Well-quasi-ordering versus clique-width *

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Abstract

Does well-quasi-ordering by induced subgraphs imply bounded clique-width for hereditary classes? This question was asked by Daligault, Rao, and Thomassé [Well-quasi-order of relabel functions. *Order*, 27(3) (2010), 301–315]. We answer this question negatively by presenting a hereditary class of graphs of unbounded clique-width which is well-quasi-ordered by the induced subgraph relation. We also show that graphs in our class have at most logarithmic clique-width and that the number of minimal forbidden induced subgraphs for our class is infinite. These results lead to a conjecture relaxing the above question and to a number of related open questions connecting well-quasi-ordering and clique-width.

1 Introduction

In this paper, we study two seemingly unrelated notions: well-quasi-ordering and clique-width.

Well-quasi-ordering (WQO) is a highly desirable property and a frequently discovered concept in mathematics and theoretical computer science [9, 14]. One of the most remarkable recent results in this area is the proof of Wagner's conjecture stating that the set of all finite graphs is well-quasi-ordered by the minor relation [18]. This, however, is not the case for the induced subgraph relation, since it contains infinite antichains, for instance, the antichain of cycles. On the other hand, the induced subgraph relation may become a well-quasi-order when restricted to graphs in particular classes. Throughout this paper, we use the notion of well-quasi-ordering with respect to the induced subgraph relation only.

Clique-width is a much younger notion than WQO. It was introduced in 1993 in [4] and it generalizes another graph parameter, tree-width, which was studied in the literature for decades. Both parameters are important in algorithmic graph theory, as graphs of "low" tree- or clique-width admit efficient solutions for many problems which are generally intractable [5]. Typically, "low" means "bounded by a constant". However, a logarithmic upper bound on clique-width does the same job, i.e., it provides polynomial-time algorithms. The notion of clique-width was also generalized from graphs to logical structures of arbitrary signature and cardinality [3].

Very little suggests that there is anything in common between these two notions, well-quasi-ordering and clique-width. One hint comes from the fact that the first non-trivial step towards the proof of Wagner's conjecture was made for graphs of bounded tree-width [17]. In the case

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of induced subgraphs, the first non-trivial result was obtained by Damaschke who showed in [8] that cographs are well-quasi-ordered by induced subgraphs. What is interesting is that cographs are precisely the graphs of clique-width at most 2 (see e.g. [6]). In [16], Petkovšek introduced an infinite family of WQO graph classes under the name k-letter graphs, and again all of them turned out to be of bounded clique-width. Recently, many new WQO classes have been discovered in the literature (see e.g. [1, 12, 13]), and the same phenomenon was observed in all of them. This discussion naturally leads to the following question:

Question 1. *Does well-quasi-ordering imply bounded clique-width?*

This question was formally stated by Daligault, Rao, and Thomassé in [7]. More precisely, they stated it for *hereditary* classes, i.e., classes closed under taking induced subgraphs. The restriction to hereditary classes is natural for graphs of bounded clique-width, since the clique-width of a graph is never smaller than the clique-width of any of its induced subgraphs [6]. An important feature of hereditary classes is that each of them can be characterized by a unique set of minimal forbidden induced subgraphs. If this set is finite, we call the class *finitely defined*.

In the present paper, we answer Question 1 negatively by exhibiting a hereditary class of graphs of unbounded clique-width, which is well-quasi-ordered by the induced subgraph relation. We call graphs in our class the *power graphs*.

Our negative result is not the end of the story about well-quasi-ordering and clique-width, as the relationship between these two notions is not exhausted by Question 1. In the same paper [7], Daligault, Rao, and Thomassé propose the following conjecture.

Conjecture 1. *Every* 2*-well-quasi-ordered hereditary class of graphs has bounded clique-width.*

The notion of 2-well-quasi-ordering deals with a restriction of the induced subgraph relation to graphs whose vertices are colored with two colors, in which case the relation is required to respect the colors. Clearly, 2-well-quasi-ordering implies well-quasi-ordering and therefore Conjecture 1 is a restriction of Question 1.

The example of power graphs does not destroy Conjecture 1, as graphs in this class are not 2-well-quasi-ordered. We derive this conclusion in a non-constructive way by showing that the class of power graphs is *not* finitely defined and combining this result with the following lemma proved by Daligault, Rao, and Thomassé in [7].

Lemma 1. *Any* 2*-well-quasi-ordered hereditary class is finitely defined.*

The above discussion suggests another restriction of Question 1: does well-quasi-ordering imply bounded clique-width for finitely defined hereditary classes? According to Lemma 1, this restriction generalizes Conjecture 1. We believe that this generalization is true and formally state it as a conjecture in the concluding section of the paper. In the same section, we discuss several related open questions connecting well-quasi-ordering with clique-width. In particular, we ask about the speed of growth of clique-width in well-quasi-ordered classes of graphs. For power graphs, we show that the speed is bounded by a logarithmic function, which keeps our class in the area of tractability for many algorithmic problems, in spite of the negative answer to Question 1.

The organization of the paper is as follows. In Section 2, we define the class $\mathcal{D}$ of power graphs and prove several useful properties of these graphs. In Section 3, we study clique-width of graphs in $\mathcal{D}$. In particular, we prove that, on the one hand, the clique-width of power graphs is not bounded by any constant (Section 3.1), and on the other hand, it is bounded by a logarithmic function of the number of vertices (Section 3.2). Section 4 is devoted to the notion of WQO and proves two results: the class $\mathcal{D}$ is well-quasi-ordered (Section 4.1) but not 2-well-quasi-ordered (Section 4.2). Section 5 concludes the paper with a discussion and a number of open problems.

2 The power graphs

Let P be a path with vertex set $\{1, \dots, n\}$ with two vertices i and j being adjacent if and only if $|i - j| = 1$. For a vertex i, the largest number of the form 2^k that divides i is called the *power* of i and is denoted by $q(i)$. For example, $q(5) = 1, q(6) = 2, q(8) = 8, q(12) = 4$.

To define the class of power graphs, we add to P edges connecting i and j whenever $q(i) = q(j)$ and denote the resulting graph by D_n. Figure 1 illustrates the graph D_{16}.

