

Computing a Group Action from the Class Field Theory of Imaginary Hyperelliptic Function Fields

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Abstract

We explore algorithmic aspects of a simply transitive commutative group action coming from the class field theory of imaginary hyperelliptic function fields. Namely, the Jacobian of an imaginary hyperelliptic curve defined over \mathbb{F}_q acts on a subset of isomorphism classes of Drinfeld modules. We describe an algorithm to compute the group action efficiently. This is a function field analog of the Couveignes-Rostovtsev-Stolbunov group action. We report on an explicit computation done with our proof-of-concept C++/NTL implementation; it took a fraction of a second on a standard computer. We prove that the problem of inverting the group action reduces to the problem of finding isogenies of fixed τ -degree between Drinfeld $\mathbb{F}_q[X]$ -modules, which is solvable in polynomial time thanks to an algorithm by Wesolowski. We give asymptotic complexity bounds for all algorithms presented in this paper.

Introduction

Context The class group $\text{Cl}(\mathbb{Q}(\sqrt{-D}))$ of an imaginary quadratic number field acts on the set of isomorphism classes of elliptic curves having complex multiplication by $\mathbb{Q}(\sqrt{-D})$. This simply transitive group action is a central object of the class field theory of imaginary quadratic number fields, as it provides an explicit way to handle their class fields and Galois groups. Drinfeld modules — initially called *elliptic modules* [15] — were introduced to create an explicit and similar class field theory for function fields. There is a comparable group action in this context; it is expressed in terms of isogenies of Drinfeld modules. Computing it is the main objective of this work.

Algorithmic questions on Drinfeld modules arose from the start of the development of the theory, see e.g. [16]. More recently, effectivity topics in Drinfeld modules were revisited from the point of view of modern computer algebra [5, 4], proposing new applications, such as factorization of univariate polynomials [14] and cryptography [2, 23]. In this paper, we aim at growing the algorithmic toolbox for isogenies of Drinfeld modules. We focus on their relationship with the class field theory of imaginary hyperelliptic function fields, and we build on the work of [5, 4, 30].

Main results Although Drinfeld modules might appear more abstract than elliptic curves, they turn out to be very convenient for concrete computations. The theory of complex multiplication of Drinfeld modules shares many similarities with that of elliptic curves. Let L/\mathbb{F}_q be a finite extension, and let ϕ be a rank-two $\mathbb{F}_q[X]$ -Drinfeld module defined over L . The characteristic polynomial of the Frobenius endomorphism of ϕ has degree 2 in $\mathbb{F}_q[X][Y]$, it defines a quadratic extension \mathbf{k} of $\mathbb{F}_q(X)$. If further $[L : \mathbb{F}_q]$ is odd, \mathbf{k} is an imaginary quadratic function field. Then, the class field theory of \mathbf{k} gives a simply transitive group action on the set S of isomorphism classes of Drinfeld modules having complex multiplication by \mathbf{k} . The underlying group G turns out to be the Galois group of an abelian extension, which is unramified at all finite places and for which the place at infinity splits completely [21, Th. 15.6]. For simplicity, we restrict our work to the case of imaginary hyperelliptic function fields; G will simply be the degree-0 Picard group $\text{Pic}^0(\mathcal{H})$ of the underlying hyperelliptic curve \mathcal{H} .

We design an efficient algorithm to compute the group action in the class field theory of hyperelliptic function field. Surprisingly, the algorithm is quite easy both to describe and to implement: it mainly relies on computing the right-GCD of two Ore polynomials. We provide an asymptotic complexity bound. We also study the difficulty of the so-called *inverse problem*, i.e. computing an element $g \in G$ such that $g \cdot x = y$, for $x, y \in S$ given. We prove that it reduces to the problem of computing isogenies of Drinfeld modules, by providing an algorithm computing the ideal in the coordinate ring of the hyperelliptic curve which corresponds to a given isogeny. Computing this isogeny can now be done efficiently using an algorithm by Wesolowski [36]. We finish our investigation by providing asymptotic complexity bounds, and we present a concrete calculation of the group action, using a C++/NTL implementation.

Related works and applications An important question is the computation of zeta functions, which are closely related to the computation of characteristic polynomials of endomorphisms of Drinfeld modules [19, Sec. 5]. Isogenies of finite Drinfeld modules are connected to this line of work by the fact that they correspond to ideals in endomorphism rings. During the last decade, the algorithmic toolbox gravitating around these topics and its interaction with computer algebra and algorithmic number theory has attracted a lot of attention, see e.g. [5, 4, 30, 14, 17, 26].

The realization of the class field theory of imaginary quadratic number fields via isogenies of elliptic curves is the cornerstone of *isogeny-based cryptography*, whose foundations were laid down by Couveignes [10], and independently by Rostovtsev and Stolbunov [32]. Their ideas paved the way towards the SIDH and the CSIDH cryptosystems [22, 8]. In [23], the authors propose Drinfeld modules analogs of SIDH and CSIDH in the supersingular setting, and they provide polynomial-time attacks on these constructions.

We originally wanted to design an analog of the Couveignes-Rostovtsev-Stolbunov (CRS) cryptosystem in the context of ordinary Drinfeld modules. In a previous preprint (<https://ia.cr/2022/349>), we replaced elliptic curves by Drinfeld modules, and the class group of an imaginary quadratic field by the Jacobian of an hyperelliptic curve. In the meantime, Wesolowski found an algorithm efficiently computing isogenies of finite Drinfeld $\mathbb{F}_q[X]$ -modules [36]. This attack hinders our potential cryptographic applications. However, this line of research is only flourishing; Drinfeld modules provide numerous useful features (complex multiplication, efficient algorithms, deep mathematical theory) and vast areas remain unexplored. For instance, a recent result showed how Carlitz modules — which are special cases of Drinfeld modules — can be used to design new cryptographic protocols [2]. This emphasizes the diversity of potential applications of Drinfeld modules, which may be discovered once a versatile algorithmic toolbox is available.

Organization of the paper Section 1 recalls the algebraic construction of Drinfeld modules, as well as basic tools. Section 2 focuses on complex multiplication and on the class field theory of imaginary hyperelliptic function fields. The main result of this section is a reduction of the group action from class field theory to our specific setting. We can then handle the group action using finite objects. In Section 3, we describe the main algorithm computing the action. We also give a method to recover the ideal class corresponding to a given isogeny of ordinary Drinfeld modules, for which we provide asymptotic complexity bounds. Finally, we present an explicit computation of the group action.

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1 Drinfeld modules

Classical textbooks on Drinfeld modules are [20], [31] and [34]. Finite Drinfeld modules are studied in depth in [19]. For algorithmic perspectives, see [30], [4] and [5].

Throughout this paper, \mathbb{F}_q is the finite field with q elements.

1.1 Ore polynomials

The core mathematical object for the algebraic construction of Drinfeld modules is the ring $K\{\tau\}$ of univariate Ore polynomials. Let $\mathbb{F}_q \hookrightarrow K$ be a field extension, and $\tau : x \mapsto x^q$ denote the Frobenius endomorphism of \overline{K} , which is \mathbb{F}_q -linear.

Definition 1.1 ([20, Def. 1.1.3]). *The ring of Ore polynomials $K\{\tau\}$ is the subring of \mathbb{F}_q -linear endomorphisms of \overline{K} of the form*

$$\sum_{0 \leq i \leq n} a_i \tau^i, \quad n \in \mathbb{Z}_{\geq 0}, a_i \in K,$$

equipped with the addition and the composition of \mathbb{F}_q -linear endomorphisms.

In $K\{\tau\}$, we write $1 = \tau^0$ for the identity endomorphism. For $i, j \in \mathbb{Z}_{\geq 0}$, $\tau^i \tau^j = \tau^{i+j}$. In Definition 1.1, and if $a_n \neq 0$, the integer n is called the τ -degree, and we say that P is *monic* if $a_n = 1$. For every $a \in K$, the equality $\tau a = a^q \tau$ holds true. Therefore, the ring $K\{\tau\}$ is not commutative as soon as $\mathbb{F}_q \neq K$. The center of $K\{\tau\}$ is $\mathbb{F}_q[\tau^{[K:\mathbb{F}_q]}]$ if K is finite, otherwise it is \mathbb{F}_q .

The ring $K\{\tau\}$ is left-Euclidean [20, Prop. 1.6.2] for the τ -degree, i.e. for every $P_1, P_2 \in K\{\tau\}$, there exist $Q, R \in K\{\tau\}$ satisfying

$$\begin{cases} P_1 = QP_2 + R, \\ \deg_\tau(R) < \deg_\tau(P_2). \end{cases}$$

We therefore define the *right-greatest common divisor*, abbreviated *rgcd*, of any non-empty subset $S \subset K\{\tau\}$ as the unique monic generator of the left-ideal generated by S in $K\{\tau\}$. The *rgcd* of two Ore polynomials can be efficiently computed using [7, Alg. 6] (altogether with [6, Prop. 3.1]), or using Euclid's algorithm (see Algorithm 2).

We say that P is *separable* if the coefficient of τ^0 is nonzero, i.e. τ does not right-divide P ; we say that P is *inseparable* if it is not separable; we say that P is *purely inseparable* if $P = \alpha \tau^i$ for some $\alpha \in K^\times, i \in \mathbb{Z}_{>0}$. Consequently, for any $P \in K\{\tau\}$ there exists $\ell \in \mathbb{Z}_{\geq 0}$ and some separable $s \in K\{\tau\}$ such that $P = \tau^\ell s$. The integer ℓ is called the *height* of P and denoted $h(P)$. Using left-Euclidean division, it can be proved that for any $P_1, P_2 \in K\{\tau\}$ such that P_1 is separable, $\text{Ker}(P_2) \subset \text{Ker}(P_1)$ if and only if P_2 right-divides P_1 .

1.2 General Drinfeld modules

Let \mathbf{k} be an algebraic function field of transcendence degree 1 over \mathbb{F}_q (i.e. a finite field extension of $\mathbb{F}_q(X)$), ∞ be a place of \mathbf{k} , and $\mathbf{A} \subset \mathbf{k}$ be the ring of functions that are regular outside ∞ . Let K/\mathbb{F}_q be a field extension equipped with a \mathbb{F}_q -algebra morphism $\gamma : \mathbf{A} \rightarrow K$. The kernel of γ is a prime ideal called the *\mathbf{A} -characteristic of K* . There are mainly two cases which are of interest for Drinfeld modules:

- (i) The field K is a finite extension of \mathbf{A}/\mathfrak{p} for some nonzero prime ideal $\mathfrak{p} \subset \mathbf{A}$, γ is the composition $\mathbf{A} \twoheadrightarrow \mathbf{A}/\mathfrak{p} \hookrightarrow K$, the \mathbf{A} -characteristic of K is \mathfrak{p} ; in this case, we will write L instead of K (see Section 1.3).
- (ii) The field K is a finite extension of \mathbf{k} and the morphism γ is injective.

By [29, Ch. 7, Cor. 2.7], quotients of \mathbf{A} by nonzero ideals \mathfrak{a} are finite-dimensional \mathbb{F}_q -vector spaces. The *degree of \mathfrak{a}* is $\deg(\mathfrak{a}) := \log_q(\#\mathbf{A}/\mathfrak{a})$. For a nonzero $a \in \mathbf{A}$, we set $\deg(a) := \deg(a\mathbf{A})$.

Definition 1.2 ([20, Def. 4.4.2], [19, Def. 1.1]). *A Drinfeld \mathbf{A} -module over K is an \mathbb{F}_q -algebra morphism $\phi : \mathbf{A} \rightarrow K\{\tau\}$ such that,*

- for all $a \in \mathbf{A}$, the coefficient of τ^0 in $\phi(a)$ is $\gamma(a)$;
- there exists $a \in \mathbf{A}$ such that $\deg_\tau(\phi(a)) > 0$.

