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On Large Games with a Bio-Social Typology*

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Abstract: We present a comprehensive theory of large non-anonymous games in which agents have a name and a determinate social-type and/or biological trait to resolve the dissonance of a (matching-pennies type) game with an exact pure-strategy Nash equilibrium with finite agents, but without one when modeled on the Lebesgue unit interval. We (i) establish saturated player spaces as both necessary and sufficient for an existence result for Nash equilibrium in pure strategies, (ii) clarify the relationship between pure, mixed and behavioral strategies via the exact law of large numbers in a framework of Fubini extension, (iii) illustrate corresponding asymptotic results.

(99 words)

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1 Introduction

The theory of large games, in both its anonymous and non-anonymous formulations, is by now well-understood.[1] Under the assumption that a player's payoffs depend, in addition to her own action, on a statistical summary, be it an average or a distribution, of the plays of everyone else in the game, the basic thrust of the theory is its focus on pure strategy equilibria. Indeed, this constitutes the *raison d'etre* of the theory and is easily justified by virtue of the fact that pure strategy equilibria do not necessarily exist in games with a finite set of players.[2] The two distinguished, and defining, features of the theory are a player's *numerical negligibility* and her *societal interdependence* in the original Nash formulation being substituted by more composite and aggregate measures of the actions of everyone else in the game. Within such a rubric, results on the existence of pure strategy equilibria, as well as their asymptotic implementability and invariance to permutations of names, have been established.[3] In addition, issues concerning measurability, purification and symmetrization have been identified and resolved in terms of decisive counterexamples and attendant theorems. The resulting theory has been shown to hinge on the cardinality of the underlying action set: if it is a finite,[4] or a countably infinite, set,[5] numerical negligibility of an individual player can be successfully formalized by an arbitrary atomless probability space; if, on the other hand, it is uncountable and compact, say even the unit interval, an arbitrary probability space does not suffice, and additional structure has to be put on the formalization of agent multiplicity. Initially, such a structure was invoked through the consideration of an atomless Loeb probability space, but recent work has identified a crucial property, namely *saturation,* and shown that agent multiplicity formalized by atomless saturated probability spaces, a more general class to which atomless Loeb probability spaces belong,[6] is not only sufficient but also necessary for the results to hold. We have thereby a viable and robust theory of large games.

A technical point of departure for the theory is the fact that the players' names do not have a natural "measure of closeness" defined on them, and that therefore the

[1] For details as to terminology and bibliographic substantiation of all claims in the ensuing paragraph, except those embodied in the last sentence, see the survey chapter [32] and their references. For saturated probability spaces mentioned in the last sentence, see [24] and [28]. We have stated two characterizations for the reader's convenience as Propositions 1 and 2 below. Other references will be furnished in the sequel as the need for them arises.

[2] The matching-pennies game is a canonical, though by no means the only, example in this regard.

[3] See [31] where the invariance to permutations is labelled as the *homogeneity* property.

[4] Various results related to pure strategy Nash equilibria in the case of finite actions follow from a general purification principle formulated by Dvoretzky-Wald-Wolfowitz in early 1950s; see [29].

[5] See [50] for the relaxation of the compactness assumption on action sets in [30].

[6] Other examples are constituted by product spaces of the form $\{0,1\}^\kappa$ where $\{0,1\}$ has the uniform measure and κ is an uncountable cardinal; see [24].

space of such names has to be conceived as an abstract rather than a topological measure space. It is by now well-appreciated that this space has necessarily to be endowed with an abstract measurable structure rather than a Borel or Baire structure that is metrically-, or more generally, topologically-generated. From a substantive point of view however, this has led to a theory that, in its most basic form, is informationally sparse. It is sparse in the sense that each agent,[7] in taking the distribution of the societal actions as given, is focussed only on the proportions of agents playing certain actions, and oblivious to any measurable social-types[8] or social or biological traits by which the players could also be distinguished. This is to say that the theory, as currently formulated, ignores players' traits more generally conceived. Thus, in the context of traffic moving through a bridge or a tunnel, an individual decision-maker deciding on a possible route, again as conceived by the theory, notes only the proportion of the traffic moving through one of the two alternative routes, and disregards as irrelevant to her decision all other information concerning traffic-traits, including the proportion of truck-drivers and the unruliness of driving patterns. In another illustrative context, in deciding which of two possible candidates to vote for, an agent, as conceived by the theory, is concerned only with the proportion of the electorate that votes for either of the candidates, and thereby neglects all other information relating to a voter's biological or socioeconomic traits, proxied by variables that may be continuous and/or discrete. In short, the theory ignores race or gender as relevant categories for the equilibrium outcome.

To be sure, even the very early extensions of the theory did consider situations where the space of players' names could be conceived as being finitely partitioned, and an agent's preferences taken to depend on the mean or the distribution of the profile of societal actions when this profile is *restricted* to each individual element of this partition.[9] Indeed, recent work has shown that the results can be extended to partitions with a countably infinite set of elements. This is of course important in that in removing the finiteness assumption, it removes an arbitrary and uniform bound on the grouping of agents.[10] There are sublists of names, finite or countably infinite, according to which a

[7]In the sequel, and especially in the informal discussion, we shall use "agent" interchangeably with "player," and "society" interchangeably with "all the players."

[8]Note that we are avoiding the word *type* on its own, using it always with its hyphenated counterpart. This is being done to keep away from the notion of *Harsanyi types* that refers primarily to player beliefs. As elaborated below, we have the sociological and, older biological, pre-game-theory usage in mind; the reader can see, for example, the first essay in [25] and track his use of the word *type,* and note that we use the word *trait* in conformity with his pre-Harsanyi notion of *type.* Hence also the composite *bio-social typology* in the title. We defer for future work, once a satisfactory formulation of the theory with a multiplicity of social-types is available, an integration of beliefs; however, see [47].

[9]This case of a finite partition goes back to Remark 2 in [42]; for subsequent work, see the references in [32] and subsequent to that survey, in [50].

[10]See Chapter 5 in [22], and note that his dissertation presents an extension in the context of both countable and uncountable compact metric action sets.

continuum of agents is grouped, and an agent is dependent on society's plays only to the extent that she is dependent on the summary statistics pertaining to the individuals in each of these exogenously classified sublists or subgroups. However, nothing is said as regards how such sublists are determined, which is to say that the grouping of names is not tethered to observable and quantitatively measurable bio-social traits of the players. The players within the same subgroup can be interpreted as having the same social-type. However, there could be cases where the number of distinct social-types is not discrete.For example, the payoffs may depend on societal summary statistics that involves a continuum of socioeconomic traits, possibly income levels in the voting context, or vehicle-tonnage in the traffic example. For example, markets fail in situations of adverse selection precisely when a player is unwilling to subsidize agents perceived by him or her to be ranked "below" her traits. Whether this perception is "objective" or universally subscribed to by all of the players in the game is hardly relevant; it figures in each individual's decision. To pursue the point a little more, rather than a finite or a countable infinite list of exogenous classifications, one wants to deal with situations when the particular subset of society whose summary statistics are seen as relevant for an agent's decisions, is decided on by the agents' socioeconomic traits in the group, possibly income levels in the voting context, or vehicle-tonnage in the traffic example.

Thus, inspite of the useful earlier extensions, the theory remains, what we are characterizing here as being, informationally sparse. The question then is whether it can be reformulated and extended to situations where an agent, in deciding on her individual action, has available to her comprehensive information not only on what players in a particular sublist are playing, but also on the variety of social-types or biological and other traits associated with the players in that sublist, as well as some conception of the social group she belongs to and the type of traits she shares. This is to ask, in other words, whether the theory can be generalized to take into account the fact that a player has a name as well as a social-type or trait,[11] the former being chosen from an abstract probability space $(I, \mathcal{I}, \lambda)$, the latter from a separable metric space T, and with a deterministic (measurable) function α connecting each element of one to the other, a social-type or trait to an individual name, as one of the essential constituents of the data of the large game. The essence of the reformulation then is to conceive of the summary of societal actions as distributions on the product space of actions and traits such that their marginal distribution on the space of traits is always identical to the given distribution of traits of the game, this distribution ρ being induced on the trait space T by the function α. This allows an individual player access to information based on traits as well as on

[11] As has already been emphasized above in Footnote 8, we are using *social-type* and *trait* as synonymous terms. As we shall see in the sequel, we use *characteristic* as a more general term that also covers a player's payoffs.

proportions, and with individual payoffs depending on the action set, on the one hand, and on an extended space of externalities on the other. One can now formally define a non-anonymous game and its Nash equilibrium in pure strategies, and proceed with an investigation of these reformulated objects. It is precisely such an extension that is being offered in this paper.

However, before getting into the technical difficulties of such an extension, it is important to note that our basic motivation, coming as it does from the theory of large games, dovetails into the rich theoretical and econometric work on identity and social interactions pioneered by Akerlof-Kranton [1].[12] They abstract their paper on "economics and identity" with the following words.

> This paper considers how identity, a person's sense of self, affects economic outcomes. In the utility function we propose, identity is associated with different social categories and how people in these categories should behave. We then construct a simple game-theoretic model showing how identity can affect individual interactions.

The point is that the reformulation that we study lifts the Akerlof-Kranton conception from the setting of a finite game to that of a large game, and even though there are important differences between the conception studied in this paper and theirs, there is an undeniable complementarity.[13] This can be most easily seen by considering the utility function of the i^{th}-player on which their analysis revolves:

$$U_i = U_i(\mathbf{a}_i, \mathbf{a}_{-i}, I_i), \text{ with } I_i = I_i(\mathbf{a}_i, \mathbf{a}_{-i}, \mathbf{c}_i, \epsilon_i, \mathbf{P}),$$

where $\mathbf{a}_i$ is the i^{th}-player's action, $\mathbf{a}_{-i}$ the actions of everyone else in the finite game, and the novel variable I_i which "shows how identity can be brought into economic analysis, allowing a new view of many economic problems." This "new type of externality" I_i depends on the particular social category $\mathbf{c}_i$ chosen from a set of exogenously-given categories $\mathbf{C}$, and on how the player's "own" given trait ϵ_i match the ideal of i's assigned category indicated by the exogenously-given prescriptions $\mathbf{P}$. In the context of the model of this paper, this "new type of externality" is given by the inclusion of τ in an individual player's utility function

$$U_i = U_i(a, \tau) \text{ with } \tau = \lambda(\alpha, f)^{-1} \in \mathcal{M}^{\rho}(T \times A),$$

[12] The reader can also see the subsequent surveys in [26] and [14], and their rich and diverse bibliographies.

[13] There are passages in [1] where their context is clearly that of a "large" society; see, for example, their "identity model of poverty and social exclusion" in which they conceive of a "a large community, normalized to size one, of individuals (page 740)." As such, the model reported here can be seen also be seen as presenting a rigorous formalization of their ideas.

where τ is a probability measure on actions and traits analogous to the individual identity variable I_j, and one which is endogenously obtained as a distribution of the Nash equilibrium f taking names into actions. In keeping with our large-game formulation, only a summary of societal actions play a role. There is of course more to be said in terms of a comparison between the two models, and we shall do so in the sequel.

The fact is that the extension of the theory to the model discussed in the above paragraph is not straightforward, and as shown in Example 1 below, there is no Nash equilibrium in a large non-anonymous game in which the set of players' names are formalized as an arbitrary, atomless probability space with the unit interval I as the space of traits even when the cardinality of the common action set is two! In particular, Example 1 below considers a large game of matching pennies with balanced players, where the space of names is modeled as the Lebesgue unit interval and with the i^{th} player who always tries to balance out those players younger than him or her in the sense that if there are more younger players who play Heads (Tails), then player i will play Tail (Head). With this counterexample, it is clear that additional assumptions will have to be made on the space of players names if the cardinality of the space of players' traits is not to be restricted.

It is worth emphasizing that the significance of this example for the theory of large games does not end with it closing the door to the existence of a pure strategy Nash equilibrium in situations where players have names as well as traits, and the space of players' names is formalized as any abstract probability space; to wit, the Lebesgue unit interval. It raises an equally interesting issue when discretized and cast in the form of a sequence of games with a large but finite set of players. It is now well understood that in general a discretized sequence of finite games has only an approximate equilibria in pure strategies, with the approximation becoming finer as the number of players becomes larger.[14] What is interesting in the sequence of finite-player games which is a discrete version of the game in Example 1 is that each of its elements has an *exact* Nash equilibrium in pure strategies (see Example 2 below)! Thus, an arbitrary atomless measure space of players' names, the Lebesgue unit interval in particular, is doubly inappropriate. There is no equilibrium for the idealized game even though there is an exact equilibrium for the elements of a sequence converging to the idealized game. It is thus clear that additional assumptions will have to be made on the space of players' names, and the example is decisive in establishing that an arbitrary atomless probability

[14]This is precisely the question of *asymptotic implementability* of an idealized limit game; see [32] for a discussion. If specific speeds of convergence are required, which is to say, the determination of the error for an arbitrarily given finite game, see [41]. These alternative ways of formalizing a particular kind of regularity for systems with a continuum of agents are now well-understood; in addition to Khan's Palgrave entries, see [16] for a more recent revisiting of this issue, see .

space simply will not do.

The point of departure for this paper is this additional assumption in the requirement that the space of players' names not only be atomless but that it satisfy the property of *saturation*.[15] The property is simple enough that it can be verbally stated even in an introduction oriented to a general audience: a probability space has the saturation property for a measure μ on a product of two complete, separable and metrizable (Polish) spaces, X and Y, if for any random variable taking values in one of these spaces, say X, and whose induced measure is the marginal of the given measure on that space, μ_X, there exists another *twin* random variable in the other space Y such that the induced measure of the pair of random variables is the given measure μ. An atomless probability measure is *saturated* if it has the saturation property for all X, Y and μ. Such a notion is natural for, and entirely amenable to, the theory of large games. Indeed, an atomless probability space is saturated if and only if every non-anonymous game based on an uncountable compact metric action space has a Nash equilibrium.[16] It is this property of a Loeb space that is responsible for the robust viability of the theory of large games with a compact metric action set;[17] and furthermore, it is necessary in the sense that the results are false without it. The fact that the Lebesgue unit interval does not satisfy this property then emerges as rather routine anti-climax.

