

Fast finite difference solvers for singular solutions of the elliptic Monge-Ampère equation

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Abstract

The elliptic Monge-Ampère equation is a fully nonlinear Partial Differential Equation which originated in geometric surface theory, and has been applied in dynamic meteorology, elasticity, geometric optics, image processing and image registration. Solutions can be singular, in which case standard numerical approaches fail.

In this article we build a finite difference solver for the Monge-Ampère equation, which converges even for singular solutions. Regularity results are used to select *a priori* between a stable, provably convergent monotone discretization and an accurate finite difference discretization in different regions of the computational domain. This allows singular solutions to be computed using a stable method, and regular solutions to be computed more accurately. The resulting nonlinear equations are then solved by Newton's method.

Computational results in two and three dimensions validate the claims of accuracy and solution speed. A computational example is presented which demonstrates the necessity of the use of the monotone scheme near singularities.

Keywords: Fully Nonlinear Elliptic Partial Differential Equations, Monge Ampère equations, Nonlinear Finite Difference Methods, Viscosity Solutions, Monotone Schemes, Convexity Constraints

1. Introduction

1 In this article we build a finite difference solver for the Monge-Ampère equation, which
2 converges even for singular solutions. Regularity results are used to select *a priori*
3 between two discretizations in different regions of the computational domain. Near possible
4 singularities, a stable, provably convergent monotone discretization is used. Elsewhere a
5 more accurate discretization is used. This allows singular solutions to be computed using
6 a stable method, and regular solutions to be computed more accurately. The resulting
7 nonlinear equations are then solved by Newton's method, which is fast, $\mathcal{O}(M^{1.3})$, where
8 M is the number of data points, independent of the regularity of the solution.
9

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10 *1.1. The setting for equation*

11 The Monge-Ampère equation is a fully nonlinear Partial Differential Equation (PDE).

12

$$\det(D^2u(x)) = f(x), \quad \text{for } x \text{ in } \Omega. \quad (\text{MA})$$

13 The Monge-Ampère operator, $\det(D^2u)$, is the determinant of the Hessian of the function
14 u . The equation is augmented by the convexity constraint

$$u \text{ is convex}, \quad (\text{C})$$

15 which is necessary for the equation to be elliptic. The convexity constraint is made
16 explicit for emphasis: it is necessary for uniqueness of solutions and it is essential for
17 numerical stability.

18 While other boundary conditions appear naturally in applications, we consider the
19 simplest boundary conditions: the Dirichlet problem in a convex bounded subset $\Omega \subset \mathbb{R}^d$
20 with boundary $\partial\Omega$,

$$u(x) = g(x), \quad \text{for } x \text{ on } \partial\Omega. \quad (\text{D})$$

21 Under suitable assumptions on the domain and the functions $f(x), g(x)$, recalled in sub-
22 section 2.1, there exist unique classical (C^2) solutions to (MA), (C). However, when
23 these conditions fail, solutions can be singular. For singular solutions, the correct notion
24 of weak solutions must be used. In this case, novel discretizations and solutions methods
25 must be used to approximate the solution.

26 *1.2. Applications*

27 The PDE (MA) is a geometric equation, which goes back to Monge and Ampère
28 (see [1]). The equation naturally arises in geometric problems of existence and uniqueness
29 of surfaces with proscribed metrics or curvatures [2, 3]. Early applications identified
30 in [4] include dynamic meteorology, elasticity, and geometric optics [5, 6, 7, 8]. For an
31 application of Monge-Ampère equations in mathematical finance, see [9].

32 The Monge-Ampère equation arises as the optimality conditions for the problem of
33 optimal mass transport with quadratic cost [1, 10, 11]. This application of the Monge-
34 Ampère equation has been used in many areas: image registration [12, 13, 14], mesh
35 generation [15, 16, 17], reflector design [18], and astrophysics (estimating the shape of
36 the early universe) [19].

37 The problem here is to find a mapping $\mathbf{g}(x)$ that moves the measure $\mu_1(x)$ to $\mu_2(y)$
38 and minimizes the transportation cost functional

$$\int_{\mathbb{R}^d} |x - \mathbf{g}(x)|^2 d\mu_1.$$

39 The optimal mapping is given by $\mathbf{g} = \nabla u$, where u satisfies the Monge-Ampère equation

$$\det(D^2u(x)) = \mu_1(x)/\mu_2(\nabla u(x)).$$

40 In this situation, the Dirichlet boundary condition (D) is replaced by the implicit bound-
41 ary condition

$$\mathbf{g}(\cdot) : \Omega_1 \rightarrow \Omega_2 \quad (1)$$

42 where the sets Ω_1 and Ω_2 are the support of the measures μ_1, μ_2 . These boundary
43 conditions are difficult to implement numerically; we are not aware of an implementation
44 using PDE methods. For many applications, both domains are squares, and a simplifying
45 assumption that edges are mapped to edges allows Neumann boundary conditions to be
46 used. In other applications, periodic boundary conditions are used.

