E(H) \text{ with endpoints } yt, \text{ and} \\ \max \lambda(E(H)) + 2 & , \text{ otherwise.} \end{cases}$$

Because $H \subseteq G$ and $\lambda \subseteq \lambda'$, note that we get $sp_{H,\lambda}(s, z) \leq sp_{G,\lambda'}(s, z)$ and $sc_{H,\lambda}(s, z) \leq sc_{G,\lambda'}(s, z)$. Therefore it suffices to prove $sp_{G,\lambda'}(s, z) \leq sp_{H,\lambda}(s, z)$ and $sc_{G,\lambda'}(s, z) \leq sc_{H,\lambda}(s, z)$. Indeed, these hold because the timefunction λ' does not allow for the existence of a temporal s, t -path not contained in H . ◀

3 Positive Results

In this section, we give XP algorithms for both SNAPSHOT s, z -CUT and SNAPSHOT DISJOINT TEMPORAL s, z -PATHS. Given the results of Section 4, unless $\text{FPT} = \text{W}[1]$ -hard, XP algorithms are best possible from the point of view of parameterized complexity. The first algorithm is quite simple and consists of the usual approach of testing all possible cuts.

► **Theorem 4.** *Given a temporal graph (G, λ) of lifetime τ , a positive integer h and $s, z \in V(G)$, we can solve $\leq h$ -SNAPSHOT s, z -CUT in $O(\tau^h \cdot (|V(G)| + |E(G)|))$.*

Proof. Let $(G, \lambda), s, z, h$ as in the hypothesis of the theorem. For each subset $S \subseteq [\tau]$ of size h , define G_S such that $V(G_S) = V(G)$ and $E(G_S) = \{e \in E(G) \mid \lambda(e) \notin S\}$. Define also $\lambda_S(e) = \lambda(e)$ for all $e \in E(G_S)$. Now, by the definition of G_S , any temporal s, z -path in (G_S, λ_S) is a temporal s, z -path in (G, λ) that does not use edges active at timesteps in S . Reciprocally, every temporal s, z -path in G that does not use edges active at timesteps in S is a temporal s, z -path in (G_S, λ_S) . As testing if there is a temporal s, z -path in (G_S, λ_S) can be done in polynomial time [19] and (G_S, λ_S) can be constructed in $O(n + m)$ time, it suffices to apply this test to (G_S, λ_S) for every $S \subseteq [\tau]$ of size h . Since there are at most τ^h such sets, the theorem follows. ◀

The next algorithm is much more involved, and uses a technique similar to the one used to find disjoint paths between given pairs of vertices in a DAG [17].

► **Theorem 5.** *Given a temporal graph (G, λ) , vertices $s, z \in V(G)$ and a positive integer k , we can solve $\geq k$ -SNAPSHOT DISJOINT TEMPORAL s, z -PATHS in time $O(m^k)$, where $m = |E(G)|$.*

Proof. We construct a digraph D with vertices s^* and z^* such that $|V(D)| = O(m^k)$ and there is an s^*, z^* -path in D if and only if there are k snapshot disjoint temporal s, z -paths in (G, λ) .

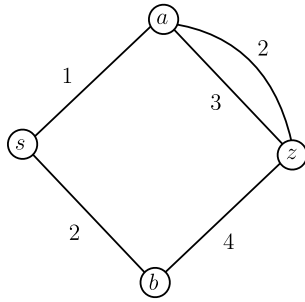
The vertex set of digraph D is equal to the k -tuples formed by edges of G , together with vertices s and z ; formally $V(D) \subseteq F^k$, where $F = E(G) \cup \{s, z\}$. Vertex s^* is set to be equal to (s, \dots, s) , while vertex z^* is set to be equal to (z, \dots, z) . Each dimension of $V(D)$ represents one of the desired k disjoint paths, and a set of snapshot disjoint temporal s, z -paths P_1, \dots, P_k will be represented by an s^*, z^* -path P in G , as previously said. So s^* represents the starting point, and z^* represents the finish point of every temporal s, z -path. Then, when an edge of D is traversed by P , we want that at least one of the k paths traverses an edge. Because we want to allow that only one of the paths gets closer to z with each step

1:8 Snapshot Disjointness in Temporal Graphs

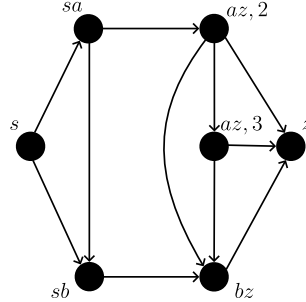
of P , there will be an edge from $\alpha \in V(D)$ to $\beta \in V(D)$ only if exactly one position of α and β differ. Not only this, but we want that, at each step of P , the path P_i that gets closer to z is the one whose last traversed edge occurs the earliest among all the P_i 's. In the next paragraph, we formally construct digraph D .

As previously said, let $F = E(G) \cup \{s, z\}$. Throughout the construction, we will be referring to Figures 4 and 5. Because we want to avoid simultaneous traversal of paths that intersect in a snapshot, we only consider elements of F^k whose each pair of coordinates are active in different snapshots. Indeed, if $k = 2$ and we allow for instance the existence of vertex (e, e') such that $t = \lambda(e) = \lambda(e')$, then this would mean that the constructed paths P_1 and P_2 intersect in timestep t . Therefore, we define V as formalized below. Observe that this implies, in Figure 5, that vertices $\{(e, e') \mid e \in E(G)\} \cup \{(az_2, sb), (sb, az_2)\}$ do not exist in $V(D)$, where az_2 denotes the edge with endpoints az active in timestep 2.

$$V = \{(u_1, \dots, u_k) \in F^k \mid \forall i, j \in [k] \text{ with } i \neq j, \text{ we have } \lambda(u_i) \neq \lambda(u_j) \text{ or } u_i = u_j \in \{s, z\}\}$$



(a) Example of temporal graph (G, λ) .



(b) Auxiliary graph M on the set $F = E(G) \cup \{s, z\}$.

■ **Figure 4** Example of construction in Theorem 5.