With the assumption of an atomless, saturated probability space, and with the counterexample thereby bypassed and made irrelevant, one can proceed with a basic extension of the theory. And with this reformulation in hand, we can now present a comprehensive extension of the theory along its standard *desiderata* emphasized more than a decade ago: a saturated probability space is sufficient for the existence of Nash equilibrium in pure strategies; that these Nash equilibria translate into their approximate counterparts for large but finite games, which is to say that the existence result is asymptotically implementable; and that these equilibria are invariant with respect to permutations of the players' names in the game. However, we can do substantially more on two counts. The first concerns the *necessity* of saturated probability spaces. The standard *desiderata* were put forward and executed in [31] in the context of Loeb probability spaces, and there was no presumption that these results would not conceivably hold for other probability spaces.[18] In short, there was no question of a Loeb measure

[15]The saturation property of probability spaces was introduced in [24], and applied to the theory of large games in [28].

[16]This is Theorem 4.5 in [28]. Note that the authors avoid the anonymous/non-anonymous classification of large games, and use the terminology game/measure-game. Their theorem is reproduced as Proposition 2 in Section 4 and underscores our use of the word *natural* in the context of large games.

[17]We refer here of course to the available theory, one that does not take the reformulation that is being pursued in this paper, into account. The fact that that the results generalize to include the reformulation is of course the objective of this paper.

[18]Also see [32] in connection with the conventional theory.

space being necessary for the results. We now close this door by showing that saturated probability spaces are necessary and sufficient for our results on the existence of pure strategy equilibria.[19] These "necessity results," in emphasizing the irrelevance to the existence theory of pure strategy Nash equilibria of *all* probability spaces whose σ-algebras are countably generated, leave alone Lebesgue spaces, are important and new to this paper.

As regards the second count, we can go further by considering randomization and coupling our discussion of Nash equilibria in pure strategies with those in behavioral and mixed strategies, all for the reformulated large games studied here. Since the relationship between these concepts has not attained a complete resolution even for the standard theory, we first present the ideas in that set-up. If I is the space of players names, and A the common action space, a pure strategy profile is simply a measurable function from I to A, and a behavioral strategy profile a measurable function from I to the distributions on A, $\mathcal{M}(A)$.[20] Other than an implicit understanding that the randomizations of individual agents underlying the distributions in a behavioral strategy are independent of each other, as befits a theory of non-cooperative games, there has been no explicit formalization of a second probability space linking these randomizations, and perhaps no need for a recourse to one. It is a concern with a viable definition of mixed strategies that one is led to a coupling of the space of players names with a probability space formalizing the independent randomizations, a concern that meets runs against the wall of the "measurability problem". As is by now understood as a result of work done in the last fifteen years, the resolution of this measurability problem requires an extension of the product probability space so that the Fubini property holds for the extension, and leads to an exact law of large numbers (ELLN) for a continuum of random variables that are essentially pairwise-independent.[21] With this conception in hand, an elegant and simple application to the theory of large games allows one to formulate a mixed strategy profile simply as an essentially pairwise-independent process from $I \times \Omega$ to A that satisfies the Nash inequalities, and allows an equivalence between behavioral and mixed strategies. Every behavioral strategy profile can be represented by a mixed strategy, and that a mixed strategy induces, in the straightforward way, a behavioral strategy profile.

In a section titled "large games with independent idiosyncratic shocks", [32] consider an essentially pairwise-independent process f as a Nash equilibrium of a large game with random payoffs. By the ELLN, the ex-post realization f_ω forms a pure strategy Nash

[19]Thus, as long as the player space is non-saturated, there exists a large game with traits that has no equilibrium in pure strategies.

[20]Such a strategy profile is also called a measure-valued profile in [38].

[21]See [44, 46] for a rigorous development of the exact law of large numbers, and for the reference in [44] to a 1996 announcement in *The Bulletin of Symbolic Logic.* In particular, [46, Corollary 2.9]) is what is needed in this paper.

equilibrium of the corresponding ex-post large game with probability one. When one considers the special case that the random payoffs are actually deterministic, the process f is very much like the conception of a mixed strategy equilibrium as mentioned in the above paragraph. Thus the ELLN implies immediately the ex-post Nash property, i.e., the ex-post realization of a mixed strategy Nash equilibrium is essentially a pure strategy Nash equilibrium.[22] The conceptual idea goes back to Cremer-McLean [18] in the context of auctions and mechanism design; also see [35] in the context of expost Walrasian equilibria.

The rest of the paper is organized as follows. We confine Section 2 to a parsimonious description of the reformulation of a large game that has been informally described above, and show how it includes earlier formulations of large games based on an exogenously-given finite or countably infinite partitions of the space of player's name. From both the technical and conceptual points of views, it is the formulation societal dependence, or externalities in the economic theory jargon, that sets the stage for the entire analysis to follow. In Section 3, we present an example for the non-existence of Nash equilibrium in large games, and show in a discretized version pertaining to a finite game, the existence of exact Nash equilibria in pure strategies. In section 4 we show the existence of a pure strategy Nash equilibria under the hypothesis that the space of players' names is a saturated probability space and the space of traits is a separable metric space, taking care to emphasize that the assumption of a saturated probability space is necessary. We also show that the reformulated game has the homogeneity property in that the equilibrium distributions are invariant to permutations of the game that leave its distribution intact. In section 5 we spell out the relationships between the three alternative solution concepts and establish the ex-post Nash property of mixed strategy equilibria. Section 6 turns to asymptotic implementation of these equilibria and specializes the existence result to a tight sequence of large but finite economies. Section 7 concludes the paper. In the Appendix, we collect for the reader's convenience some relevant results from the earlier literature, and also relegate to it some rather technical results that are required in Sections 3 and 4.

2 A Reformulation

It is by now conventional to see a large non-anonymous game as being constituted by two basic objects: an abstract atomless probability space $(I, \mathcal{I}, \lambda)$ representing the space

[22] See [38] for an earlier discussion of relevant issues with mixed strategies in a large game setting, and [27] for an asymptotic version of the ex-post Nash property for large finite games, and following him, [17].

of player names, and a compact metric space A representing a common action space. The common action space is then used to build a space of payoff characteristics, thereby leading to a definition of a game and its Nash equilibrium. When endowed with its Borel σ-algebra, the action set leads to the Borel measurable space $(A, \mathcal{B}(A))$, and through it, to the space $\mathcal{M}(A)$ of all probability measures on A endowed with its weak topology.[23] The space $\mathcal{M}(A)$ is then also a compact metric space, and it represents the distributions of possible plays in the game. In terms of the vocabulary used in the introduction, it represents "externalities" or "society's plays". The space of players' payoffs $\mathcal{U}_A$ is then given by the space of all continuous functions on the product space $(A \times \mathcal{M}(A))$, and based on its sup-norm topology and endowed with its resulting Borel σ-algebra, it can also be conceived as a measurable space $(\mathcal{U}_A, \mathcal{B}(\mathcal{U}_A))$ of players' characteristics. Note that the space of players' names I does not figure in the space of players' characteristics. A large *non-anonymous* game is then a random variable in $\mathcal{U}_A$ and its *Nash equilibrium* a random variable in A. More formally,

Definition 1. *A large non-anonymous game is a measurable function $\mathcal{G}^0$ from I to $\mathcal{U}_A$. A Nash equilibrium of a game $\mathcal{G}^0$ is a measurable function $f : I \longrightarrow A$, such that for λ-almost all $i \in I$, and with u_i abbreviated for $\mathcal{G}^0(i)$,*

$$u_i\left(f(i), \lambda f^{-1}\right) \geq u_i\left(a, \lambda f^{-1}\right) \text{ for all } a \in A.$$

All this is now standard.[24]

The reformulation of a large non-anonymous game that we study in this paper rests on four, rather than two, basic objects: in addition to space of players' names I and the common action set A, we work with a complete, separable metrizable (Polish) space T representing a space of possible player traits, and endowed with a probability measure ρ on the Borel σ-algebra $\mathcal{B}(T)$ induced by the topology on T. The game and its Nash equilibria are then built up from these four basic objects. The first important element in the reformulation is that the externalities or society's plays are now conceived as a probability measure on the product space $(T \times A)$, with the latter endowed with its product Borel σ-algebra,[25] and such that the marginal of the measure on T is identical to the given measure ρ on T. Formally, let $\mathcal{M}(T \times A)$ be the space of Borel probability distributions on $T \times A$, and $\mathcal{M}^\rho(T \times A)$ be the subspace of $\mathcal{M}(T \times A)$ such that for any $\tau \in \mathcal{M}^\rho(T \times A)$, its marginal probability on T, $\tau_{T} = \rho$. Note that unlike the space

[23] We conform to standard usage and forgo referring to this as the weak*-topology, the formally correct designation.

[24] In addition to [32], see [46] and [28].

[25] Since we are assuming separability everywhere, the product Borel σ-algebra is the same as the Borel σ-algebra of the product of the spaces T and A.

$\mathcal{M}(A)$, the space $\mathcal{M}(T \times A)$ is not necessarily compact in the weak topology. However it is a standard result that the space $\mathcal{M}^\rho(T \times A)$ is indeed compact.

Now, just as in the conventional theory described in the first paragraph of this section, the space of players' payoffs $\mathcal{V}_{(A,T,\rho)}$ is then given by the space of all continuous functions on the product space $(A \times \mathcal{M}^\rho(T \times A))$, and based on its sup-norm topology and endowed with its resulting Borel σ-algebra, it can also be conceived as a measurable space $(\mathcal{V}_{(A,T,\rho)}, \mathcal{B}(\mathcal{V}_{(A,T,\rho)}))$ of players' payoffs. The point to be made here is that in the generalized setting with socioeconomic traits, this does not exhaust the set of player characteristics which is now defined as the product $T \times \mathcal{V}_{(A,T,\rho)}$. Note that here again the space of players' names I does not figure in the space of players' characteristics. We can now present a reformulation of a large non-anonymous game that again involves a random variable, but one taking values in a richer target space.

Definition 2. *A large non-anonymous game with traits is a measurable function $\mathcal{G}$ from I to $T \times \mathcal{V}_{(A,T,\rho)}$ such that $\lambda\mathcal{G}_1^{-1} = \rho$, where $\mathcal{G}_i$ is the projection of $\mathcal{G}$ on its i^{th}-coordinate, $i = 1, 2$. A Nash equilibrium of a game $\mathcal{G}$ is a measurable function $f : I \longrightarrow A$, such that for λ-almost all $i \in I$, and with v_i abbreviated for $\mathcal{G}_2(i)$, and $\alpha : I \longrightarrow T$ abbreviated for $\mathcal{G}_1$,*

$$v_i\left(f(i), \lambda(\alpha, f)^{-1}\right) \geq v_i\left(a, \lambda(\alpha, f)^{-1}\right) \text{ for all } a \in A.$$

On decomposing the target space $(T \times \mathcal{V}_{(A,T,\rho)})$, we see that in our reformulation, a large non-anonymous game, is really a pair of random variables, one a measurable function $\alpha = \mathcal{G}_1$ from I to T associating each player $i \in I$ with her traits, or rather an array of traits $\alpha(i) \in T$, and the other, a measurable function $\mathcal{G}_2$ associating each player $i \in I$ with her payoff $\mathcal{G}_2(i) \in \mathcal{V}_{(A,T,\rho)}$, one that we are also referring to her trait.[26]

In the context of games with exogenously-given, finite or countably-infinite, partitions of the space of names, the earlier work of [30, 31], and its generalizations in [22], has already been mentioned in the introduction. We conclude this section by showing how the reformulation presented above subsumes these efforts as it moves forward. Specifically, we show how conventional formulations available in the literature are special cases of Definition 2. The issue revolves around moving back and forth from the conventional form of a large game where the externalities parameters are based on the Cartesian product of probability measures on the action space, the index of the product running across the index of a countable, possible infinite, partition of the space of names, to one where it is based on a joint probability measure on the space of actions and traits. Since the space of names and the common action set are shared by both formulations, this reduces to moving back and forth between the payoff functions.

[26] See Footnotes 8 and 11 in this connection.

Towards this end, let $\{t_k\}_{k\in K}$ be a list of all the elements of T, where K is an index set which is at most countable.[27] We have to relate a function $v \in \mathcal{V}_{(A,T,\rho)}$ to another function $u \in \mathcal{U}_A^T$ where[28] $\mathcal{U}_A^T$ is the space of real-valued continuous functions on $A \times \prod_{t\in T} \mathcal{M}(A)$. Define a function

$$\Phi : \mathcal{M}^\rho(T \times A) \longrightarrow \prod_{t\in T} \mathcal{M}(A) \text{ such that } \Phi(\tau) = \{\mu_t\}_{t\in T},$$

where for any t in T,

$$\mu_t(B) = \tau(\{t\} \times B)/\rho(t) \text{ for all } B \in \mathcal{B}(A).$$

Now consider the function

$$\Psi : \prod_{t\in T} \mathcal{M}(A) \longrightarrow \mathcal{M}^\rho(T \times A) \text{ such that } \Psi(\{\mu_t\}_{t\in T}) = \tau,$$

where for all $C \in \mathcal{B}(T \times A)$,

$$\tau(C) = \sum_{t\in T} \rho(t)\mu_t(C_t),\ C_t = \{a \in A : (t,a) \in C\}.$$

It is now easy to check that Ψ is the inverse of Φ, and that therefore the latter is a bijective mapping. And since Φ takes a compact set into a compact set, we can claim that it is also a homeomorphism if we show that Φ is continuous. We turn to this.

Suppose $\{\tau^m\}$ weakly converges to τ^0. Denote $\Phi(\tau^m)$ by $\{\mu_t^m\}_{t\in T}$ and $\Phi(\tau^0)$ by $\{\mu_t^0\}_{t\in T}$. Now for any $k \in K$, pick any $\mu_{t_k}^0$-continuity set $B \in \mathcal{B}(A)$. Then it is clear that $(\{t_k\} \times B)$ is a τ^0-continuity set in $\mathcal{B}(T \times A)$, and hence $\tau^m(\{t_k\} \times B)$ converges to $\tau^0(\{t_k\} \times B)$, and hence $\tau^m(\{t_k\} \times B)/\rho(t_k)$ converges $\tau^0(\{t_k\} \times B)/\rho(t_k)$, and hence $\mu_{t_k}^m(B)$ converges to $\mu_{t_k}^0(B)$. The assertion of our claim that Φ is a homeomorphism is complete.

And now we can define a function $\bar{\Phi} : \mathcal{U}_A^T \longrightarrow \mathcal{V}_{(A,T,\rho)}$ such that, for all $a \in A$,

$$v(a,\tau) = \bar{\Phi}(u)(a,\tau) = u\left(a, \Phi(\tau)\right),$$

$$u(a, \{\mu_t\}_{t\in T}) = \bar{\Phi}^{-1}(v)(a, \{\mu_t\}_{t\in T}) = v\left(a, \Psi(\{\mu_t\}_{t\in T})\right),$$

which allow one to go back and forth from the conventional formulation to the one

[27] Without loss of generality, assume that $\rho(\{t_k\}) > 0$ for all $k \in K$.