47 In other problems, the Monge-Ampère operator appears in an *inequality constraint* in
48 a variational problem for optimal mappings where the cost is no longer the transportation
49 cost. Here the operator has the effect of restricting the local area change on the set of
50 admissible mappings, see [20] or [21].

51 1.3. Related numerical works

52 Despite the number of applications, until recently there have been few numerical
53 publications devoted to solving the Monge-Ampère equation. We make a distinction
54 between numerical approaches with optimal transportation type boundary conditions (1)
55 and the standard Dirichlet boundary conditions (D). In the latter case, a number of
56 numerical methods have been recently proposed for the solution of the Monge-Ampère
57 equation.

58 An early work is [4], which presents a discretization which converges to the Aleksandrov
59 solution in two dimensions. Another early work by Benamou and Brenier [22]
60 used a fluid mechanical approach to compute the solution to the optimal transportation
61 problem.

62 For the problem with Dirichlet boundary conditions which is treated here, a series of
63 papers have recently appeared by two groups of authors, Dean and Glowinski [23, 24, 25],
64 and Feng and Neilan, [26, 27]. The methods introduced by these authors perform best
65 in the regular case and can break down in the singular case. See [28] a more complete
66 discussion.

67 We also mention the works [29], in the periodic case, and [15] for applications to
68 mappings. The method of [30] treats the problem of periodic boundary conditions in
69 odd dimensional space.

70 1.4. Numerical challenges

71 When the conditions for regularity are satisfied, classical solutions can be approx-
72 imated successfully using a range of standard techniques (see, for example works such
73 as [23, 24, 25], and [26, 27]). However, for singular solutions, standard numerical methods
74 break down: either by becoming unstable, poorly conditioned, or by selecting the wrong
75 (non-convex) solution.

76 Weak solutions

77 For singular solutions, the appropriate notion of weak (viscosity or Aleksandrov)
78 solutions must be used. Numerical methods have been developed which capture weak
79 solutions: Oliker and Prussner, in [4], presents a method which converges to the Aleksan-
80 drov solution. One of the authors introduced a finite difference method which converges
81 to the viscosity solution in [31]. Both of these methods were restricted to two dimensions.
82 However, methods which are provably convergent may have lower accuracy or slower so-
83 lution methods than other methods which are effective for regular solutions. In [32] we
84 introduced a monotone discretization which is valid in arbitrary dimensions. A proof

85 of convergence to viscosity solutions is provided, as well as a proof of convergence of
86 Newton's method.

87 *Convexity*

88 The convexity constraint is necessary for both uniqueness and stability. In partic-
89 ular, the equation (MA) fails to be elliptic if u is non-convex (see subsection 2.5). so
90 instabilities can arise if the convexity condition (C) is violated, as demonstrated in sub-
91 section 8.1. Any approximation of (MA) requires some selection principle to choose the
92 convex solution. This selection principle can be built in to the discretization, as in [31],
93 or built in to the solution method, as in [28].

94 *Accuracy*

95 The convergent monotone schemes of [31] and [32] use a wide stencil, and the accuracy
96 of the scheme depends on the *directional resolution*, which depends on the width of the
97 stencil. As we demonstrate below, for highly singular solutions, such as (17), the direc-
98 tional resolution error can dominate. On the other hand, more accurate discretizations,
99 such as standard finite differences, can be unstable for singular solutions.

100 *Fast solvers*

101 Previous work by the authors and a coauthor [28] investigated fast solvers for (MA).
102 An explicit method was presented which was moderately fast, independent of the solution
103 time. For regular solutions, a faster (by an order of magnitude) semi-implicit solution
104 method was introduced (see subsection 6.2) but this method was slower (by an order of
105 magnitude) on singular solutions.

106 **2. Analysis and weak solutions**

107 In this section we present regularity results and background analysis which inform
108 the numerical approach taken in this work. In particular, the regularity results of sub-
109 section 2.1 are used to determine the discretization used in section 5.

110 The definition of viscosity solutions and Aleksandrov solutions presented in subsec-
111 tion 2.2-2.3 are used to make sense of the weak solutions (15) and (17), respectively.

112 *2.1. Regularity*

113 Under the following conditions, the Monge-Ampère equation is guaranteed to have
114 a unique $C^{2,\alpha}$ solution. Regularity results for the Monge-Ampère equation have been
115 established in [33, 34, 35]. We refer to the book [36] for the following result.

$$\begin{cases} \text{The domain } \Omega \text{ is strictly convex with boundary } \partial\Omega \in C^{2,\alpha}. \\ \text{The boundary values } g \in C^{2,\alpha}(\partial\Omega). \\ \text{The function } f \in C^\alpha(\Omega) \text{ is strictly positive.} \end{cases} \quad (2)$$

116 **Remark 1.** In the extreme case, with $f(x) = 0$ for all $x \in \Omega$, the equation (MA),(C)
117 reduces to the computation of the convex envelope of the boundary conditions [37, 38].
118 In this case, solutions may not even be continuous up to the boundary and can also be
119 non-differentiable in the interior.