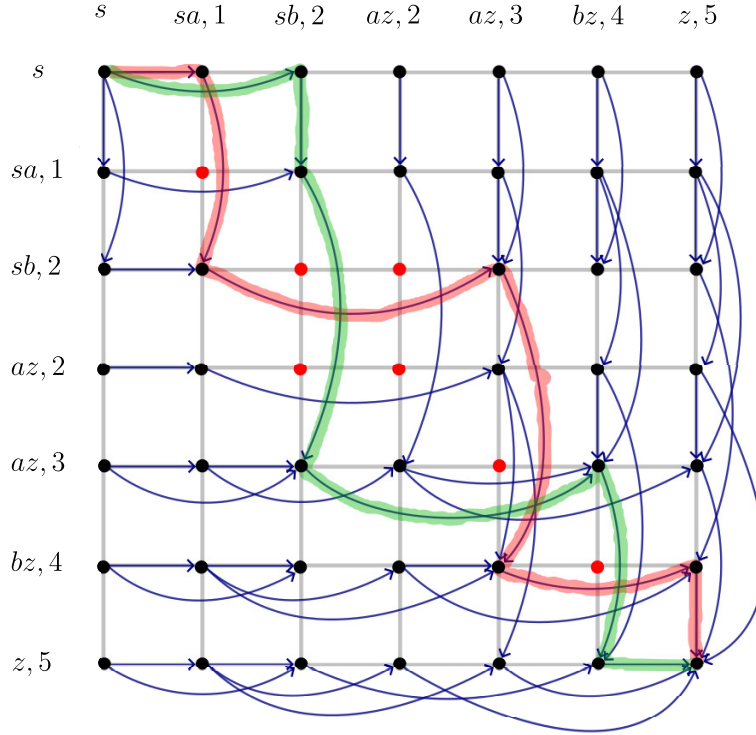
Now, we define the edge set of D . For this, we first construct an auxiliary graph M whose vertex set is equal to F ; observe Figure 4b to follow the construction. First of all, we want that a traversal of an edge in D translates into a valid traversal in (G, λ) . Therefore, for every pair $e, f \in F$, add to D an edge from e to f only if e can be followed by f in a temporal s, z -path in (G, λ) . Formally, add ef in the following cases:

- For every $e \in E(G)$, and every $f \in E(G)$ adjacent to e such that $\lambda(e) \leq \lambda(f)$;
- For $e = s$ and every $f \in E(G)$ incident to s ; and
- For every $e \in E(G)$ incident to z and $f = z$.

Finally, as previously said, we want that at each edge traversal of an s^*, z^* -path in D , the path in (G, λ) that is getting closer to z is that one whose last used edge is the earliest (one with the smallest value of λ) among all the other paths. To help with this, we also define $\lambda(s)$ to be equal to 0, and $\lambda(z)$ to be equal to $\tau + 1$, where τ is equal to the lifetime of (G, λ) . This means intuitively that we give always priority to leave s , and that, once we reach z in any dimension, then we cannot depart from z anymore. So, given a vertex $\alpha = (u_1, \dots, u_k) \in V(D)$, we add an edge from α to $\beta \in V(D)$ if and only if:

- β differ from α in exactly one position, i ;
- i is such that $\lambda(u_i) \leq \min_{j \in [k]} \lambda(u_j)$; and
- By letting u'_i be the value in the i -th position of β , we have that $u_i u'_i$ is a valid move, i.e., that $u_i u'_i \in E(M)$.

Observe Figure 5. Another way of seeing this construction is by starting with copies of M on each row and column of D , then removing the vertices that do not belong to D , and finally removing from row/column e any edge leaving f with $\lambda(f) > \lambda(e)$.



■ **Figure 5** Digraph D related to the temporal graph in Figure 4a; value $k = 2$ is being used, which means that $V(D) \subseteq F^2$. Each row and column is labeled with an element e of F , together with the value $\lambda(e)$; this will help in the construction. A vertex (e, f) of D is represented in the intersection of row e and column f . Red dots in the figure represent the fact that the related pair $(row, column)$ is not a vertex in D . The snapshot disjoint temporal s, z -paths $P_1 = (s, 1, a, 3, z)$ and $P_2 = (s, 2, b, 4, z)$ can be obtained either through the red or the green s^*, z^* -path.

Now, we prove that there are k snapshot disjoint temporal s, z -paths in (G, λ) if and only if there is an s^*, z^* -path in D . In what follows, given a vertex $\alpha \in V(D)$, we denote by (a_1, \dots, a_k) the tuple related to α . Recall that $s^* = (s, \dots, s)$ and $z^* = (z, \dots, z)$. Suppose P_1, \dots, P_k is a set of snapshot disjoint temporal s, z -paths in (G, λ) . For each $i \in [k]$, let $e_i^1, \dots, e_i^{p_i}$ be the sequence of edges used in P_i in order of traversal and define $e_i^0 = s$ and $e_i^{p_i+1} = z$. By induction, we define a sequence of vertices of D , $(s^* = \alpha^1, \dots, \alpha^q = \alpha)$, that forms an s^*, α -path for some α with the following property:

- (P) For each dimension $i \in [k]$ and each $\ell \in [q]$, the sequence of edges traversed in dimension i is a subpath of P_i . Formally, by removing s and repetitions of edges from the sequence (a_i^1, \dots, a_i^q) , we obtain a subsequence of $e_i^1, \dots, e_i^{p_i}$.

First, we define $\alpha_1 = s^*$; clearly property (P) holds as all paths start in s . Now suppose that sequence $\alpha_1, \dots, \alpha_q$ satisfying Property (P) is obtained, $q \geq 1$. Let $i \in [k]$ be such that $\lambda(a_i^q) = \min_{j \in [k]} \lambda(a_j^q)$. By Property (P), observe that either $a_i^q = s$, or a_i^q is an edge of P_i , or $a_i^q = z$. If the latter occurs, then we have that P_i is an s^*, z^* -path in D , since $\lambda(z) > \lambda(e)$ for every $e \in F \setminus \{z\}$, i.e., the only way $\lambda(z)$ is minimum is if all other positions are also equal to z . So suppose one of the other cases occurs. Note that it means that there is some edge

following a_i^q in P , say e_i^ℓ . By definition of temporal path, $\lambda(e_i^\ell) \geq \lambda(a_i^q)$; hence $a_i^q e_i^\ell \in E(M)$. Define α^{q+1} to be equal to α^q except that in position i we have e_i^ℓ instead of a_i^q , and note that $(\alpha^1, \dots, \alpha^{q+1})$ is a path in D that satisfies Property (P).

Now suppose the existence of an s^*, z^* -path in D , $(\alpha^1, \dots, \alpha^q)$. We construct a set of k snapshot disjoint temporal s, z -paths in (G, λ) . For this, for each $i \in [k]$, let P_i be a sequence of edges obtained from dimension i , i.e., from (a_i^1, \dots, a_i^q) by removing occurrences of s and z , and repetitions of edges. Because each transition respects M , we trivially get that P_i defines a temporal s, z -path in (G, λ) . It remains to show that such paths are snapshot disjoint. Suppose otherwise, and let i, j be such that there are edges e_i in P_i and e_j in P_j such that $\lambda(e_i) = \lambda(e_j) = \ell$. Let ℓ_i be the smallest index such that $a_i^{\ell_i} = e_i$, and ℓ_j be the smallest index such that $a_j^{\ell_j} = e_j$. By the definition of $V(D)$, we have that $\ell_i \neq \ell_j$. Indeed no vertex of D can contain two elements of F with same value of λ , and recall that $i \neq j$ as P_i, P_j are distinct paths. So, we can suppose, without loss of generality, that $\ell_i < \ell_j$. Observe that this means that α^{ℓ_i-1} differ from α^{ℓ_i} in exactly position i ; additionally, it means that $\lambda(e_i) \leq \min_{h \in [k]} \lambda(a_h^{\ell_i-1})$. In particular, we have that $\ell = \lambda(e_i) \leq \lambda(a_j^{\ell_i-1})$. But observe that, in a fixed dimension, the values of λ can only increase, i.e., since $\ell_i < \ell_j$, we get $\lambda(a_j^{\ell_i-1}) \leq \lambda(a_j^{\ell_j}) = \lambda(e_j) = \ell$. We get a contradiction as in this case vertex α^{ℓ_i-1} should not be defined as it contains two elements with the same value of λ , namely $a_j^{\ell_i-1}$ and e_i .