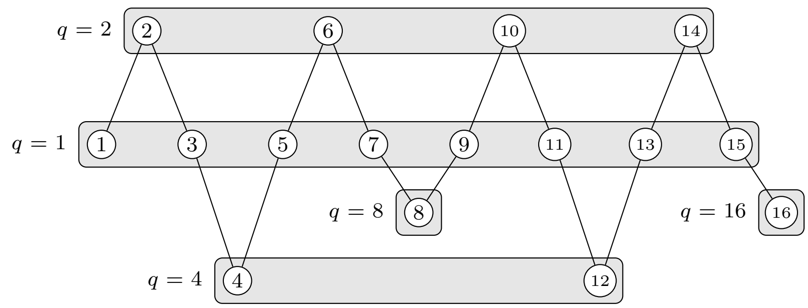

Figure 1: The graph D_{16}. To avoid shading the picture with many edges, the power cliques are represented as gray rectangular boxes.

By definition, the edges $E(D_n) \setminus E(P)$ form a set of disjoint cliques each consisting of vertices of the same power. We call these cliques *power cliques*, and say that a power clique Q *corresponds to* 2^k if it consists of vertices of power 2^k. We call P the *body* of D_n, the edges of $E(P)$ the *path edges*, and the edges of $E(D_n) \setminus E(P)$ the *clique edges*.

Definition 1. *We define $\mathcal{D}$ to be the class of all graphs $D_n, n \in \mathbb{N}$, and all their induced subgraphs and call graphs in $\mathcal{D}$ the power graphs.*

Given a graph G isomorphic to a graph in $\mathcal{D}$, among all possible sets of integers yielding an induced subgraph of some graph D_n isomorphic to G, we pick one arbitrarily and identify $V(G)$ with this set.

Any set of consecutive integers will be called an *interval* and any subgraph of D_n induced by an interval will be called a *factor*. The number of elements in an interval inducing a factor is the *length* of the factor. The vertex set of every graph $G \in \mathcal{D}$ can be split into maximal intervals and we call the subgraphs of G induced by these intervals *factor-components* of G. We say that a vertex u of a factor F is *maximal* if $q(u) \geq q(v)$ for each vertex v of F different from u.

The following statements show that every factor F has exactly one maximal vertex, moreover the power of any other vertex v of F is bounded by the length of F, and is uniquely determined by the difference between v and the maximal vertex.

Lemma 2. *Every factor F contains exactly one maximal vertex.*

Proof. Clearly, F contains at least one maximal vertex. Suppose that F contains two maximal vertices $2^k p$ and $2^k(p + r)$ for some odd number p and even number $r \geq 2$. Then F also contains the vertex $2^k(p+1)$. Clearly $p+1$ is an even number and hence $q(2^k(p+1)) \geq 2^{k+1}$, which contradicts the maximality of 2^k. $\square$

Lemma 3. *Let F be a factor of length at most c. If v is a vertex of F different from its maximal vertex m, then $q(v) = q(|m-v|)$. In particular, $q(v) < c$.*

Proof. Assume that $v > m$. Let k_1, p_1, k_2, p_2 be such that $m = 2^{k_1}p_1$ and $v - m = 2^{k_2}p_2$, with p_1, p_2 being odd numbers. Observe that $k_2 < k_1$, since otherwise $v = 2^{k_1}p_1 + 2^{k_2}p_2 = 2^{k_1}(p_1 + 2^{k_2-k_1}p_2)$, where $p_1 + 2^{k_2-k_1}p_2$ is a natural number. Therefore, $q(v) \geq 2^{k_1} = q(m)$, which contradicts either the maximality of m or Lemma 2. Consequently, $v = 2^{k_1}p_1 + 2^{k_2}p_2 = 2^{k_2}(2^{k_1-k_2}p_1 + p_2)$, where $2^{k_1-k_2}p_1 + p_2$ is an odd number. Hence, $q(v) = 2^{k_2} = q(v-m)$. Finally, since the length of F is at most c, we conclude that $v - m < c$, and therefore $q(v) = q(v-m) < c$.

The case when $v < m$ is proved similarly. $\square$

Corollary 1. *Let F be a factor of length at most c. If m is a vertex of F with $q(m) \geq c$, then m is the maximal vertex of F.*

3 Clique-width of power graphs

The clique-width of a graph G, denoted $cwd(G)$, is the minimum number of labels needed to construct the graph by means of the four graph operations:

- creation of a new vertex with a label,
- disjoint union of two labeled graphs,
- connecting vertices with specified labels i and j,
- renaming label i to label j.

Every graph G can be constructed by means of these four operations, and the process of the construction can be described either by an algebraic expression or by a rooted binary decomposition tree, whose leaves correspond to the vertices of G, the root corresponds to G, and the internal nodes correspond to the union operations. In Section 3.1, we show that the clique-width of graphs in $\mathcal{D}$ is unbounded, i.e., there is no constant bounding the clique-width of graphs in $\mathcal{D}$. On the other hand, in Section 3.2 we show that the clique-width of graphs in $\mathcal{D}$, as a function of the number of vertices, grows at most logarithmically.

3.1 Clique-width is unbounded in $\mathcal{D}$

Given a graph G and a subset $U \subset V(G)$, we denote by $\overline{U}$ the set $V(G) - U$. We say that two vertices $x, y \in U$ are *U-similar* if $N(x) \cap \overline{U} = N(y) \cap \overline{U}$, i.e., if x and y have the same neighbourhood outside of U. Clearly, the U-similarity is an equivalence relation and we denote the number of similarity classes of U by $\mu_G(U)$. Also, we denote

$$\mu(G) = \min_{\frac{1}{3}n \leq |U| \leq \frac{2}{3}n} \mu_G(U),$$

where $n = |V(G)|$. Our proof of the main result of this section is based on the following lemma.

Lemma 4. *For any graph G, $\mu(G) \leq cwd(G)$.*

Proof. Let T be a decomposition tree constructing G with $cwd(G)$ labels, t a node of T, and U_t the set of vertices of G that are leaves of the subtree of T rooted at t. It is known (see e.g. [15]) that $cwd(G) \geq \mu_G(U_t)$ for any node t of T. According to a well known folklore result, binary tree T has a node t such that $\frac{1}{3}|V(G)| \leq |U_t| \leq \frac{2}{3}|V(G)|$, in which case $\mu_G(U_t) \geq \mu(G)$. Hence the lemma. □

Let $U \subseteq V(D_n)$, and let P be the body of D_n. We denote by P^U the subgraph of P induced by U. In other words, P^U is obtained from $D_n[U]$ by removing the clique edges. Since P is a path, P^U is a graph every connected component of which is a path.