[28] Note that Definition 1, as presented above deals with the case of T being a singleton; its extension to the case when T has a finite or a countably infinite set of elements is straightforward; see the end of this section for the payoffs in this general setting.

studied here.

3 Two Examples

In the classical example of a 2-player game of matching pennies, the players try to outplay each other; one player wants the two pennies to match, while the other does not. There is no pure-strategy Nash equilibrium for such a game. In this section, we shall first consider a large game of matching pennies in which the players are balanced and with a multiplicity of traits. It has no Nash equilibrium in pure-strategies. We then consider a corresponding discrete version of the game with exact Nash equilibria in finite games with traits.

Example 1: We consider a large game that is a natural generalization of the matching pennies with two players. Let the space of players be the Lebesgue unit interval $(I, \mathcal{I}, \lambda)$ and the space of actions A be the set $\{\bar{H}, \bar{T}\}$, representing Head and Tail respectively. Let the space of traits T be the unit interval I. Assume that the players' traits are uniformly distributed on I; and thus let α be the identity mapping on I and $\rho = \lambda$. We can interpret the trait of an individual players as normalized age (or complexion or income). For concreteness, let us use age.

For any $a \in A$ and $\tau \in \mathcal{M}^\rho(T \times A)$, let $v_i(a, \tau) = -\int_{T \times A} 1_{[0,i) \times \{a\}}(t, x)\ d\tau$ for all $i \in I$, where 1_C is the indicator function of set C.[29] Let f be a strategy profile for the game, $\tau = \lambda(\alpha, f)^{-1}$, $\tau_{\bar{H}}([0, i)) = \lambda\left((\alpha, f)^{-1}([0, i) \times \{\bar{H}\})\right)$ and $\tau_{\bar{T}}([0, i)) = \lambda\left((\alpha, f)^{-1}([0, i) \times \{\bar{T}\})\right)$. Namely, $\tau_{\bar{H}}([0, i))$ $(\tau_{\bar{T}}([0, i)))$ is the proportion of all players younger than i who play Heads (Tails). It is easy to see that $v_i(\bar{H}, \tau) = -\tau_{\bar{H}}([0, i))$ and $v_i(\bar{T}, \tau) = -\tau_{\bar{T}}([0, i))$. Hence, player i's optimal response is respectively Tail if $\tau_{\bar{H}}([0, i)) > \tau_{\bar{T}}([0, i))$, Head if $\tau_{\bar{T}}([0, i)) > \tau_{\bar{H}}([0, i))$, and Head or Tail if $\tau_{\bar{H}}([0, i)) = \tau_{\bar{T}}([0, i))$. It means that the i^{th} player always tries to balance out those players younger than him or her. If there are more younger players who play Heads (Tails), then player i will play Tail (Head).

Suppose that f^* is an equilibrium in pure strategies. Then for every $i \in I$, the respective sets of younger players who play Heads and Tails must have the same measure. Let $F = f^{*-1}(\{\bar{H}\})$ be the set of all the players who play Heads. Then the measurable set F cuts through $[0, i)$ into half for every $i \in I$, which is impossible. The main conclusion is summarized into the following claim, whose proof is given in the Appendix.

Claim 1. *Let $\mathcal{G}$ be a function satisfying $\mathcal{G}(i) = (\alpha(i), v_i)$. Then, (1) $\mathcal{G}$ is a measurable function from I to $T \times \mathcal{V}_{(A,T,\rho)}$, thus, a game which falls within the purview of Definition*

[29] This payoff function is motivated by the example in Remark 3 of [42].

2; (2) there does not exist a Nash equilibrium for the game $\mathcal{G}$.

We now show that the above game can be obtained as the limit of a sequence of finite-player games, each of which has an exact Nash equilibrium.

Example 2: Fix a positive integer n and let $I_n = \{k/2^n : k = 0, 1, \ldots, 2^n\}$. Let λ_n denote the counting measure on I_n, i.e., $\lambda_n(i) = 1/(2^n + 1)$ for every $i \in I_n$. Let $A = \{\bar{H},\ \bar{T}\}$ be the set of actions. Let $T_n = I_n$, $\alpha_n : I_n \longrightarrow T_n$ be $\alpha_n(i) = i$ and $\rho_n = \lambda_n \alpha_n^{-1} = \lambda_n$. For any $a \in A$ and $\tau_n \in \mathcal{M}^{\rho_n}(T_n \times A)$, let $v_i^n(a, \tau_n) = -\int_{T_n \times A} 1_{[0,i)\times\{a\}}(t, x)\ d\tau_n$. Notice that when $i = 0$, $v_i(\cdot, \cdot)$ is identically zero. Define $\mathcal{G}_n : I_n \longrightarrow T_n \times \mathcal{V}_{(A,T_n,\rho_n)}$ as $\mathcal{G}_n(i) = (\alpha_n(i), v_i^n)$ for for all $i \in I_n$. Given any $a \in A$ and any measurable function f_n from I_n to A,

$$v_i^n\left(a, \lambda_n(\alpha_n, f_n)^{-1}\right) = -\int_{T_n \times A} 1_{[0,i)\times\{a\}}(t, x)\ d\lambda_n(\alpha_n, f_n)^{-1} = -\lambda_n\left([0, i) \cap f_n^{-1}(a)\right).$$

Let $f_n^* : I_n \longrightarrow A$ be $f_n^*(k/2^n) = \bar{H}$ when k is even, and equals $\bar{T}$ otherwise. We will show that f_n^* is a Nash equilibrium of $\mathcal{G}_n$. Player 0 is always indifferent between $\bar{H}$ and $\bar{T}$. So, $f_n^*(0) = \bar{H}$ is a best response for player 0. For player $i = k/2^n$ where k is positive and even, $\lambda_n\left([0, i) \cap f_n^{*-1}(\bar{H})\right) = \lambda_n\left([0, i) \cap f_n^{*-1}(\bar{T})\right)$. So, $f_n^*(i) = \bar{H}$ is a best response for i. For player $i = k/2^n$ where k is odd, $\lambda_n\left([0, i) \cap f_n^{-1}(\bar{H})\right) > \lambda_n\left([0, i) \cap f_n^{-1}(\bar{T})\right)$. So, $\bar{T}$ is the best response for i. Hence, f_n^* is a Nash equilibrium of $\mathcal{G}_n$.

We note that the sequence $\mathcal{G}_n$ of games is simply a discrete version of the game $\mathcal{G}$. However, $\mathcal{G}_n$ has a Nash equilibrium while the limit game does not have a Nash equilibrium.

We are now ready to turn to the resolution of the discrepancy between the limiting and idealized limit cases.

4 Saturated Probability Spaces and Existence Theorems

We now turn to the issue of an idealized limit game for which there exist Nash equilibria. We shall show that the relevant condition for the atomless space of players' names is that it be a *saturated* probability space, and that such a requirement is both sufficient *and* necessary. We begin with the basic definitions for the convenience of the reader.

The following definition is taken from [24, Definition 5.1] and already discussed in the introduction.

Definition 3. *A probability space $(I, \mathcal{I}, \lambda)$ is said to be saturated if for any two Polish spaces X and Y, any Borel probability measure $\tau \in \mathcal{M}(X \times Y)$ with marginal probability* *measure τ_X on X, and any measurable mapping g from $(I, \mathcal{I}, \lambda)$ to X with distribution τ_X, there exists a measurable mapping $h : (I, \mathcal{I}, \lambda) \longrightarrow Y$ such that the measurable mapping $(g, h) : (I, \mathcal{I}, \lambda) \longrightarrow X \times Y$ has distribution τ.*

Analytically useful as this definition is, it does not quite address the substantive property that leads the saturation property to be both necessary and sufficient when dealing with non-anonymous games with non-denumerable action sets. An equivalent characterization presented in [24, Corollary 4.5] offers the clarification. First, we need the following notation for a restricted probability space. Given a probability space $(I, \mathcal{I}, \lambda)$, for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, denote by $(S, \mathcal{I}^S, \lambda^S)$ the probability space restricted to S. Here $\mathcal{I}^S := \{S \cap S' : S' \in \mathcal{I}\}$ and λ^S is the probability measure re-scaled from the restriction of λ to $\mathcal{I}^S$. A probability space is said to be *countably generated* if its σ-algebra can be generated by a countable number of subsets (modulo all the null subsets). It is *not countably generated* if the σ-algebra $\mathcal{I}$ can not be generated by any countable number of subsets (modulo all the null subsets).[30]

Proposition 1. *A probability space $(I, \mathcal{I}, \lambda)$ is saturated if and only if it is nowhere countably generated, i.e., for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, the restricted probability space $(S, \mathcal{I}^S, \lambda^S)$ is not countably generated, modulo all the null subsets.*[31]

This proposition stipulates that a saturated probability space admits as measurable functions a richer variety of non-cooperative behavior. For example, the Lebesgue unit interval, i.e., the interval $[0, 1]$ associated with the the σ-algebra of Lebesgue measurable sets and the Lebesgue measure, is a countably generated probability space; it is thus not a saturated probability space. In comparison, any atomless Loeb probability space is saturated. The genesis of the idea goes to Maharam's work in the forties whose techniques can be used to show that a probability space is saturated if and only if its measure algebra is a countable convex combination of measure algebras of uncountable powers of the Borel σ-algebra on $[0, 1]$; see [20] for details. In any case, armed with this intuition, we can present our first principal result.[32]

Theorem 1. *Every large non-anonymous game with traits $\mathcal{G} : I \longrightarrow T \times \mathcal{V}_{(A,T,\rho)}$ has a Nash equilibrium if either of the following two (sufficient) conditions hold:*

[30] That is, the least cardinality of the collection of subsets that generates $\mathcal{I}$ (modulo all the null subsets) is greater than the cardinality of all the natural numbers, $\mathbb{N}$.

[31] This proposition is available in [24, Corollary 4.5]. Throughout the paper, we refer to results previously available in the literature as "propositions." The reader is also warned about the proliferation of terminology related to the saturation property: the condition is originally called $\aleph_1$-*atomless* in [24], and subsequently referred to as *nowhere separable, super-atomless* and *nowhere countably generated;* see [39] and his references.

[32] Note that we work with the standing hypothesis that the space of players' names $(I, \mathcal{I}, \lambda)$ is an atomless probability space, that the common action set A is a compact metric space, and that T is a Polish space endowed with a probability measure ρ.

(i) T and A are both countable spaces,

(ii) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.

Proof: Given any $\tau \in \mathcal{M}^\rho(T \times A)$, let the best response set $B(i, \tau)$ of player i be

$$B(i, \tau) = \text{argmax}_{a \in A} v_i(a, \tau),$$

where $v_i = \mathcal{G}_2(i)$ for all $i \in I$. Since v_i is continuous on $A \times \mathcal{M}^\rho(T \times A)$, we can appeal to Berge's maximum theorem to guarantee that $B(i, \cdot)$ is upper hemicontinuous. In particular, for any given (i, τ), $B(i, \tau)$ is a closed set. Furthermore, for each $\tau \in \mathcal{M}^\rho(T \times A)$, since $v_{(\cdot)}(\cdot, \tau)$ is a measurable function on I, and a continuous function on A, we can apply measurable maximum theorem (see, for example, Theorem 18.19 in [3]) to assert that there exists a measurable selection from the correspondence $B(\cdot, \tau)$. Let $\tilde{B}(i, \tau) = \{\alpha(i)\} \times B(i, \tau)$ for all $i \in I$ and for all $\tau \in \mathcal{M}^\rho(T \times A)$ where $\alpha(i) = \mathcal{G}_1(i)$ for all $i \in I$. It is easy to see that $\tilde{B}(i, \cdot)$ is also upper hemicontinuous on $\mathcal{M}^\rho(\mathcal{T} \times \mathcal{A})$ for each i and $\tilde{B}(i, \tau)$ is closed-valued for any given (i, τ). Denote the correspondence $\tilde{B}(\cdot, \tau) : I \longrightarrow T \times A$ by $\tilde{B}_\tau$. Now define a correspondence $\Phi : \mathcal{M}^\rho(T \times A) \longrightarrow \mathcal{M}^\rho(T \times A)$ by letting $\Phi(\tau) = \mathcal{D}_{\tilde{B}_\tau}$ where $\mathcal{D}_{\tilde{B}_\tau} = \{\lambda \tilde{f}^{-1} : \tilde{f}$ is a measurable selection of $\tilde{B}_\tau\}$. We now show that Φ is a nonempty, closed and convex valued, upper hemicontinuous correspondence from a non-empty convex compact subset of a locally convex space into itself. For any given $\tau \in \mathcal{M}^\rho(T \times A)$, there exists a measurable selection $f : I \longrightarrow A$ where $f(i) \in B(i, \tau)$. Let $h(i) = (\alpha(i), f(i))$ for all $i \in I$. It is clear that $h(i)$ is a measurable selection of $\tilde{B}_\tau$ and $\lambda h^{-1} \in \mathcal{M}^\rho(T \times A)$. Thus, Φ is nonempty-valued. If either (i) or (ii) is satisfied, we can apply Proposition A.P1 in the Appendix to assert that Φ is convex also. Moreover, because $\tilde{B}_\tau$ is closed-valued, hence, compact-valued, and $\tilde{B}(i, \cdot)$ is upper hemicontinuous on $\mathcal{M}^\rho(T \times A)$, we can apply Proposition A. P2 and Proposition A. P4 in the Appendix respectively to assert that Φ is closed and upper hemicontinuous. Thus, we can apply the Fan-Glicksberg fixed-point theorem to assert that there exists a $\tau^* \in \Phi(\tau^*)$, and thus, a measurable selection $f^* : I \longrightarrow A$ such that $\tau^* = \lambda(\alpha, f^*)^{-1}$. It is clear that f^* is a Nash equilibrium. ∎

Remark 1: It is clear that the procedure that relates the conventional form of a large non-anonymous game to the reformulated version studied here, and delineated at the end of Section 2, can be used to derive as straightforward corollaries the principal results of [30] and [22]. The interested reader can also check that the compactness of the action set A can be relaxed with less restrictive assumptions that allow each player to choose his action from a compact subset of the complete countable metric space, as shown in [50].

Remark 2: Examples showing that a large non-anonymous game does not have a Nash equilibrium if the action set is uncountable are by now well-known; see [32] for references. These examples can be used to show that Theorem 1(i) is false if the countability hypothesis on A is relaxed without any additional assumptions. What is more important, of course, is that the result does not extend to the case where the trait space T is uncountable even with a finite action set. Indeed, the example discussed in the last section is based on an action set with only two elements.