To finish the proof just recall that $|V(D)| \leq (m+2)^k$, where $m = |E(G)|$, and that deciding if there is a path between two vertices in D can be made in time $O(|V(D)|^2)$. So, deciding if there are k snapshot disjoint temporal s, z -paths in (G, λ) can be done in time $O(m^k)$. ◀

4 Negative Results

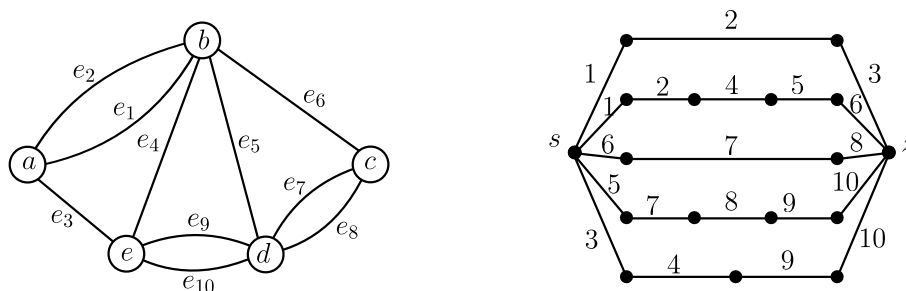
In this section, we prove that the algorithms presented in Section 3 are best possible, i.e., that $\geq k$ -SNAPSHOT DISJOINT TEMPORAL s, z -PATHS and $\leq h$ -SNAPSHOT s, z -CUT are $W[1]$ -hard when parameterized by k and h , respectively.

► **Theorem 6.** $\geq k$ -SNAPSHOT DISJOINT TEMPORAL s, z -PATHS is $W[1]$ -hard when parameterized by k .

Proof. We make a parameterized reduction from $\geq k$ -INDEPENDENT SET when parameterized by k . Such problem has as input a simple graph G and an integer k , and the question is whether G has an independent set of size at least k . This is known to be $W[1]$ -hard (see e.g. [6]).

Consider an instance G, k of $\geq k$ -INDEPENDENT SET and let $|V(G)| = n$. Observe Figure 6 to follow the construction. First, add to G' vertices s and z . Then, for each $u \in V(G)$, add to G' an s, z -path on $d(u)$ edges; denote such path by Q_u . Now, consider any ordering e_1, \dots, e_m of $E(G)$, and denote the edges incident to a vertex $u \in V(G)$ by $\delta(u)$. We can define $\lambda: E(G') \rightarrow \mathbb{N} \setminus \{0\}$ in a way that each Q_u is a temporal s, z -path using the orders of the edges in $\delta(u)$. Formally, for each $u \in V(G)$, let $\delta(u) = \{e_{i_1}, \dots, e_{i_q}\}$ with $i_1 < \dots < i_q$, and define $\lambda(E(Q_u))$ to be equal to $\{i_1, \dots, i_q\}$ in a way that Q_u is a temporal path.

We show that (G', λ) has k snapshot disjoint temporal s, z -paths if and only if G has an independent set of size at least k . By the definition of G' , all temporal s, z -paths are of type Q_u for some $u \in V(G)$. Therefore, it suffices to show that a subset $S \subseteq V(G)$ is an independent set of G if and only if $\{Q_u \mid u \in S\}$ is a set of snapshot disjoint temporal s, z -paths in (G', λ) . Suppose first that S is an independent set, and suppose by contradiction that $u_1, u_2 \in S$ are such that Q_{u_1} and Q_{u_2} are not snapshot disjoint. Then there exists



■ **Figure 6** To the left, graph G , and to the right, the constructed temporal graph (G', λ) . In G' , paths P_a, P_b, P_c, P_d, P_e are depicted from top to bottom, in this order.

$e_1 \in E(Q_1)$ and $e_2 \in E(Q_2)$ such that $\lambda(e_1) = \lambda(e_2)$. By construction, this means that $e_1 = e_2$, and since $u_1 \neq u_2$, we get that actually this edge has endpoints $u_1 u_2$, a contradiction as S is an independent set. Thus, $\{Q_u \mid u \in S\}$ is a set of snapshot disjoint temporal s, z -paths. Finally, observe that if $e_i = vw \in E(G)$, then $\lambda(Q_v) \cap \lambda(Q_w) = \{i\}$, which directly implies that if $\{Q_u \mid u \in S\}$ is a set of snapshot disjoint temporal s, z -paths, then S cannot contain any pair of adjacent vertices. ◀

Now, we prove the analogous result for the cut problem.

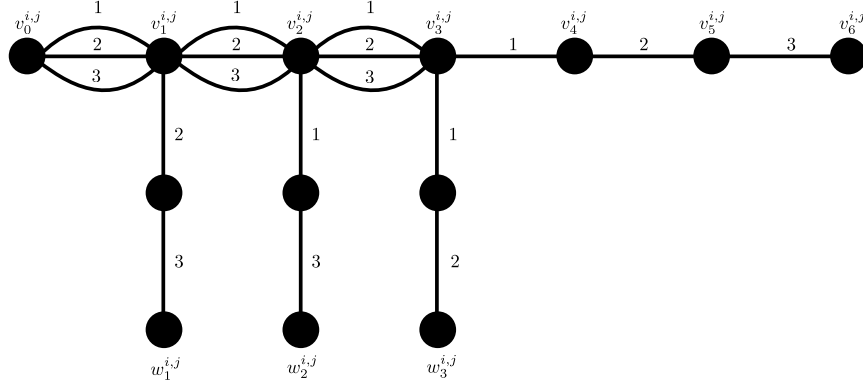
► **Theorem 7.** $\leq h$ -SNAPSHOT s, z -CUT is $W[1]$ -hard when parameterized by h .

Proof. We make a reduction from MULTICOLORED k -CLIQUE, when parameterized by k , known to be $W[1]$ -hard [6]. Such problem has as input a simple graph G , an integer k , and a partition of $V(G)$ into k independent sets (alternatively, a proper k -coloring), and the question is whether G has a (multicolored) clique of size k . So let G be a graph and $\{X_1, \dots, X_k\}$ be a proper k -coloring of G . By adding artificial vertices and edges if necessary, we can suppose that the number of edges between X_i and X_j is equal to a value m , for every pair $i, j \in [k]$. So, for $i, j \in [k], i \neq j$, denote the set of such edges by $E_{i,j} = \{e_1^{i,j}, \dots, e_m^{i,j}\}$. We make this assumption in order to make presentation simpler.