Let ϕ be a Drinfeld \mathbf{A} -module over K . For any $a \in \mathbf{A}$, the image $\phi(a)$ is denoted ϕ_a . An important feature of Drinfeld modules is that there exists an integer $r \in \mathbb{Z}_{>0}$, called *the rank of ϕ* , such that $\deg_\tau(\phi_a) = r \deg(a)$ for any $a \in \mathbf{A}$ [20, Def. 4.5.4]. We let $\text{Dr}_r(\mathbf{A}, K)$ denote the set of Drinfeld \mathbf{A} -modules over K with rank r . A special case of interest is when $\mathbf{A} = \mathbb{F}_q[X]$, in which ϕ is uniquely determined by ϕ_X and its rank is $\deg_\tau(\phi_X)$.

A Drinfeld \mathbf{A} -module ϕ induces a \mathbf{A} -module law on \overline{K} , defined by $(a, x) \mapsto \phi_a(x)$, where $a \in \mathbf{A}, x \in \overline{K}$. When $\mathbf{A} = \mathbb{F}_q[X]$, this structure of $\mathbb{F}_q[X]$ -module on \overline{K} can be viewed as an analog of the \mathbb{Z} -module law on the group of points $\mathcal{E}(\overline{K})$ of an elliptic curve defined over K .

Let ψ be another Drinfeld \mathbf{A} -module over K . A *morphism of Drinfeld modules* $\iota : \phi \rightarrow \psi$ is an Ore polynomial $\iota \in K\{\tau\}$ such that $\iota\phi_a = \psi_a\iota$ for all $a \in \mathbf{A}$. An *isogeny* is a nonzero morphism. If K'/K is a field extension, a *K' -morphism* (resp. *K' -morphism*) $\phi \rightarrow \psi$ is a morphism that lives in $K'\{\tau\}$. Throughout this paper, isogenies between ϕ and ψ are by default K -isogenies. The endomorphisms of ϕ form a ring denoted $\text{End}(\phi)$ which always contains \mathbb{F}_q and elements of the form $\phi_a, a \in \mathbf{A}$. Said otherwise, \mathbf{A} is isomorphic to a subring of $\text{End}(\phi)$. When $\mathbf{A} = \mathbb{F}_q[X]$, endomorphisms ϕ_a are analogs of integer multiplication on elliptic curves.

1.3 Finite Drinfeld modules

In this section, we specialize in Drinfeld \mathbf{A} -modules over a finite field L , which are also called *finite Drinfeld modules*. In that case, we fix $\mathfrak{p} \subset \mathbf{A}$ a nonzero prime ideal and L a finite extension of \mathbf{A}/\mathfrak{p} , equipped with the canonical morphism $\gamma : \mathbf{A} \rightarrow \mathbf{A}/\mathfrak{p} \hookrightarrow L$. Finite Drinfeld modules have a special endomorphism $\tau_L := \tau^{[L:\mathbb{F}_q]}$, called the *Frobenius endomorphism*. It is worth noticing that for any Drinfeld \mathbf{A} -module ϕ over L , $\text{End}(\phi)$ contains $\mathbb{F}_q[\tau_L]$. Any isogeny $\iota : \phi \rightarrow \psi$ can be written $\tau^\ell \text{deg}(\mathfrak{p}) s$ for some $\ell \in \mathbb{Z}_{\geq 0}$ and a separable $s \in L\{\tau\}$ [19, §(1.4), Eq. (ii)]. Furthermore, the endomorphism ϕ_a is separable if and only if a is not contained in $\text{Ker}(\gamma) = \mathfrak{p}$.

We define now the norm of an isogeny $\iota : \phi \rightarrow \psi$ of finite Drinfeld modules. By [28, Th. III.8.1], there exists a map χ , called the *Euler-Poincaré characteristic*, which sends finite \mathbf{A} -modules to ideals in \mathbf{A} and which satisfies the following properties for any finite \mathbf{A} -modules M_1, M_2, M_3 :

- (i) $\chi(0) = \mathbf{A}$ and $\chi(\mathbf{A}/\mathfrak{q}) = \mathfrak{q}$ if \mathfrak{q} is prime,
- (ii) $\chi(M_1) = \chi(M_2)$ if $M_1 \simeq M_2$,
- (iii) $\chi(M_1) = \chi(M_2)\chi(M_3)$ if $0 \rightarrow M_2 \rightarrow M_1 \rightarrow M_3 \rightarrow 0$ is a short exact sequence.

The *norm of ι* is defined in [19, §(3.9)] as $\mathfrak{n}(\iota) := \mathfrak{p}^{\text{h}(\iota)/\text{deg}(\mathfrak{p})}\chi(\text{Ker}(\iota))$. See [19, Lem. 3.10] for its properties. As for elliptic curves, “being isogenous” is an equivalence relation. For $a \in \mathbf{A}$, we say that $\iota : \phi \rightarrow \psi$ is an *a -isogeny* if ι right-divides ϕ_a in $L\{\tau\}$. We emphasize that a may not generate the norm of the isogeny. In fact, ι is an a -isogeny if and only if a belongs to the norm of ι , which need not be a principal ideal. Every isogeny is an a -isogeny for some nonzero $a \in \mathbf{A}$. If $a \notin \mathfrak{p}$ and ι is an a -isogeny, then ι is separable and there exists another separable a -isogeny $\hat{\iota} : \psi \rightarrow \phi$, called the *dual a -isogeny*, such that $\hat{\iota} \cdot \iota = \phi_a$ and $\iota \cdot \hat{\iota} = \psi_a$. See e.g. [12, §(4.1)].

Drinfeld modules have an analog for Vélú’s formula. Let $\iota \in L\{\tau\}$ be nonzero. There exists a finite Drinfeld \mathbf{A} -module ψ defined over L such that ι is an isogeny $\phi \rightarrow \psi$ if and only if $\text{Ker } \iota$ is an \mathbf{A} -submodule of \overline{L} (endowed with the \mathbf{A} -module structure $(a, x) \mapsto \phi_a(x)$ for $a \in \mathbf{A}, x \in \overline{L}$) and $\text{deg}(\mathfrak{p})$ divides $\text{h}(\iota)$ [19, §(1.4)]. We emphasize that for any $a \in \mathbf{A}$, the Ore polynomial ψ_a can be explicitly computed. The τ -degrees of ϕ_a and ψ_a are equal. By equating the coefficients of $\iota \cdot \phi_a$ and $\psi_a \cdot \iota$, we obtain simple formulas for computing iteratively the coefficients of ψ_a . For instance, if ι is separable, by writing

$$\begin{cases} \iota = \sum_{0 \leq i \leq \text{deg}_\tau(\iota)} \iota_i \tau^i, \\ \phi_a = \sum_{0 \leq i \leq \text{deg}_\tau(\phi_a)} \lambda_i \tau^i, \\ \psi_a = \sum_{0 \leq i \leq \text{deg}_\tau(\psi_a)} \mu_i \tau^i, \end{cases}$$

we obtain the following formulas for $i \in \llbracket 0, \deg_\tau(\phi_a) \rrbracket$:

$$\mu_i = \frac{1}{\iota_0^{q^i}} \left(\sum_{0 \leq j \leq i} \iota_j \lambda_{i-j}^{q^j} - \sum_{0 \leq j \leq i-1} \mu_j \iota_{i-j}^{q^j} \right). \quad (1.1)$$

2 Class field theory of imaginary hyperelliptic function fields

2.1 Complex multiplication for rank-two finite Drinfeld modules

Rank-two Drinfeld modules over finite fields enjoy a theory of complex multiplication which shares many similarities with that of elliptic curves defined over finite fields. The main difference is that imaginary quadratic number fields are replaced by imaginary quadratic function fields, namely quadratic extensions of $\mathbb{F}_q(X)$ for which the place at infinity associated to the discrete valuation ring

$$\{f/g \mid f, g \in \mathbb{F}_q[X], \deg(g) \geq \deg(f)\} \subset \mathbb{F}_q(X)$$

ramifies.

We start by fixing a nonzero prime ideal $\mathfrak{p} \subset \mathbb{F}_q[X]$ and by considering Drinfeld modules in $\text{Dr}_2(\mathbb{F}_q[X], L)$, where L is a finite extension of $\mathbb{F}_q[X]/\mathfrak{p}$ endowed with the canonical map $\gamma : \mathbb{F}_q[X] \rightarrow \mathbb{F}_q[X]/\mathfrak{p} \hookrightarrow L$. A Drinfeld module $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)$ is completely described by the image ϕ_X of X , which is an Ore polynomial of the form

$$\phi_X = \Delta\tau^2 + g\tau + \gamma(X), \quad g \in L, \Delta \in L^\times.$$

The Frobenius endomorphism $\tau_L \in \text{End}(\phi)$ satisfies a quadratic equation [19, Cor. 3.4] [30, Th. 1]; for any $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)$, there is a unique monic polynomial ξ in $\mathbb{F}_q[X][Y]$ of the form

$$\xi = Y^2 + h(X)Y - f(X) \in \mathbb{F}_q[X][Y],$$

such that

$$\begin{cases} \xi(\phi_X, \tau_L) = 0, \\ \deg(f) = [L : \mathbb{F}_q], \\ \deg(h) \leq [L : \mathbb{F}_q]/2. \end{cases} \quad (2.1)$$

The polynomial ξ is called the *characteristic polynomial of the Frobenius endomorphism*.

Definition 2.1 ([18, Lemma (5.2) and Satz (5.3)]). *Let $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)$, and let $\xi = Y^2 + h(X)Y - f(X)$ be the characteristic polynomial of its Frobenius endomorphism. Then ϕ is called supersingular if $h \in \mathfrak{p}$, otherwise ϕ is called ordinary.*

If $[L : \mathbb{F}_q]$ is odd and the affine curve defined by ξ is nonsingular, then the degree bounds in (2.1) imply that ξ defines a imaginary hyperelliptic curve over \mathbb{F}_q [9, Def. 14.1]. As ξ is a polynomial of degree 2 in Y , the affine curve it defines is nonsingular if and only if its discriminant with respect to Y is squarefree. In this case, the endomorphism ring is maximal and completely described:

Proposition 2.2. *Assume $[L : \mathbb{F}_q]$ is odd, let $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)$ be an ordinary rank-2 Drinfeld module, and assume that ξ defines an imaginary hyperelliptic curve \mathcal{H} . Then $\text{End}_{\overline{L}}(\phi) = \text{End}_L(\phi)$. Writing $\mathbf{A}_{\mathcal{H}} = \mathbb{F}_q[X][Y]/(\xi)$, the \mathbb{F}_q -algebras $\text{End}_L(\phi)$ and $\mathbf{A}_{\mathcal{H}}$ are isomorphic via*

$$\begin{array}{ccc} \mathbf{A}_{\mathcal{H}} & \rightarrow & \text{End}_L(\phi) \\ \overline{X} & \mapsto & \phi_X \\ \overline{Y} & \mapsto & \tau_L. \end{array}$$

Proof. By [19, Lem. 3.3], the minimal polynomial of τ_L over $\mathbb{F}_q[X]$ is ξ , which implies that the kernel of the map

$$\begin{aligned} \mathbb{F}_q[X][Y] &\rightarrow \text{End}_L(\phi) \\ X &\mapsto \phi_X \\ Y &\mapsto \tau_L \end{aligned}$$

is the ideal generated by ξ and hence $\mathbb{F}_q[\phi_X, \tau_L]$ is isomorphic to $\mathbf{A}_{\mathcal{H}}$.