Next, we turn to the necessity of saturated spaces when Nash equilibria exist in large non-anonymous games. The relevant benchmark result is due to [28, Theorem 4.7].

Proposition 2. *Let $(I, \mathcal{I}, \lambda)$ be an atomless probability space, and A an uncountable compact metric space. Then $(I, \mathcal{I}, \lambda)$ is saturated if and only if every game $\mathcal{G}^0$ in terms of Definition 1 with player space $(I, \mathcal{I}, \lambda)$ and action space A has a Nash equilibrium.*

We can now present our main result on the necessity of saturated spaces then Nash equilibria exist in our reformulation of large games, namely, large non-anonymous games with traits.

Theorem 2. *An atomless probability space $(I, \mathcal{I}, \lambda)$ is saturated if and only if every large non-anonymous game with traits $\mathcal{G} : I \longrightarrow T \times \mathcal{V}_{(A,T,\rho)}$ has a Nash equilibrium provided one of the following two conditions hold*

(i) A is uncountable,

(ii) T is uncountable and ρ is atomless.

Proof: Suppose that $(I, \mathcal{I}, \lambda)$ is saturated. For both (i) and (ii), by Theorem 1, there exists a Nash equilibrium.

We now prove the necessary parts. First, suppose that condition (i) holds but $(I, \mathcal{I}, \lambda)$ is not saturated. Thus, by Proposition 2, when A is an uncountable compact metric space, there must exist a game in terms of Definition 1 with $(I, \mathcal{I}, \lambda)$ as the name space that does not have any Nash equilibrium. Let this game be $\mathcal{G}^0$. Thus, $\mathcal{G}^0$ is a measurable function from I to $\mathcal{U}_A$ such that there does not exist any measurable function $f : I \longrightarrow A$ that satisfies for λ-almost all $i \in I$, $u_i(f(i), \lambda f^{-1}) \geq u_i(a, \lambda f^{-1})$ for all $a \in A$, where $u_i = \mathcal{G}^0$.

Let $\mathcal{G}$ be a function from I to $T \times \mathcal{V}_{(A,T,\rho)}$ such that for all $i \in I$, $\lambda \mathcal{G}_1^{-1} = \rho$ and $\mathcal{G}_2(i) = v_i$ where v_i is defined as $v_i(a, \tau) = u_i(a, \tau_A)$ with τ_A being the marginal of τ on A for any $a \in A$ and any $\tau \in \mathcal{V}_{(A,T,\rho)}$. It is easy to see that $\mathcal{G}$ is a non-anonymous game with traits that satisfies condition (i). Now suppose that any game with the structure described in Theorem 2 with condition (i) has a Nash equilibrium. Then there must exist a Nash equilibrium f^* for this constructed game $\mathcal{G}$. That is, there is a measurable function $f^* : I \longrightarrow A$ such that for λ-almost all $i \in I$,

$$v_i(f^*(i), \lambda(\mathcal{G}_1, f^*)^{-1}) \geq v_i(a, \lambda(\mathcal{G}_1, f^*)^{-1}) \text{ for all } a \in A.$$

Then, by the construction of the game $\mathcal{G}$, the measurable function f^* also satisfies, for λ-almost all $i \in I$,

$$u_i(f^*(i), \lambda f^{*-1}) \geq u_i(a, \lambda f^{*-1}) \text{ for all } a \in A.$$

This is a contradiction. Hence, $(I, \mathcal{I}, \lambda)$ must be saturated if condition (i) holds.

Now suppose condition (ii) holds but $(I, \mathcal{I}, \lambda)$ is not saturated. Suppose that every game with traits under condition (ii) has a Nash equilibrium. However, Lemma 2 in the Appendix shows that when T is uncountable, ρ is atomless, and $A = \{-1, 1\}$, there is a large non-anonymous game with traits $\mathcal{G} : I \longrightarrow T \times \mathcal{V}_{(A,T,\rho)}$ which does not have any Nash equilibrium. This is a contradiction. Thus, $(I, \mathcal{I}, \lambda)$ must be saturated as well if condition (ii) holds. ∎

5 Mixed and Behavioral Strategies: A Relationship

So far, we have had no occasion to emphasize the distinction between *pure* and *mixed* strategies of a large non-anonymous game, be it with or without traits. Indeed, as noted in the introduction, the interest in a large game arises precisely from the fact that under an attendant externality notion summarizing the state of society's plays, such games posses Nash equilibria without any linear structure on the (common) action set. For games with traits, such existence theorems have been presented in the previous section based on Definition 2 that simply involves a measurable function from the space of players' names $(I, \mathcal{I}, \lambda)$ to the set of actions A. However, in games with complete information, as are considered in this paper, rationality leads one to investigate strategies in which players form probability distributions over the actions of other players, and then take actions in keeping with expectations with respect to such distributions.[33] As we shall see, the basic notions, once formulated correctly, take a particularly satisfactory form.

For the convenience of the reader, and also given the proliferation of confusing terminologies, it is easiest to begin with a taxonomy presented in [36] in the context of finite-player games with incomplete information, one that is heavily influenced by earlier

[33] For a discussion of Bayesian rationality and its manifestation as a correlated equilibrium in the context of finite games with incomplete information, see [11].

work of Aumann's. In the context of such games, they write:

> Each player i observes an informational variable (or *type* t_i) whose values lie in some complete separable metric space T_i. After observing his type, player i selects an action a_i from some compact metric space A_i of feasible actions. The conventional analysis of games involves three types of strategies: pure, mixed and behavioral. A *pure strategy* is a measurable function $p_i : T_i \longrightarrow A_i$. This has the interpretation that when player i learns his type t_i, he selects the action $p_i(t_i)$. Aumann has observed that to define a mixed strategy properly (when T_i is "large") a randomizing device must be introduced for each player.[34] Thus, let $\tilde{s}_i$ be uniformly distributed on $[0, 1]$. A *mixed strategy* for player i is a measurable function $\sigma : [0, 1] \times T_i \longrightarrow A_i$. The interpretation is that when player i observes his type t_i and his randomizing variable s_i, he selects the action $\sigma(s_i, t_i)$. Let $\mathcal{B}(A_i)$ be the collection of Borel subsets of A_i. A *behavioral strategy* is a function $\beta_i : \mathcal{B}(A_i) \times T_i \longrightarrow [0, 1]$ with these two properties: (i) For every $B \in \mathcal{B}(A_i)$, the function $\beta_i(B, \cdot) : T_i \longrightarrow [0, 1]$ is measurable, (ii) for every $t_i \in T_i$, the function $\beta_i(\cdot, t_i) : \mathcal{B}(A_i) \longrightarrow [0, 1]$ is a probability measure.[35] The interpretation of a behavioral strategy is that when player i observes t_i, he selects an action in A_i according to the measure $\beta_i(\cdot, t_i)$.

This extended quotation serves as a point of departure for the development of analogous notions, and in particular, the following formal complement to Definition 1 above for a large game without traits.

Definition 4. *A Nash equilibrium in behavioral strategies of $\mathcal{G}^0$ is a measurable function $h^* : I \longrightarrow \mathcal{M}(A)$, with the latter being endowed Borel σ-algebra generated by the weak topology, such that for λ-almost all $i \in I$,*

$$\int_A u_i\left(a, \int_I h^*(j)d\lambda\right) dh_i^* \geq \int_A u_i\left(a, \int_I h^*(j)d\lambda\right) d\nu \tag{1}$$

for all $\nu \in \mathcal{M}(A)$.[36]

However before enlisting the conceptual vocabulary of [36], and its attendant taxonomy, to the context of a large non-anonymous game, with or without traits, some preliminary discussion is warranted. There is a technical similarity in that in the notion

[34] The relevant paper referred to here is [9] and the relevant passage from this paper is quoted in [36, p. 624]. Also see the introduction to [10].

[35] From a technical point of view, this is nothing but a transition probability in standard probability theory.

[36] Note that the measurability of the mapping h^* is equivalent to the measurability of $h^*(\cdot)(B) : I \longrightarrow [0, 1]$, for any given $B \in \mathcal{A}$; see, for example, Lemma 1 in [23]. The societal aggregate $\int_I h^*(j)d\lambda$ is well-defined as a probability measure whose measure on any given $B \in \mathcal{A}$ is $\int_I h^*(j)(B)d\lambda$.

of a pure strategy, as used in Definitions 1 and 2, and in that of a behavioral strategy in Definition 4, one simply substitutes the space of players' names $(I, \mathcal{I}, \lambda)$ for the finite set of all the players, and the common action set A for the individual action set A_i. However it is important to be clear that from an interpretive point of view, which is to say, the game-theoretic substantive register, the similarity is facile at best, and misleading at worst. The point is that we investigate a game with complete information in which the *type* of a player i is represented simply by his name i in a game without traits, and by the pair $(i, \alpha(i)) \in I \times T$, and, in either case, is completely known. To be more specific, whereas Milgrom and Weber deal with the strategy of an individual player, be they pure, mixed or behavioral, our transcribing of these notions[37] in the forms of Definitions 1, 2 and 4 concern strategy *profiles,* which is to say that they pertain to the society at large rather than to an individual player. Since there has been some confounding of this issue, a further elucidation is useful.

The interesting issue here relates to the externality notion that is involved. Each player chooses a probability distribution on the common action space, and randomizes over his payoff with respect to such a distribution taking as given the "aggregate" of the distributions of all the other players in the game.

This notion of a Nash equilibrium in behavioral strategies was used by Schmeidler in the context of large games but without the attendant externality notion. In a setting with a finite number of actions embedded in the unit-simplex of a finite-dimensional Euclidean space, Schmeidler investigated Nash equilibria of large games where individual payoffs depend on the elements of the simplex and the *entire* profile of actions. He referred to these equilibria as equilibria in mixed strategies, and used a "purification" argument to show the existence of a Nash equilibrium as in Definition 1 above.[38] Invoking [10], [38] argued as follows:

> The failure of the law of large numbers for a continuum of independent randomizations implies that Schmeidler's (1973) concept of a measure-valued profile function in equilibrium might not coincide with the concept of a mixed strategies equilibrium of a nonatomic game. It casts some doubt on the significance of Schmeidler's concept of equilibrium for a nonatomic game.[39]

In this connection it is important to note that Schmeidler's primary interest was in showing the existence of a Nash equilibrium in pure strategies, and his use of the concept

[37] This is of course not to suggest that this transcription in the context of a large game without traits, as is conventional treated and surveyed in [32], either necessarily follows [36] or is novel to this paper.

[38] Rath's direct proof that circumvents this procedure is by now well-known; see [32] for details and discussion.

[39] See the abstract and the first paragraph of the introduction in [38]; also [37].

of a Nash equilibria in behavioral strategies was simply a technical device to accomplish this.[40] As such, one does not quite know what to make of the suggestion relating to significance. To repeat, the point is that in the game formulated by Schmeidler, and in games with more general action sets, as reported in [32], a mixed strategy is very much in the classical footsteps of Nash and Fan-Glicksberg, with the notion of a mixed strategy profile simply extending from a finite-player setting to one with a continuum of players.[41] [38] and his followers take the Bayesian notion of a mixed strategy, as defined by Aumann and others for games with incomplete information, and apply it to such games of complete information, and conceive of a so-called problem of consistency and reconciliation that is not a problem to begin with.

The issue then is an appropriate definition of the notion of a mixed strategy Nash equilibrium for a large non-anonymous game, one that accommodates a continuum of independent randomizations and joint measurability. It is especially here that we break new ground, but before turning to it, we return to Milgrom and Weber.

> These conventional characterizations of strategies are not well suited to our purposes. Instead we define a *distributional strategy* for player i as a probability measure on $\mathcal{B}(T_i) \otimes \mathcal{B}(A_i)$ for which the marginal distribution on T_i is the given probability distribution. In the case where T_1 is uncountable, it was observed by Aumann that a mixed strategy cannot be acceptably defined as a measure on the set of pure strategies. Our approach of defining a (distributional) strategy as a measure on $T_i \times A_i$ providing another way of avoiding measurability problems.

And so the question is: "what is this measurability problem"? In the notation being followed here, [9, pp. 507-508] writes:

> A *mixed* strategy, then, should be a probability measure on $A_i^{T_i}$, [the space of measurable functions from T_i to A_i] the latter having been endowed with an appropriate measurable structure R. But as we have shown elsewhere there is no structure R for which this is so; *no* structure on $A_i^{T_i}$, is "appropriate"! [42]

As brought out in the first quote from Milgrom and Weber, a mixed strategy for player i is a function of two variables: the realization of the randomizing variable s_i and the player's type t_i. In other words, a *process*, defined over $[0,1] \times T_i$. There is no reason for Milgrom and Weber to work with a mixed strategy: their equilibrium notion requires

[40] In private correspondence, David Schmeidler has observed to the authors that his reference to the "combination of strategies as a T-strategy, and not as a mixed strategy" was a considered one given Aumann's earlier usage.

[41] See [32] and their references.

[42] This point is exposited in [8], having been announced in [6] and detailed in [7].

that only the distribution of the players' actions enter as arguments in the payoffs of each individual player, and therefore allows them to work with a distributional strategy for *each* player, represent it as a disintegrated behavioral strategy, and then with the assumption of a finite action space, *purify* this distintegration to a pure strategy. Indeed, [40] make no use of distributional strategies, and neither do [30, 31] for an identical game-theoretic situation.[43]

It is in the context of a large non-anonymous game, with or without traits, that the "measurability problem" arises in the notion of a mixed strategy profile. This is a *process* defined over a sample space that consolidates all the independent randomizations into one "large" space Ω and takes this space and the space of players' names into the space of common actions. In short, to construct a sample space $(\Omega, \mathcal{F}, P)$ such that a mixed strategy profile can be conceived as a measurable function $F : I \times \Omega \longrightarrow A$ such that for any realization $\omega \in \Omega$, we obtain a pure strategy $F(\omega, \cdot) : I \longrightarrow A$, and for any two players' i and j in I, the random variables $F(\cdot, i)$ and $F(\cdot, j)$ are independent. It is this independence that goes to the heart of non-cooperative game theory – the players are not coordinating their randomizations and acting "out of concert" so to speak. It is this that causes the "measurability problem," a difficulty whose sharpest articulation is furnished by a result that we develop next.