120 **Remark 2.** While it is usual to perform numerical solutions on a rectangle, regularity
 121 can break down in particular convex polygons [11, 39].

122 2.2. Viscosity solutions

123 We recall the definition of viscosity solutions [40], which are defined for the Monge-
 124 Ampère equation in [36].

125 **Definition 1.** Let $u \in C(\Omega)$ be convex and $f \geq 0$ be continuous. The function u is
 126 a *viscosity subsolution (supersolution)* of the Monge-Ampère equation in Ω if whenever
 127 convex $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that $(u - \phi)(x) \leq (\geq)(u - \phi)(x_0)$ for all x in a
 128 neighbourhood of x_0 , then we must have

$$\det(D^2\phi(x_0)) \geq (\leq)f(x_0).$$

129 The function u is a *viscosity solution* if it is both a viscosity subsolution and supersolution.

130 **Example 1 (Viscosity solution of Monge-Ampère).** We consider an example which
 131 will later be solved numerically in two and three dimensions (sections 8-9). Consider (MA)
 132 with solution and f given by

$$u(\mathbf{x}) = \frac{1}{2}(|\mathbf{x}| - 1)^+, \quad f(\mathbf{x}) = (1 - 1/|\mathbf{x}|)^+.$$

133 This function, although it is not a classical C^2 solution of the Monge-Ampère equa-
 134 tion, is a viscosity solution.

135 2.3. Aleksandrov solutions

136 Next we turn our attention to the Aleksandrov solution, which is a more general weak
 137 solution than the viscosity solutions. Here f is generally a measure [36]. We begin by
 138 recalling the definition of the normal mapping or subdifferential of a function.

139 **Definition 2.** The *normal mapping (subdifferential)* of a function u is the set-valued
 140 function ∂u defined by

$$\partial u(x_0) = \{p : u(x) \geq u(x_0) + p \cdot (x - x_0)\}, \quad \text{for all } x \in \Omega.$$

141 For a set $V \subset \Omega$, we define $\partial u(V) = \bigcup_{x \in V} \partial u(x)$.

142 Now we want to look at a measure generated by the Monge-Ampère operator.

143 **Definition 3.** Given a function $u \in C(\Omega)$, the *Monge-Ampère measure* associated with
 144 u is defined as

$$\mu(V) = |\partial u(V)|$$

145 for any set $V \subset \Omega$.

146 This measure naturally leads to the notion of the generalized or Aleksandrov solution
 147 of the Monge-Ampère equation.

148 **Definition 4.** Let μ be a Borel measure defined in a convex set $\Omega \in \mathbb{R}^d$. Then the
 149 convex function u is an *Aleksandrov solution* of the Monge-Ampère equation

$$\det(D^2u) = \mu$$

150 if the Monge-Ampère measure associated with u is equal to the given measure μ .

151 **Example 2 (Aleksandrov solution).** As an example, we consider the cone and the
 152 scaled Dirac measure

$$u(\mathbf{x}) = |\mathbf{x}|, \quad \mu(V) = \pi \int_V \delta(\mathbf{x}) d\mathbf{x}.$$

153 2.4. A PDE for convexity

154 The convexity constraint (C) is necessary for uniqueness, since without it, $-u$ is also
 155 a solution of (MA).

156 For a twice continuously differentiable function u , the convexity restriction (C) can be
 157 written as D^2u is positive definite. Since we wish to work with less regular solutions, (C)
 158 can be enforced by the equation

$$\lambda_1(D^2u) \geq 0,$$

159 understood in the viscosity sense [37, 38], where $\lambda_1[D^2u]$ is the smallest eigenvalue of the
 160 Hessian of u .

161 The convexity constraint can be absorbed into the operator by defining

$$\det^+(M) = \prod_{j=1}^d \lambda_j^+ \tag{3}$$

162 where M is a symmetric matrix, with eigenvalues, $\lambda_1 \leq \dots, \leq \lambda_n$ and

$$x^+ = \max(x, 0).$$

163 Using this notation, (MA),(C) becomes

$$\det^+(D^2u(x)) = f(x), \quad \text{for } x \text{ in } \Omega \tag{MA}^+$$

164 **Remark 3.** Notice that there is a trade off in defining (3): the constraint (C) is elimi-
 165 nated but the operator becomes non-differentiable near singular matrices.

166 2.5. Linearization and ellipticity

167 The linearization of the determinant is given by

$$\nabla \det(M) \cdot N = \text{trace}(M_{adj}N)$$

168 Where M_{adj} is the adjugate [41], which is the transpose of the cofactor matrix. The
 169 adjugate matrix is positive definite if and only if M is positive definite. When the matrix
 170 M is invertible, the adjugate, M_{adj} , satisfies

$$M_{adj} = \det(M)M^{-1} \tag{4}$$

171 We now apply these considerations to the linearization of the Monge-Ampère opera-
 172 tor. When $u \in C^2$ we can linearize this operator as

$$\nabla_M \det(D^2u) \cdot v = \text{trace}((D^2u)_{adj}D^2v). \tag{5}$$

173 **Example 3.** In two dimensions we obtain

$$\nabla_M \det(D^2u)v = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}$$

174 which is homogenous of order one in D^2u . In dimension $d \geq 2$, the linearization is
 175 homogeneous order $d - 1$ in D^2u .