In order to use Lemma 4 for proving the main result of the section we will show that $\mu_{D_n}(U)$ is 'large' whenever both U and $\overline{U}$ are 'large'. Note that if P^U has c connected components, then $P^{\overline{U}}$ has at least $c-1$ connected components. This allows us to distinguish between two cases: 1) both P^U and $P^{\overline{U}}$ have many connected components; 2) both P^U and $P^{\overline{U}}$ have a limited number of connected components. The former case is considered in the following lemma.

Lemma 5. *If P^U has $c+1$ connected components, then $\mu_{D_n}(U) \geq c/2$.*

Proof. In the i-th connected component of P^U, $i \leq c$, we choose the last vertex (listed along the path P) and denote it by u_i. The next vertex of P, denoted $\overline{u}_i$, belongs to $\overline{U}$. This creates a matching of size c with edges $(u_i, \overline{u}_i)$. Note that none of $(u_i, \overline{u}_j)$ is a path edge for $i < j$. Among the chosen vertices of U at least half have the same parity. Their respective matched vertices of $\overline{U}$ have the opposite parity. Since the clique edges connect only the vertices of the *same* parity, we conclude that at least $c/2$ vertices of U have pairwise different neighbourhoods in $\overline{U}$, i.e., $\mu_{D_n}(U) \geq c/2$. □

Now we consider the case where both P^U and $P^{\overline{U}}$ have a limited number of connected components. Taking into account the definition of $\mu(G)$ and Lemma 4 we can assume that both U and $\overline{U}$ are 'large', and hence each of P^U and $P^{\overline{U}}$ has a 'large' connected component. In order to address this case we use the following lemma which states that a large number of power cliques intersecting both U and $\overline{U}$ implies a large value of $\mu_{D_n}(U)$.

Lemma 6. *If there exist c different power cliques $Q_1, \ldots, Q_c$ each of which*

(1) corresponds to a power of 2 greater than 1 and

(2) intersects both U and $\overline{U}$,

then $\mu_{D_n}(U) \geq c$.

Proof. Let u_i and $\overline{u}_i$ be some vertices in Q_i, which belong to U and $\overline{U}$, respectively. Since all the vertices in $M = \{u_1, \overline{u}_1, \ldots, u_c, \overline{u}_c\}$ are even and two even vertices are adjacent in D_n if and only if they belong to the same power clique, M induces a matching in D_n with edges $(u_i, \overline{u}_i)$, $i = 1, \ldots, c$. This implies that $u_1, \ldots, u_c$ have pairwise different neighbourhoods in $\overline{U}$, that is $\mu_{D_n}(U) \geq c$. □

The only remaining ingredient to prove the main result of this section is the following lemma.

Lemma 7. *Let c be a positive integer and P' a subpath of P with at least 2^{c+1} vertices. Then P' intersects each of the power cliques corresponding to $2^1, \ldots, 2^c$.*

Proof. The statement easily follows from the fact that for any $k \in \{1, \ldots, c\}$, vertices v with $q(v) = 2^k$ are of the form $v = 2^k(2p+1)$. In other words, they occur in P with period 2^{k+1}. $\square$

Now we are ready to prove the main result of this section.

Theorem 1. *Let n and c be natural numbers such that $n \geq 3((2c+1)(2^{c+1}-1)+1)$. Then $cwd(D_n) \geq c$ and hence the clique-width of graphs in $\mathcal{D}$ is unbounded.*

Proof. Let U be an arbitrary subset of vertices of D_n, such that $\frac{n}{3} \leq |U| \leq \frac{2n}{3}$. Note that the choice of U implies that the cardinalities of both U and $\overline{U}$ are at least $\frac{n}{3} \geq (2c+1)(2^{c+1}-1)+1$.

If P^U has at least $2c+1$ connected components, then by Lemma 5 $\mu_{D_n}(U) \geq c$. Otherwise P^U has less than $2c+1$ connected components and $P^{\overline{U}}$ has less than $2c+2$ connected components. By the pigeonhole principle, each of the graphs has a connected component of size at least 2^{c+1}. Clearly, these connected components are disjoint subpaths of P. By Lemma 7, the power cliques corresponding to $2^1, \ldots, 2^c$ intersect both U and $\overline{U}$, and hence, by Lemma 6, $\mu_{D_n}(U) \geq c$.

Since U has been chosen arbitrarily, we conclude that $\mu(D_n) \geq c$, and therefore, by Lemma 4, $cwd(D_n) \geq c$, as required. $\square$

3.2 Power graphs have at most logarithmic clique-width[1]

In this section, we show that for any n-vertex graph G in $\mathcal{D}$ the clique-width of G is bounded from above by $2\lceil \log n \rceil + 8$. We start with two auxiliary lemmas.

Lemma 8. *For any natural n, the clique-width of D_n is at most $\lceil \log(n+1) \rceil + 2$.*

Proof. We obtain graph D_n by constructing consecutively labeled graphs $H_1, ..., H_n$, where H_i is isomorphic to D_i. During the construction process of H_i we only use labels from the set $\{a \mid 2^a \leq i, a \in \mathbb{N}_0\}$ and two more auxiliary labels x, y. Moreover, if $i < n$, then vertex i of H_i is labeled by x, otherwise a vertex j of H_i is labeled by a, where $q(j) = 2^a$.

Let H_1 be the graph with $V(H_1) = \{1\}$, and the unique vertex of H_1 is labeled by x. For every $i = 2, \ldots, n$, we consecutively perform the following steps:

1. create vertex i with label y;
2. define H_i to be a disjoint union of H_{i-1} and vertex i;
3. in H_i add an edge between the only vertex with label x (i.e., vertex $i-1$) and the only vertex with label y (i.e., vertex i);
4. in H_i assign to vertex $i-1$ label a, where $q(i-1) = 2^a$;
5. in H_i add the edges between the only vertex with label y (i.e., vertex i) and all the vertices with label a, where $q(i) = 2^a$;
6. if $i < n$, then assign to vertex i label x, otherwise assign to i label a, where $q(i) = 2^a$.