By [29, Chap. 5, Th. 10.8], $\mathbf{A}_{\mathcal{H}}$ is the integral closure of $\mathbb{F}_q[X]$ in the function field $\mathbb{F}_q(\mathcal{H})$. Let \mathcal{O} be an $\mathbb{F}_q[X]$ -order in $\mathbb{F}_q(\mathcal{H})$. Since the canonical field extension $\mathbb{F}_q(X) \hookrightarrow \mathbb{F}_q(\mathcal{H}) = \text{Frac}(\mathbf{A}_{\mathcal{H}})$ has degree 2, \mathcal{O} must be a rank-2 $\mathbb{F}_q[X]$ -module. Let $1, \alpha \in \mathbb{F}_q(\mathcal{H})$ be an $\mathbb{F}_q[X]$ -basis of \mathcal{O} . Then $\alpha^2 = a + b\alpha$ for some $a, b \in \mathbb{F}_q[X]$, which implies that α belongs to the $\mathbb{F}_q[X]$ -integral closure of $\mathbb{F}_q[X]$ in $\mathbb{F}_q(\mathcal{H})$, which is $\mathbf{A}_{\mathcal{H}}$. This implies that $\mathcal{O} \subset \mathbf{A}_{\mathcal{H}}$. Hence, $\mathbf{A}_{\mathcal{H}}$ is maximal.

Since τ_L is not in the image of the map $g \mapsto \phi_g$, $\mathbb{F}_q[\phi_X, \tau_L] \subset \text{End}_L(\phi)$ is a 2-dimensional $\mathbb{F}_q[X]$ -module in $\text{End}_L(\phi) \otimes \mathbb{F}_q(X)$. By [4, Th. 6.4.2.(iii)], $\text{End}_L(\phi) \otimes \mathbb{F}_q(X)$ is an imaginary quadratic function field and $\text{End}_L(\phi)$ is an $\mathbb{F}_q[X]$ -order in it. Therefore, $\mathbb{F}_q[\phi_X, \tau_L]$ is an $\mathbb{F}_q[X]$ -order in $\text{End}_L(\phi) \otimes \mathbb{F}_q(X) \simeq (\mathbb{F}_q[X][Y]/\xi) \otimes \mathbb{F}_q(X) \simeq \mathbb{F}_q(\mathcal{H})$. Finally, notice that $\text{End}_L(\phi)$ contains the maximal order $\mathbb{F}_q[\phi_X, \tau_L]$, so it must be equal to it. It remains to prove that $\text{End}_{\overline{L}}(\phi) = \text{End}_L(\phi)$. For any finite extension L' of L , by [4, Th. 6.4.2.(iii)], $\text{End}_L(\phi)$ is a sub-order of $\text{End}_{L'}(\phi)$. As $\text{End}_L(\phi)$ is maximal, $\text{End}_L(\phi) = \text{End}_{L'}(\phi)$. \square

The *j-invariant* of ϕ , denoted $j(\phi)$, is the quantity g^{q+1}/Δ [4, Def. 5.4.1]. For every $j \in L$, there exists a Drinfeld module $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)$ whose *j-invariant* is j ; it is defined by $j^{-1}\tau^2 + \tau + \gamma(X)$ if $j \neq 0$, and $\tau^2 + \gamma(X)$ otherwise. The *j-invariant* and the characteristic polynomial serve as classifying criterion [4, Rem. 5.4.2], [19, Th. 3.5]; two Drinfeld modules $\phi, \psi \in \text{Dr}_2(\mathbb{F}_q[X], L)$ are:

- (i) \overline{L} -isomorphic if and only if they have the same *j-invariant*,
- (ii) L -isogenous if and only if they have the same characteristic polynomial.

The next proposition is an analog for Drinfeld modules of a classical property of endomorphism rings of ordinary elliptic curves defined over finite fields [11, Prop. 4.19], see also [25, Thm. 3.3].

Proposition 2.3. *Two ordinary Drinfeld modules in $\text{Dr}_2(\mathbb{F}_q[X], L)$ are L -isomorphic if and only if they are L -isogenous and \overline{L} -isomorphic.*

Proof. Let $\phi, \psi \in \text{Dr}_2(\mathbb{F}_q[X], L)$ be two ordinary Drinfeld modules which are L -isogenous and \overline{L} -isomorphic. Let $\lambda : \phi \rightarrow \psi$ be an \overline{L} -isomorphism and $\iota : \phi \rightarrow \psi$ be an L -isogeny, then $\lambda^{-1}\iota \in \text{End}_{\overline{L}}(\phi)$. By Proposition 2.2, $\text{End}_{\overline{L}}(\phi) = \text{End}_L(\phi)$, so $\lambda^{-1}\iota \in L\{\tau\}$, and therefore $\lambda \in L$.

Reciprocally, an L -isomorphism is an L -isogeny and an \overline{L} -isomorphism. \square

We recall that throughout this paper and unless stated otherwise, isogenies are L -isogenies and Drinfeld modules are called isogenous if they are L -isogenous.

2.2 Rank-one Drinfeld modules on imaginary hyperelliptic curves

Let $d \geq 5$ be an odd integer and let m be a positive divisor of d . Let $p \in \mathbb{F}_q[X]$ be a monic irreducible polynomial of degree d/m and let $f = \alpha p(X)^m \in \mathbb{F}_q[X]$ for some $\alpha \in \mathbb{F}_q^\times$. Finally, let $h \in \mathbb{F}_q[X]$ be a nonzero polynomial of degree at most $(d-1)/2$ which is not divisible by p . This assumption on h is especially important as it ensures that we will encounter only ordinary Drinfeld modules, see Definition 2.1. Fix $\xi = Y^2 + h(X)Y - f(X)$ and assume that ξ defines an imaginary hyperelliptic curve \mathcal{H} , i.e. the curve is smooth in the affine plane [9, Def. 14.1]. As in Proposition 2.2, set $\mathbf{A}_{\mathcal{H}} = \mathbb{F}_q[X][Y]/(\xi)$. The ring $\mathbf{A}_{\mathcal{H}}$ is isomorphic to the ring of functions of \mathcal{H} regular outside the place at infinity. Let \mathfrak{p} be the prime ideal $\langle p(X), Y \rangle$, which has degree d . Let L be a degree- m extension of $\mathbb{F}_q[X][Y]/\mathfrak{p}$; notice that $[L : \mathbb{F}_q] = d$ is odd; this will have several technical consequences. Set $\gamma : \mathbf{A}_{\mathcal{H}} \rightarrow \mathbf{A}_{\mathcal{H}}/\mathfrak{p} \simeq \mathbb{F}_q[X]/(p) \hookrightarrow L$.

The aim of this section is to prove the following correspondence:

Proposition 2.4. *There is a bijection between the set of \overline{L} -isomorphism classes in $\text{Dr}_2(\mathbb{F}_q[X], L)$ containing a representative whose characteristic polynomial of the Frobenius endomorphism is ξ , and the set of \overline{L} -isomorphism classes in $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$.*

This bijection sends the class of a Drinfeld module $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)$ whose characteristic polynomial of the Frobenius endomorphism is ξ to the class of $\psi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ where $\psi_{\overline{X}} = \phi_X$ and $\psi_{\overline{Y}} = \tau_L$.

The proof of Proposition 2.4 is postponed to the end of this section.

By using Proposition 2.4, we can define the j -invariant of a Drinfeld module in $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ as the j -invariant of its corresponding \overline{L} -isomorphism class in $\text{Dr}_2(\mathbb{F}_q[X], L)$.

Lemma 2.5. *Any rank-1 Drinfeld module $\phi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ has the following form:*

$$\begin{cases} \phi_{\overline{X}} = \Delta\tau^2 + g\tau + \gamma(\overline{X}) \\ \phi_{\overline{Y}} = \beta\tau_L, \end{cases}$$

where $\Delta \in L^\times$, $g \in L$, $\beta \in \mathbb{F}_q^\times$. Moreover, β is a nonzero square root of $\alpha \text{Norm}_{L/\mathbb{F}_q}(\Delta)$ and it is uniquely determined by Δ and g .

Proof. Since \overline{X} has degree 2 in $\mathbf{A}_{\mathcal{H}}$ and ϕ has rank 1, $\phi_{\overline{X}}$ must be an Ore polynomial of τ -degree 2. Therefore, $\phi_{\overline{X}} = \Delta\tau^2 + g\tau + \gamma(\overline{X})$ for some $\Delta \in L^\times$, $g \in L$.

Next, we show that $\phi_{\overline{Y}} = \beta\tau_L$ for some $\beta \in \mathbb{F}_q^\times$. We start by noticing that since ϕ has rank 1 and \overline{Y} has degree d , we must have $\deg_\tau(\phi_{\overline{Y}}) = d$. As $\phi_{\overline{Y}}$ has constant coefficient zero, [19, (1.4), Eq. (ii)] implies that $\tau^{d/m}$ right-divides $\phi_{\overline{Y}}$. Therefore $\phi_{\overline{Y}} = \alpha\phi_{\overline{Y}}^m$ is right-divisible by $\tau^d = \tau_L$. Since \overline{Y} has degree $2d$ and ϕ has rank 1, this implies that $\phi_{\overline{Y}} = w\tau^d$ for some $w \in L\{\tau\}$ of τ -degree d , and consequently $\phi_{\overline{Y}}\phi_{\overline{Y}+\overline{h}} = \phi_{\overline{Y}} = w\tau_L$. Since h is not divisible by p , $\overline{Y} + \overline{h} \notin \mathfrak{p}$ and therefore $\phi_{\overline{Y}+\overline{h}}$ is separable. Consequently, $\phi_{\overline{Y}+\overline{h}} = w/\beta$ for some $\beta \in L^\times$ and $\phi_{\overline{Y}} = \beta\tau_L$.

By examining the coefficient of τ^{2d} in the equation $\phi_{\overline{Y}}^2 + \phi_{\overline{Y}}\phi_{\overline{h}} = \phi_{\overline{Y}}^2$ we obtain that $\beta^2 = \alpha \text{Norm}_{L/\mathbb{F}_q}(\Delta)$ (as $[L : \mathbb{F}_q]$ is odd, τ^2 is a generator of $\text{Gal}(L/\mathbb{F}_q)$). Since d is odd, there is no subfield of L of degree 2 over \mathbb{F}_q , and hence $\beta \in \mathbb{F}_q^\times$. We then prove that only one square root β of $\alpha \text{Norm}_{L/\mathbb{F}_q}(\Delta)$ is suitable. If q is a power of 2, then there is only one square root. Therefore, let us assume now that q is odd, and let $\pm\delta$ be the two distinct square roots of $\alpha \text{Norm}_{L/\mathbb{F}_q}(\Delta)$. By contradiction, assume that there exists Drinfeld modules $\psi, \psi' \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ such that $\psi_{\overline{X}} = \psi'_{\overline{X}} = \Delta\tau^2 + g + \gamma(\overline{X})$ and $\psi_{\overline{Y}} = \delta\tau_L$, $\psi'_{\overline{Y}} = -\delta\tau_L$. Then $0 = \psi_{\overline{Y}^2+\overline{hY}-\overline{f}} - \psi'_{\overline{Y}^2+\overline{hY}-\overline{f}} = 2\delta\psi'_{\overline{h}}\tau_L = 0$, which contradicts the fact that $h \neq 0$. \square

Lemma 2.6. *Any $\phi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ is \overline{L} -isomorphic to a Drinfeld module $\psi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ such that $\psi_{\overline{Y}} = \tau_L$.*

Proof. By Lemma 2.5, $\phi_{\overline{Y}} = \beta\tau_L$ for some $\beta \in \mathbb{F}_q^\times$. Let $\mu \in L^\times$ be an element such that $\text{Norm}_{L/\mathbb{F}_q}(\mu) = \beta$ and let $\lambda \in \overline{L}^\times$ be a $(q-1)$ th-root of μ . Then $\lambda^{q^d-1} = (\lambda^{q-1})^{1+q+q^2+\dots+q^{d-1}} = \text{Norm}_{L/\mathbb{F}_q}(\mu) = \beta$. Direct computations show that the Drinfeld module $\psi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ defined for all $a \in \mathbf{A}_{\mathcal{H}}$ by $\psi_a = \mu\phi_a\mu^{-1}$ satisfies the desired property. \square

Proof of Proposition 2.4. To a Drinfeld module $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)$ with characteristic polynomial ξ , we associate a Drinfeld module $\psi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ defined by $\psi_{\overline{X}} = \phi_X$ and $\psi_{\overline{Y}} = \tau_L$.