176 The linear operator

$$L[u] \equiv \text{trace}A(x)D^2u$$

177 is *elliptic* if the coefficient matrix $A(x)$ is positive definite.

178 **Lemma 1.** *Let $u \in C^2$. The linearization of the Monge-Ampère operator, (5) is elliptic*
 179 *if D^2u is positive definite or, equivalently, if u is (strictly) convex.*

180 **Remark 4.** When the function u fails to be strictly convex, the linearization can be
 181 degenerate elliptic, which affects the conditioning of the linear system (5). When the
 182 function u is nonconvex, the linear system can be unstable.

183 The definition of a nonlinear elliptic PDE operator generalizes the definition of linear
 184 elliptic operator. It also allows for the operators to be non-differentiable.

185 **Definition 5.** Let the PDE operator $F(M)$ be a continuous function defined on sym-
 186 metric matrices. Then $F(M)$ is *elliptic* if it satisfies the monotonicity condition

$$F(M) \leq F(N) \text{ whenever } M \leq N,$$

187 where for symmetric matrices $M \leq N$ means $x^T M x \leq x^T N x$ for all x .

188 **Example 4.** The operator $\det^+(M)$ is a non-decreasing function of the eigenvalues, so
 189 it is elliptic.

190 3. The standard finite difference discretization

191 We begin by considering the standard finite difference discretization of the Monge-
 192 Ampère equation. For brevity, we describe the discretization in two dimensions, but this
 193 is easily generalized to higher dimensions.

194 This discretization does not enforce the convexity condition (C), so it can lead to
 195 instabilities. In particular, we show in subsection 8.1 that Newton's method can become
 196 unstable if this discretization is used.

197 The Monge-Ampère operator has a particularly simple form in two dimensions:

$$\det(D^2u) = \frac{\partial^2u}{\partial x^2} \frac{\partial^2u}{\partial y^2} - \left(\frac{\partial^2u}{\partial x \partial y} \right)^2, \quad \text{in } \Omega \subset \mathbb{R}^2.$$

198 In two dimensions, the natural discretization of the operator is given by

$$MA^N[u] \equiv (\mathcal{D}_{xx}u)(\mathcal{D}_{yy}u) - (\mathcal{D}_{xy}u)^2 \quad (MA)^N$$

where, writing h for the spatial resolution of the grid,

$$\begin{aligned} [\mathcal{D}_{xx}u]_{ij} &= \frac{1}{h^2} (u_{i+1,j} + u_{i-1,j} - 2u_{i,j}) \\ [\mathcal{D}_{yy}u]_{ij} &= \frac{1}{h^2} (u_{i,j+1} + u_{i,j-1} - 2u_{i,j}) \\ [\mathcal{D}_{xy}u]_{ij} &= \frac{1}{4h^2} (u_{i+1,j+1} + u_{i-1,j-1} - u_{i-1,j+1} - u_{i+1,j-1}). \end{aligned}$$

199 **Remark 5.** There is no reason to assume that the standard discretization converges. In
 200 fact, the two dimensional scheme has multiple solutions. In [28] this discretization was
 201 used, but the the solvers were designed to select the convex solution.

202 4. Convergent monotone discretization

203 The method of [31] describes a discretization of the two-dimensional Monge-Ampère
 204 equation that converges to the viscosity solution. In [32] we introduced another dis-
 205 cretization, which generalized to higher dimensions, and also converged to the viscosity
 206 solution. Both methods require the use of a wide stencil scheme, which has an additional
 207 discretization parameter, the *directional resolution*, explained below.

208 In addition to being monotone, which means it is provably convergent, the latter
 209 method discretizes the convexified version of the equation, $(MA)^+$, which is enough to
 210 ensure convergence of Newton’s method. The proof of this result can be found in [32].

211 In this section we present the convergent discretization, which will be used to build
 212 the hybrid solver.

213 4.1. Wide stencils

214 When we discretize the operator on a finite difference grid, we approximate the second
 215 derivatives by centred finite differences (spatial discretization). In addition, we consider
 216 a finite number of possible directions ν that lie on the grid (directional discretization).

217 We consider the finite difference operator for the second directional derivative in the
 218 direction ν , which lies on the finite difference grid. These directional derivatives are
 219 discretized by simply using finite differences on the grid

$$\mathcal{D}_{\nu\nu}u_i = \frac{1}{|\nu|h^2} (u(x_i + \nu h) + u(x_i - \nu h) - 2u(x_i)).$$

220 Depending on the direction of the vector ν , this may involve a wide stencil. At points
 221 near the boundary of the domain, some values required by the wide stencil will not be
 222 available; see Figure 1. In these cases, we use interpolation at the boundary to construct
 223 a (lower accuracy) stencil for the second directional derivative; see [31] for more details.