For any two probability spaces $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$, we write $\mathcal{I} \otimes \mathcal{F}$ as the usual product σ-algebra (including all the null subsets) generated by $\{S \times T : S \in \mathcal{I}, T \in \mathcal{F}\}$, and write $\lambda \otimes P$ as the product probability measure on $\mathcal{I} \otimes \mathcal{F}$. Given any mapping F from $I \times \Omega$ to a Polish space X, for any $i \in I$ and $\omega \in \Omega$, let F_i denote the marginal mapping $F(i, \cdot)$ on Ω, and F_ω the marginal mapping $F(\cdot, \omega)$ on I. The following concept is from [44, 46].

Definition 5. *A process F is said to be* essentially pairwise independent *if for λ-almost all $i \in I$, F_i and $F_{i'}$ are independent for λ-almost all $i' \in I$.*[44]

We shall construct an essentially pairwise independent process as follows. Let $[0, 1]$ be the unit interval endowed with the Borel σ-algebra $\mathcal{B}_{[0,1]}$ and the uniform distribution. For an atomless probability space $(I, \mathcal{I}, \lambda)$, let $\Omega = [0, 1]^I$ represent the space of all functions from I to the unit interval $[0, 1]$. By the Kolmogorov's extension theorem, we can consider the continuum product probability space $(\Omega, \mathcal{F}', P')$, where $\mathcal{F}'$ is the σ-algebra generated by cylinders of the form $\{\omega \in \Omega : \omega(i) \in B\}$ for all $B \in \mathcal{B}_{[0,1]}$, and P'

[43] All this is now routine with the clarification in [29] and its further extensions by Loeb-Sun; also see [34, Remark 4]. Distintegrations in this context seem to have been first emphasized in a 1989 paper of Khan's, see [32].

[44] Given that $(I, \mathcal{I}, \lambda)$ is an atomless probability space, essential pairwise independence is more general than the usual pairwise and mutual independence.

is the continuum product probability measure on $(\Omega, \mathcal{F}')$. Next define π to be a process from $I \times \Omega$ to $[0,1]$ by letting $\pi(i,\omega) := \omega(i)$ for all $(i,\omega) \in I \times \Omega$. Here the marginal function π_i is the i-th coordinate function on $(\Omega, \mathcal{F}', P')$. It is clear that π_i induces the uniform distribution on $[0,1]$ for any $i \in [0,1]$, and π_i, π_j are independent for $i \neq j$. Accordingly, the process π is an essentially pairwise independent process. However, it is well-known that this process π is not $\mathcal{I} \otimes \mathcal{F}'$-measurable.[45] Indeed, as shown in the following proposition, the essential pairwise independence and the joint measurability of a process with respect to the usual product σ-algebra are never compatible with each other except for the trivial case that almost all random variables are essentially constant.

Proposition 3. *Let f be a function from $I \times \Omega$ to a Polish space X. If f is jointly measurable on the product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$, and if f is essentially pairwise independent, then, for λ-almost all $i \in I$, f_i is a constant random variable.*[46]

And so, with this result, we have a precise and clear articulation of the "measurability problem": the usual continuum product guaranteed by Kolomogorov construction will simply not work.[47] But this does not imply that that we need to relinquish the conceptual vocabulary; it simply necessitates extending the usual product space. In other words, to overcome the above non-compatibility problem of measurability and independence, we need to work with the framework of *Fubini extension*. It is an enrichment of the usual product probability space on which the Fubini property is retained. The following definition is taken from [46, Definitions 2.2 and 5.1].

Definition 6. *A probability space $(I \times \Omega, \mathcal{W}, Q)$ is said to be a Fubini extension of the usual product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if for any real-valued Q-integrable function F on $(I \times \Omega, \mathcal{W})$,*

(i) F_i is P-integrable on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$, and F_ω is λ-integrable on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$;

(ii) $\int_\Omega F_i \, \mathrm{d}P$ and $\int_I F_\omega \, \mathrm{d}\lambda$ are integrable on $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ respectively, in addition, $\int_{I \times \Omega} F \, \mathrm{d}Q = \int_I \left(\int_\Omega F_i \, \mathrm{d}P \right) \mathrm{d}\lambda = \int_\Omega \left(\int_I F_\omega \, \mathrm{d}\lambda \right) \mathrm{d}P$.

A Fubini extension $(I \times \Omega, \mathcal{W}, Q)$ is said to be rich if there is a $\mathcal{W}$-measurable process G from $I \times \Omega$ to the interval $[0,1]$, such that G is essentially pairwise independent, and

[45] See [44, 48] for references to Doob's consideration of the special case that $(I, \mathcal{I}, \lambda)$ is the Lebesgue unit interval.

[46] See [46, Proposition 2.1]. The result is valid even when λ has atoms since the essential pairwise independence condition implies the essential constancy of the random variables f_i for λ-almost all $i \in A$, and therefore on the atom.

[47] For an extended discussion that also includes difficulties of working with finitely-additive measures, see [46, Section 6].

G_i induces the uniform distribution on $[0,1]$ for λ-almost all $i \in I$. We say that such a rich Fubini extension is based on $(I, \mathcal{I}, \lambda)$, and the process G witnesses the richness of the Fubini extension.[48]

In a Fubini extension $(I \times \Omega, \mathcal{W}, Q)$, note that the marginal probability measures of Q on $(I, \mathcal{I})$ and $(\Omega, \mathcal{F})$ are λ and P respectively. To reflect this property, we follow the attendant literature and denote the Fubini extension $(I \times \Omega, \mathcal{W}, Q)$ by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

Next, we connect the existence of a rich Fubini extension to the saturation property of a probability space which is formalized in Definition 3, and with which we have been working so far. The following result is from [46, Proposition 5.6] and [39, Theorem 1] and summarized as [49, Corollary 1].

Proposition 4. *The probability space $(I, \mathcal{I}, \lambda)$ is saturated if and only if there is a rich Fubini extension based on it.*

Note that this result is phrased in terms of the single probability space $(I, \mathcal{I}, \lambda)$, and whereas this is no impediment for the sufficiency part of the result, the requisite sample space has to be constructed for the necessity part. And so the necessity claim, when elaborated, comes down to asserting the existence of a probability space $(\Omega, \mathcal{F}, P)$ extending $(\Omega, \mathcal{F}', P')$, as defined after Definition 5 above, such that there exists a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which the process of coordinate functions π is $\mathcal{I} \boxtimes \mathcal{F}$-measurable and witnesses the richness of the Fubini extension.[49] Finally, we also record a convenient universality property of a rich Fubini extension based on a saturated probability space. A rich Fubini extension satisfies the universality property in the sense that one can construct processes on it with essentially pairwise independent random variables that have any given variety of distributions on a general Polish space. The following result is available in [46, Proposition 5.3].

Proposition 5. *Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich Fubini extension, X be a Polish space, and f a measurable mapping from $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(X)$. Then there exists an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $F : I \times \Omega \to X$ such that the process F is essentially pairwise independent and $f(i)$ is the induced distribution by F_i, for λ-almost all $i \in I$.*

This now formalizes the fact that, unlike the Lebesgue unit interval, saturated probability spaces are hospitable to independence *and measurability,* and that too in a

[48]For the existence of a rich Fubini extension, see Theorem 6.2 of [44], Theorem 5.6 of [46], Theorem 1 of [48] and of [39].

[49]This is precisely the content of [49, Lemma 1] who then use it, in conjunction with their Lemma 4 to prove their Corollary 1, reported as Lemma 1 here.

strong sense that they admit processes whose random variables have a full and arbitrarily-given variety of distributions.[50]

The next result is taken from Corollary 2.9 of [46], which provides a version of the ELLN in the framework of Fubini extension.

Proposition 6. *Assume that* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *is a Fubini extension. If* F *is an essentially pairwise independent and* $\mathcal{I} \boxtimes \mathcal{F}$*-measurable process, then the sample distribution* λF_ω^{-1} *is the same as the distribution* $(\lambda \boxtimes P)\, F^{-1}$ *for* P*-almost all* $\omega \in \Omega$*.*

Having seen the technical necessity of moving from a single probability space to a product, we now turn to the game-theoretic substance. First, consider any large non-anonymous game $\mathcal{G}^0 : I \longrightarrow \mathcal{U}_A$ that fits Definition 1. From now on, let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich Fubini extension of the product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. For the first time in this paper in terms of the formalities, we shift from the nomenclature of a strategy to that of a strategy profile.[51]

Definition 7. *A mixed strategy profile of a game* $\mathcal{G}^0$ *is a* $\mathcal{I} \boxtimes \mathcal{F}$*-measurable function* $g : I \times \Omega \longrightarrow A$ *where the process* g *is assumed to be essentially pairwise independent.*[52] *A Nash equilibrium in mixed strategies of* $\mathcal{G}^0$ *is a mixed strategy profile* g^**, such that for* λ*-almost all* $i \in I$*,*

$$\int_\Omega u_i(g_i^*(\omega), \lambda g_\omega^{*-1}) dP \geq \int_\Omega u_i(\eta(\omega), \lambda g_\omega^{*-1}) dP \tag{2}$$

for all random variables $\eta : \Omega \longrightarrow A$*.*

And now that we have developed the necessary background to inquire into the relationship between a Nash equilibrium in mixed strategies and one in behavioral strategies, we can present the following result whose proof is natural and straightforward.[53]

Theorem 3. *The following equivalence holds for a large non-anonymous game* $\mathcal{G}^0$*.*

(i) Every Nash equilibrium in mixed strategies induces a Nash equilibrium in behavioral strategies, and

[50] This arbitrary nature is only modulated by the fact that the distributions are stitched together by a measurable function; the function f in Proposition 5.

[51] The attentive reader has surely noted that we could have been explicitly mentioned the notion of a *pure strategy profile* in Definitions 1 and 2, and a *behavioral strategy profile* in Definition 4 but chose not to do so.

[52] Though the Lebesgue unit interval itself cannot be the player space, it is shown in [48] that some rich extension of it can the player space.

[53] In the statement of the theorem, we have not formally defined the words *induce* and *lift*: we feel it would be pedantic to do so given that their meaning is clear from the context, and especially from the rather straightforward proof.

(ii) every Nash equilibrium in behavioral strategies can be lifted to a Nash equilibrium in mixed strategies.

Proof: (i) Suppose g^* is a Nash equilibrium in mixed strategies of a game $\mathcal{G}^0$. Let $h^*(i) = Pg_i^{*-1}$ for all $i \in I$. By the ELLN in Proposition 6, we have

$$\int_I h^*(i)d\lambda = \int_I Pg_i^{*-1}d\lambda = \lambda g_\omega^{*-1}. \tag{3}$$

Because g^* is a Nash equilibrium in mixed strategies, (2) holds for g^*. It is clear that for any random variable $\eta : \Omega \longrightarrow A$, $P\eta^{-1} \in \mathcal{M}(A)$. Moreover, given that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is rich, P is atomless. Hence, for any $\nu \in \mathcal{M}(A)$, there exists a random variable $\eta : \Omega \longrightarrow A$ such that $\nu = P\eta^{-1}$. Thus, by (3) and the change of variable theorem, (2) is equivalent to (1) for h^* where $h^*(i) = Pg_i^{*-1}$ for all $i \in I$. By Definition 4, the measurable function h^* is a Nash equilibrium in behavioral strategies for the game $\mathcal{G}^0$.

(ii) Now suppose h^* is a Nash equilibrium in behavioral strategies of a game $\mathcal{G}^0$. That is, h^* is a measurable function from I to $\mathcal{M}(A)$ that satisfies (1). Given that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a rich Fubini extension, by Proposition 5, there is a $\mathcal{I} \boxtimes \mathcal{F}$-measurable process g^* from $I \times \Omega$ to A such that g^* is an essentially pairwise independent process, and the distribution Pg_i^{*-1} is the given distribution $h^*(i)$ for λ-almost all $i \in I$. By the ELLN, (3) holds for such a g^* and h^*. Thus, this $\mathcal{I} \boxtimes \mathcal{F}$-measurable process g^* from $I \times \Omega$ to A also satisfies (2) by (1) and the change of variables theorem. Hence, g^* is a Nash equilibrium in mixed strategies for the same game $\mathcal{G}^0$. ∎

The remainder of this section is an elaboration of these concepts to the setting of a large non-anonymous game with traits. In this context, we have a richer notion of externalities, one that embodies distributions over actions as well as over traits. We begin with definitions analogous to these of Definition 4 and Definition 7.

Definition 8. *A behavioral strategy profile of a large non-anonymous game with traits $\mathcal{G}$ is a measurable function $h : I \longrightarrow \mathcal{M}(A)$, with the latter being endowed Borel σ-algebra generated by the weak topology. A Nash equilibrium in behavioral strategies of $\mathcal{G}$ is a behavioral strategy profile $h^* : I \longrightarrow \mathcal{M}(A)$ such that for λ-almost all $i \in I$,*

$$\int_A v_i(a, \int_I \delta_{\alpha(j)} \otimes h^*(j)d\lambda)dh_i^* \geq \int_A v_i(a, \int_I \delta_{\alpha(j)} \otimes h^*(j)d\lambda)d\nu \tag{4}$$

for all $\nu \in \mathcal{M}(A)$, where δ_t is the distribution on T with mass one on some $t \in T$.

Definition 9. *A mixed strategy profile of a large non-anonymous game with traits $\mathcal{G}$ is a $\mathcal{I} \boxtimes \mathcal{F}$-measurable function $g : I \times \Omega \longrightarrow A$, where g is essentially pairwise independent.*

A Nash equilibrium in mixed strategies of $\mathcal{G}$ is a mixed strategy profile, such that λ-almost all i,

$$\int_\Omega v_i(g_i^*(\omega), \lambda(\alpha, g_\omega^*)^{-1})dP \geq \int_\Omega v_i(\eta(\omega), \lambda(\alpha, g_\omega^*)^{-1})dP \tag{5}$$

for all random variables $\eta : \Omega \longrightarrow A$.

We are now ready to show the relationship between a Nash equilibrium in mixed strategies and one in behavioral strategies for games with traits.

Theorem 4. *Theorem 3 holds for $\mathcal{G}$, a large non-anonymous game with traits.*

Proof: Suppose g^* is a mixed strategy Nash equilibrium of a large non-anonymous game with traits $\mathcal{G}$. (5) holds for such a g^* of the game $\mathcal{G}$. Let $h_i^* = Pg_i^{*-1}$ for all $i \in I$. As the random variables g_i^* are essentially pairwise independent, it is clear that the random variables $(\alpha(i), g_i^*(\omega))$ are essentially pairwise independent. Thus we can appeal to the ELLN to assert that, for P-almost all $\omega \in \Omega$,

$$\lambda(\alpha, g_\omega^*)^{-1} = \int_I P(\alpha(i), g_i^*)^{-1} d\lambda = \int_I \delta_{\alpha(i)} \otimes Pg_i^{*-1} d\lambda, \tag{6}$$

which shows that (4) holds for the profile h^* in which $h_i^* = Pg_i^{*-1}$ for all $i \in I$ by the change of variables theorem and the fact that P is atomless. Hence, by Definition 8, h is a Nash equilibrium in behavioral strategies for the game $\mathcal{G}$.