Let $\phi' = \alpha\phi\alpha^{-1} \in \text{Dr}_2(\mathbb{F}_q[X], L)$, $\alpha \in \overline{L}$, be a Drinfeld module \overline{L} -isomorphic to ϕ . Note that the characteristic polynomial of the Frobenius endomorphism of ϕ' need not be ξ . We prove that ψ' defined by $\psi'_{\overline{X}} = \phi'_X$ and $\psi'_{\overline{Y}} = \alpha\tau_L\alpha^{-1} = \alpha^{1-q^d}\tau_L$ is a Drinfeld module in $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$. Writing $\phi_X = \Delta\tau^2 + g\tau + \gamma(X)$, we must have $g \neq 0$ since otherwise ϕ would have j -invariant 0; ϕ would be supersingular [1, Lem. 3.2], which contradicts our assumption that h is not divisible by p (see Definition 2.1). Since the coefficient of τ in ϕ'_X equals $\alpha^{q-1}g$ and is in L , we obtain that $\alpha^{q-1} \in L$. Then $\alpha^{1-q^d} \in L$ as a power of $\alpha^{q-1} \in L$. Therefore, $\psi' \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$. Notice that if $\xi(\phi'_X, \tau_L) = 0$, then $\alpha \in L$ (Proposition 2.3), so that $\alpha\tau_L\alpha^{-1} = \tau_L$. The Drinfeld modules ψ and ψ' are \overline{L} -isomorphic, and we extend our association to a well-defined map from the set \overline{L} -isomorphism classes of Drinfeld modules in $\text{Dr}_2(\mathbb{F}_q[X], L)$ containing a representative whose characteristic

polynomial of the Frobenius endomorphism is ξ , to the set of \overline{L} -isomorphism classes of Drinfeld modules in $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$. It remains to prove that this map is bijective. Injectivity comes easily and surjectivity is a direct consequence of Lemma 2.6. \square

2.3 A group action from class field theory

Our object of study is a group action of $\text{Cl}(\mathbf{A})$ on the set of isomorphism classes of $\text{Dr}_r(\mathbf{A}, K)$, where \mathbf{A} and K are as in Section 1.2. Indeed, if $\mathfrak{a} \subset \mathbf{A}$ is a nonzero ideal, we define

$$\iota_{\mathfrak{a}} = \text{rgcd}(\{\phi_f : f \in \mathfrak{a}\}).$$

By [21, Sec. 4], $\iota_{\mathfrak{a}}$ is a well-defined isogeny from ϕ to some Drinfeld module denoted $\mathfrak{a} \star_K \phi \in \text{Dr}_r(\mathbf{A}, K)$. This map actually has multiplicative properties, and principal ideals lead to isogenies that are endomorphisms. Therefore this map can be extended to a group action of $\text{Cl}(\mathbf{A})$ on the set of K -isomorphism classes of Drinfeld modules in $\text{Dr}_r(\mathbf{A}, K)$. A similar group action in fact appears to be one of the main motivations in the landmark paper by Drinfeld for making explicit the class field theory of function fields, see [15, Th. 1].

In this section, $d, m, p, h, f, \xi, \mathcal{H}, \mathbf{A}_{\mathcal{H}}, \mathfrak{p}, L$ are as in Section 2.2.

Theorem 2.7. *If $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ is nonempty, then the set of \overline{L} -isomorphism classes of Drinfeld modules in $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ is a principal homogeneous space for $\text{Cl}(\mathbf{A}_{\mathcal{H}})$ under the \star_L action.*

The proof of Theorem 2.7 is postponed to the end of this section.

Theorem 2.7 can be seen as a reduction modulo prime ideals of the following general theorem, which might itself be seen as an function field analog of [33, Prop. 2.4, Lem. 2.5.1]. We emphasize that [21, Th. 9.3] holds in greater generality than what we need here; it holds for any function field and it is not restricted to hyperelliptic curves.

Theorem 2.8 ([21, Th. 9.3]). *Let \mathbf{k} be the function field of \mathcal{H} , \mathbf{K} be the completion of \mathbf{k} at the place ∞ , and \mathbf{C} be the completion of an algebraic closure $\overline{\mathbf{K}}$. Then the set of \mathbf{C} -isomorphism classes of Drinfeld modules in $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, \mathbf{C})$ is a principal homogeneous space for $\text{Cl}(\mathbf{A}_{\mathcal{H}})$ under the $\star_{\mathbf{C}}$ action.*

Our strategy to prove Theorem 2.7 is to use the reduction and lifting properties of ordinary Drinfeld modules [21, Sec. 11][1, Th. 3.4].

Let K be a finite extension of \mathbf{k} . Let \mathfrak{P} be a place of K above $\mathfrak{p} \subset \mathbf{A}_{\mathcal{H}}$ and $\mathcal{O}_{\mathfrak{P}}$ be the associated discrete valuation ring, with the associated reduction morphism $\text{red}_{\mathfrak{P}} : \mathcal{O}_{\mathfrak{P}} \rightarrow \mathcal{O}_{\mathfrak{P}}/\mathfrak{P}$. An Ore polynomial $f \in K\{\tau\}$ is said to be *defined over $\mathcal{O}_{\mathfrak{P}}$* if its coefficients lie in $\mathcal{O}_{\mathfrak{P}}$ and its leading coefficient is invertible in $\mathcal{O}_{\mathfrak{P}}$. A Drinfeld $\mathbf{A}_{\mathcal{H}}$ -module ϕ over K is said to be *defined over $\mathcal{O}_{\mathfrak{P}}$* if for all $a \in \mathbf{A}_{\mathcal{H}}$, ϕ_a is defined over $\mathcal{O}_{\mathfrak{P}}$. Let $\text{Dr}_{r, \mathfrak{P}}(\mathbf{A}_{\mathcal{H}}, K)$ be the set of Drinfeld modules defined over $\mathcal{O}_{\mathfrak{P}}$. By considering the morphism $\gamma : \mathbf{A}_{\mathcal{H}} \rightarrow \mathbf{A}_{\mathcal{H}}/\mathfrak{p} \hookrightarrow \mathcal{O}_{\mathfrak{P}}/\mathfrak{P}$, the reduction map $\text{red}_{\mathfrak{P}}$ extends canonically to a map $\text{Dr}_{r, \mathfrak{P}}(\mathbf{A}_{\mathcal{H}}, K) \rightarrow \text{Dr}_r(\mathbf{A}_{\mathcal{H}}, \mathcal{O}_{\mathfrak{P}}/\mathfrak{P})$.

Lemma 2.9. *For any $\phi \in \text{Dr}_{r, \mathfrak{P}}(\mathbf{A}_{\mathcal{H}}, K)$ and any ideal $\mathfrak{a} \subset \mathbf{A}_{\mathcal{H}}$, the Drinfeld module $\mathfrak{a} \star_K \phi$ is defined over $\mathcal{O}_{\mathfrak{P}}$ and*

$$\text{red}_{\mathfrak{P}}(\mathfrak{a} \star_K \phi) = \mathfrak{a} \star_{(\mathcal{O}_{\mathfrak{P}}/\mathfrak{P})} \text{red}_{\mathfrak{P}}(\phi).$$

Proof. The Drinfeld module $\mathfrak{a} \star_K \phi$ is defined over $\mathcal{O}_{\mathfrak{P}}$ by [21, Prop. 11.2], hence $\text{red}_{\mathfrak{P}}(\mathfrak{a} \star_K \phi)$ is well-defined. Let $\iota_{\mathfrak{a}}$ be the monic generator of the left-ideal in $K\{\tau\}$ generated by $\{\phi_g : g \in \mathfrak{a}\}$. Since ϕ is defined over $\mathcal{O}_{\mathfrak{P}}$, we deduce that $\iota_{\mathfrak{a}}$ must have coefficients in $\mathcal{O}_{\mathfrak{P}}$ and that its reduction generates the left-ideal in $(\mathcal{O}_{\mathfrak{P}}/\mathfrak{P})\{\tau\}$ generated by $\{\text{red}_{\mathfrak{P}}(\phi_g) : g \in \mathfrak{a}\}$. Consequently, $\text{red}_{\mathfrak{P}}(\iota_{\mathfrak{a}})$ is the isogeny associated to $\mathfrak{a} \star_{(\mathcal{O}_{\mathfrak{P}}/\mathfrak{P})} \text{red}_{\mathfrak{P}}(\phi)$, which concludes the proof. \square

Proof of Theorem 2.7. Using the correspondence of Proposition 2.4 between \overline{L} -isomorphism classes of Drinfeld modules, we can associate to any Drinfeld module in $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ a Drinfeld module in $\text{Dr}_2(\mathbb{F}_q[X], L)$ whose characteristic polynomial of the Frobenius endomorphism is ξ . Throughout this proof, we fix a place \mathfrak{P} of $\overline{\mathbb{F}_q(X)}$ above \mathfrak{p} . Such a place defines a compatible discrete valuation ring $\mathcal{O}_{\mathfrak{P}}^{(K)}$ in any finite extension K of $\mathbb{F}_q(X)$.

Let us prove the transitivity of the action. Let $j_1, j_2 \in L$ be the j -invariants of two Drinfeld modules $\phi, \psi \in \text{Dr}_2(\mathbb{F}_q[X], L)$, whose characteristic polynomial of the Frobenius is ξ . Since the ideal $\langle p(\overline{X}) \rangle$ splits in $\mathbf{A}_{\mathcal{H}}$ ($\langle p(\overline{X}) \rangle = \langle p(\overline{X}), \overline{Y} \rangle \cdot \langle p(\overline{X}), \overline{Y} + h(\overline{X}) \rangle$), Deuring's lifting theorems for Drinfeld modules [1, Th. 3.4, Th. 3.5] (see [27, Ch. 13, §4] for the analogs for elliptic curves) imply that there exists a finite extension K of $\mathbb{F}_q(X)$ and two \mathbf{C} -isomorphism classes of Drinfeld modules in $\text{Dr}_2(\mathbb{F}_q[X], K)$, whose j -invariants reduce to j_1, j_2 modulo \mathfrak{P} . Those j -invariants are algebraic integers in \mathbf{C} [18, §(4.3)]. Moreover, those classes contain Drinfeld modules $\phi', \psi' \in \text{Dr}_2(\mathbb{F}_q[X], K)$ whose endomorphism rings are isomorphic to $\text{End}(\phi) \simeq \text{End}(\psi) \simeq \mathbf{A}_{\mathcal{H}}$. Therefore those Drinfeld modules can be regarded as Drinfeld modules in $\text{Dr}_{1, \mathfrak{P}}(\mathbf{A}_{\mathcal{H}}, K)$. Since \star_K acts on $\text{Dr}_{1, \mathfrak{P}}(\mathbf{A}_{\mathcal{H}}, K)$ [21, Prop. 11.2], and the group action associated to $\star_{\mathbf{C}}$ is transitive (Theorem 2.8), there is an ideal $\mathfrak{a} \subset \mathbf{A}_{\mathcal{H}}$ such that $\mathfrak{a} \star_K \phi'$ is isomorphic to ψ' . Consequently, the j -invariants $\mathfrak{a} \star_K \phi'$ and ψ' are equal, and therefore their reduction modulo \mathfrak{P} equals j_2 . Using Lemma 2.9, the j -invariant of $\mathfrak{a} \star_K \phi'$ reduces modulo \mathfrak{P} to the j -invariant of $\mathfrak{a} \star_{\mathcal{O}_{\mathfrak{P}}^{(K)}/\mathfrak{P}} \phi$, which therefore also equals j_2 . Hence \mathfrak{a} sends the \overline{L} -isomorphism class of ϕ to that of ψ via the $\star_{\overline{L}}$ action (which is the same as the \star_L -action on ϕ , since ϕ is defined over L).