Now, for each $i, j \in [k], i \neq j$, we construct a gadget denoted by $F_{i,j}$. Observe Figure 7 to follow the construction. First add to $F_{i,j}$ the set of vertices $V_{i,j} = \{v_0^{i,j}, \dots, v_{2m}^{i,j}\}$, making the first $m + 1$ of them form a path of multiplicity m , and the latter $m + 1$ form a path of multiplicity 1. Formally, for each $\ell \in \{0, \dots, m - 1\}$, add m edges with endpoints $v_\ell^{i,j} v_{\ell+1}^{i,j}$. Also, for each $\ell \in \{m, \dots, 2m - 1\}$, add 1 edge with endpoints $v_\ell^{i,j} v_{\ell+1}^{i,j}$. Now, for each $\ell \in [m]$, add vertex $w_\ell^{i,j}$ and join such vertex with $v_\ell^{i,j}$ by a path with $m - 1$ edges and denote such path by $P_\ell^{i,j}$. We say that vertex $w_\ell^{i,j}$ of our gadget is associated with edge $e_\ell^{i,j}$ of $E_{i,j}$. The timefunction is defined only later.

Now, we finish the construction of our temporal graph. For this, take the union of all graphs $F_{i,j}$ and identify all vertices $v_0^{i,j}$, calling the obtained vertex s , and identify all vertices $v_{2m}^{i,j}$, calling the obtained vertex z . Also, for each $i, j \in [k], i \neq j$, and $\ell \in [m]$, we add two edges between $w_\ell^{i,j}$ and z . Denote by G' the obtained graph, and by W the set $\{w_\ell^{i,j} \mid i, j \in [k], i \neq j, \ell \in [m]\}$. Observe that G' contains $O(k^2 \cdot m)$ vertices and edges.

Now we define λ . The idea is that each $F_{i,j}$ will be active during its own dedicated time window. Formally, define $\Delta_{i,j} = [f_{i,j} + 1, f_{i,j} + m]$, for each pair $i, j \in [k], i \neq j$, in a way that $\Delta_{i,j} \cap \Delta_{i',j'} = \emptyset$ whenever $\{i, j\} \neq \{i', j'\}$. Now, consider $i, j \in [k]$ with $i \neq j$. For each $\ell \in \{0, \dots, m - 1\}$, we define λ in a way that every value in $\Delta_{i,j}$ appears in some edge with endpoints $v_\ell^{i,j} v_{\ell+1}^{i,j}$. Also, for each $\ell \in [m]$, we let $\lambda(v_{m-1+\ell}^{i,j} v_{m+\ell}^{i,j}) = \{f_i + \ell\}$, and we define



■ **Figure 7** A representation of $F_{i,j}$ with labels of λ where $m = 3$ and $\Delta_{i,j} = \{1, 2, 3\}$.

$\lambda(E(P_\ell^{i,j}))$ to be equal to $\Delta_{i,j} \setminus \{f_{i,j} + \ell\}$ and in a way that $P_\ell^{i,j}$ is a temporal $v_\ell^{i,j}, w_\ell^{i,j}$ -path. Finally, the only edges that remain unlabelled are the edges between z and vertices of type w . For such edges, we reserve a time window of size $n = |V(G)|$, that we denote by Δ_V , where any timestep in such set is greater than any timestep we used to define λ so far. Moreover, we associate each vertex $v \in V(G)$ with a timestep $t_v \in \Delta_V$. Let $w_\ell^{i,j} \in W$ and recall that such vertex is associated with $e_\ell^{i,j} \in E(G)$. Suppose $e_\ell^{i,j}$ have endpoints xy , and let the two edges of G' with endpoints $w_\ell^{i,j}z$ be active in timesteps $\{t_x, t_y\}$.

Now, we prove that G has a clique of size k if and only if (G', λ) has a snapshot s, z -cut of size at most $\binom{k}{2} + k$. Consider first a clique C of G of size k , and let $\{e_{\ell_1}^{i_1, j_1}, \dots, e_{\ell_a}^{i_a, j_a}\}$ be the set of edges of G between vertices of C . Notice that, because C has a vertex from each part, we get that $a = \binom{k}{2}$. Define $S = \{f_{i_b, j_b} + \ell_b \mid b \in \{1, \dots, a\}\} \cup \{t_v \mid v \in C\}$. We prove that S is a snapshot s, z -cut. By contradiction, suppose that P is a temporal s, z -path not passing by S , i.e., such that $\lambda(E(P)) \cap S = \emptyset$. Since $a = \binom{k}{2}$ and all edges incident to s are active in timesteps $\bigcup_{i,j \in [k], i \neq j} \Delta_{i,j}$, we can define $b \in [a]$ to be the index related to the first edge in P , i.e., P starts in an edge of F_{i_b, j_b} , say the one active in timestep $f_{i_b, j_b} + \ell_b$. Observe that the value $f_{i_b, j_b} + \ell_b$ is within the temporal s, z -path contained in F_{i_b, j_b} , and that it also separates s and $w_\ell^{i_b, j_b}$ for every $\ell \in [m] \setminus \{\ell_b\}$. Hence, P must start with the temporal $s, w_{\ell_b}^{i_b, j_b}$ -path contained in F_{i_b, j_b} . However, as $e_{\ell_b}^{i_b, j_b}$ is incident to vertices of the clique, say x and y , then we have that P uses timestep t_x or t_y , a contradiction as $\{t_x, t_y\} \subseteq S$.

Now, suppose that S is a minimum snapshot s, z -cut in (G', λ) and that it has size at most $\binom{k}{2} + k$. Let $V_S = \{x \in V(G) \mid t_x \in S\}$. We prove that V_S is a clique of G of size k . Denote by O the set of pairs $\{(i, j) \mid i, j \in [k], i < j\}$. We say that $(i, j) \in O$ is *open* if $\Delta_{i,j} \cap S = \{f_{i,j} + \ell\}$ for some $\ell \in [m]$, and we say that $e_\ell^{i,j}$ is the *open edge of* (i, j) . The following simple facts will be useful:

1. For every $i, j \in [k], i \neq j$, we have $\Delta_{i,j} \cap S \neq \emptyset$: this is due to the fact that there is a temporal s, z -path using only timesteps in $\Delta_{i,j}$;
2. If $\ell \in [m]$ is such that $\Delta_{i,j} \cap S = \{f_{i,j} + \ell\}$, then $\{x, y\} \subseteq V_S$, where xy are the endpoints of $e_\ell^{i,j}$: this is because there exists a temporal $s, w_\ell^{i,j}$ -path not using any timestep in S , and hence such path can be extended to a temporal s, z -path by using an edge with endpoints $w_\ell^{i,j}$ either in timestep t_x or in timestep t_y ;
3. For every $i, j \in [k], i \neq j$, we have $|\Delta_{i,j} \cap S| \leq 2$: it suffices to see that any two timesteps in $\Delta_{i,j}$ intersects all temporal paths between s and any vertex in $\{w_\ell^{i,j} \mid \ell \in [m]\} \cup \{z\}$;
4. If $x \in V_S$, then x is incident to some open edge: indeed, if x is not incident to any open edge, then $w_\ell^{i,j}$ is separated from s by $S \setminus \{t_x\}$ for every edge $e_\ell^{i,j}$ incident in x , and since timestep t_x contains only edges incident to some such $w_\ell^{i,j}$, it follows that $S \setminus \{t_x\}$ is also a snapshot s, z -cut, contradicting the minimality of S .