Now suppose h^* is a Nash equilibrium in behavioral strategies of $\mathcal{G}$. That is to say, (4) holds for h^*. By arguments similar to (ii) in the proof of Theorem 3, together with (6) and (4), we can then lift h^* to an essentially pairwise independent process g^* where $Pg^{*-1} = h^*(i)$ for λ-almost all $i \in I$ such that g^* is a Nash equilibrium in mixed strategies of $\mathcal{G}$. ■

6 Mixed and Pure Strategies: An Ex-Post Relationship

In a section titled "large games with independent idiosyncratic shocks", [32] observed that the notion of externalities in the form of a distribution of the actions of all players – a distinguishing characteristic of the theory of large games – allows one to make a rather novel claim: this is the assertion that in a setting of idiosyncratic shocks, "in equilibrium, societal responses do not depend on a particular sample realization, and each player is *justified* in ignoring other players' risks."[54] We begin this section by transcribing Theorem

[54]See Section 11 in [32]. The quote is taken from Section 11.3 on page 1792 where we substitute "player" for "agent." In this connection, also see Assumption C and its discussion in Cremer and

7 in [32] in the vocabulary of a rich Fubini extension developed in [46] and used essentially in this paper. We begin with the following definition.

Definition 10. *A large non-anonymous game with idiosyncratic uncertainty is a measurable function* $\mathcal{G}^U$ *from* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *to* $\mathcal{U}_A$ *such that* $\mathcal{G}^U$ *is essentially pairwise independent.*

We can now present

Proposition 7. *Let* $\mathcal{G}^U$ *be a large non-anonymous game with idiosyncratic uncertainty. Then there is a process* $f : I \times \Omega \longrightarrow A$ *such that* f *is a Nash equilibrium of the game* $\mathcal{G}^U$*, such that the random strategies* $f(i, \cdot)$ *are essentially pairwise independent, and for* P*-almost all* $\omega \in \Omega$*,* $f(\cdot, \omega)$ *is an equilibrium of the large game* $\mathcal{G}^U(\cdot, \omega)$ *with constant societal distribution* $(\lambda \boxtimes P)f^{-1}$.

A two-line proof of this proposition is furnished in [32]. The basic idea is straightforward. One can regard the game $\mathcal{G}^U$ as a large non-anonymous game modeled on the space of players' names to be the joint space $I \times \Omega$, and given joint measurability on such a space, deduce the existence of a Nash equilibrium $g : T \times \Omega \longrightarrow A$ from the standard result. This is to assert that there exists a measurable function such that

$$\mathcal{G}^U_{(i,\omega)}(g(i,\omega),\ (\lambda \boxtimes P)g^{-1}) \geq \mathcal{G}^U_{(i,\omega)}(a,\ (\lambda \boxtimes P)g^{-1}) \text{ for all } a \in A.$$

The point is that this measurable function is a selection from the set-valued process

$$(t,\omega) \longrightarrow F(t,\omega) = \text{argmax}_{a\in A}\mathcal{G}^U_{(t,\omega)}(a,\ (\lambda \boxtimes P)g^{-1}).$$

And we can now finish the proof by appealing to the following proposition (which is Theorem 2 of [45]) and to the ELLN as stated in Proposition 6.

Proposition 8. *Let* F *be a set-valued process from* $I \times \Omega$ *to a complete separable metric space* A*. Assume that* $F(i, \cdot)$ *are essentially pairwise independent. Let* g *be a selection of* F *with distribution* μ*. Then there is another selection* f *of* F *such that the distribution of* f *is* μ*, and* $f(t, \cdot)$ *is essentially pairwise independent.*

And so rather than the proof, it is the interpretation of the theorem that is of interest. The context is one of exogenous uncertainty whereby the individual payoffs, as well as the individual randomized strategies, are independent, and the theorem rigorously develops the intuition that once uncertainty is resolved, a player has no incentive to depart *ex-post* from her optimal strategy taken in the *ex-ante* game when she finds

Mclean (1985, p. 346); also Footnote 48 below.

herself in the realized *ex-post* game. In this connection, but in the context of a large but finite games, [27, p. 1632] writes:

> A particular modeling difficulty of noncooperative game theory is the sensitivity of Nash equilibrium to the rules of the game, e.g., the order of the players' moves and the information structure. Since such details are often not available to the modeler or even to the players of the game, equilibrium prediction may be unreliable. For this purpose, we define a Nash equilibrium of a game to be extensively robust[55] if it remains a Nash equilibrium in *all extensive versions* of the simultaneous-move game. Extensive robustness means in particular that an equilibrium must be *ex-post Nash.* Even with perfect hindsight knowledge of the types and selected actions of all of his opponents, no player regrets, or has an incentive to revise, his own selected action.

One can look on the dependence of the payoffs on the space Ω in Proposition 7 as the proxy for the variety of phenomena emphasized by Kalai and not explicitly modeled.[56]

To be sure, what makes Proposition 7 work is the existence of a rich Fubini extension and the ELLN. But the point can be sharpened still if rather than work with a large game with idiosyncratic uncertainty $\mathcal{G}^U$, one works with a deterministic non-anonymous large game as in Definition 1. In this case, the uncertainty underlying a mixed strategy arises only from the uncertainty regarding the moves, randomized or otherwise, of everyone else' plays. To put the matter another way, a natural question concerns the possibility of such a claim in situations when there is no exogenous parametric uncertainty, but one introduced as a result of players' playing mixed strategies based on independent randomizations, as befits a non-cooperative game setting. If all these independent randomizations can be consolidated in "one large" space Ω with the mixed strategy profile again being a process from $I \times \Omega$ to the space of actions, we are back in the situation considered in [32], but with the underlying space of uncertainty being generated *only* from the independent randomized strategies of the players. One can then again ask whether each player is *justified* in ignoring other players' risks and has no incentive to depart *ex-post* from her optimal strategy taken in the *ex-ante* game when all the randomizations of each player have been individually realized. In other words, this is to ask, in the terminology adopted by [27], whether a mixed strategy equilibrium has

[55] [27, p. 1632] emphasizes, "This is a new notion of robustness, different from other robustness notions used in economics or game theory. While of course accepting the validity of this statement, one can usefully connect it to [18, p. 347] who write "Then we utilize an equilibrium concept, called ex-post Nash equilibrium, which states that, after seeing the bids of others, buyers will not want to revise their bids."

[56] Referring to *extensive versions* of the simultaneous-move game, Kalai refers to "wide flexibility in the order of players' moves, as well as information leakage, commitment and revision possibilities, cheap talk, and more."

an *ex-post* purification. Taking this terminology, and Kalai's approximate result as their point of departure, [17] investigate approximate existence issues concerning a "Bayesian equilibrium in pure strategies that is also expost stable." Since the question is being posed in a deterministic large non-anonymous game, an affirmative answer is even easier to obtain than in the situation considered in Proposition 7 above. The definition of the ex-post property of a mixed strategy profile of a game, with or without traits, is provided as follows.

Definition 11. *A mixed strategy profile g^* of a game is said to have the ex-post Nash property if for P-almost all $\omega \in \Omega$, g^*_ω is a pure strategy Nash equilibrium for the same game with the empirical action distribution λg^{*-1}_ω.*

And now we can present the following benchmark result for games without traits $\mathcal{G}^0$.

Theorem 5. *A mixed strategy profile of $\mathcal{G}^0$ is a Nash equilibrium in mixed strategies if and only if it has ex-post Nash property.*

Proof: Suppose g^* is a Nash equilibrium in mixed strategies. We shall show that g^* has ex-post Nash property. Towards this end, first note that by the ELLN as stated in Proposition 6, given any mixed strategy profile g, we have, for P-almost all ω,

$$\lambda g_\omega^{-1}(\cdot) = \int_I P g_i^{-1}(\cdot) d\lambda. \tag{7}$$

Let $\xi = \int_I P g_i^{*-1}(\cdot) d\lambda$. Because g^* is a Nash equilibrium in mixed strategies, (2) holds. By (7), (2) can be rewritten as, for λ-almost all $i \in I$,

$$\int_\Omega u_i\left(g_i^*(\omega), \xi\right) dP \geq \int_\Omega u_i\left(\eta(\omega), \xi\right) dP \text{ for all random variables } \eta : \Omega \longrightarrow A,$$

which implies, for λ-almost all $i \in I$, for P-almost all $\omega \in \Omega$,

$$u_i(g_i^*(\omega), \xi) = \max_{a \in A} u_i(a, \xi).$$

By the Fubini property of a Fubini extension, we have, for P-almost all $\omega \in \Omega$, λ-almost all $i \in I$,

$$u_i\left(g_\omega^*(i), \xi\right) = \max_{a \in A} u_i(a, \xi).$$

By the ELLN again, hence, for P-almost all $\omega \in \Omega$, λ-almost all $i \in I$,

$$u_i\left(g_\omega^*(i), \lambda g_\omega^{*-1}\right) = \max_{a \in A} u_i\left(a, \lambda g_\omega^{*-1}\right).$$

This means, for P-almost all $\omega \in \Omega$, g^*_ω is a pure strategy Nash equilibrium, and therefore, g^* has the ex-post Nash property.

Now, suppose that a mixed strategy profile g has ex-post Nash property, which is to say, for P-almost all $\omega \in \Omega$, λ-almost all $i \in I$,

$$u_i\left(g_\omega(i), \lambda g_\omega^{-1}\right) = \max_{a \in A} u_i(a, \lambda g_\omega^{-1}).$$

By (7) and the Fubini property of a Fubini extension, we have, for λ-almost all $i \in I$, P-almost all $\omega \in \Omega$,

$$u_i\left(g_i(\omega), \lambda g_\omega^{-1}\right) = \max_{a \in A} u_i\left(a, \lambda g_\omega^{-1}\right).$$

Hence, for any random variable $\eta : \Omega \longrightarrow A$, we have, for λ-almost all $i \in I$, P-almost all $\omega \in \Omega$,

$$u_i\left(g_i(\omega), \lambda g_\omega^{-1}\right) \geq u_i\left(\eta(\omega), \lambda g_\omega^{-1}\right),$$

which implies that for λ-almost all $i \in I$,

$$\int_\Omega u_i\left(g_i(\omega), \lambda g_\omega^{-1}\right) dP \geq \int_\Omega u_i\left(\eta(\omega), \lambda g_\omega^{-1}\right) dP, \text{ for any random variable } \eta : \Omega \longrightarrow A.$$

This verifies that g is a Nash equilibrium in mixed strategies. ∎

As mentioned above, Kalai works with an increasing sequence of large but finite games and emphasizes an approximate ex-post Nash notion and an equicontinuity property of payoffs that plays no role in the result presented here. However, his discussion of the property itself is illuminating.[57]

> An immediate consequence of the ex-post Nash property is a *purification* property in large games. First, for normal-form games the ex-post Nash property provides stronger conclusions than Schmeidler's (1973) on the role of pure strategy equilibria in large anonymous games. Working in the limit with a continuum of players, Schmeidler shows that every "mixed strategy" equilibrium may be "purified." This means that for any mixed strategy equilibrium one can construct a pure strategy equilibrium with the same individual payoffs. The ex-post Nash theorem ... shows (asymptotically) that in large semi-anonymous games there is no need to purify since it is done for us automatically by the laws of large numbers.[58] So every mixed strategy may be thought of as a "self-purifying device."

The point is that these words go with, and underscore, Proposition 7. The technical vocabulary which constitutes Theorem 5 presented above, and which delivers what needs

[57] See, for example, [27, Lemma 6.1] and related discussion.

[58] [27] has no asymptotic analog of the equivalence result as stated in Theorem 5.

to be substantively delivered, was simply not available to [38] and to [27].[59]

We now turn to large non-anonymous games with traits $\mathcal{G}$.

Theorem 6. *Theorem 5 holds for $\mathcal{G}$, a large non-anonymous game with traits.*

Proof: For any given mixed strategy profile g of $\mathcal{G}$, let $G : I \times \Omega \longrightarrow T \times A$ be a function satisfying $G(i, \omega) = (\alpha(i), g_i(\omega))$. It is easy to see that G is an essentially pairwise independent process. By the ELLN, we have, for P-almost all $\omega \in \Omega$,

$$\lambda G_\omega^{-1} = \lambda(\alpha, g_\omega)^{-1} = \int_I P(\alpha(i), g_i)^{-1} d\lambda.$$

Hence, for P-almost all $\omega \in \Omega$,

$$\lambda G_\omega^{-1} = \int_I \delta_{\alpha(i)} \otimes P g_i^{-1} d\lambda. \tag{8}$$

Suppose g^* is a Nash equilibrium in mixed strategies for $\mathcal{G}$. That is, (5) holds for g^*. Let $\xi' = \int_I \delta_{\alpha(i)} \otimes P g_i^{*-1} d\lambda$. Thus, by (8), (5) can be written as, for λ-almost all $i \in I$,

$$\int_\Omega v_i(g_i^*(\omega), \xi') dP = \int_\Omega v_i(\eta(\omega), \xi') dP \text{ for all random variable } \eta : \Omega \longrightarrow A.$$

Hence, for λ-almost all $i \in I$, P-almost all $\omega \in \Omega$, $v_i(g_i^*(\omega), \xi') = \max_{a \in A} v_i(a, \xi')$. By the Fubini property of a Fubini extension, for P-almost all $\omega \in \Omega$, λ-almost all $i \in I$, $v_i(g_\omega^*(i), \xi') = \max_{a \in A} v_i(a, \xi')$. Therefore, by the ELLN, for P-almost all $\omega \in \Omega$, for λ-almost all $i \in I$,

$$v_i(g_\omega^*(i), \lambda G_\omega^{*-1}) = \max_{a \in A} v_i(a, \lambda G_\omega^{*-1}).$$

This means for P-almost all $\omega \in \Omega$, g_ω^* is a pure strategy Nash equilibrium, which ensures that g^* has the ex-post Nash property.