It is easy to verify that H_n is equal to D_n. Moreover, the only used labels in the above procedure are x, y and integers from the set $\{a \mid 2^a \leq n, a \in \mathbb{N}_0\}$. Since the latter set has $\lceil \log(n+1) \rceil$ elements, we obtain the desired result. Notice that in H_n two vertices have the same power if and only if they have the same label. $\square$

[1] All logarithms in this paper are of base 2.

Lemma 9. *Let F be a factor of length n. Then the clique-width of F is at most $\lceil \log n \rceil + 4$.*

Proof. Taking into account Lemma 8 and the fact that the clique-width of an induced subgraph of a graph does not exceed the clique-width of the graph, it is sufficient to show that F is an induced subgraph of D_{3n}.

Let i, j and m be the first, the last, and the maximal vertices of F, respectively. Let also $c_1 = m - i$, $c_2 = j - m$, $c = \max\{c_1, c_2\}$, and let 2^r be the smallest power of 2 exceeding c. We claim that function $f : V(F) \to V(D_{2^r+c_2})$, given by

$$f(m + v) = 2^r + v \text{ for } v \in \{-c_1, \ldots, -1, 0, 1, \ldots, c_2\}$$

is a *subgraph isomorphism*[2] from F to $D_{2^r+c_2}$. Note that by Lemma 2 each of F and $D_{2^r+c_2}$ has a unique maximal vertex. Further, by the definition, f maps consecutive vertices of F to consecutive vertices of $D_{2^r+c_2}$, and the maximal vertex of F to the maximal vertex of $D_{2^r+c_2}$. Moreover, by Lemma 3 function f preserves powers of non-maximal vertices. Now, since in a power graph two vertices are adjacent if and only if either they are consecutive integers, or they have the same power, we conclude that f is a subgraph isomorphism from F to $D_{2^r+c_2}$. Finally, since $2^r \leq 2c$, we have $2^r + c_2 \leq 3c \leq 3n$, and the result follows. □

Theorem 2. *Let G be an n-vertex graph from $\mathcal{D}$. Then the clique-width of G is at most $2\lceil \log n \rceil + 8$.*

Proof. Denote by t the length of a longest factor-component in G. We will show that G can be constructed by the clique-width operations using at most $2(\lceil \log t \rceil + 4)$ different labels. Since $t \leq n$, this will give the result. By Lemma 3, powers of all vertices in G, except possibly the maximal vertices of some factor-components, are less than t. In particular, in the construction of a factor-component of G, provided by Lemmas 8 and 9, all non-maximal vertices are labeled by non-negative integers less than $\lceil \log t \rceil$, and without loss of generality we will assume that if the unique maximal vertex has power at least t then it is always labeled by $\lceil \log t \rceil$. We will distinguish between two types of maximal vertices:

1. maximal vertices of power less than t. These vertices may have non-maximal neighbours outside their factor-components. We will treat them as any other non-maximal vertex, that is, for a vertex of power 2^i we will use label i;

2. maximal vertices of power at least t. It follows from Corollary 1 that outside their factor-components these vertices are adjacent only to other maximal vertices of the same power. For all of them we will use one common label $\lceil \log t \rceil$. To do this we will need to construct independently each of the subgraphs induced by vertices of factor-components with maximal vertices of the same power, and successively combine these subgraphs with each other.

For convenience, let $k = \lceil \log t \rceil + 4$.

Let us first assume that the maximal vertices of all factor-components in G have the same power. We will prove by induction on the number s of factor-components that G can be constructed using $2k$ labels in such a way that at the end of the procedure the vertices of G are assigned labels from the set $\{0, \ldots, \lceil \log t \rceil\}$ with a vertex of power 2^i being assigned label i if

[2]For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ an injective function $f : V_1 \to V_2$ is called a subgraph isomorphism from G_1 to G_2 if $(v, u) \in E_1$ if and only if $(f(v), f(u)) \in E_2$.

$2^i < t$, and label $\lceil \log t \rceil$ otherwise. If G has only one factor, then it can be constructed using at most k labels by Lemma 9. Moreover, it follows from the proof of Lemma 8 that the construction possesses the desirable properties. Let now $s > 1$. Assume that we can properly construct every graph with less than s factor-components, and let G has s factor-components. Denote by F a factor-component of G, and let G' be the subgraph of G induced by the vertices outside of F. Using the induction hypothesis and an appropriate relabeling of the vertices of graph F, we can assume that both G' and F have been constructed independently using at most $2k$ different labels in such a way that the vertices of G' are assigned labels from the set $\{0, \dots, \lceil \log t \rceil\}$, vertices of F are assigned labels from the set $\{k, \dots, k + \lceil \log t \rceil\}$, and for every $i \in \{0, \dots, \lceil \log t \rceil\}$ the vertices of G' with label i have the same power as the vertices of F with label $k+i$. Now, to construct G we take the disjoint union of G' and F, and for every $i \in \{0, \dots, \lceil \log t \rceil\}$ successively do the following operations:

1. add all edges between the vertices with label i and the vertices with label $k + i$;
2. rename label $k + i$ to label i.