By Fact 3, if (i, j) is not open, then $\Delta_{i,j} \cap S = \{f_{i,j} + \ell_1, f_{i,j} + \ell_2\}$ for some pair of values $\ell_1, \ell_2 \in [m]$. In such case, we say that edges $e_{\ell_1}^{i,j}$ and $e_{\ell_2}^{i,j}$ are *chosen* for (i, j) . We show how to modify S in order to decrease the number of chosen edges.

▷ **Claim 8.** If (i, j) is not open, then we can suppose that $V_S \cap (X_i \cup X_j) = \emptyset$.

Proof of Claim 8. Let $e_{\ell_1}^{i,j}$ and $e_{\ell_2}^{i,j}$ be the chosen edges for (i, j) . Denote by $x_1 y_1$ and $x_2 y_2$ the endpoints of $e_{\ell_1}^{i,j}$ and $e_{\ell_2}^{i,j}$, respectively. Suppose without loss of generality that $\{x_1, x_2\} \subseteq X_i$ and $\{y_1, y_2\} \subseteq X_j$. Now suppose that there exists $x \in X_i \cap S$. If $x \in \{x_1, x_2\}$, say $x = x_1$, then replace $f_{i,j} + \ell_2$ in S by y_1 , obtaining S' . Observe that in this case $e_{\ell_1}^{i,j}$ becomes an open edge, but such that $\{t_{x_1}, t_{y_1}\} \subseteq S'$. Hence S' is still a snapshot s, z -cut containing fewer chosen edges. And if $x \notin \{x_1, x_2\}$, then let xy be any edge incident in x such that $y \in X_j$. We can suppose that such edge exists as otherwise x cannot be in any multicolored k -clique and then we can remove it from G . Let $\ell \in [m]$ be such that $e_{\ell}^{i,j}$ has endpoints xy . Observe that Fact 3 also tells us that $S' = (S \setminus \{f_{i,j} + \ell_2\}) \cup \{f_{i,j} + \ell\}$ is a snapshot s, z -cut. We can then apply the previous argument to replace $f_{i,j} + \ell_1$ by t_y to again obtain a snapshot s, z -cut with fewer chosen edges. ◀

▷ **Claim 9.** We can suppose that every pair is open.

Proof of Claim 9. We can clearly suppose that G is connected, as otherwise the answer to MULTICOLORED k -CLIQUE is trivially “no”. Now, let I_1 be the set of indices $\{i \in [k] \mid V_S \cap X_i = \emptyset\}$ and $I_2 = S \setminus I_1$. Suppose first that $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$, and consider $i_1 \in I_1$ and $i_2 \in I_2$. By Claim 8 and because $i_2 \in I_2$, we know that (i_1, i_2) is open. So let $e_{\ell}^{i_1, i_2}$ be the open edge for (i_1, i_2) , and let xy be its endpoints, with $x \in X_{i_1}$ and $y \in X_{i_2}$. By Fact 2, we get that $\{t_x, t_y\} \subseteq S$, i.e., $\{x, y\} \subseteq V_S$, contradicting Claim 8 as $x \in X_{i_1}$ and $i_1 \in I_1$.

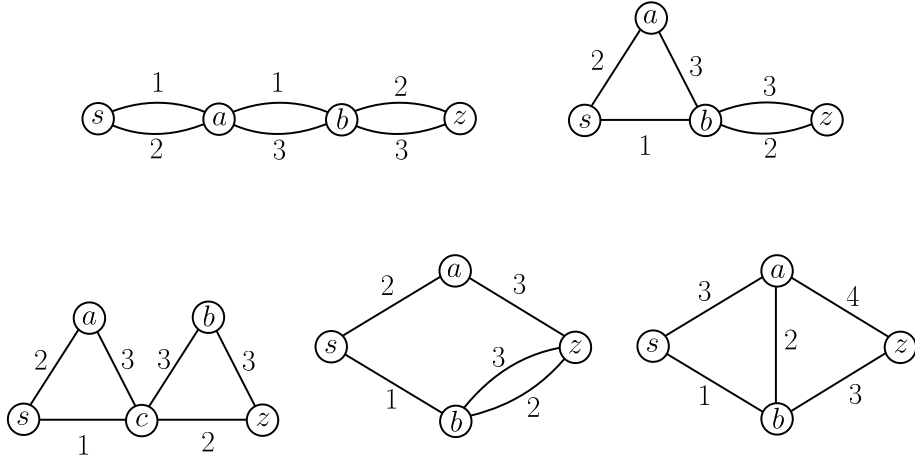
So either $I_1 = \emptyset$ or $I_2 = \emptyset$. Observe that if $I_1 = \emptyset$, then $V_S \cap X_i \neq \emptyset$, for every $i \in [k]$, and the claim follows from Claim 8. And if $I_2 = \emptyset$, then by Fact 3 we get that S contains two edges for every pair i, j , totalling $|S| = 2\binom{k}{2}$. Since $|S| \leq \binom{k}{2} + 2$, we get that this happens only if $k \leq 3$, in which case MULTICOLORED k -CLIQUE is polynomial-time solvable. ◀

Finally, observe that the set of open edges, E^* , contains exactly $\binom{k}{2}$ edges, by definition of open edge and by Claim 9. We then get that $|S| = |E^*| + |V_S| = \binom{k}{2} + |V_S|$. It follows that $|V_S| \leq k$. Additionally, by Fact 2 we know that E^* forms a subgraph of G with vertex set V_S . Because G is a simple graph, E^* contains $\binom{k}{2}$ edges, and V_S contains at most k vertices, the only way this can be possible is if V_S contains exactly k pairwise adjacent vertices, i.e., V_S is a clique of size k , as we wanted to prove. ◀

5 Characterization and recognition of Mengerian graphs

For a temporal graph (G, λ) and pair of vertices s, z , let \mathcal{P} be a set of snapshot disjoint temporal s, z -paths and S a snapshot s, z -cut. By definition, for each path $P \in \mathcal{P}$, there is an edge in P active at a timestep t , for some $t \in S$; choose $\alpha(P)$ to be one such timestep. As the paths in \mathcal{P} are snapshot disjoint, we have that $\alpha(P) \neq \alpha(Q)$ for every $Q \in \mathcal{P}$ different from P . Therefore $|\mathcal{P}| \leq |S|$ and the inequality $sp_{G,\lambda}(s, z) \leq sc_{G,\lambda}(s, z)$ follows. In Proposition 10 we show that there are temporal graphs for which $sp_{G,\lambda}(s, z) < sc_{G,\lambda}(s, z)$, and after this we prove Theorem 1. In our characterization, we have 5 graphs as forbidden structures, M_1, M_2, M_3, M_4, M_5 , that are represented in Figure 2. Let \mathcal{M} be the set of such graphs.