224 Since the discretization considers only a finite number of directions ν , there will be
 225 an additional term in the consistency error coming from the angular resolution $d\theta$ of the
 226 stencil. This angular resolution will decrease and approach zero as the stencil width is
 227 increased.

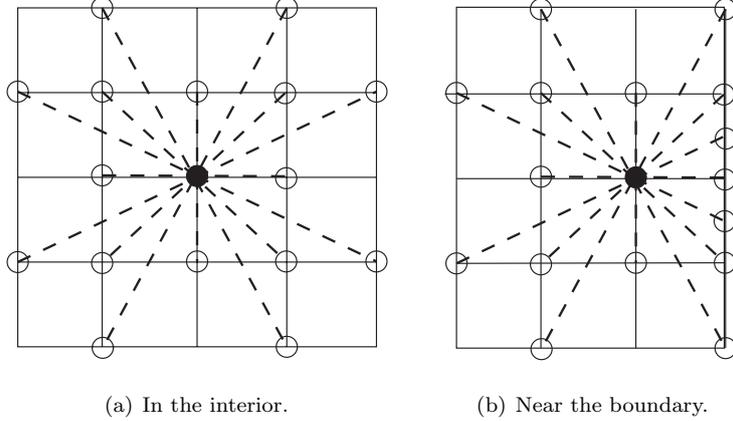


Figure 1: Wide stencils on a two dimensional grid.

228 *4.2. Discretization of the convexified Monge-Ampère operator*

229 In two dimensions, the largest and smallest eigenvalues of a symmetric matrix can be
 230 represented by the variational formula

$$\lambda_1[A] = \min_{|\nu|=1} \nu^T A \nu, \quad \lambda_2[A] = \max_{|\nu|=1} \nu^T A \nu.$$

231 This formula was used in [31] to build a monotone scheme for the Monge-Ampère oper-
 232 ator, which is the product of the eigenvalues of the Hessian, by replacing the min, max
 233 over all directions, by a finite number of grid directions.

234 In higher dimensions, the formula above does not generalize naturally. Instead, in [32],
 235 we used another characterization, which applied to positive definite matrices.

236 **Lemma 2** (Variational characterization of the determinant). *Let A be a $d \times d$ symmetric*
 237 *positive definite matrix with eigenvalues λ_j and let V be the set of all orthonormal bases*
 238 *of \mathbb{R}^d :*

$$V = \{(\nu_1, \dots, \nu_d) \mid \nu_j \in \mathbb{R}^d, \nu_i \perp \nu_j \text{ if } i \neq j, \|\nu_j\|_2 = 1\}.$$

239 *Then the determinant of A is equivalent to*

$$\prod_{j=1}^d \lambda_j = \min_{(\nu_1, \dots, \nu_d) \in V} \prod_{j=1}^d \nu_j^T A \nu_j.$$

240 We use Lemma 2 to characterize the determinant of the Hessian of a convex C^2
 241 function ϕ in terms of second directional derivatives of ϕ .

$$\det(D^2 \phi) = \min_{(\nu_1, \dots, \nu_d) \in V} \prod_{j=1}^d \nu_j^T D^2 \phi \nu_j = \min_{(\nu_1, \dots, \nu_d) \in V} \prod_{j=1}^d \frac{\partial^2 \phi}{\partial \nu_j^2}.$$

242 The convexified Monge-Ampère operator $(MA)^+$ can then be represented by simply
 243 enforcing positivity of the eigenvalues, which leads to the following,

$$\det^+(D^2\phi) = \min_{\{\nu_1, \dots, \nu_d\} \in V} \prod_{j=1}^d \left(\frac{\partial^2 \phi}{\partial \nu_j^2} \right)^+.$$

244 To discretize the operator on a finite difference grid, restrict to the set of orthogonal
 245 vectors, \mathcal{G} , available on the given stencil. Then the convexified Monge-Ampère opera-
 246 tor $(MA)^+$ is approximated by

$$MA^M[u] \equiv \min_{\{\nu_1, \dots, \nu_d\} \in \mathcal{G}} \prod_{j=1}^d (\mathcal{D}_{\nu_j \nu_j} u)^+ \quad (MA)^M$$

247 **Theorem 3** (Convergence to Viscosity Solution). *Let the PDE (MA) have a unique*
 248 *viscosity solution. Then the solutions of the scheme $(MA)^M$, converges to the viscosity*
 249 *solution of (MA) as $h, d\theta, \delta \rightarrow 0$.*

250 The proof of convergence follows from verifying consistency and degenerate ellipticity
 251 and can be found in [32].

252 5. A hybrid discretization

253 In this section we propose a hybrid discretization of the Monge-Ampère equation
 254 which takes advantage of the best features of each of the previous discretizations. We
 255 want to make use of the natural discretization $(MA)^N$ wherever possible in order to take
 256 advantage of its simplicity and higher accuracy. However, we wish to use the monotone
 257 discretization $(MA)^M$ in regions where the solution may be singular in order to prop-
 258 erly capture the behaviour of the viscosity solution. In this way we hope to achieve the
 259 second-order accuracy of the simple discretization in smooth regions and the monotonic-
 260 ity necessary to capture the behaviour of the viscosity solution in non-smooth regions.