Finally, let us prove the freeness of the action. Let $\phi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ be a Drinfeld module, and set $\psi = \mathfrak{a} \star_L \phi$. Assume that ϕ and ψ are \overline{L} -isomorphic. Since ϕ and ψ are L -isogenous, by Proposition 2.3 they must be L -isomorphic. Let $\alpha \in L$ be such an isomorphism, i.e. $\alpha \phi \alpha^{-1} = \psi$. Using [1, Th. 3.4] as above, the lifting procedure provides us with $\phi' \in \text{Dr}_{1, \mathfrak{P}}(\mathbf{A}_{\mathcal{H}}, K)$ which reduces to ϕ modulo \mathfrak{P} . Then set $\psi' = \mathfrak{a} \star_K \phi'$, and let $\iota_{\mathfrak{a}}$ be the associated isogeny. By the same argument as in the proof of Lemma 2.9, we obtain that $\iota_{\mathfrak{a}}$ is defined over $\mathcal{O}_{\mathfrak{P}}^{(K)}$ and that $\text{red}_{\mathfrak{P}}(\iota_{\mathfrak{a}}) = \alpha$ which implies that $\iota_{\mathfrak{a}} \in K$, and therefore ϕ' and ψ' are isomorphic. Consequently, \mathfrak{a} is principal (Theorem 2.8), and hence the group action associated to \star_L is free. \square

3 Algorithms

In this section, $d, m, p, h, f, \xi, \mathcal{H}, \mathbf{A}_{\mathcal{H}}, \mathfrak{p}, L$ are as in Section 2.2. We also fix $\omega := \gamma(X) \in L$.

3.1 Computation of the group action

Before describing the algorithm for computing the group action in Theorem 2.7, we need data structures to represent elements in $\text{Cl}(\mathbf{A}_{\mathcal{H}})$ and \overline{L} -isomorphism classes. Thanks to Proposition 2.4, we can use j -invariants — which are elements of L — to represent \overline{L} -isomorphism classes of Drinfeld modules in $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$. For representing elements in $\text{Cl}(\mathbf{A}_{\mathcal{H}})$, we use Mumford coordinates [9, Th. 14.5]: in our case $\text{Cl}(\mathbf{A}_{\mathcal{H}})$ is isomorphic to $\text{Pic}^0(\mathcal{H})$.

Lemma 3.1. *The ring $\mathbf{A}_{\mathcal{H}}$ is a Dedekind domain, and $\text{Cl}(\mathbf{A}_{\mathcal{H}}) \simeq \text{Pic}^0(\mathcal{H})$.*

Proof. The ring $\mathbf{A}_{\mathcal{H}}$ is a Dedekind domain because \mathcal{H} is smooth in the affine plane [29, Ch. 7, Cor. 2.7]. The isomorphism $\text{Cl}(\mathbf{A}_{\mathcal{H}}) \simeq \text{Pic}^0(\mathcal{H})$ comes from the fact that there is a unique degree-1 place ∞ at infinity. Indeed, the group of affine divisors $\text{Div}(\mathbf{A}_{\mathcal{H}})$ (i.e. the subgroup of divisors whose valuation at infinity is 0) is isomorphic to the group of degree-0 divisors in $\text{Div}_0(\mathcal{H})$ via the map which sends a divisor D in $\text{Div}(\mathbf{A}_{\mathcal{H}})$ to $D - \deg(D)\infty$. Next, we notice that D is principal in $\text{Div}(\mathbf{A}_{\mathcal{H}})$ if and only if its image in $\text{Div}_0(\mathcal{H})$ is principal. We conclude by using the isomorphism in [29, Ch. 7, Prop. 7.1], which shows that the quotient of $\text{Div}(\mathbf{A}_{\mathcal{H}})$ by principal divisors is isomorphic to $\text{Cl}(\mathbf{A}_{\mathcal{H}})$. \square

Since \mathcal{H} has genus $\lfloor (d-1)/2 \rfloor$, elements in $\text{Pic}^0(\mathcal{H})$ can be represented by *Mumford coordinates* [9, Th. 14.5], which are pairs of polynomials $(u, v) \in \mathbb{F}_q[X]^2$ such that:

- (i) u is a nonzero monic polynomial of degree at most $(d-1)/2$,
- (ii) $\deg(v) < \deg(u)$,
- (iii) u divides $\xi(X, v(X))$.

Mumford coordinates (u, v) encode the ideal class of $\langle u(\overline{X}), \overline{Y} - v(\overline{X}) \rangle \subset \mathbf{A}_{\mathcal{H}}$.

Algorithm 1: GROUPACTION

Input:

- A j -invariant $j \in L$ encoding an isomorphism class \mathcal{C} of Drinfeld modules in $\text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$.
- Mumford coordinates $(u, v) \in \mathbb{F}_q[X]^2$ for a divisor class $[D]$ in $\text{Pic}^0(\mathcal{H})$.

Output: The j -invariant obtained by making $[D]$ act on \mathcal{C} by the \star_L action.

- 1 $\tilde{u} \leftarrow u(j^{-1}\tau^2 + \tau + \omega) \in L\{\tau\}$;
 - 2 $\tilde{v} \leftarrow v(j^{-1}\tau^2 + \tau + \omega) \in L\{\tau\}$;
 - 3 $\iota \leftarrow \text{rgcd}(\tilde{u}, \tau_L - \tilde{v})$;
 - 4 $\hat{g} \leftarrow \iota_0^{-q}(\iota_0 + \iota_1(\omega^q - \omega))$;
 - 5 $\hat{\Delta} \leftarrow j^{-q^{\deg_{\tau}(\iota)}}$;
 - 6 **return** $\hat{g}^{q+1}/\hat{\Delta}$.
- /* $\iota = \sum_{0 \leq k \leq \deg_{\tau}(\iota)} \iota_k \tau^k$ */
-

Proposition 3.2. *Algorithm 1 (GROUPACTION) is correct.*

Proof. A representative of the class in $\text{Cl}(\mathbf{A}_{\mathcal{H}}) \simeq \text{Pic}^0(\mathcal{H})$ represented by the Mumford coordinates (u, v) is the ideal $\langle u(\overline{X}), \overline{Y} - v(\overline{X}) \rangle \subset \mathbf{A}_{\mathcal{H}}$. A representative of the isomorphism class of Drinfeld modules represented by the j -invariant j is a Drinfeld module $\phi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ such that $\phi_{\overline{X}} = j^{-1}\tau^2 + \tau + \omega$ and $\phi_{\overline{Y}} = \beta\tau_L$ for some $\beta \in \mathbb{F}_q^{\times}$ (see Section 2.2). Note that $j \neq 0$ by [1, Lem. 3.2]. We shall prove that $\langle u(\overline{X}), \overline{Y} - v(\overline{X}) \rangle \star_L \phi = \psi$, where $\psi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ is the Drinfeld module such that $\psi_{\overline{X}} = \hat{\Delta}\tau^2 + \hat{g}\tau + \omega$ and $\psi_{\overline{Y}} = \beta\tau_L$.

The Ore polynomial ι computed at Step 3 is $\text{rgcd}(\phi_{u(\overline{X})}, \tau_L - \phi_{v(\overline{X})})$, which is by construction the monic Ore polynomial defining the isogeny. Since we need to invert the coefficient ι_0 (at Step 4), we need to prove that ι is separable. This is indeed true: ι right-divides $\phi_{u(\overline{X})}$, which is separable because $\deg(u) < d$. Hence u cannot be a multiple of p , which is a generator of $\text{Ker}(\gamma)$.

Since ι is an isogeny [21, Cor. 5.10], there exists $\psi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$ such that $\iota \cdot \phi_{\overline{X}} = \psi_{\overline{X}} \cdot \iota$ where $\psi_{\overline{X}}$ has τ -degree 2. It remains to prove that $\psi_{\overline{X}} = \hat{\Delta}\tau^2 + \hat{g}\tau + \omega$. This is done by extracting as in Equations (1.1) the coefficients of τ and $\tau^{\deg_{\tau}(\iota)+2}$ in the equality $\iota \cdot \phi_{\overline{X}} = \psi_{\overline{X}} \cdot \iota$, which provides us with:

$$\begin{cases} \iota_0 g + \iota_1 \omega^q &= \hat{g} \iota_0^q + \omega \iota_1, \\ j^{-q^{\deg_{\tau}(\iota)}} &= \hat{\Delta}. \end{cases}$$

There is only one pair $(\hat{\Delta}, \hat{g}) \in L^2$ which satisfies these two equalities, and the associated Drinfeld module has j -invariant $\hat{g}^{q+1}/\hat{\Delta}$. \square

Algorithm 2: OREEUCLIDEANDIVISION

Input: Two Ore polynomials $a, b \in L\{\tau\}$.

Output: Ore polynomials $q, r \in L\{\tau\}$ such that $a = qb + r$ and $\deg_{\tau}(r) < \deg_{\tau}(b)$.

- 1 $q \leftarrow 0$;
 - 2 $r \leftarrow a$;
 - 3 **while** $\deg_{\tau}(r) \geq \deg_{\tau}(b)$ **do**
 - 4 $\varepsilon \leftarrow \text{lc}(r) \cdot \tau^{\deg_{\tau}(r) - \deg_{\tau}(b)} \cdot (\text{lc}(b))^{-1}$;
 - 5 $q \leftarrow q + \varepsilon$;
 - 6 $r \leftarrow r - \varepsilon \cdot b$;
 - 7 **return** (q, r) .
-

Lemma 3.3. *Algorithm 2 (OREEUCLIDEANDIVISION) terminates and is correct.*

Proof. As the degree of r decreases at each such call, the algorithm must terminate. At each recursive call, the relation $a = qb + r$ holds, which implies that the algorithm is correct □

We finish this section by studying the asymptotic complexity of Algorithm 1. For Ore Euclidean division and right-greatest common divisor computation, we use Algorithms 2 and 3, mimicking naïve algorithms for usual (commutative) univariate polynomials. In fact, using the fastest known algorithms for Ore polynomial multiplication and Euclidean division in [7, 6] would not improve our complexity bound for Algorithm 1, see Remark 3.14.

Here and subsequently, if f and g are two functions defined on $\mathbb{Z}_{\geq 0}^2$, with values in $\mathbb{R}_{>0}$, we write $f = O(g)$ if there exists $M > 0$ such that for every $(x, y) \in \mathbb{Z}_{\geq 0}^2$, $f(x, y) \leq Mg(x, y)$. Also, by "application of the Frobenius endomorphism", we mean computing λ^q given $\lambda \in L$.

Lemma 3.4. *Assuming $\deg_\tau(a) > \deg_\tau(b)$, Algorithm 2 (OREEUCLIDEANDIVISION) requires $O(\deg_\tau(b)(\deg_\tau(a) - \deg_\tau(b)))$ arithmetic operations in L and $O(\deg_\tau(b)(\deg_\tau(a) - \deg_\tau(b)))$ applications of the Frobenius endomorphism.*

Proof. First, we notice that in the worst-case scenario, the algorithm needs to compute $\tau^{\deg_\tau(r) - \deg_\tau(b)} \text{lc}(b)^{-1}$ and $\tau^{\deg_\tau(r) - \deg_\tau(b)} \text{lc}(b)^{-1} b$ for $\deg_\tau(r)$ ranging from $\deg_\tau(b)$ to $\deg_\tau(a)$. This can be precomputed for $O(\deg_\tau(b)(\deg_\tau(a) - \deg_\tau(b)))$ operations in L and applications of the Frobenius endomorphism.