In Figure 8, we present timefunctions for the graphs in \mathcal{M} that turn the inequality $sp_{G,\lambda}(s, z) \leq sc_{G,\lambda}(s, z)$ strict. This is formally stated in the next proposition.



■ **Figure 8** Graphs in the set \mathcal{M} with timefunctions such that $sp_{G,\lambda}(s, z) < sc_{G,\lambda}(s, z)$.

► **Proposition 10.** *Let (G, λ) be one of the temporal graphs depicted in Figure 8. Then $sp_{G,\lambda}(s, z) < sc_{G,\lambda}(s, z)$.*

Proof. To observe that $sc_{G,\lambda}(s, z) > 1$, one just needs to verify that for each timestep, there is a temporal s, z -path not using this timestep. Now, for the cases $G \in \{M_1, M_2, M_3, M_4\}$, suppose $sp_{G,\lambda}(s, z) = sc_{G,\lambda}(s, z)$ and let Q, J be snapshot disjoint temporal s, z -paths. In each of those cases, there are only two edges incident to s , one active at timestep 1 and other at timestep 2. Hence, one of these paths, say Q , starts at timestep 2. Observe that in this case, Q can only finish through edges active at timestep 3. Therefore, J cannot use timestep 2 nor 3, however, all edges incident to z are active at timesteps 2 or 3, a contradiction as there is no temporal s, z -path contained in the first snapshot. Finally, suppose $G = M_5$; we will apply a similar argument. So, let Q, J be snapshot disjoint temporal s, z -paths. Note that one of them, say Q , must use the edge incident to s active at timestep 3 and, therefore, finishes using the edge incident to z active at timestep 4. It follows that J is not allowed to use timestep 3 nor 4, a contradiction as all edges incident to z are only active at such timesteps. ◀

We now prove that the equality between the parameters always holds if each timestep contains at most one active edge, i.e., λ is injective.

► **Proposition 11.** *Let (G, λ) be a temporal graph such that λ is injective. Then $sp_{G,\lambda}(s, z) = sc_{G,\lambda}(s, z)$ for every $s, z \in V(G)$. Moreover, we can compute such value in polynomial time.*

Proof. If P and Q are snapshot disjoint temporal s, z -paths, then, for $e \in E(P)$ and $f \in E(Q)$, we have that $\lambda(e) \neq \lambda(f)$, therefore $e \neq f$. On other hand, if P and Q are such that $E(P) \cap E(Q) = \emptyset$, then we have that $\lambda(E(P)) \cap \lambda(E(Q)) = \emptyset$. Thus, P and Q are snapshot disjoint if and only if they are edge disjoint. Therefore, the maximum size of edge disjoint temporal s, z -paths is equal $sp_{G,\lambda}(s, z)$. Moreover notice that $S \subseteq E(G)$ is a set such that every temporal s, z -path uses an edge of S , then every temporal s, z -path uses an edge active at timestep $\{\lambda(e) \mid e \in S\}$. If $S^T \subseteq [\tau]$ is such that every temporal s, z -path uses an edge active at timestep $\alpha \in S^T$, then it uses the only edge in $\lambda^{-1}(\alpha)$. Therefore, the size of a minimum set of edges such that every temporal s, z -path intersects such set is equal to $sc_{G,\lambda}(s, z)$. Using a result proved in [3] we conclude that $sp_{G,\lambda}(s, z) = sc_{G,\lambda}(s, z)$, and that both parameters can be found in polynomial time. ◀

Proof of Theorem 1. We first prove that if G has M as a m -topological minor, for some $M \in \mathcal{M}$, then G is not Mengerian. In Proposition 10, we already get that each $M \in \mathcal{M}$ is not Mengerian. To finish necessity, we need to prove that adding vertices, adding edges and m -subdividing edges in H does not lead to a Mengerian graph. Propositions 2 and 3 do this work.

Now, suppose by contradiction that G does not contain any M in \mathcal{M} as m -topological minor and that G is non-Mengerian. Hence, there exists $\lambda: E(G) \rightarrow \mathbb{N} \setminus \{0\}$ and $s, z \in V(G)$ such that $sp_{G,\lambda}(s, z) < sc_{G,\lambda}(s, z)$. Suppose, without loss of generality, that among all such graphs and timefunctions we choose those that minimize $|V(G)| + |E(G)| + \tau(\lambda)$. We first show that there are no edges connecting s and z . Suppose otherwise, that there is an edge e with endpoints and let $\lambda(e) = \alpha$. Let G' be the graph obtained from removing all edges in $\lambda^{-1}(\alpha)$ and λ' , the restriction of λ to $E(G) \setminus \lambda^{-1}(\alpha)$. Because (s, e, z) is a temporal s, z -path, observe that $sp_{G',\lambda'}(s, z) = sp_{G,\lambda}(s, z) - 1$ and $sc_{G',\lambda'}(s, z) = sc_{G,\lambda}(s, z) - 1$. This contradicts the fact that (G, λ) minimizes $|V(G)| + |E(G)| + \tau(\lambda)$. Now, suppose that there is a cycle C containing s and z . As $sz \notin E(G)$, we have that such cycle has size at least 4. As G has no M_5 as m -topological minor, there are no paths between the vertices of C that are not contained in C . Also, as G has no M_4 as m -topological minor, all the multiedges of C have multiplicity 1. In other words, the 2-connected component containing C is formed just by C itself, with each multiedge of C having multiplicity 1. In particular, there are only two paths connecting s and z . If these two paths are snapshot disjoint temporal s, z -paths, we take the timesteps of the two edges incident to z to obtain a snapshot s, z -cut as it kills the only two temporal s, z -paths. This is a contradiction as we then get $sp_{G,\lambda}(s, z) = sc_{G,\lambda}(s, z)$. So, if both paths between s and z are temporal paths, they must intersect at a timestep α . But in this case, the set $\{\alpha\}$ would be a snapshot s, z -cut, again a contradiction. Therefore, we can suppose that there is no cycle containing s and z . This means that, if we consider the decomposition of G in bi-connected components B_1, \dots, B_k , then we have that s and z are in different components. Moreover, as we are supposing $(G, \lambda), s, z$ that minimize $|V(G)| + |E(G)| + \tau(\lambda)$, the graph induced by the decomposition is a path and we can suppose that $s \in V(B_1)$ and $z \in V(B_k)$. Suppose that there is $i \in [k]$, such that B_i contains at least 3 vertices. Let $v \in V(B_i) \cap V(B_j)$ for some $j \in \{i-1, i+1\}$ and C_1 be a cycle of B_i containing v . If B_j has at least 3 vertices, then we can find another cycle C_2 contained in B_j such that $V(C_1) \cap V(C_2) = \{v\}$, this contradicts the fact that G has no M_3 as m -topological minor. So, we can suppose that $|V(B_j)| = 2$. If the multiedge contained in B_j has multiplicity 1, then let α be the timestep of such edge. We have that $\{\alpha\}$ is a snapshot s, z -cut, and then $sp_{G,\lambda}(s, z) = sc_{G,\lambda}(s, z)$ a contradiction. Therefore we can assume that the multiedge of B_j has multiplicity at least 2; however the graph induced by such multiedge and C_1 is an m -subdivision of the graph M_2 , again a contradiction. Thus, we can assume that bi-connected component B_i contains a cycle, for every $i \in \{1, \dots, k\}$. In other words, $U(G)$ is an s, z -path. If some of the multiedges of G has multiplicity 1, then $sc_{G,\lambda}(s, z) \leq 1$ and we have the equality, a contradiction. We can say also that $|U(G)| = 3$, as otherwise G would have M_1 as m -topological minor. Now, we show that in such case G must be Mengerian, thus finishing the proof.