The general case is proved similarly, except that the induction goes over the number p of different powers of maximal vertices in factor-components of G. Specifically, we will show by induction on p that the clique-width of G is at most $2k$, and G can be constructed using at most $2k$ labels in such a way that at the end of the procedure a vertex of power 2^i is assigned label i if $2^i < t$, and every (maximal) vertex of power at least t is assigned label $\lceil \log t \rceil$. The above discussion provides the base case, when $p = 1$. Let now $p > 1$. Suppose that all graphs in which powers of maximal vertices take less than p different values admit the desirable construction, and consider a graph G whose powers of maximal vertices of factor-components take exactly p different values $q_1, \dots, q_p$. Let G_1 be the subgraph of G induced by the vertices of those factor-components whose maximal vertices have one of the powers $q_1, \dots, q_{p-1}$. Similarly, let G_2 be the subgraph of G induced by the vertices of those factor-components whose maximal vertices have power q_p. Using the induction hypothesis and an appropriate relabeling of the vertices of graph G_2, we can assume that both G_1 and G_2 have been constructed independently using at most $2k$ different labels in such a way that the vertices of G_1 are assigned labels from the set $\{0, \dots, \lceil \log t \rceil\}$ with $\lceil \log t \rceil$ being the label of the maximal vertices of power at least t, and the vertices of G_2 are assigned labels from the set $\{k, \dots, k + \lceil \log t \rceil\}$ with $k + \lceil \log t \rceil$ being the label of the maximal vertices of power at least t, and for every $i \in \{0, \dots, \lceil \log t \rceil - 1\}$ the vertices of G_1 with label i and the vertices of G_2 with label $k+i$ have the same power 2^i. Now, to construct G in a suitable way we first take the disjoint union of the constructed graphs G_1 and G_2. Then rename label $k + \lceil \log t \rceil$ to label $\lceil \log t \rceil$, and for every $i \in \{0, \dots, \lceil \log t \rceil - 1\}$ successively do the following operations:

1. add all edges between the vertices with label i and the vertices with label $k + i$;
2. rename label $k + i$ to label i.

□

4 Power graphs and well-quasi-ordering

A binary relation $\leq$ on a set W is a *quasi-order* (also known as *preorder*) if it is reflexive and transitive. Two elements $x, y \in W$ are said to be comparable with respect to $\leq$ if either $x \leq y$ or

$y \leq x$. Otherwise, x and y are incomparable. A set of pairwise comparable elements is called a *chain* and a set of pairwise incomparable elements an *antichain*. A quasi-order $(W, \leq)$ is a *well-quasi-order* if it contains neither infinite strictly decreasing chains nor infinite antichains. Since we deal with the induced subgraph relation on finite graphs, infinite strictly decreasing chains are impossible. Therefore, a class of graphs is well-quasi-ordered by the induced subgraph relation if and only if it contains no infinite antichains with respect to this relation. In Section 4.1, we show that graphs in $\mathcal{D}$ are well-quasi-ordered.

In Section 4.2, we deal with a more restrictive version of well-quasi-ordering known as *k-well-quasi-ordering*. A class of graphs $\mathcal{X}$ is said to be k-well-quasi-ordered if the set consisting of all vertex k-colored[3] graphs from $\mathcal{X}$ is well-quasi-ordered by the induced subgraph relation respecting the colors. In other words, when we embed a graph H into a graph G as an induced subgraph we must map the vertices of H to vertices of G of the *same* color. In Section 4.2, we show that the class $\mathcal{D}$ is not 2-well-quasi-ordered.

4.1 $\mathcal{D}$ is well-quasi-ordered

In the proof of the main result of this section, we apply a celebrated lemma due to Higman [10], which can be stated as follows. For an arbitrary set M, let M^* be the set of all finite sequences of elements of M. Any quasi-order $\leq$ on M defines a quasi-order $\preceq$ on M^* as follows: $(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_n)$ if and only if there exists a strictly increasing mapping $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $a_i \leq b_{f(i)}$ for each $i = 1, \ldots, m$.

Lemma 10. [10] *If $(M, \leq)$ is a* WQO*, then $(M^*, \preceq)$ is a* WQO.

Since the induced subgraph relation contains no infinite strictly decreasing chains, to prove the main result we need to show that for each infinite sequence $\mathcal{G} = G_1, G_2 \ldots$ of graphs in $\mathcal{D}$ there are i, j such that G_i is an induced subgraph of G_j. First we prove the following auxiliary lemma.

Lemma 11. *Let G be a graph in $\mathcal{D}$. Then there exists an integer $t = t(G)$ such that for any $n \geq t$ every factor of D_n of length at least t contains G as an induced subgraph.*

Proof. Let s be the smallest number such that G is an induced subgraph of D_s. We will show that $t = 5s$ satisfies the lemma. To this end it is enough to prove that any factor F of D_n of length at least t contains D_s as an induced subgraph. By the transitivity of the induced subgraph relation, this will imply that G is an induced subgraph of F.

Let 2^k be the smallest power of 2 larger than s. Clearly, $2^{k+1} \leq 4s$. Hence, by Lemma 7, among the first $4s$ vertices of F there is a vertex y with $q(y) = 2^k$. Let F' be the factor induced by the vertices of F starting at $y + 1$. Since F is of length at least $5s$ and y is among the first $4s$ vertices of F, the length of F' is at least s. Thus we can define an injective function $f : V(D_s) \to V(F')$ as follows: $f(z) = y + z$ for $1 \leq z \leq s$. We claim that f is a subgraph isomorphism from D_s to a subgraph of F. Clearly, $f(z + 1) = f(z) + 1$ for $1 \leq z < s$, hence it remains to verify that adjacencies and non-adjacencies are preserved for vertices z_1, z_2 of D_s such that $z_2 > z_1+1$. Clearly, in this case z_1 and z_2 are adjacent if and only if $q(z_1) = q(z_2)$. Moreover, since $f(z_2) > f(z_1)+1$, $f(z_2)$ and $f(z_1)$ are adjacent if and only if $q(f(z_1)) = q(f(z_2))$. Below we show that $q(f(z)) = q(z)$ for $1 \leq z \leq s$ and hence $q(z_1) = q(z_2)$ if and only if $q(f(z_1)) = q(f(z_2))$, implying the lemma.

[3]By a vertex k-coloring of a graph $G = (V, E)$ we mean an arbitrary mapping $V \to \{1, \ldots, k\}$.

Indeed, $f(z) = y + z = 2^k p + 2^{k_1} p_1$, where $2^{k_1} = q(z)$ and p, p_1 are odd numbers. Since $2^{k_1} \leq s < 2^k$, $k_1 < k$ and hence $y + z = 2^{k_1}(2^{k-k_1} p + p_1)$, where $2^{k-k_1} p + p_1$ is an odd number. Consequently, $q(y + z) = 2^{k_1} = q(z)$, as required. □

Lemma 12. *If $\mathcal{G}$ contains graphs with arbitrarily long factor-components, then $\mathcal{G}$ is not an antichain.*

Proof. Pick an arbitrary G_i in $\mathcal{G}$. By assumption, $\mathcal{G}$ contains a graph G_j with a factor-component F of length at least $t(G_i)$, where $t(G_i)$ is given by Lemma 11. By the same lemma, the graph G_i is an induced subgraph of F, and therefore is an induced subgraph of G_j. □

From now on, we assume that the length of factor-components of graphs in $\mathcal{G}$ is bounded by some constant $c = c(\mathcal{G})$. In what follows we prove that in this case $\mathcal{G}$ is not an antichain as well.