261 We propose the following hybrid scheme.

262 Discretize (MA) using a weighted average of the two discretizations:

$$MA^H = w(x)MA^N + (1 - w(x))MA^M \quad (MA)^H$$

263 where $w : \Omega \rightarrow [0, 1]$ is a weight function defined *a priori* from the data as follows.

264 We first identify Ω^s which is a neighborhood of the possible singular set of u on Ω ,
 265 using conditions (2).

$$\Omega^s = \{x \in \Omega \mid \epsilon < f(x) < 1/\epsilon\} \cup \{x \in \partial\Omega \mid \partial\Omega \text{ flat or } g(x) \notin C^{2,\alpha}\} \quad (6)$$

266 where ϵ is a small parameter, which we can take equal to h , the spatial resolution.

267 Then define $w(x)$ to be a differentiable function which is zero in an h -neighborhood
 268 of Ω^s , and which goes to 1 elsewhere.

269 **Remark 6.** The hybrid scheme will sometimes be less accurate than the standard finite
 270 differences when the solution is C^2 , because it will lose some accuracy at the flat bound-
 271 ary. While this might seem conservative, there are examples, (see [28]), where the flat
 272 boundary causes blow up in the Hessian, so the monotone scheme is needed.

273 **6. Explicit and semi-implicit solution methods**

274 Any discretization of (MA) leads to a system of nonlinear equations which must be
 275 solved in order to obtain the approximate solution.

276 *6.1. Explicit solution methods for monotone schemes*

277 Using a monotone discretization $F[u]$ of the Monge-Ampère operator, the simplest
 278 way to solve the Monge-Ampère equation is by solving the parabolic version of the
 279 equation using forward Euler. That is, we perform the iteration

$$u^{n+1} = u^n + dt(F[u^n] - f).$$

280 Explicit iterative methods have the advantage that they are simple to implement, but the
 281 number of iterations required suffers from the well known CFL condition (which applies
 282 in a nonlinear form to monotone discretizations, as explained in [42]). This approach is
 283 slow because for stability it requires a small time step dt , which depends on the spatial
 284 resolution h . The time step, which can be computed explicitly, is $\mathcal{O}(h^2)$. This was the
 285 approach used in [31].

286 *6.2. A semi-implicit solution method*

287 The next method we discuss is a semi-implicit method, which involves solving the
 288 Laplace equation at each iteration. In [28] we used the identity (8) to build a semi-implicit
 289 solution method. We showed that the method is a contraction, but the strictness of the
 290 contraction requires strict positivity of f . In practice, this meant that the iteration was
 291 fast for regular solutions, but degenerated to become slower than the explicit method
 292 when f was zero in large parts of the domain.

293 The conditioning of the linearized equation (5), which affects solution time, depends
 294 on the strict convexity of the solution, see lemma 1. The convexity, in turn depends of
 295 strict positivity of f , see subsection 2.1. This explains why solution time of the semi-
 296 implicit solver depends on regularity.

297 Next, we describe a generalization of the semi-implicit method to higher dimensions.
 298 We won't be using the method to solve (MA). Instead, we will use one iteration to set
 299 up the initial value for Newton's method.

300 Begin with the following identity for the Laplacian in two dimensions,

$$|\Delta u| = \sqrt{(\Delta u)^2} = \sqrt{u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}}. \quad (7)$$

So if u solves the Monge-Ampère equation, then

$$|\Delta u| = \sqrt{u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 + 2f} = \sqrt{|D^2u|^2 + 2f}$$

301 This leads to a semi-implicit scheme for solving the Monge-Ampère equation, used in [28].

302

$$\Delta u^{n+1} = \sqrt{2f + |D^2u^n|^2} \quad (8)$$

To generalize this to \mathbb{R}^d , we can write the Laplacian in terms of the eigenvalues of the Hessian: $\Delta u = \sum_{i=1}^d \lambda_i [D^2 u]$. Taking the d -th power, and expanding, gives the sum of all possible products of d eigenvalues.

$$(\Delta u)^d = d! \prod_{i=1}^d \lambda_i + P(\lambda_1, \dots, \lambda_d),$$

303 where $P(\lambda)$ is a d -homogeneous polynomial, which we won't need explicitly.

304 The result is the semi-implicit scheme

$$\Delta u^{n+1} = \sqrt{d!f + P(\lambda_1[D^2 u^n], \dots, \lambda_d[D^2 u^n])}. \quad (9)$$

305 A natural initial value for the iteration is given by the solution of

$$\Delta u^0 = \sqrt{d!f}. \quad (10)$$

306 7. Implementation of Newton's method

307 To solve the discretized equation

$$MA^H[u] = f$$

308 we use a damped Newton iteration

$$u^{n+1} = u^n - \alpha v^n$$

309 for some $0 < \alpha < 1$. The damping parameter α is chosen at each step to ensure that
 310 the residual $\|MA^H(u^n) - f\|$ is decreasing. (In practice we can often take $\alpha = 1$, but
 311 damping is sometimes needed.)