We show that, for a graph H such that $U(H)$ is a path of size 3 between s and z and any timefunction λ in H , we have that $sp_{G,\lambda}(s, z) = sc_{G,\lambda}(s, z)$. Let $V(H) = \{s, w, z\}$, which means that the multiedges of H are sw and wz . We use induction on the number of edges of H . The base of induction is when sw and wz both have multiplicity 1. Then, either $\lambda(sw) \leq \lambda(wz)$, in which case $sp_{H,\lambda}(s, z) = sc_{H,\lambda}(s, z) = 1$, or $\lambda(sw) > \lambda(wz)$, in which

1:16 Snapshot Disjointness in Temporal Graphs

case $sp_{H,\lambda}(s, z) = sc_{H,\lambda}(s, z) = 0$. Now, suppose valid when $|E(H')| \leq m$ and consider $|E(H)| = m + 1$. By Proposition 11, we can suppose that there are two edges appearing at same timestep α , say f and g . As each snapshot is a simple graph, we get that f, g form a temporal s, z -path. Let $H' = H - \{f, g\}$. By induction hypothesis, we have that $sp_{H',\lambda}(s, z) = sc_{H',\lambda}(s, z)$. Now let \mathcal{P} be a set of snapshot disjoint temporal s, z -paths and S be a snapshot s, z -cut in (H', λ) . Then, $\mathcal{P} \cup \{(s, f, w, g, z)\}$ is a set of snapshot disjoint temporal s, z -paths in (H, λ) and $S \cup \{\alpha\}$ is a snapshot s, z -cut in (H, λ) . Therefore, $sc_{H,\lambda}(s, z) = sc_{H',\lambda}(s, z) + 1 = sp_{H',\lambda}(s, z) + 1 = sp_{H,\lambda}(s, z)$. ◀

Now we turn our attention to the recognition of Mengerian graphs, showing that it can be done in polynomial time. We observe that the proof of characterization of Mengerian graphs helps us to construct an algorithm of recognition of Mengerian graphs. We make the proper adaptation and prove the next theorem.

► **Theorem 12.** *One can decide in polynomial time whether a graph G has a graph in \mathcal{M} as m -topological minor.*

Proof. First, we find a decomposition of G in 2-connected components, B_1, \dots, B_k . This can be done in $O(m + n)$ (see e.g. [18]).

We consider the set of all components B_i such that $|V(B_i)| = 2$ and the multiedge contained in B_i has multiplicity at least 2, let D be such set. Then, we test if some component B_i in D has vertex in common if a component B_j of size at least 3. This takes $O(k^2)$ times. If the answer is positive, then we would have that G has M_2 as m -topological minor. Then we can suppose the following:

2. No component in D share a vertex with other component that has at least 3 vertices.

Now, we separate all components that have at least three vertices and test if two of them share a vertex, this step takes $O(k^2)$ times. If the answer is true for some two components, then G has M_3 as m -topological minor. Therefore, we can suppose:

3. Components B_i and B_j of size at least 3 do not share vertex.

Now, we look to the components B_i such that $|V(B_i)| \geq 4$. As B_i is two connected, it has a cycle C that we can find in $O(n^2)$. Then we check if there is an edge $e \in E(B_i) \setminus E(C)$. If the answer is positive, then as B_i is 2-connected, there is a chord in C containing e . This implies that G has M_5 as m -topological minor. So, we have the following property.

4. Each component B_i of size at least 4 is such that $U(B_i)$ is a cycle.

With such observation, we now can test if the components B_i of size at least 4 contains some multiedge with multiplicity at least 2. If the answer is positive we are done as it would lead to B_i having M_4 as m -topological minor. So, we can suppose otherwise:

5. Each component B_i of size at least 4 contains no multiedges of multiplicity at least 2.

Now we show that the properties 2-5 implies that G has no graphs in \mathcal{M} as m -topological minors. One just need to observe that every cycle is contained in a 2-connected component. So the properties 2-5, assure us that G has no M_2, M_3, M_4 or M_5 as m -topological minors.

Now, we only need to test if G has M_1 as m -topological minor. We can consider the graph G' obtained from G but excluding the multiedges with multiplicity 1. Then, we test if G' has a path with at least 4 vertices. The answer is positive if and only if G has M_1 as m -topological minor. This finishes the recognition. ◀

6 Multiedge Cut

Finally, in this section we study another version of the cut problem. Recall that two temporal s, z -paths are multiedge disjoint if they do not share any multiedge, and that a multiedge temporal s, z -cut is a set S of multiedges of G intersecting every temporal s, z -path. In this section, we investigate the following problem.

$\leq h$ -MULTIEDGE TEMPORAL S, Z -CUT

Input. A temporal graph (G, λ) , vertices $s, z \in V(G)$, and an integer h .

Question. Is there a multiedge temporal s, z -cut in (G, λ) of size at most h ?

► **Theorem 13.** $\leq h$ -MULTIEDGE TEMPORAL S, Z -CUT is NP-complete, even if $\tau = 2$.

Proof. We make a reduction from Vertex Cover, which consists of, given a simple graph G and a positive integer k , deciding whether there exists a subset $S \subseteq V(G)$ such that $|S| \leq k$ and every $e \in E(G)$ is incident to some $u \in S$; such a set is called a *vertex cover* (of size at most k). So, consider $\mathcal{I} = (G, k)$ an instance of Vertex Cover. We construct a graph G' with vertex set $V(G') = \{s, z\} \cup \{x_v^1, x_v^2, x_v^3, x_v^4 : v \in V(G)\} \cup \{f_{vw} \mid vw \in E(G)\}$. One can use Figure 9 to follow the construction. Then, we add edges from s to x_v^1 and x_v^2 , and from x_v^3 and x_v^4 to z , for every $v \in V(G)$. Also, let $(x_v^1, x_v^2, x_v^3, x_v^4)$ form a path, and add, for each edge $vw \in E(G)$, edges $x_v^1 f_{vw}$ and $f_{vw} x_w^4$. More formally, we have:

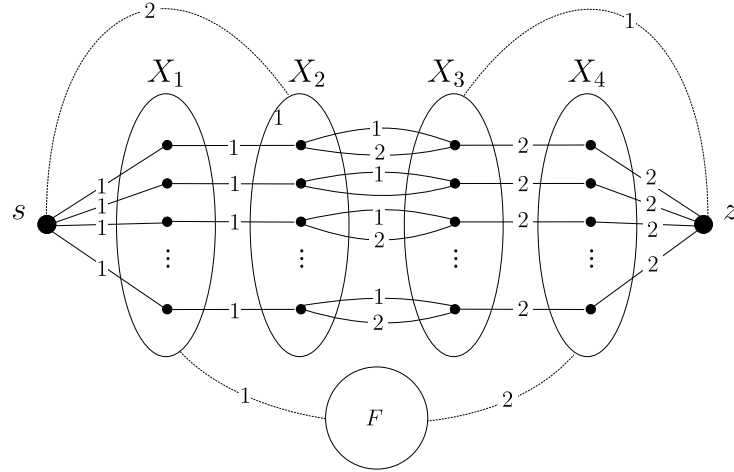
$$\begin{aligned} E(G') = & \{sx_v^1, sx_v^2, x_v^3 z, x_v^4 z \mid v \in V(G)\} \\ & \cup \{x_v^i x_v^{i+1} \mid i \in \{1, 2, 3\}, v \in V(G)\} . \\ & \cup \{x_v^1 f_{vw}, f_{vw} x_w^4 \mid vw \in E(G)\} \end{aligned}$$

Finally, add a second edge with endpoints $x_v^2 x_v^3$, for each $v \in V(G)$. Since these are the only edges with multiplicity greater than 1, we will generally denote an edge by its endpoints, with the exception of these, which we denote by e_v^1, e_v^2 . Now for each $i \in \{1, \dots, 4\}$ define $X_i = \{x_v^i : v \in V\}$, let $F = \{f_{vw} \mid vw \in E(G)\}$ and consider a timefunction λ such that:

$$\lambda(e) = \begin{cases} 1 & , \text{ if } e \in (\{s\} \times X_1) \cup (X_1 \times X_2) \cup (X_3 \times \{z\}) \cup (X_1 \times F) \\ 2 & , \text{ if } e \in (\{s\} \times X_2) \cup (X_3 \times X_4) \cup (X_4 \times \{z\}) \cup (F \times X_4), \text{ and} \\ i & , \text{ if } e = e_v^i \text{ for some } v \in V(G). \end{cases}$$

We prove that G has a vertex cover of size at most k if and only if (G', λ) has a multiedge temporal s, z -cut of size at most $n + k$. Given a solution S of Vertex Cover, we can define $S' = \{sx_v^1, x_v^4 z : v \in S\} \cup \{x_v^2 x_v^3 : v \notin S\}$. It remains to argue that S' separates s from z , or more formally, that there is no temporal s, z -path in $(G' - S', \lambda)$, where S' contains every $e \in E(G')$ such that the endpoints of e are in S' . Notice that if a temporal s, z -path does not use edges between X_1 and F , then it contains one of the following paths: $(s, x_v^1, x_v^2, x_v^3, x_v^4, z)$, $(s, x_v^2, x_v^3, x_v^4, z)$, or $(s, x_v^1, x_v^2, x_v^3, z)$. In any case such path intersects S' . Now suppose that a temporal s, z -path, P , uses an edge $x_v^1 f_{vw}$, which implies that it also uses $f_{vw} x_w^4$ and therefore it arrives in z at timestep 2. Note that the only edges active at timestep 2 incident to x_w^4 are incident either to F , or to x_w^3 , or to z . In the former case, we hit a dead end because f_{uw} has only one edge incident to it in timestep 2, for every $u \in N(w)$. This also happens in the second case, since from x_w^3 one can only go to x_w^2 , hitting again a dead end. We then get that if P contains $x_v^1 f_{vw}$, then it must also contain $x_w^4 z$. By a similar argument one can also show that it must contain $s x_v^1$ too. As $vw \in E(G)$, at least one of v and w are in S , and by construction we get that P uses some multiedge of S' , as we wanted to show.

Now let S' be a multiedge temporal s, z -cut of size at most $n + k$. For each $v \in V(G)$, let the set of multiedges $\{s x_v^1, x_v^1 x_v^2, x_v^2 x_v^3, x_v^3 x_v^4, x_v^4 z\}$ be denoted by A_v , and let $S'_v = S' \cap A_v$.



■ **Figure 9** A representation of temporal graph (G', λ) . The dotted lines represent all the edges between the corresponding set of vertices.

Because A_v forms a temporal s, z -path, we know that $S'_v \neq \emptyset$. One can notice that every temporal s, z -path using $x_v^1 x_v^2$ also uses $s x_v^1$, and in the same way every temporal s, z -path using $x_v^3 x_v^4$ uses $x_v^4 z$. Then by changing S' if necessary, we can suppose that S'_v is a non-empty subset of $\{s x_v^1, x_v^2 x_v^3, x_v^4 z\}$. Now we show that we can actually suppose that $S' \cap A_v$ is either $\{s x_v^1, x_v^4 z\}$ or $\{x_v^2 x_v^3\}$. We do it by analysing the cases where this does not happen.

- $S'_v = \{s x_v^1\}$ or $S'_v = \{x_v^4 z\}$. We just solve the first subcase as the second is similar. Notice that $(s, x_v^2, x_v^3, x_v^4, z)$ is a temporal s, z -path, and that the only edge in this path that can be in S' is $s x_v^2$ because of the case being analyzed. So, removing $s x_v^2$ and adding x_v^4 to S' maintains the property of being a multiedge temporal s, z -cut and turns S'_v into $\{s x_v^1, x_v^4 z\}$.
- $S'_v = \{s x_v^1, x_v^2 x_v^3\}$ or $S'_v = \{x_v^2 x_v^3, x_v^4 z\}$. In both subcases we can remove $x_v^2 x_v^3$ and add either $x^4 z$ (in the first case) or $s x_v^1$ (in the second one).

Now we can define $S = \{v \in V : S_v = \{s x_v^1, x_v^4 z\}\}$. The desired property $|S| \leq k$ follows from the fact that $1 \leq |S'_v| \leq 2$ for every $v \in V(G)$, and that $|S'| \leq n + k$. Finally, suppose that $vw \in E(G)$ is such that $S \cap \{v, w\} = \emptyset$. Then (s, x_v^1, x_v^4, z) is a temporal s, z -path not passing through the edges in S' , a contradiction. ◀

7 Conclusion

We have introduced the concept of snapshot disjointness and proved that the related paths and cut problems, when parameterized by the size of the solution, are both $W[1]$ -hard and XP -time solvable. We then adapted to our context the definition of Mengerian graph introduced by Kempe, Kleinberg and Kumar [13], giving also a characterization in the lines of the ones given in [13] and [11], as well as a polynomial-time recognition algorithm. Since all our results concern only non-strict temporal paths, one can ask whether they also hold for strict paths.

Further open problems can be extracted from Table 1. In particular, we ask whether the results for node departure disjoint paths and cuts presented in [16] for strict paths also hold for non-strict paths.

Finally, while Menger’s Theorem is known to hold for edge disjoint paths [3, 16], it is also known not to hold for multiedge disjoint paths [11]. We reinforce the question posed in [11] about the characterization of Mengerian graphs in the context of multiedge disjoint paths.

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1:20 Snapshot Disjointness in Temporal Graphs

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