Let F be a factor. In light of Lemma 2, we denote the unique maximal vertex of F by $m(F)$. Also, let $s(F)$ be the smallest vertex of F. Now we define two equivalence relations on the set of factor graphs as follows. We say that two factors F_1 and F_2 are

- t-equivalent if they are of the same length and $m(F_1) - s(F_1) = m(F_2) - s(F_2)$,
- ℓ-equivalent if $q(m(F_1)) = q(m(F_2))$.

For a non-negative integer i we denote by L_i the ℓ-equivalence class such that $q(m(F)) = 2^i$ for every factor F in this class. We also order the t-equivalence classes (arbitrarily) and denote by T_j the j-th class in this order.

Lemma 13. *Let F_1, F_2 be two t-equivalent factors. Then there exists an isomorphism f from F_1 to F_2 such that:*

(a) $f(m(F_1)) = m(F_2)$;

(b) $q(f(v)) = q(v)$ for all $v \in V(F_1)$ except possibly for $m(F_1)$.

Proof. We claim that the function f that maps the i-th vertex of the factor F_1 (starting from the smallest) to the i-th vertex of the factor F_2 is the desired isomorphism. Indeed, property (a) follows from the condition that the factors are t-equivalent. Now property (a) together with Lemma 3 imply property (b). Finally, since adjacency between vertices in a factor is completely determined by their adjacency in the body and by their powers, we conclude that f is, in fact, an isomorphism. □

For a graph $G \in \mathcal{D}$, we denote by $G_{i,j}$ the set of factor-components of G in $L_i \cap T_j$, and define a binary relation $\leq$ on graphs of $\mathcal{D}$ as follows: $G \leq H$ if and only if $|G_{i,j}| \leq |H_{i,j}|$ for all i and j (clearly in this definition one can be restricted to non-empty sets $G_{i,j}$).

Finally, for a constant $c = c(\mathcal{G})$ we slightly modify the definition of $\leq$ to $\leq_c$ as follows. We say that a mapping $h : \mathbb{N}_0 \to \mathbb{N}_0$ is c-preserving if it is injective and $h(i) = i$ for all $i \leq \lfloor \log c \rfloor$. Then $G \leq_c H$ if and only if there is a c-preserving mapping h such that $|G_{i,j}| \leq |H_{h(i),j}|$ for all i and j. The importance of the binary relation $\leq_c$ is due to the following lemma.

Lemma 14. *Suppose the length of factor-components of G and H is bounded by c and $G \leq_c H$. Then G is an induced subgraph of H.*

Proof. We say that a factor F is *low-powered* if $F \in L_i$, for some $i \leq \lfloor \log c \rfloor$, i.e., if $q(m(F)) \leq c$. Also for a graph G we denote by $\mathcal{F}(G)$ the set of all its factor-components.

It can be easily checked that the definition of $\leq_c$ implies the existence of an injective function $\phi : \mathcal{F}(G) \to \mathcal{F}(H)$ that possesses the following properties:

(1) ϕ maps each of the factors in $\mathcal{F}(G)$ to a t-equivalent factor in $\mathcal{F}(H)$;

(2) $F \in \mathcal{F}(G)$ is a low-powered factor if and only if $\phi(F)$ is;

(3) ϕ preserves power of the maximal vertex for each of the low-powered factors, i.e., $q(m(F)) = q(m(\phi(F)))$ for every low-powered factor $F \in \mathcal{F}(G)$;

(4) for any two factors $F_1, F_2 \in \mathcal{F}(G)$, $q(m(F_1)) = q(m(F_2))$ if and only if $q(m(\phi(F_1))) = q(m(\phi(F_2)))$.

To show that G is an induced subgraph of H we define a witnessing function that maps vertices of a factor $F \in \mathcal{F}(G)$ to vertices of $\phi(F) \in \mathcal{F}(H)$ according to an isomorphism described in Lemma 13. This mapping guarantees that a factor F of G is isomorphic to the factor $\phi(F)$ of H. Therefore it remains to check that adjacency relation between vertices in different factors is preserved under the defined mapping.

Note that adjacency between two vertices in different factors is determined entirely by the powers of these vertices. Moreover, Lemma 13 and property (3) of ϕ imply that our mapping preserves powers of all vertices except possibly maximal vertices of power more that c. Therefore in order to complete the proof we need only to make sure that in graph G a maximal vertex m of a factor F with $q(m) > c$ is adjacent to a vertex v in a factor different from F if and only if the corresponding images of m and v are adjacent in H.

Taking into account Corollary 1 we derive that a maximal vertex m with $q(m) > c$ is adjacent to a vertex v in a different factor if and only if v is maximal in that factor and $q(m) = q(v)$. Now the desired conclusion follows from Lemma 13 and properties (2) and (4) of function ϕ. □

Lemma 15. *The set of graphs in $\mathcal{D}$ in which factor-components have length at most c is well-quasi-ordered by the relation $\leq_c$.*

Proof. We associate with each graph $G \in \mathcal{D}$ containing no factor-component of length greater than c a matrix $M_G = m(i,j)$ with $m(i,j) = |G_{i,j}|$, where $i \in \mathbb{N}_0$, and $j \in \mathbb{N}$.

Each row of this matrix corresponds to an ℓ-equivalence class and we delete any row corresponding to L_i with $i > \lfloor \log c \rfloor$ which is empty (contains only 0s). This leaves a finite amount of rows (since G is finite).

Each column of M_G corresponds to a t-equivalence class and we delete all columns corresponding to t-equivalence classes containing factors of length greater than c (none of these classes has a factor-component of G). This leaves precisely $\binom{c+1}{2}$ columns in M_G.