312 The corrector v^n solves the linear system

$$(\nabla_u MA^H[u^n]) v^n = MA^H[u^n] - f. \quad (11)$$

313 To set up the equation (11), the Jacobian of the scheme is needed. Since the hybrid
 314 discretization is a weighted average of the monotone and standard discretization, and
 315 the weight function, $w(x)$, is determined *a priori*, the Jacobian of the hybrid scheme will
 316 simply be a weighted average of the corresponding Jacobians.

317 The Jacobian of the Monge-Ampère operator, discretized using standard finite differ-
 318 ences, is given by

$$\nabla_u MA^N[u] = (\mathcal{D}_{xx}u)\mathcal{D}_{yy} + (\mathcal{D}_{yy}u)\mathcal{D}_{xx} - 2(\mathcal{D}_{xy}u)\mathcal{D}_{xy}, \quad (12)$$

319 which is a discrete version of the linearization of Monge-Ampère (5)

320 The Jacobian for the monotone discretization is obtained by using Danskin's Theo-
 321 rem [43] and the product rule.

$$\nabla_u MA^M[u] = \sum_{j=1}^d \text{diag} \left(\prod_{k \neq j} \mathcal{D}_{\nu_k^* \nu_k^*} u \right) \mathcal{D}_{\nu_j^* \nu_j^*}$$

322 where the ν_j^* are the directions active in the minimum in $(MA)^M$.

Thus the corrector is obtained by solving the weighted average of the two linearizations

$$\begin{aligned} (w(x)\nabla_u MA^N[u^n] + (1-w(x))\nabla_u MA^M[u^n])v^n \\ = w(x)MA^N[u^n] + (1-w(x))MA^M[u^n]. \end{aligned} \quad (13)$$

323 In order for the linear equation (11) to be well-posed, we require the coefficient matrix
324 to be positive definite. As observed in lemma Theorem 1, this condition can fail if the
325 iterate u^n is not strictly convex.

326 7.1. Initialization of Newton's method

327 Newton's method requires a good initialization for the iteration. Since we need the
328 resulting linear system to be well posed it is essential that the initial iterate: (i) be
329 convex, (ii) respect the boundary conditions, (iii) be close to the solution.

330 In order to do this, we first: use one step of the semi-implicit scheme (9), to obtain a
331 close initial value. This amounts to solving (10) along with consistent Dirichlet boundary
332 conditions (D). Then convexify the result, using the method of [37]. Since both the steps
333 can be performed on a very coarse grid, and interpolated onto the finer grid, the cost of
334 the initialization is low.

335 7.2. Preconditioning

336 In degenerate examples, the PDE for v^n (13) may be degenerate, which can lead
337 to an ill-conditioned or singular Jacobian. To get around this problem, we regularize
338 the Jacobian to make sure the linear operator is strictly negative definite; this will not
339 change the fixed points of Newton's method. We accomplish this by replacing the second
340 directional derivatives $u_{\nu\nu}$ with

$$\tilde{u}_{\nu\nu} = \max\{u_{\nu\nu}, \epsilon\}$$

341 Here ϵ is a small parameter. In the computations of section 8, we take $\epsilon = \frac{1}{2dx^2} \times 10^{-8}$.

342 8. Computational results in two dimensions

343 In this section, we summarize the results of a number of two-dimensional examples
344 solved using the hybrid scheme described in section 5. In particular, we are interested
345 in comparing the computation time for Newton's method with the time required by the
346 methods proposed in [28]. We also visualize the map generated by the gradient of the
347 solution.

348 These computations are performed on an $N \times N$ grid on the square $[0, 1]^2$. The
349 monotone scheme used a 17 point stencil.

350 When needed as part of the initialization, the convex envelope is computed on a
351 coarse grid using the discretization described in [37]. Since the solution can be computed
352 on a coarse grid, and interpolated, the added computational time is negligible.

353 *8.1. Failure of Newton's method for natural finite differences*

354 In this section, we give an example where Newton's method breaks down when stan-
 355 dard finite differences are used.

356 We chose an example which is only singular at one point, on the boundary. Never-
 357 theless, this mild singularity is enough for Newton's method to break down.

358 Consider the solution of (MA) in $[0, 1]^2$, given by

$$u(\mathbf{x}) = -\sqrt{2 - |\mathbf{x}|^2}, \quad f(\mathbf{x}) = 2(2 - |\mathbf{x}|^2)^{-2}$$

359 The gradient of the solution is unbounded on $|\mathbf{x}| = 2$, in particular at the point $(1, 1)$.
 360 The singularity arises from the fact that f is unbounded there.

361 Due to the singularity, there is an instability in Newton's method if the natural finite
 362 difference method is used. The iteration is initialized with the exact solution. The result
 363 after performing two iterations of Newton's method along with the gradient map, is
 364 illustrated in Figure 2. The correct computed solution is presented in Figure 3(g)-3(h).