At each step of the loop, computing ε and $q + \varepsilon$ costs a constant number of operations, and computing r costs $O(\deg_\tau(b))$ operations. In total, this amounts to $\deg_\tau(b)(\deg_\tau(a) - \deg_\tau(b))$ operations in L and the same upper bound for the number of applications of the Frobenius endomorphism. □

Algorithm 3: EUCLIDRGCD

Input: Two Ore polynomials $a = \sum_{0 \leq i \leq \deg_\tau(a)} a_i \tau^i$, $b = \sum_{0 \leq i \leq \deg_\tau(b)} b_i \tau^i$ in $L\{\tau\}$, such that $a \neq 0$.

Output: The right-gcd of a and b .

```

1 if  $b = 0$  then
2   return  $a$ .
3 if  $\deg_\tau(b) > \deg_\tau(a)$  then
4   return EUCLIDRGCD( $b, a$ ).
5  $(q, r) \leftarrow$  OREEUCLIDEANDIVISION( $a, b$ );
6 return EUCLIDRGCD( $r, b$ ).

```

Lemma 3.5. *Algorithm 3 (EUCLIDRGCD) terminates and is correct.*

Proof. This algorithm is the classical Euclid's algorithm for computing a gcd and its proof of correctness is similar to the classical case. □

The following lemma yields a uniform complexity bound for the rgcd in terms of all the parameters $q, d, \deg_\tau(a), \deg_\tau(b)$.

Lemma 3.6. *Algorithm 3 (EUCLIDRGCD) requires at most $O(\deg_\tau(a) \deg_\tau(b))$ arithmetic operations in L and $O(\deg_\tau(a) \deg_\tau(b))$ applications of the Frobenius endomorphism.*

Proof. This complexity is proved by using the standard methods to evaluate the complexity of Euclid's algorithm from the complexity of the Euclidean division in $O(\deg_\tau(b)(\deg_\tau(a) - \deg_\tau(b)))$ operations (Lemma 3.4), see e.g. [35, Th. 3.11]. □

Proposition 3.7. *Algorithm 1 requires $O(d^2)$ operations in L and $O(d^2)$ applications of the Frobenius endomorphism.*

Proof. Writing $u = u_\ell X^\ell + \dots + u_0$ and $\phi_X = j^{-1}\tau^2 + \tau + \omega$, we have $\ell \leq (d-1)/2$ and $\tilde{u} = u_\ell \phi_X^\ell + \dots + u_0$. In order to compute \tilde{u} , we can first compute $\phi_X^2, \dots, \phi_X^\ell$ iteratively. Let $n \in \llbracket 1, \ell-1 \rrbracket$ and write $\phi_X^n = \sum_{i=0}^{2n} a_i \tau^i$. Then

$$\phi_X \phi_X^n = \sum_{i=0}^{2n} \left(a_i \omega \tau^i + g a_i^q \tau^{i+1} + \Delta a_i^{q^2} \tau^{i+2} \right).$$

Knowing ϕ_X^n , the computation of ϕ_X^{n+1} requires $O(n)$ additions, multiplications, q -exponentiations and q^2 -exponentiations, which is $O(n)$ operations in L and $O(n)$ applications of the Frobenius endomorphism of L/\mathbb{F}_q . Consequently, $O(d^2)$ operations in L and $O(d^2)$ applications of the Frobenius endomorphism are required to compute $\phi_X^2, \dots, \phi_X^\ell$.

The last operation that will affect the asymptotic complexity is the `rgcd`, which we perform using Algorithm 3. We have $\deg(u) = \ell$, $\deg(v) < \ell$, so that \tilde{v} and $\tau_L - \tilde{u}$ respectively have τ -degree at most d . By Lemma 3.6, this algorithm requires $O(d^2)$ operations in L and $O(d^2)$ applications of the Frobenius endomorphism. □

3.2 Computation of the ideal corresponding to an isogeny

In this section, we make explicit the transitivity of the group action. Given two Drinfeld modules $\phi, \psi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$, our goal is to compute Mumford coordinates $(u, v) \in \mathbb{F}_q[X]^2$ such that the class of $\langle u(\overline{X}), \overline{Y} - v(\overline{X}) \rangle \subset \mathbf{A}_{\mathcal{H}}$ sends the \overline{L} -isomorphism class of ϕ to that of ψ , via \star_L . We emphasize that, given $\phi, \psi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$, computing an isogeny ι between ϕ and ψ can be achieved efficiently by using Wesolowski's method [36]. Our algorithm then converts such an isogeny ι into the desired Mumford coordinates (u, v) . For simplicity, we shall assume that the norm of the isogeny is coprime to \mathfrak{p} , in order to avoid separability issues. In the general case, once the part of the isogeny which is coprime to the characteristic has been treated, the part whose norm is a power of \mathfrak{p} can be computed easily since it is either a power of the Frobenius or a power of its dual, and these cases can be easily discriminated.

We use the shorthand notation $\text{Dr}_2(\mathbb{F}_q[X], L)_\xi$ to denote the subset of Drinfeld modules in $\text{Dr}_2(\mathbb{F}_q[X], L)$ whose characteristic polynomial of the Frobenius endomorphism is ξ . By Proposition 2.4, to any $\phi \in \text{Dr}_1(\mathbf{A}_{\mathcal{H}}, L)$, we can associate a Drinfeld module $\phi' \in \text{Dr}_2(\mathbb{F}_q[X], L)_\xi$. Notice that \star_L leaves $\text{Dr}_2(\mathbb{F}_q[X], L)_\xi$ globally invariant. Hence, by slight abuse of notation, we shall use the \star_L notation to also denote the corresponding action of nonzero ideals in $\mathbf{A}_{\mathcal{H}}$ over $\text{Dr}_2(\mathbb{F}_q[X], L)_\xi$. Another useful remark is that computing Mumford coordinates for the class of a given ideal in $\mathbf{A}_{\mathcal{H}}$ can be done efficiently by using the reduction step of Cantor's algorithm [9, Algo. 14.7]. Therefore, our main algorithmic task is to construct the ideal in $\mathbf{A}_{\mathcal{H}}$ corresponding to a given isogeny.

We start by the following lemma, which establishes a correspondence between ideals in $\mathbf{A}_{\mathcal{H}}$ and isogenies:

Lemma 3.8. *Let $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)_\xi$ be an ordinary Drinfeld module. Then there is a one-to-one correspondence between monic isogenies with domain ϕ and nonzero ideals in $\mathbf{A}_{\mathcal{H}}$. Moreover, let $\phi_1, \phi_2, \phi_3 \in \text{Dr}_2(\mathbb{F}_q[X], L)_\xi$ be Drinfeld modules and $\iota_1 : \phi_1 \rightarrow \phi_2$, $\iota_2 : \phi_2 \rightarrow \phi_3$ be isogenies; the ideal associated to $\iota_2 \circ \iota_1$ in $\mathbf{A}_{\mathcal{H}}$ is the product of the ideals associated to ι_1 and ι_2 .*

Proof. To any monic isogeny $\iota : \phi \rightarrow \psi$, we associate the nonzero ideal $\text{Hom}(\psi, \phi)\iota \subset \text{End}(\phi) \simeq \mathbf{A}_{\mathcal{H}}$. Notice that $\psi \in \text{Dr}_2(\mathbb{F}_q[X], L)_\xi$ (Section 2). Reciprocally, to any nonzero ideal $\mathfrak{a} \subset \mathbf{A}_{\mathcal{H}}$ corresponds the isogeny which is the monic generator of the left-ideal in $L\{\tau\}$ generated by $\{g(\phi_X, \tau_L) : g \in \mathfrak{a}\}$. We refer to [19, §(3.6)] for more details.

To prove the second statement, we start by letting Ξ denote the isomorphism between $\text{End}(\phi)$ and $\text{End}(\psi)$ which sends $g(\phi_X, \tau_L)$ to $g(\psi_X, \tau_L)$ for any $g \in \mathbf{A}_{\mathcal{H}}$. Let $\hat{\iota}$ be a u -dual isogeny for ι , for some $u \in \mathbb{F}_q[X]$ such that ι right-divides ϕ_u (see Section 1.3). Notice that for all $g \in \mathbf{A}_{\mathcal{H}}$, ϕ_u right-divides $g(\phi_X, \tau_L) \cdot \hat{\iota}$ if and only if ψ_u left-divides $\hat{\iota} \cdot g(\psi_X, \tau_L)$. Said otherwise, Ξ sends the ideal $\text{Hom}(\psi, \phi)\iota \subset \text{End}(\phi)$ to the ideal $\iota \text{Hom}(\psi, \phi) \subset \text{End}(\psi)$. By considering the isomorphism $\Xi_{1,2} : \text{End}(\phi_1) \rightarrow \text{End}(\phi_2)$ and by using the

commutativity of $\text{End}(\phi_2)$, we obtain

$$\begin{aligned}
\text{Hom}(\phi_3, \phi_2)\iota_2 \cdot \Xi_{1,2}(\text{Hom}(\phi_2, \phi_1)\iota_1) &= (\text{Hom}(\phi_3, \phi_2)\iota_2) \cdot (\iota_1 \text{Hom}(\phi_2, \phi_1)) \\
&= (\iota_1 \text{Hom}(\phi_2, \phi_1)) \cdot (\text{Hom}(\phi_3, \phi_2)\iota_2) \\
&= \Xi_{1,2}(\text{Hom}(\phi_3, \phi_2) \text{Hom}(\phi_2, \phi_1)\iota_2\iota_1) \\
&\subset \Xi_{1,2}(\text{Hom}(\phi_3, \phi_1)\iota_2\iota_1).
\end{aligned}$$

To conclude, we use the properties of the norm of isogenies: the norm is multiplicative [19, Lem. 3.10.(i)] and it corresponds to the norm of the associated ideal in $\mathbf{A}_{\mathcal{H}}$ [19, Lem. 3.10.(iv)]. Consequently, the norms on both sides of the inclusion are equal. This implies that the last inclusion is in fact an equality. \square

Algorithm 5 (ISOGENYTOIDEAL) computes prime factors of the ideal in $\mathbf{A}_{\mathcal{H}}$ corresponding to the given isogeny (of norm coprime to \mathfrak{p}), in order to recover the full factorization. Each prime non-principal factor is treated independently by the subroutine PRIMEISOGENYTOPRIMEIDEAL (Algorithm 4).

Algorithm 4: PRIMEISOGENYTOPRIMEIDEAL

Input:

- An ordinary Drinfeld module $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)_{\xi}$,
- A monic prime $r \in \mathbb{F}_q[X]$ such that $r \notin \mathfrak{p}$,
- An r -isogeny $\iota : \phi \rightarrow \psi$ between $\phi, \psi \in \text{Dr}_2(\mathbb{F}_q[X], L)_{\xi}$.

Output: A polynomial $v \in \mathbb{F}_q[X]$ such that the left-ideal $\langle \phi_r, \tau_L - \phi_v \rangle \subset L\{\tau\}$ is generated by ι .

- 1 $y \leftarrow$ remainder in the right-division of τ_L by ι ;
 - 2 $\iota^{(0)} \leftarrow 1$;
 - 3 **for** $1 \leq n \leq \deg(r)$ **do**
 - 4 $\left[\iota^{(n+1)} \leftarrow$ remainder in the right-division $\phi_T \cdot \iota^{(n)}$ by ι ;
 - 5 using linear algebra, find $(v_0, \dots, v_{\deg(r)-1}) \in \mathbb{F}_q^{\deg(r)}$ such that
 $y - (v_0\iota^{(0)} + \dots + v_{\deg(r)-1}\iota^{(\deg(r)-1)}) = 0$;
 - 6 **return** $v_0 + v_1X + \dots + v_{\deg(r)-1}X^{\deg(r)-1}$.
-

Algorithm 5: ISOGENYTOIDEAL

Input:

- An ordinary Drinfeld module $\phi \in \text{Dr}_2(\mathbb{F}_q[X], L)_{\xi}$,
- A (non-necessarily prime) monic polynomial $u \in \mathbb{F}_q[X]$, such that $u \notin \mathfrak{p}$,
- A u -isogeny $\iota : \phi \rightarrow \psi$ between ordinary Drinfeld modules in $\text{Dr}_2(\mathbb{F}_q[X], L)_{\xi}$,

Output: A factorization of the ideal $\mathfrak{a} \subset \mathbb{F}_q[X, Y]/(\xi)$ associated to ι in Lemma 3.8.