We define the relation $\preceq$ on the set $\mathcal{M}$ of matrices constructed in this way as follows. For $M_1, M_2 \in \mathcal{M}$ we say that $M_1 \preceq M_2$ if and only if there exists a strictly increasing mapping β from the index set of the rows of M_1 to the index set of the rows of M_2 such that $m_1(i,j) \leq m_2(\beta(i), j)$ for all i and j. In addition, if β is c-preserving, then we say that $M_1 \preceq_c M_2$. Note that if both M_1 and M_2 have exactly $\lfloor \log c \rfloor$ rows, then $M_1 \preceq M_2$ is equivalent to $M_1 \preceq_c M_2$.

It is not difficult to see that if $M_{G_1} \preceq_c M_{G_2}$, then $G_1 \leq_c G_2$. Therefore, if $\preceq_c$ is a well-quasi-order, then $\leq_c$ is a well-quasi-order too. The well-quasi-orderability of matrices follows by repeated applications of Higman's lemma. First, we split each matrix $M \in \mathcal{M}$ into two submatrices M' and M'' so that M' contains the first $\lfloor \log c \rfloor$ rows and M'' contains the remaining rows. Let $\mathcal{M}' = \{M' | M \in \mathcal{M}\}$ and $\mathcal{M}'' = \{M'' | M \in \mathcal{M}\}$.

To see that the set of matrices $\mathcal{M}'$ is WQO by $\preceq$ we apply Higman's Lemma twice. First, the set of rows is WQO by $\preceq$ since all of them are finite words of equal length over the alphabet of non-negative integers (which is WQO by the ordinary arithmetic $\leq$ relation). Second, the set of matrices $\mathcal{M}'$ is WQO by $\preceq$ since each of them is a finite word over the alphabet of rows. Similarly, the set of matrices $\mathcal{M}''$ is WQO by $\preceq$. Note that in both applications of Lemma 10 to $\mathcal{M}'$ and in the first application to $\mathcal{M}''$, we considered sets of sequences of the *same length*. Hence, in this case, Higman's Lemma in fact implies the existence of two sequences one of which is *coordinate-wise* smaller than the other, exactly what we need in these cases.

Finally, since each matrix in $\mathcal{M}$ can be considered as a word of two letters over the alphabet $\mathcal{M}' \cup \mathcal{M}''$, which is WQO by $\preceq$, and the relations $\preceq$ and $\preceq_c$ are equivalent on the set of matrices $\mathcal{M}'$, we conclude that $\mathcal{M}$ is WQO by $\preceq_c$. □

Combining Lemmas 12, 14, and 15, we obtain the main result of this section.

Theorem 3. *The class $\mathcal{D}$ is well-quasi-ordered by the induced subgraph relation.*

4.2 $\mathcal{D}$ is not 2-well-quasi-ordered

In this section, we prove that $\mathcal{D}$ is not 2-well-quasi-ordered. We obtain this result by showing that the class $\mathcal{D}$ is not finitely defined. The desired conclusion that $\mathcal{D}$ is not 2-well-quasi-ordered will then follow from Lemma 1.

To prove that the class $\mathcal{D}$ is not finitely defined, we show that there are infinitely many minimal graphs that are not in $\mathcal{D}$. For every integer $k \geq 3$, we define B_k as a graph obtained from a subgraph of $D_{3 \cdot 2^k}$ induced by $\{1, 2, \ldots, 2^k, 5 \cdot 2^{k-1}, 5 \cdot 2^{k-1} + 1, \ldots, 3 \cdot 2^k\}$ by adding one new edge $(1, 3 \cdot 2^k)$, which we call the *binding* edge of B_k. Similarly to graphs D_n, an edge of B_k connecting two consecutive vertices, i.e., an edge of the form $(i, i+1)$, is called a *path edge*. In what follows, we show that the graphs $B_k, k \geq 3$, are minimal forbidden induced subgraphs for the class $\mathcal{D}$. We start with technical lemmas.

Lemma 16. *Let i and j be two different integers of the same power 2^k. Then $|i - j| \geq 2^{k+1}$.*

Proof. As $q(i) = q(j) = 2^k$ and $i \neq j$, we conclude that $i = r_1 2^k$ and $j = r_2 2^k$, where r_1 and r_2 are two different odd integers. Then $|r_1 2^k - r_2 2^k| = 2^k |r_1 - r_2| \geq 2^{k+1}$. □

Lemma 17. *Let i and j be two integers with $q(i) = 2^k$ and $q(j) = 2^s$, where $k > s$. Then $i + j$ and $|i - j|$ both have power 2^s.*

Proof. Let $i = r_1 2^k$ and $j = r_2 2^s$, where r_1, r_2 are odd integers. Then $i + j = 2^s(r_1 2^{k-s} + r_2)$ and $|i - j| = 2^s |r_1 2^{k-s} - r_2|$, and the lemma follows from the fact that both $r_1 2^{k-s} + r_2$ and $|r_1 2^{k-s} - r_2|$ are odd. □

Now we will show that none of the graphs $B_k, k \geq 3$, belongs to $\mathcal{D}$.

Lemma 18. *For every $k \geq 3$, the graph B_k does not belong to $\mathcal{D}$.*

Proof. Suppose to the contrary that there is an integer n and a function $f : V(B_k) \to V(D_n)$ such that f is a subgraph isomorphism from B_k to D_n. Note that the vertices of any clique of size at least 4 in B_k or in D_n have the same power. Therefore, since a clique of B_k is mapped by f to a clique of D_n, the images of all odd vertices of B_k have the same power. Moreover, since every even vertex in B_k has at most two odd neighbours, the power of its image is different from that of images of odd vertices. This means that all the path edges and the binding edge of B_k are mapped to path edges in D_n. Hence, the image of $V(B_k)$ forms an interval, and we denote by F the factor of D_n induced by this interval. Note that B_k has a unique largest clique, namely the clique formed by the odd vertices. Similarly, because F is a factor of length at least $|V(B_k)| \geq |V(B_3)| = 13$, it has a unique largest clique, namely the clique formed by its odd vertices. Therefore, as the largest clique is an invariant, the odd vertices of B_k are mapped to the odd vertices of F, and, hence, f preserves parity of the vertices. Furthermore, since in both B_k and F even vertices are adjacent if and only if they have the same power, we conclude that $v, u \in V(B_k)$ have the same power if and only if $f(v), f(u)$ have the same power.