Now, suppose that there is a mixed strategy profile g of $\mathcal{G}$ has ex-post Nash property. By arguments similar to the second paragraph in the proof of Theorem 5 and the above argument, it is easy to check that g is a Nash equilibrium in mixed strategies of the same game $\mathcal{G}$. ■

[59] In this connection, [27, Footnote 11] is confusing. This states, "As Schmeidler points out in his paper, it is difficult to define a "real mixed strategy" equilibrium due to failings of law of large numbers in the case of continuously many random variables." As discussed above, the problem in Schmeidler's paper is simply of nomenclature: he refers to what we (and [36]) are calling a behavioral strategy as a mixed strategy. In particular he has no reference to the law of large numbers or to the *independence* condition. In fact his statement that "in many real gamelike situations a mixed strategy has no meaning" refers to difficulties in reality rather than those for modeling.

7 An Illustrative Result for Large but Finite Games

Theorems 1 to 6 presented above all concern an idealized limit game based on a saturated probability space interpreted as a space of player's names (the thrust of the word *non-anonymous* in the various definitions). A traditional question in economic theory, dating at least to the late sixties, is the relevance of these results to a finite-agent setting, in the first instance, and to a large but finite set-up in the second. In short, this is to ask for the asymptotic implementation, with or without speeds of convergence, of the limit results.[60] The answer to this is clear once one appreciates that the saturation property of a probability space, as formalized in Definition 3 above, is shared by atomless Loeb probability spaces, and therefore, by Loeb counting spaces.[61] However, once the results are pushed down to a setting where a Loeb counting space renders service as a space of player's names instead of a general saturated probability space, the methodological procedures are well-laid out and well-understood since the comprehensive surveys of [4], and essentially go back to Brown-Robinson (see [4]: one pushes down the Loeb space results to a result for a nonstandard internal game, and then, under the "tightness" hypothesis, transfers to a setting of large but finite games. It would be tedious to present asymptotic implementations of each of the eight theorems, and we only present a translation of Theorem 1 (ii) for illustration and the convenience of the reader.

Let I^n be the set of the first n positive integers and with the counting probability measure λ^n on its power set $\mathcal{I}^n$ and $\mathcal{V}$ the space of all continuous functions on the product space $A \times \mathcal{M}(T \times A)$ based on its sup-norm topology and endowed with its resulting Borel σ-algebra. We shall need the following definition.

Definition 12. *Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of measurable mappings from a probability space $(Y, \mathcal{Y}, \mu_Y)$ to a Polish space X equipped with its Borel σ-algebra $\mathcal{B}(X)$. It is said to be tight if for any $\epsilon > 0$, there exists a compact subset K_ϵ of X such that for all $n \in \mathbb{N}$, $\mu_Y\left(g_n^{-1}(K_\epsilon)\right) > 1 - \epsilon$.*

Theorem 7. *For each $n \geq 1$, let a finite game $\mathcal{G}^n$ be a mapping from I^n into $T \times \mathcal{V}$ with $\alpha^n(i) \equiv \mathcal{G}_1^n(i)$ and $v_i^n \equiv \mathcal{G}_2^n(i)$ for each $i \in I^n$. Assume that the sequence of finite games is tight. Then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there exists*

[60] This is one of the three criteria for a viable name space that [31] adduce in favor of Loeb spaces.

[61] Indeed this was one of the points of entry for such a property as is evident in [24], and the subsequent text of [20].

$f_n : I^n \longrightarrow A$ and $I^n_\epsilon \subseteq I_n$ with $\lambda^n(I^n_\epsilon) > 1 - \epsilon$, such that for all $i \in I^n_\epsilon$, for all $a \in A$,[62]

$$v^n_i(f^n(i), \lambda^n(\alpha^n, f^n)^{-1}) \geq v^n_i(a, \lambda^n(\alpha^n, f^n)^{-1}) - \epsilon.$$

Proof: Fix any real number $\epsilon > 0$. We transfer the sequence of finite games to the nonstandard universe to obtain a sequence $\{\mathcal{G}^n\}_{n \in {}^*\mathbb{N}}$ of internal games on the associated sequence of $\{(I^n, \mathcal{I}^n, \lambda^n)\}_{n \in {}^*\mathbb{N}}$ of internal probability spaces. The tightness assumption on the original sequence $\{\mathcal{G}^n\}_{n \in \mathbb{N}}$ implies that for each $n \in {}^*\mathbb{N}_\infty$, $\mathcal{G}^n$ is near standard in the sense that for λ-almost all $i \in I$ and $\mathcal{G}^n(i)$ has a standard part ${}^\circ(\mathcal{G}^n(i))$ in $T \times \mathcal{V}$.

Fix any $n \in {}^*\mathbb{N}_\infty$. Let $(I, \mathcal{I}, \lambda)$ be the corresponding Loeb space of $(I^n, \mathcal{I}^n, \lambda^n)$.[63] For $i \in I$, let $\alpha(i) = {}^\circ(\alpha^n(i))$, $v_i = {}^\circ(v^n_i)$, and $\mathcal{G}(i) = (\alpha(i), v_i)$. Since $(I, \mathcal{I}, \lambda)$ is a saturated probability space, it follows from Theorem 1 (ii) that there exists a measurable function $f : I \longrightarrow A$ such that for λ-almost all $i \in I$,

$$v_i(f(i), \lambda(\alpha, f)^{-1}) \geq v_i(a, \lambda(\alpha, f)^{-1}) \text{ for all } a \in A.$$

By Theorem 5.2.4 in [33], we can obtain an internal lifting $f^n : I^n \longrightarrow {}^*A$ of f such that for λ-almost all $i \in I, {}^\circ(f^n(i)) = f(i)$. Thus, for λ-almost all $i \in I$, we have,

$$v^n_i(f^n(i), \lambda^n(\alpha^n, f^n)^{-1}) \simeq v_i(f(i), \lambda(\alpha, f)^{-1}).$$

On the other hand, for any ${}^*a \in {}^*A$ and $i \in I$,

$$v^n_i({}^*a, \lambda^n(\alpha^n, f^n)^{-1}) \simeq v_i({}^{\circ *}a, \lambda(\alpha, f)^{-1}).$$

Comparing the above two equations with the first equation in this proof, we can assert that there exists a $I^n_\epsilon \in \mathcal{I}^n$ with $\lambda^n(I^n_\epsilon) > 1 - \epsilon$, such that for all $i \in I^n_\epsilon$, for all ${}^*a \in^* A$,

$$v^n_i(f^n(i), \lambda^n(\alpha^n, f^n)^{-1}) \geq v^n_i({}^*a, \lambda^n(\alpha^n, f^n)^{-1}) - \epsilon.$$

Since the above equation holds for all $n \in {}^*N_\infty$, the conclusion follows the spill-over principle; for the latter, see Theorem 2.8.11 in [33]. ∎

[62] Here we note that there is a trivial correction to be made to each of the statements of Theorems 6 and 7 in [31]. One needs to use "for all $t \in T^n_\epsilon$ with $\lambda^n(T^n_\epsilon) > 1 - \epsilon$" instead of "for all $t \in T$", where λ^n is the counting probability on T^n. Similarly, one needs to replace "for all $t \in T$" in the statements of Theorems 9 and 10 by "for almost all $t \in T$".

[63] We abbreviate an entire procedure here, and the interested reader can see [33] and the specific details to a game-theoretic context in [31].

8 Conclusion

In this paper, the standard theory of large non-anonymous games, as surveyed in [32], and the subject of several recent extensions and applications,[64] is given an alternative cast that allows a treatment of large games in which individual players have names as well as traits, and a player's dependence on society is formulated as a joint probability measure on the space of actions *and* traits. The key property of *saturation,* originally due to [24], and discussed in the context of the conventional formulation of the space of names in the theory of large games, is identified, and shown to be sufficient *and* necessary for a reformulated and comprehensive theory. It addresses the, by now well-known, difficulty that there is, in general, no Nash equilibrium if one works with a large game with the Lebesgue unit interval as the space of players' names and the interval [-1, 1] as the action space. To be sure, [31] established that this difficulty can be resolved if one works with an appropriate player space that captures the asymptotic properties of a sequence of large finite games, the Loeb counting space, it took the suggestion of [28, Theorem 4.6] that if one works with an uncountable action space, it is a necessary and sufficient for a robust theory that the player space be saturated. This suggestion is now developed and now placed in a broader rubric of the seven theorems presented in this paper. It is hoped that the analytically rigorous and rich formalization of traits that they shape will be more relevant in terms of applications.

We end this paper with two open questions, one possibly straightforward given the arguments and results reported above, but the other certainly not so. The first concerns the exploration of our reformulation from an non-anonymous to an anonymous formulation, as formulated by [34]. To be more specific, what we have in mind is an identification of the role of the saturation property to the theory of large anonymous games with uncountable compact metric action sets. As is well-understood, this is a setting in which the space of players' names plays no part, and a large game and its Nash equilibria are probability measures on the space of characteristics represented by the space of payoffs, and this space is "built up" solely out of the action space A. It is this simplicity of conception that makes possible the equivalence theorems presented in [28]. However, as we have seen in Section 2, in the reformulation studied in this paper, the space of characteristics represented by the space of payoffs involves the triple (T, A, ρ), and it is natural to ask whether the theory delineated here extends to probability measures on this space, and in particular, the equivalence theorems of [28] extend to this richer setting.

The second question asks whether the reformulation reported here can be pursued

[64] For the extensions of the theory, see, for example, [16], [22] and [50]; for applications, in addition to [1, 2], [15] and [19], see [5] and their references.

in a stochastic setting in that the function α from the the space of players' names to the space of traits is conceived of as Young measure, which is to say, a function to the probabilities on the space of traits rather a (deterministic and identifiable) point in the space. From a substantive game-theoretic point of view, this asks whether the theory can be generalized to situations where only a player's names is known with certainty, but her individual trait is random, and thereby moving from a large game of complete information to one with incomplete information. Even a cursory perusal of both applied and theoretical work identifies this as an important question; see [5] and their references for the rich variety of applications, and the discussion of Bayesian rationality in a finite-player setting in [12].[65] We hope to return to both questions in subsequent work.

9 Appendix

We begin this Appendix by collecting recent results on the distributions of correspondences; we present a composite result culled from the theorems in [30] and [28].

Proposition A. *Let X be a compact metric space and $(\Omega, \mathcal{A}, P)$ an atomless probability space. Then the following results are valid if, in addition, (i) X is a countable, or (ii) $(\Omega, \mathcal{A}, P)$ is a saturated probability space.*

P1: For any correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F = \{Pf^{-1} : f$ is a measurable selection of $F\}$ is convex.

P2: For any closed-valued correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F$ is closed.

P3: For any compact-valued correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathcal{D}_F$ is compact.

P4: Let F be a compact-valued correspondence from $(\Omega, \mathcal{A}, P)$ to X. Suppose that Y is a metric space and G is a closed-valued correspondence from $\Omega \times Y$ to X such that:

(a) For all $(\omega, y) \in \Omega \times Y$, $G(\omega, y) \subseteq F(\omega)$.

(b) For each fixed $y \in Y$, $G(\cdot, y)$ (denoted by G_y) is a measurable correspondence from $(\Omega, \mathcal{A}, P)$ to X.

(c) For each fixed $\omega \in \Omega$, $G(\omega, \cdot)$ is upper hemicontinuous from Y to X.

Then the correspondence $H(y) = \mathcal{D}_{G_y}$ is upper hemicontinuous from Y to $\mathcal{M}(X)$.

Proof of Claim 1: (1) We first show that for any given i, $v_i(\cdot, \cdot)$ is a continuous function. Since A has only two points, it is enough to verify the continuity with respect to τ. Suppose $\{\tau_n\}$ converges weakly to τ in $\mathcal{M}^\rho(T \times A)$, where $\tau_n \in \mathcal{M}^\rho(T \times A)$ for each n. Let D_c be the set of discontinuity of the function $1_{[0,i) \times \{a\}}(t, x)$. It is clear that

[65] For a preliminary exploration based on a process rather than a conditional probability, see [47].

$\tau(D_c) = 0$. By Theorem 25.8 in [13], $\lim_{n\to\infty} v_i(a,\tau_n) = v_i(a,\tau)$ for any given i and a, and the proof of the claim is complete.[66]

Next, notice that when $i = 0$, $v_i(\cdot,\cdot)$ is identically zero. We show that $v_i(\cdot,\cdot)$ is a continuous function of i. $v_i(a,\tau) = -\int_{T\times A} 1_{[0,i)\times\{a\}}(t,x)\, d\tau$. Assume that $j > i$. Then

$$\begin{aligned} v_i(a,\tau) - v_j(a,\tau) &= \int_{T\times A} 1_{[0,j)\times\{a\}}(t,x)\, d\tau - \int_{T\times A} 1_{[0,i)\times\{a\}}(t,x)\, d\tau \\ &= \int_{T\times A} 1_{[i,j)\times\{a\}}(t,x)\, d\tau \\ &= \tau\left([i,j)\times\{a\}\right) \\ &\leq j - i. \end{aligned}$$

Therefore, for any i and j in I, $|v_i(\cdot,\cdot) - v_j(\cdot,\cdot)| \leq |j - i|$, which shows that $v_i(\cdot,\cdot)$ is a continuous function of i. Hence, $\mathcal{G}$ is a measurable function from I to $T \times \mathcal{V}_{(A,T,\rho)}$.

(2) Assume that f^* is a Nash equilibrium of the game $\mathcal{G}$. We first show that for all $i > 0$, $v_i\left(\bar{H}, \lambda(\alpha, f^*)^{-1}\right) = v_i\left(\bar{T}, \lambda(\alpha, f^*)^{-1}\right)$. Suppose that for some $i > 0$, $v_i\left(\bar{H}, \lambda(\alpha, f^*)^{-1}\right) < v_i\left(\bar{T}, \lambda(\alpha, f^*)^{-1}\right)$. For this fixed i, let

$$S = \left\{ r \in [0,i) : v_r\left(\bar{H}, \lambda(\alpha, f^*)^{-1}\right) = v_r\left(\bar{T}, \lambda(\alpha, f^*)^{-1}\right) \right\}.$$

S is nonempty since it contains 0, so $s^* = \sup S$ exists. The continuity of $v_r(\cdot,\cdot)$ in r implies that $v_{s^*}\left(\bar{H}, \lambda(\alpha, f^*)^{-1}\right) = v_{s^*}\left(\bar{T}, \lambda(\alpha, f^*)^{-1}\right)$. Note that given any $a \in A$ and any measurable function f from I to A,

$$v_i\left(a, \lambda(\alpha, f)^{-1}\right) = -\int_{T\times A} 1_{[0,i)\times\{a\}}(t,x)\, d\lambda(\alpha, f)^{-1} = -\lambda\left([0,i) \cap f^{-1}(a)\right).$$

Thus, we have, $\lambda([0,s^*) \cap f^{*-1}(\bar{H})) = \lambda([0,s^*) \cap f^{*-1}(\bar{T}))$. Furthermore, $s^* < i$. For any $y \in (s^*, i)$, $v_y(\bar{H}, \lambda(\alpha, f^*)^{-1}) < v_y(\bar{T}, \lambda(\alpha, f^*)^{-1})$ by the continuity of $v_y(\cdot,\cdot)$ in y. Since f^* is a Nash equilibrium, we know $f^*(y) = \bar{T}$ for all $y \in (s^*, i)$. Therefore,

$$\lambda\left([0,i) \cap f^{*-1}(\bar{T})\right) = \lambda\left([0,s^*) \cap f^{*-1}(\bar{T})\right) + \lambda\left([s^*,i) \cap f^{*-1}(\bar{T})\right) > \lambda\left([0,s^*) \cap f^{*-1}(\bar{T})\right)$$

and

$$\lambda\left([0,i) \cap f^{*-1}(\bar{H})\right) = \lambda\left([0,s^*) \cap f^{*-1}(\bar{H})\right) + \lambda\left([s^*,i) \cap f^{*-1}(\bar{H})\right) = \lambda\left([0,s^*) \cap f^{*-1}(\bar{H})\right).$$

[66] The reader is warned that rather than the statement of the theorem in [13], we appeal to a statement in its proof.