- 1 **if** $u = 1$ **then**
 - 2 $\left[$ **return** $\mathbb{F}_q[X, Y]/(\xi)$.
 - 3 $r \leftarrow$ a nonconstant monic prime factor of u ;
 - 4 $\tilde{\iota} \leftarrow \text{rgcd}(\iota, \phi_r)$;
 - 5 **if** $\tilde{\iota} = 1$ **then**
 - 6 $\left[$ **return** $\text{ISOGENYTOIDEAL}(\phi, u/r^{\text{val}_r(u)}, \iota)$.
 - 7 **else if** $\tilde{\iota} = \lambda\phi_r$ for some $\lambda \in L^{\times}$ **then**
 - 8 $\left[$ **return** $\langle r(\overline{X}) \rangle \cdot \text{ISOGENYTOIDEAL}(\phi, u/r, \iota \cdot \phi_r^{-1})$.
 - 9 **else**
 - 10 $v \leftarrow \text{PRIMEISOGENYTOPRIMEIDEAL}(\phi, r, \tilde{\iota})$;
 - 11 $\tilde{\phi} \leftarrow$ the codomain of $\tilde{\iota}$, computed from ϕ and $\tilde{\iota}$ with Formulas (1.1);
 - 12 $\left[$ **return** $\langle u(\overline{X}), \overline{Y} - v(\overline{X}) \rangle \cdot \text{ISOGENYTOIDEAL}(\tilde{\phi}, u/r, \iota \cdot \tilde{\iota}^{-1})$.
-

In what follows, θ is a feasible exponent for matrix multiplication in L , satisfying $2 \leq \theta \leq 3$.

Algorithm 5 involves the factorization of a polynomial $u \in \mathbb{F}_q[X]$. We choose to use the Cantor-Zassenhaus algorithm [3], a Las Vegas probabilistic algorithm with expected complexity bounded above by $\tilde{O}(\delta^2 + \delta \log q)$, where δ is the degree of the input. Another possibility was to use Berlekamp's algorithm, which is deterministic. However, with a complexity dominated by δ^θ , its use would severely hinder the overall complexity of the algorithm. Finally, the complexities for Algorithms 4 and 5 will be expressed in terms of d , q , and the degree of the input polynomial r (resp. u). As ι is an r -isogeny (resp. u), its degree is bounded by that of r (resp. u).

Before proving the correctness of Algorithm 4, we need the following technical lemma:

Lemma 3.9. *If there exists an isogeny of norm $r \notin \mathfrak{p}$ between two finite Drinfeld \mathbf{A} -modules ϕ and ψ , then $\text{rgcd}(\text{Hom}(\psi, \phi)) = 1$.*

Proof. Let $f : \phi \rightarrow \psi$ be an r -isogeny, with $r \notin \mathfrak{p}$. Set $V = \bigcap_{u \in \text{Hom}(\psi, \phi)} \text{Ker}(u)$, and let g be an isogeny in $\text{Hom}(\psi, \phi)$. The sequence of A -modules $0 \rightarrow V \rightarrow \text{Ker}(g) \rightarrow \text{Ker}(g)/V \rightarrow 0$ is exact, so that $\chi(V)$ divides $\chi(\text{Ker}(g))$, where χ is the Euler-Poincaré characteristic, see Section 1.3. Consequently, $\chi(V)\mathfrak{p}^{h(g)/\deg(\mathfrak{p})}\mathfrak{n}(f)$ divides $\mathfrak{n}(fg)$. In particular, $\chi(V)\mathfrak{n}(f) \mid \mathfrak{n}(fg)$. By [19, Lem. 3.10.(iv)], we have $\sum_{g \in \text{Hom}(\psi, \phi)} \mathfrak{n}(fg) = \mathfrak{n}(f)$. Since $\mathfrak{n}(f) \neq (0)$, $\chi(V)$ must equal \mathbf{A} and hence $V = 0$.

Then $\text{Ker}(\text{rgcd}(\text{Hom}(\psi, \phi))) = V$ is trivial, which implies that $\text{rgcd}(\text{Hom}(\psi, \phi))$ divides $\tau^{\deg(\mathfrak{p})^\ell}$ for some $\ell \in \mathbb{Z}_{\geq 0}$. Since $r \notin \mathfrak{p}$, the r -dual \hat{f} of f is separable (it has norm $r \notin \mathfrak{p}$), hence $\text{rgcd}(\text{Hom}(\psi, \phi)) = 1$. \square

Proposition 3.10. *Algorithm 4 (PRIMEISOGENYTOPRIMEIDEAL) is correct.*

Proof. First, we notice that since r is prime, the norm of ι must be the ideal $(r) \subset \mathbb{F}_q[X]$, and hence $\deg_\tau(\iota) = \deg(r)$. Since ι is an r -isogeny, $\phi_r \in \text{Hom}(\psi, \phi)\iota$. Since $\mathbf{A}_{\mathcal{H}}$ is a Dedekind ring in a quadratic extension of $\mathbb{F}_q(X)$, the ideal $\text{Hom}(\psi, \phi)\iota$ — regarded as an ideal in $\mathbf{A}_{\mathcal{H}}$ by Lemma 3.8 — contains the prime r . Therefore, it can only be either the full ring $\mathbf{A}_{\mathcal{H}}$, the principal ideal $\langle r \rangle$, or a prime ideal of degree 1 above $\langle r \rangle$.

By Lemma 3.9, the left-ideal in $L\{\tau\}$ generated by elements in $\text{Hom}(\psi, \phi)\iota$ equals $L\{\tau\}\iota$, which is neither the full ring $L\{\tau\}$, nor $L\{\tau\}\phi_r$ since $\deg_\tau(\phi_r) = 2 \deg(r) > \deg(\iota)$. Consequently, using the correspondence in Lemma 3.8, $\text{Hom}(\psi, \phi)\iota$ must be a degree-1 prime ideal above the principal ideal associated to r . Said otherwise, the polynomial $Y^2 + h(X)Y - f(X)$ factors over $(\mathbb{F}_q[X]/(r))[Y]$, and a prime ideal above $\langle r \rangle$ in $\mathbf{A}_{\mathcal{H}}$ has the form $\langle r(\overline{X}), Y - v(\overline{X}) \rangle$, where $v \in \mathbb{F}_q[X]$ satisfies $\xi(\overline{X}, \overline{v}) = 0$ in $\mathbb{F}_q[X]/(r)$. Note that up to reducing v modulo r , we can assume that $\deg(v) < \deg(r)$; under this assumption, v is uniquely defined.

We now prove that the coefficients of v satisfy the equality in Step 5, so that it can indeed be computed via linear algebra. To this end, we need to prove that ι right-divides $\tau_L - \phi_v$. This is a direct consequence of the fact that the ideal $\text{Hom}(\psi, \phi)\iota \subset \text{End}(\phi)$ corresponds to the ideal $\langle r(\overline{X}), Y - v(\overline{X}) \rangle \subset \mathbf{A}_{\mathcal{H}}$. \square

Algorithm 5 needs as input a polynomial $u \in \mathbb{F}_q[X]$ such that ι right-divides ϕ_u . It can be found by looking for a non-trivial \mathbb{F}_q -linear relation between the remainders of $\phi_{X^0}, \phi_{X^1}, \dots, \phi_{X^\ell}$ in the right-division by ι . When $\ell \geq \deg_\tau(\iota)$, such a non-trivial linear combination exists.

Proposition 3.11. *Algorithm 5 (ISOGENYTOIDEAL) terminates and is correct.*

Proof. The proof is done by induction on the degree of u . The termination comes from the fact that the degree of u decreases in each recursive call.

By Lemma 3.8, there is a uniquely defined ideal $\mathfrak{a} \subset \mathbf{A}_{\mathcal{H}}$ corresponding to ι . Since $\mathbf{A}_{\mathcal{H}}$ is Dedekind (Lemma 3.1), \mathfrak{a} factors as a product of prime ideals. For $r \in \mathbb{F}_q[X]$ an irreducible polynomial, we let \mathfrak{a}_r denote the product of all primes in the factorization of \mathfrak{a} which contain $\overline{r} \in \mathbf{A}_{\mathcal{H}}$. Consequently, since $\overline{u} \in \mathfrak{a}$, we have

$$\mathfrak{a} = \prod_{\substack{r \text{ prime} \\ r \text{ divides } u}} \mathfrak{a}_r.$$

Let r be a prime factor of u . Then there are three possible cases, depending on whether r is inert, splits, or ramifies in $\mathbf{A}_{\mathcal{H}}$.

If r is inert, then $\mathfrak{a}_r = \langle \bar{r} \rangle^\ell$ for some $\ell \geq 0$. If $\ell = 0$ then $\mathfrak{a}_r = \mathbf{A}_{\mathcal{H}}$. In this case, if $u \neq 1$, then $\bar{r} \notin \mathfrak{a}$ and therefore $\text{rgcd}(\iota, \phi_r) = 1$. Consequently, \bar{r} is invertible in \mathfrak{a} , and therefore $\bar{u}/\bar{r}^{\text{val}_r(u)}$ belongs to \mathfrak{a} and we can apply our induction hypothesis. If $\ell > 0$, then \bar{r} divides all elements in \mathfrak{a} . Therefore ϕ_r right-divides ι and hence $\tilde{\iota} = \lambda\phi_r$ for some $\lambda \in L^\times$. Since ϕ_r is an endomorphism of ϕ , $\iota \cdot \phi_r^{-1}$ is a well-defined isogeny between ϕ and ψ and its corresponding ideal in $\mathbf{A}_{\mathcal{H}}$ is $\{g : g \in \mathbf{A}_{\mathcal{H}} \mid g \cdot \bar{r} \in \mathfrak{a}\}$. This ideal contains \bar{u}/\bar{r} , hence we can apply our induction hypothesis.

If r splits then the ideal $\langle \bar{r} \rangle \subset \mathbf{A}_{\mathcal{H}}$ factors as a product $\mathfrak{p}_1 \cdot \mathfrak{p}_2$ of two distinct prime ideals. Therefore, $\mathfrak{a}_r = \mathfrak{p}_1^\alpha \cdot \mathfrak{p}_2^\beta$ for some $\alpha, \beta \geq 0$. First, if both α and β are nonzero, then $\mathfrak{a}_r = \langle \bar{r} \rangle \cdot \mathfrak{p}_1^{\alpha-1} \mathfrak{p}_2^{\beta-1}$. Consequently, ι is right-divisible by ϕ_r , $\tilde{\iota} = \lambda\phi_r$ for some $\lambda \in L^\times$ and we can apply our induction hypothesis on the isogeny $\iota \cdot \phi_r^{-1}$. Now, we study the case where either α or β is zero. Without loss of generality, let us assume that $\beta = 0$. Then $\mathfrak{a}_r = \mathfrak{p}_1^\alpha$. In this case, $\tilde{\iota}$ cannot be right-divisible by ϕ_r : this would contradict the fact that $\langle \bar{r} \rangle$ does not divide \mathfrak{a} . On the other hand, $\tilde{\iota}$ cannot equal 1 since for any element $g \in \mathfrak{p}_1$, $g(\phi_X, \tau_L)$ must right-divide both ϕ_r and ι . Since ι is an isogeny, $\text{Ker}(\iota)$ is an $\mathbb{F}_q[X]$ -submodule of \bar{L} (for the module law induced by ϕ), and hence so is $\text{Ker}(\tilde{\iota}) = \text{Ker}(\iota) \cap \text{Ker}(\phi_r)$. Consequently, $\tilde{\iota}$ is an isogeny from ϕ to some other Drinfeld module $\phi' \in \text{Dr}_2(\mathbb{F}_q[X], L)_\xi$. The Drinfeld module ϕ' can be computed using Formulas (1.1), and the ideal corresponding to this isogeny can be computed using Algorithm 4, which is correct by Proposition 3.10. To apply the induction hypothesis on ℓ , it remains to prove that $\iota' := \iota \cdot \tilde{\iota}^{-1}$ defines an isogeny $\iota' : \phi' \rightarrow \psi$ which right-divides $\phi'_{u/r}$. To this end, let ι_{dual} denote the dual u -isogeny of ι , and let $\tilde{\iota}_{\text{dual}}$ be the dual r -isogeny of $\tilde{\iota}$. We have

$$\begin{aligned} \phi'_u \phi'_r &= \tilde{\iota} \cdot \tilde{\iota}_{\text{dual}} \cdot \phi'_u &= \tilde{\iota} \cdot \phi_u \cdot \tilde{\iota}_{\text{dual}} &= \tilde{\iota} \cdot \iota_{\text{dual}} \cdot \iota \cdot \tilde{\iota}_{\text{dual}} \\ &= \tilde{\iota} \cdot \iota_{\text{dual}} \cdot \iota' \cdot \tilde{\iota} \cdot \tilde{\iota}_{\text{dual}} &= \tilde{\iota} \cdot \iota_{\text{dual}} \cdot \iota' \cdot \phi'_r. \end{aligned}$$