Now, since B_k has $k+1$ cliques corresponding to different powers with at least two vertices in each clique, there are two different vertices $v, u \in V(B_k)$ whose images $f(v)$ and $f(u)$ have the same power 2^r with $r \geq k$. Finally, as the image of $V(B_k)$ forms an interval of length $3 \cdot 2^{k-1} + 1$, we have $|f(v) - f(u)| \leq 3 \cdot 2^{k-1} < 2^{k+1}$, which contradicts Lemma 16. □

In the following lemma, we show that all graphs $B_k, k \geq 3$, are in fact minimal forbidden.

Lemma 19. *For every $k \geq 3$, the graph B_k is a minimal forbidden induced subgraph for the class $\mathcal{D}$.*

Proof. Taking into account Lemma 18, it is sufficient to prove that for every vertex $v \in V(B_k)$ the graph $G_v = B_k \setminus \{v\}$ belongs to $\mathcal{D}$. Clearly, this is true if v is one of the ends of the binding edge, i.e if $v \in \{1, 3 \cdot 2^k\}$. For all other vertices v, we will provide a subgraph isomorphism f from G_v to some graph in $\mathcal{D}$.

First, assume that $v \in \{2, \ldots, 2^k\}$. We define $f : V(G_v) \to V(D_{2^{k+2}})$ as follows:

$$\begin{aligned} f(i) &= 3 \cdot 2^k + i, \text{ for } i \in \{1, \ldots, v-1\}; \\ f(i) &= i, \text{ for all other } i \in V(G_v). \end{aligned}$$

Since $v - 1 < 2^k$, it follows from Lemma 17 that f preserves powers. Also, it is easy to see that f maps consecutive vertices of G_v to consecutive ones. Moreover, only two non-consecutive vertices of G_v, namely 1 and $3 \cdot 2^k$, become consecutive under f, which corresponds to mapping of the binding edge $(1, 3 \cdot 2^k)$ to a path edge $(3 \cdot 2^k, 3 \cdot 2^k + 1)$. Therefore, f is a subgraph isomorphism from G_v to $D_{2^{k+2}}$.

Now let $v \in \{5 \cdot 2^{k-1}, 5 \cdot 2^{k-1} + 1, \ldots, 3 \cdot 2^k - 1\}$. We define $f : V(G_v) \to V(D_{13 \cdot 2^{k-1}})$ as follows:

$$\begin{aligned} f(i) &= 3 \cdot 2^{k-1} + i, \text{ for } i \in \{1, \ldots, 2^k\}; \\ f(i) &= i - 3 \cdot 2^{k-1}, \text{ for } i \in \{v+1, \ldots, 3 \cdot 2^k\}; \\ f(i) &= 7 \cdot 2^{k-1} + i, \text{ for } i \in \{5 \cdot 2^{k-1}, \ldots, v-1\}. \end{aligned}$$

Note that the powers of the vertices of G_v are at most 2^k. By Lemma 17, the function f preserves powers that are at most 2^{k-2}. There are at most two vertices of power 2^{k-1}, namely 2^{k-1} and $5 \cdot 2^{k-1}$, and they are mapped by f, respectively, to vertices 2^{k+1} and $3 \cdot 2^{k+1}$ of power 2^{k+1}. Also there are exactly two vertices of power 2^k, namely 2^k and $3 \cdot 2^k$, and they are mapped by f, respectively, to vertices $5 \cdot 2^{k-1}$ and $3 \cdot 2^{k-1}$ of power 2^{k-1}. Therefore, f preserves power cliques. Further, it is easy to see that consecutive vertices of G_v are mapped by f to consecutive vertices. Moreover, only two non-consecutive vertices of G_v, namely 1 and $3 \cdot 2^k$, become consecutive under f, which corresponds to mapping of the binding edge $(1, 3 \cdot 2^k)$ to a path edge $(3 \cdot 2^{k-1}, 3 \cdot 2^{k-1}+1)$. Therefore, f is a subgraph isomorphism from G_v to $D_{13 \cdot 2^{k-1}}$. $\square$

Combining Lemmas 18 and 19 with Lemma 1, we derive the following conclusion.

Theorem 4. *The class $\mathcal{D}$ is not finitely defined and hence is not 2-well-quasi-ordered.*

5 Concluding remarks and open problems

In this paper, we introduced a new hereditary class of graphs, the power graphs, and derived a number of properties of these graphs. In particular, we proved that

(1) the clique-width of power graphs is not bounded by any constant,

(2) the clique-width of power graphs is at most logarithmic in the number of vertices,

(3) the class of power graphs is well-quasi-ordered by the induced subgraph relation,

(4) the class of power graphs is not finitely defined and hence is not 2-well-quasi-ordered by induced subgraphs.

This sequence of results implies several conclusions. First of all, it provides a negative answer to Question 1 posed in [7]. Let us observe that this question is not trivial at all and the area where the answer to this question is positive includes a variety of graph classes. In particular, it includes all hereditary graph classes where the number of edges is bounded from above by a subquadratic function [2], or, equivalently, all hereditary graph classes forbidding some complete bipartite graph as a (not necessarily induced) subgraph. In view of these results, identifying the “area of positivity” of Question 1 becomes an important open problem. We believe that, in addition to the above classes, this area includes all finitely defined hereditary classes and propose this idea as a conjecture.

Conjecture 2. *Well-quasi-ordering implies bounded clique-width for finitely defined hereditary classes of graphs.*

According to Lemma 1, Conjecture 2 generalizes Conjecture 1. Whether this generalization is proper (i.e., strictly stronger) is another open question:

Question 2. *Are there finitely defined hereditary classes which are well-quasi-ordered but not 2-well-quasi-ordered by induced subgraphs?*

We conclude the paper with one more open question suggested by our results.

Question 3. *What is the maximal speed of growth of clique-width in well-quasi-ordered classes of graphs?*

A logarithmic bound on clique-width would have strong algorithmic consequences. In particular, many computational problems intractable for general graphs would be solvable in polynomial time for well-quasi-ordered classes. This is the case for power graphs, as we showed in Section 3.2. Finding a bound valid for all well-quasi-ordered classes remains a challenging open problem.

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