Thus, we have

$$v_i\left(\bar{H}, \lambda(\alpha, f^*)^{-1}\right) = -\lambda\left([0,i) \cap f^{*-1}(\bar{H})\right) > -\lambda\left([0,i) \cap f^{*-1}(\bar{T})\right) = v_i\left(\bar{T}, \lambda(\alpha, f^*)^{-1}\right),$$

a contradiction. Similarly, for any $i > 0$, $v_i\left(\bar{H}, \lambda(\alpha, f^*)^{-1}\right) < v_i\left(\bar{T}, \lambda(\alpha, f^*)^{-1}\right)$ cannot hold. Therefore, for every $i > 0$,

$$v_i\left(\bar{H}, \lambda(\alpha, f^*)^{-1}\right) = v_i\left(\bar{T}, \lambda(\alpha, f^*)^{-1}\right),$$

which is equivalent to

$$\lambda\left([0,i) \cap f^{*-1}(\bar{H})\right) = \lambda\left([0,i) \cap f^{*-1}(\bar{T})\right).$$

Furthermore, since $\lambda\left([0,i) \cap f^{*-1}(\bar{H})\right) + \lambda\left([0,i) \cap f^{*-1}(\bar{T})\right) = \lambda\left([0,i)\right) = i$, we know that

$$\lambda\left([0,i) \cap f^{*-1}(\bar{H})\right) = \lambda\left([0,i) \cap f^{*-1}(\bar{T})\right) = i/2.$$

Let $F = \left\{i \in I : f^*(i) = \bar{H}\right\}$. Let $g(i) = i$ on F, $g(i) = -i$ on $I \setminus F$. Then for any $c \in (0,1]$, $\lambda g^{-1}\left([0,c]\right) = \lambda g^{-1}\left([-c,0]\right) = c/2$. So, λg^{-1} is the uniform distribution on $[-1,1]$. We will show that this itself is a contradiction. Clearly, $\lambda(F) > 0$. Since F is a subset of [0, 1], $\lambda g^{-1}(F) = (1/2)\lambda(F)$. On the other hand, g is the identity on F, and so, $\lambda g^{-1}(F) = \lambda(F)$. Thus, $0 \neq (1/2)\lambda(F) = \lambda(F)$. This contradiction establishes that the game does not have a Nash equilibrium. ■

Lemma 1. *If an atomless probability space $(I, \mathcal{I}, \lambda)$ is not saturated, then there exists a large non-anonymous game with traits $\mathcal{G} : (I, \mathcal{I}, \lambda) \longrightarrow T \times \mathcal{V}_{(A,T,\rho)}$ that does not have a Nash equilibrium, where $A = \{-1, 1\}$, $T = [0, b]$ for some $b > 1$ and $\rho = \lambda\mathcal{G}_1^{-1}$, a uniform distribution on $[0, b]$.*

Proof: First, let $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ be the Lebesgue unit interval, $\hat{T}$ be $[0,1]$ and $\hat{\alpha}$ be a function from $\hat{I}$ to $\hat{T}$ satisfying $\hat{\alpha}(j) = j$ for each $j \in \hat{I}$. For all $j \in \hat{I}$, let $\hat{v}_j(a, \hat{\tau}) = -\int_{\hat{T}\times A} 1_{[0,j)\times\{a\}}(t,x)d\hat{\tau}$ for all $a \in A$ and for all $\hat{\tau} \in \mathcal{M}^{\hat{\lambda}\hat{\alpha}^{-1}}(T \times A)$. Let $\hat{\mathcal{G}}$ be a function satisfying $\hat{\mathcal{G}}(j) = (\hat{\alpha}(j), \hat{v}_j)$ for all $j \in \hat{I}$. As we only replace the action set $\{\bar{H}, \bar{T}\}$ in Example 1 of Section 3 with $\{-1, 1\}$, it is easy to see that $\hat{\mathcal{G}}$ is a large non-anonymous game without any Nash equilibrium. That is to say, there is no measurable function $\hat{g} : \hat{I} \longrightarrow A$, such that for $\hat{\lambda}$-almost all $j \in \hat{I}$, $\hat{v}_j(\hat{g}(j), \hat{\lambda}(\hat{\alpha}, \hat{g})^{-1}) \geq \hat{v}_j(a, \hat{\lambda}(\hat{\alpha}, \hat{g})^{-1})$ for all $a \in A$.

Now, suppose that $(I, \mathcal{I}, \lambda)$ is not saturated. By Proposition 1, we know that there is a set $C \in \mathcal{I}$ with $\lambda(C) = \beta$, $0 < \beta < 1$ such that $(C, \mathcal{I}^C, \lambda^C)$ is countably generated, where $\mathcal{I}^C = \{E \in \mathcal{I}, E \subseteq C\}$ and $\lambda^C(E) = \lambda(E)/\beta$ for all $E \in \mathcal{I}^C$. By

Maharam's theorem (see [21, 20], we know that any two countably generated atomless measure algebras are isomorphic. Thus, since the Lebesgue unit interval is countable generated, there exists an isomorphism from the measure algebra of $(C, \mathcal{I}^C, \lambda^C)$ to the measure algebra of the Lebesgue unit interval. By [21, Theorem 4.12], there exists a measurable mapping $h : C \longrightarrow [0,1]$ such that h induces such an isomorphism. Let $b = 2-\beta$ and $T = [0,b]$. Note that $\lambda(I \setminus C) = 1-\beta$. There exists a measurable function $\alpha^0 : I \setminus C \longrightarrow (1,b]$, such that for any Borel set B in $(1,b]$, $\lambda(\alpha^0)^{-1}(B) = \hat{\lambda}(B)$; here we continue to use $\hat{\lambda}$ to denote the Lebesgue measure on $(1,b]$. Construct a function α from I to T as follows:

$$\alpha(i) = \begin{cases} h(i) & \text{for } i \in C \\ \alpha^0(i) & \text{for } i \in I \setminus C. \end{cases}$$

It is clear that α is measurable. Let $\rho = \lambda\alpha^{-1}$. By construction, ρ is a uniform distribution on $[0,b]$. Now, for all $a \in A$ and for all $\tau \in \mathcal{M}^\rho(T \times A)$, let

$$v_i(a,\tau) = \begin{cases} -\int_{T\times A} 1_{[0,h(i))\times\{a\}}(t,x)d\tau & \text{for } i \in C \\ a & \text{for } i \in I \setminus C. \end{cases}$$

By the proof of Claim 1, it is now easy to see that $v_i \in \mathcal{V}_{(A,T,\rho)}$ for any $i \in I$. Let $\mathcal{G}(i) = (\alpha(i), v_i)$ for all $i \in I$. By construction, $\mathcal{G}$ is a measurable function from I to $T \times \mathcal{V}_{(A,T,\rho)}$, and thus, a large non-anonymous game with traits.

We now show that $\mathcal{G}$ does not have a Nash equilibrium. Suppose that $\mathcal{G}$ has a Nash equilibrium $f^* : I \longrightarrow A$. It is obvious that $f^*(i) = 1$, for $i \notin C$. For $i \in C$, we first show that there exists a Borel measurable function g such that $f^*(i) = g(h(i))$ for λ-almost all $i \in C$. Let $C_1 = \{i \in C : f^*(i) = 1\}$. Since h induces an isomorphism from the measure algebra of $(C, \mathcal{I}^C, \lambda^C)$ to the measure algebra of the Lebesgue unit interval, there exists a Borel set $B_1 \subseteq [0,1]$, such that, $\lambda^C\left(C_1 \Delta h^{-1}(B_1)\right) = 0$, where Δ is the symmetry difference on $\mathcal{I}^C$, and hence, $\lambda\left(C_1 \Delta h^{-1}(B_1)\right) = 0$. Thus, g can be constructed as $g(i) = 1$ for $i \in B_1$ and $g(i) = -1$ for $i \in [0,1] \setminus B_1$. Now we show that there will be a contradiction. Since f^* is an equilibrium, thus, for λ-almost all $i \in C$,

$$v_i\left(f^*(i), \lambda(\alpha, f^*)^{-1}\right) \geq v_i\left(a, \lambda(\alpha, f^*)^{-1}\right) \text{ for all } a \in A,$$

which is to say, for λ-almost all $i \in C$,

$$-\int_{T\times A} 1_{[0,h(i))\times\{f^*(i)\}}(t,x)d\lambda(\alpha, f^*)^{-1} \geq -\int_{T\times A} 1_{[0,h(i))\times\{a\}}(t,x)d\lambda(\alpha, f^*)^{-1} \text{ for all } a \in A.$$

Because $1_{[0,h(i))\times\{f^*(i)\}}$ is zero on $(1,b]\times A$, the above equation can be written as

$$-\int_{[0,1]\times A} 1_{[0,h(i))\times\{f^*(i)\}}(t,x)d\lambda(\alpha,f^*)^{-1} \geq -\int_{[0,1]\times A} 1_{[0,h(i))\times\{a\}}(t,x)d\lambda(\alpha,f^*)^{-1} \text{ for all } a\in A.$$

Dividing both sides of the above equation by β and thus, normalizing, we have for λ-almost all $i\in C$,

$$v_{h(i)}\left(g\left(h(i)\right),\lambda^C\left(h,g\left(h(i)\right)\right)^{-1}\right) \geq v_{h(i)}\left(a,\lambda^C\left(h,g\left(h(i)\right)\right)^{-1}\right) \text{ for all } a\in A.$$

Hence, for λ-almost all $j\in\hat{I}$, the measurable function g satisfies

$$\hat{v}_j\left(g(j),\hat{\lambda}\left(\hat{\alpha},g\right)^{-1}\right) \geq \hat{v}_j\left(a,\hat{\lambda}\left(\hat{\alpha},g\right)^{-1}\right) \text{ for all } a\in A.$$

This is a contradiction to the first paragraph in the proof. Hence, the constructed game $\mathcal{G}$ does not have a Nash equilibrium. ■

Lemma 2. *If an atomless probability space $(I,\mathcal{I},\lambda)$ is not saturated and T' is uncountable complete separable metric space with a atomless probability measure ρ', there exists a large non-anonymous game with traits $\mathcal{G}': I \longrightarrow T'\times\mathcal{V}_{(A,T',\rho')}$ that has no Nash equilibrium where $A=\{-1,1\}$.*

Proof: Throughout this proof, we reserve the notation (T,ρ) for the space of traits, $(I,\mathcal{I},\lambda)$ for the space of players, A for the action set, $\mathcal{G}=(\alpha,v)$ for the game, respectively, as in the proof of Lemma 1.

Consider any given uncountable complete separable metric space T' endowed with an atomless probability measure ρ' on the Borel σ-algebra $\mathcal{B}(T')$ induced by the topology on T'. By Theorem 1 in [43], there exists a Borel measurable bijection $F: T'\longrightarrow T$ such that F is measure preserving between $(T',\mathcal{B}(T'),\rho')$ and $(T,\mathcal{B}(T),\rho)$, and continuous ρ'-almost everywhere, and F^{-1} is continuous ρ-almost everywhere. Now consider a mapping $\mathcal{G}'=(\alpha',v')$ which satisfies that $\alpha'=F^{-1}(\alpha)$ and for each $i\in I$, $v'_i(a,\tau')=v_i(a,\tau'(F,id_A)^{-1})$ for all $a\in A$ and $\tau'\in\mathcal{M}^{\rho'}(T'\times A)$, where id_A stands for the identity map on A. In order to show $\mathcal{G}'$ is a well-defined large non-anonymous game with traits, we need to show that for any given $a\in A$, $v'_i(a,\cdot)$ is continuous for any i. Suppose that a sequence $\{\tau'^n\}$ in $\mathcal{M}^{\rho'}(T'\times A)$ converges weakly to $\tau'^0\in\mathcal{M}^{\rho'}(T'\times A)$. Since F is continuous ρ'-almost everywhere, (F,id_A) is continuous τ'^0-almost everywhere. Hence, $\{\tau'^n(F,id_A)^{-1}\}$ converges weakly to $\tau'^0(F,id_A)^{-1}$ as well. Therefore, for any $a\in A$, the continuity of $v_i(a,\cdot)$ implies $v'_i(a,\cdot)$ is continuous for any i. Thus, $\mathcal{G}'$ is a large non-anonymous game with traits.

We now show that the game $\mathcal{G}'$ does not have an equilibrium. Suppose it does have an equilibrium, say, f'. Then, for λ-almost all i,

$$v'_i\left(f'(i), \lambda(\alpha', f')^{-1}\right) \geq v'_i\left(a, \lambda(\alpha', f')^{-1}\right) \text{ for all } a \in A,$$

which is to say,

$$v'_i\left(f'(i), \lambda(F^{-1}(\alpha), f')^{-1}\right) \geq v'_i\left(a, \lambda(F^{-1}(\alpha), f')^{-1}\right) \text{ for all } a \in A.$$

Hence, we have for λ-almost all i,

$$v_i\left(f'(i), \lambda(\alpha, f')^{-1}\right) \geq v_i\left(a, \lambda(\alpha, f')^{-1}\right) \text{ for all } a \in A,$$

which shows that f' is a Nash equilibrium of $\mathcal{G}$, the game which is constructed in Lemma 1. This is a contradiction. Therefore, the constructed game $\mathcal{G}'$ does not have any equilibrium. ■

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