By dividing on the right by ϕ'_r , we obtain that ι' divides ϕ_u and that it is the u -dual of the composed isogeny $\tilde{\iota} \cdot \iota_{\text{dual}}$. This proves that ι' is a well-defined isogeny. By using the second statement in Lemma 3.8, we obtain that the ideal associated to ι' is

$$\mathfrak{p}_1^{\alpha-1} \cdot \prod_{\substack{r' \text{ prime} \\ r' \text{ divides } u \\ r' \neq r}} \mathfrak{a}_{r'},$$

which contains \bar{u}/\bar{r} , so that we can apply our induction hypothesis.

Finally, the ramified case is proved similarly than the split case. The main difference is that $\mathfrak{p}_1 = \mathfrak{p}_2$, so that $\mathfrak{a}_r = \langle r \rangle^\ell \cdot \mathfrak{p}_1^\alpha$, for some $\ell \geq 0$ and $\alpha \in \{0, 1\}$; this does not change the proof. \square

Proposition 3.12. *Let m denote the degree of r . Algorithm 4 (PRIMEISOGENYTOPRIMEIDEAL) requires $O(dm^\theta)$ operations in L and $O(dm + m^2)$ applications of the Frobenius endomorphism.*

Proof. Computing the first remainder costs $O(dm)$ operations in L , and $O(dm)$ applications of the Frobenius endomorphism. The other remainders are computed recursively. Knowing $\iota^{(n)}$, computing $\iota^{(n+1)} = \phi_T \cdot \iota^{(n)}$ requires $O(m)$ operations in L , and the same number of Frobenius applications. This Ore polynomial has degree at most $\deg_\tau(\iota) + 1$. By Lemma 3.4, computing this remainder requires $O(\deg_\tau(\iota)) = O(m)$ operations in L , and as much applications of the Frobenius. Consequently, computing all elements in the loop requires $O(m^2)$ operations in L and $O(m^2)$ applications of the Frobenius endomorphism.

The last costly step is solving a linear system. More precisely, the algorithm finds a solution of an affine system over \mathbb{F}_q , whose associated matrix has less than dm rows, and m columns. Solving such a system requires $O(dm^\theta)$ operations in L . In total, we get $O(dm^\theta)$ operations in L , and $O(dm + m^2)$ applications of the Frobenius endomorphism. \square

Proposition 3.13. *Let m denote the degree of u . Using the Cantor-Zassenhaus algorithm for polynomial factorization, Algorithm 5 (ISOGENYTOIDEAL) is a probabilistic Las Vegas algorithm requiring $\tilde{O}(dm^\theta + m^3 + m \log(q))$ expected operations in L and $O(dm + m^3)$ expected applications of the Frobenius endomorphism.*

Proof. Step 3 is performed using the Cantor-Zassenhaus algorithm, with expected cost bounded by $\tilde{O}(m^2 + m \log(q))$. Notice also that the initial factorization of u may be performed only once for this cost, at the first call of the algorithm. Then Step 4 performs an Ore Euclidean division, which costs $O(m^2)$ operations in L and $O(m^3)$ applications of the Frobenius endomorphism using Euclid's algorithm (Lemma 3.6).

If $\tilde{\iota}$ is 1 or $\lambda\phi_r$, we just need to compute a polynomial division, and $\iota \cdot \phi_r^{-1}$ in the latter case. These computations do not exceed the complexity of Step 4. No other computation is performed and the algorithm is recursively called on a smaller instance. If $\tilde{\iota}$ is neither 1 or $\lambda\phi_r$, then Algorithm 4 is called. Let $u = r_1 \cdots r_\ell$ be a factorization of u (r_i prime, not necessarily distinct), and let $k_i := \deg(r_i)$. We can assume that the prime factors are ordered as the algorithm processes them. Then, counting all the recursive calls of the algorithm, we get that the total expected cost is bounded above by

$$\begin{aligned} & \tilde{O}(m^2 + m \log(q)) + \sum_{i=1}^{\ell} \left(O \left(\sum_{j=i}^{\ell} k_j^2 \right) + O(d k_i^\theta) \right) \text{ operations in } L, \\ & \sum_{i=1}^{\ell} \left(O \left(\sum_{j=i}^{\ell} k_j^2 \right) + O(d k_i + k_i^2) \right) \text{ applications of the Frobenius endomorphism.} \end{aligned}$$

Since $\sum_{i=1}^{\ell} k_i = m$, we obtain that these formulas are bounded above by $\tilde{O}(dm^\theta + m^3 + m \log(q))$ expected operations in L and $O(dm + m^3)$ expected applications of the Frobenius endomorphism. \square

Remark 3.14. *One could ask whether the complexities of Propositions 3.7, 3.12 and 3.13 may be enhanced by using more efficient algorithmic primitives for the arithmetic of Ore polynomials. In [7] and [6], Caruso and Le Borgne provide new algorithms for Ore polynomial multiplication, Euclidean division and right-greatest common divisor; in many applications, those algorithms yield substantial speed-ups. We highlight that the authors work under the hypothesis that q is fixed and that operations in L as well as applications of the Frobenius and L cost $\tilde{O}(d)$ operations in \mathbb{F}_q .*

Let us first ask ourselves if we can enhance Algorithm 1 by using asymptotically fast Ore Euclidean division at Step 3. Per [6, Prop. 3.1], computing ι would cost $\tilde{O}(\text{SM}^{\geq 1}(d, d))$ operations in \mathbb{F}_q , where $\text{SM}^{\geq 1}$ is a function introduced in [6, Sec. 3]. Using the values provided by the authors¹, we get $\text{SM}^{\geq 1}(d, d) = d^{\frac{9-\theta}{5-\theta}}$. Even if we used $\theta = 2$, we would get $\frac{9-\theta}{5-\theta} = \frac{7}{3}$, and the computation of ι would then be outweighed by the computations of \tilde{u} and \tilde{v} , which both cost $\tilde{O}(d^3)$ operations in \mathbb{F}_q in the complexity model of loc. cit. Therefore, using the algorithmic primitives of [7] and [6] does not at the moment improve the complexity bound in Proposition 3.7. To benefit from those, one would need to enough reduce the cost of computing \tilde{u} and \tilde{v} . Our attempts to do so were unsuccessful.

The situation for Algorithm 4 is quite similar. The first step requires computing the remainder in the Euclidean division of τ_L by ι . As ι has degree $O(m)$ and τ_L has degree d , the computation would require $\tilde{O}(\text{SM}^{\geq 1}(d + m, d))$ operations in \mathbb{F}_q . With the formula for $\text{SM}^{\geq 1}$ in [6, Sec. 3], this is $\tilde{O}((d + m)d^{\frac{4}{5-\theta}})$ operations in \mathbb{F}_q . Adding to that the $O(d^2 m^\theta)$ operations in \mathbb{F}_q required to solve the system, there is no benefit in using the algorithms of [6]. In fact, doing so would actually worsen the asymptotic complexity with respect to the variable d . This is due to the fact that the bound is linear with respect to $d + m$ for fixed d , but it has a costly dependence with respect to d .

The complexity of Algorithm 5 depends on that of Algorithm 4, and our conclusion is the same.

3.3 An explicit computation

To demonstrate the practical effectivity of Algorithm 1 (GROUPACTION), we have implemented the group action for a hyperelliptic curve of genus 260 defined over \mathbb{F}_2 .

¹Private discussions with the authors revealed that the critical exponent $\frac{5-\theta}{2}$ was mistyped in [6], and should instead be $\frac{2}{5-\theta}$.

Our C++/NTL code is available at <https://gitlab.inria.fr/pspaenle/crs-drinfeld-521>.

Set $L = \mathbb{F}_2[X]/\mathfrak{p}$, where \mathfrak{p} is the ideal generated by $X^{521} + X^{32} + 1 \in \mathbb{F}_2[X]$. We encode polynomials using the hexadecimal NTL notation: for instance, `0x4bc` denotes $X^2 + X^4 + X^5 + X^7 + X^{10} + X^{11} \in \mathbb{F}_2[X]$. By extension, we also denote elements in L by the NTL hexadecimal convention, implicitly using the reduction modulo the ideal \mathfrak{p} . Our isomorphism class of Drinfeld modules has j -invariant (in L)

$$j_0 = \begin{array}{l} \text{0xb985b4ce23bd9cf992f1176e17c27dab7ae67270131} \\ \text{12a2804cb64abccc7cce061e12786bb3248809922da} \\ \text{35d3b624d67d08087e07c260fcaa9807a420ca83fa95.} \end{array}$$

The coefficients of the characteristic polynomial of the Frobenius endomorphism of the Drinfeld module $\phi \in \text{Dr}_2(\mathbb{F}_2[X], L)$ defined by $\phi_X = j_0^{-1}\tau^2 + \tau + \omega$ are:

$$\begin{cases} h = \begin{array}{l} \text{0xb1ffea4ab7e58b96adf4e4972d7db918} \\ \text{4821c1d64b375df52669c60973bb80dee} \end{array} \in \mathbb{F}_2[X], \\ f = X^{521} + X^{32} + 1 \in \mathbb{F}_2[X]. \end{cases}$$

The polynomial $Y^2 + h(X)Y - f(X)$ defines a genus-260 hyperelliptic curve \mathcal{H} over \mathbb{F}_2 , whose Picard group $\text{Pic}^0(\mathcal{H})$ is cyclic and has almost-prime order

$$2 \times 315413182467545672604116316415047743350494962889744865259442943656024073295689.$$

This group order was computed using the Magma implementation of the Denef-Kedlaya-Vercauteren algorithm [24, 13]. This computation costs 53 hours on a Intel(R) Xeon(R) CPU E7-4850.

We ran experiments for computing the group action on a laptop (Intel i5-8365U@1.60GHz CPU, 8 cores, 16 GB RAM). We chose an element of $\text{Pic}^0(\mathcal{H})$ at random such that the u -polynomial in the Mumford coordinates is irreducible and has degree 35. The most costly step in practice is the first step of Euclid's algorithm: it starts by computing τ^{521} modulo ϕ_u , which has τ -degree 70. Unfortunately, in our non-commutative setting we cannot use binary exponentiation to speed-up this step: $(P_1 + L\{\tau\}Q) \cdot (P_2 + L\{\tau\}Q)$ need not equal $P_1 P_2 + L\{\tau\}Q$ for $P_1, P_2, Q \in L\{\tau\}$. Therefore, we implemented a parallelized subroutine specialized for this task. By using the 8 cores of the laptop, computing this group action takes 24 ms.

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