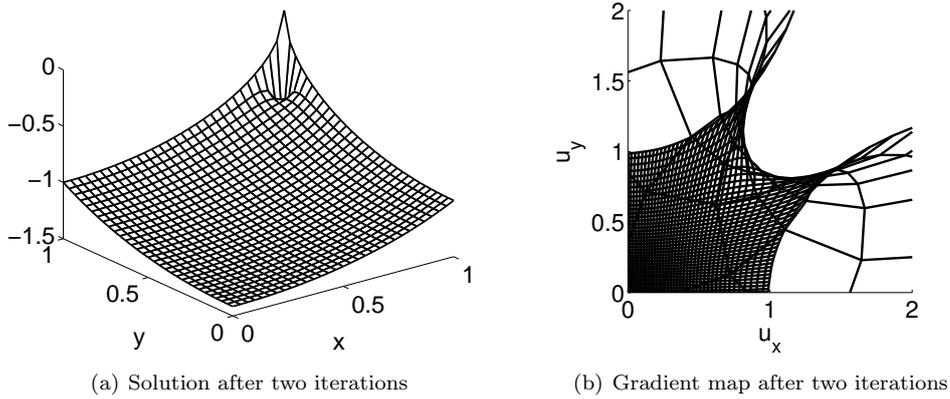


Figure 2: Failure of Newton's method using standard finite differences: the solution oscillates and becomes non-convex.

365 *8.2. Four representative examples*

366 We have tested the hybrid method on a number of examples of varying regularity;
 367 the results are summarized in subsection 8.4-8.3. To illustrate these results, we present
 368 more detailed results for four representative examples.

369 Write $\mathbf{x} = (x, y)$, and $\mathbf{x}_0 = (.5, .5)$ for the center of the domain.

370 The first example solution, which is smooth and radial, is given by

$$u(\mathbf{x}) = \exp\left(\frac{|\mathbf{x}|^2}{2}\right), \quad f(\mathbf{x}) = (1 + |\mathbf{x}|^2) \exp(|\mathbf{x}|^2). \quad (14)$$

371 The second example, which is C^1 , is given by

$$u(\mathbf{x}) = \frac{1}{2} ((|\mathbf{x} - \mathbf{x}_0| - 0.2)^+)^2, \quad f(\mathbf{x}) = \left(1 - \frac{0.2}{|\mathbf{x} - \mathbf{x}_0|}\right)^+. \quad (15)$$

372 The third example is the one which was used in subsection 8.1 to demonstrate that
 373 Newton's method for standard finite differences is unstable. The solution is twice differ-
 374 entiable in the interior of the domain, but has an unbounded gradient near the boundary
 375 point $(1, 1)$. The solution is given by

$$u(\mathbf{x}) = -\sqrt{2 - |\mathbf{x}|^2}, \quad f(\mathbf{x}) = 2(2 - |\mathbf{x}|^2)^{-2}. \quad (16)$$

376 This final is example solution is the cone, which was discussed in subsection 2.3. It
 377 is Lipschitz continuous.

$$u(\mathbf{x}) = \sqrt{|\mathbf{x} - \mathbf{x}_0|}, \quad f = \mu = \pi \delta_{\mathbf{x}_0} \quad (17)$$

378 In order to approximate the solution on a grid with spatial resolution h , using viscosity
 379 solutions, we approximate the measure μ by its average over the ball of radius $h/2$, which
 380 gives

$$f^h = \begin{cases} 4/h^2 & \text{for } |\mathbf{x} - \mathbf{x}_0| \leq h/2, \\ 0 & \text{otherwise.} \end{cases}$$

381 8.3. Visualization of solutions and gradient maps

382 In Figure 3 the solutions and the gradient maps for the three representative examples
 383 are presented. For example (17) the gradient map is too singular to illustrate. To
 384 visualize the maps, the image of a Cartesian mesh under the mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{D}_x u \\ \mathcal{D}_y u \end{pmatrix}$$

385 is shown, where $(\mathcal{D}_x u, \mathcal{D}_y u)$ is the numerical gradient of the solution of the Monge-
 386 Ampère equation. The image of a circle is plotted for visualization purposes, the equation
 387 is solved on a square. For reference, the identity mapping is also displayed.

388 In each case, the maps agree with the maps obtained using the gradient of the exact
 389 solution.

390 8.4. Computation time

391 The computation times for the four representative examples is presented in Table 1.
 392 The computations time are compared to those for the Gauss-Seidel and Poisson iterations
 393 described in [28]. The Newton solver is faster in terms of absolute solution time in each
 394 case. Table 2 presents of order of magnitude solutions times. The order of magnitude
 395 solution time for Newton's method is independent of the regularity of the solutions and
 396 faster than both of the other methods.

397 8.5. Accuracy

398 Numerical errors are presented in Table 3. We compare the accuracy of the hybrid
 399 scheme to the standard finite difference discretization, (using the results of [28]) and to
 400 the monotone scheme which was also solved using Newton's method.

401 We discuss each example in turn.

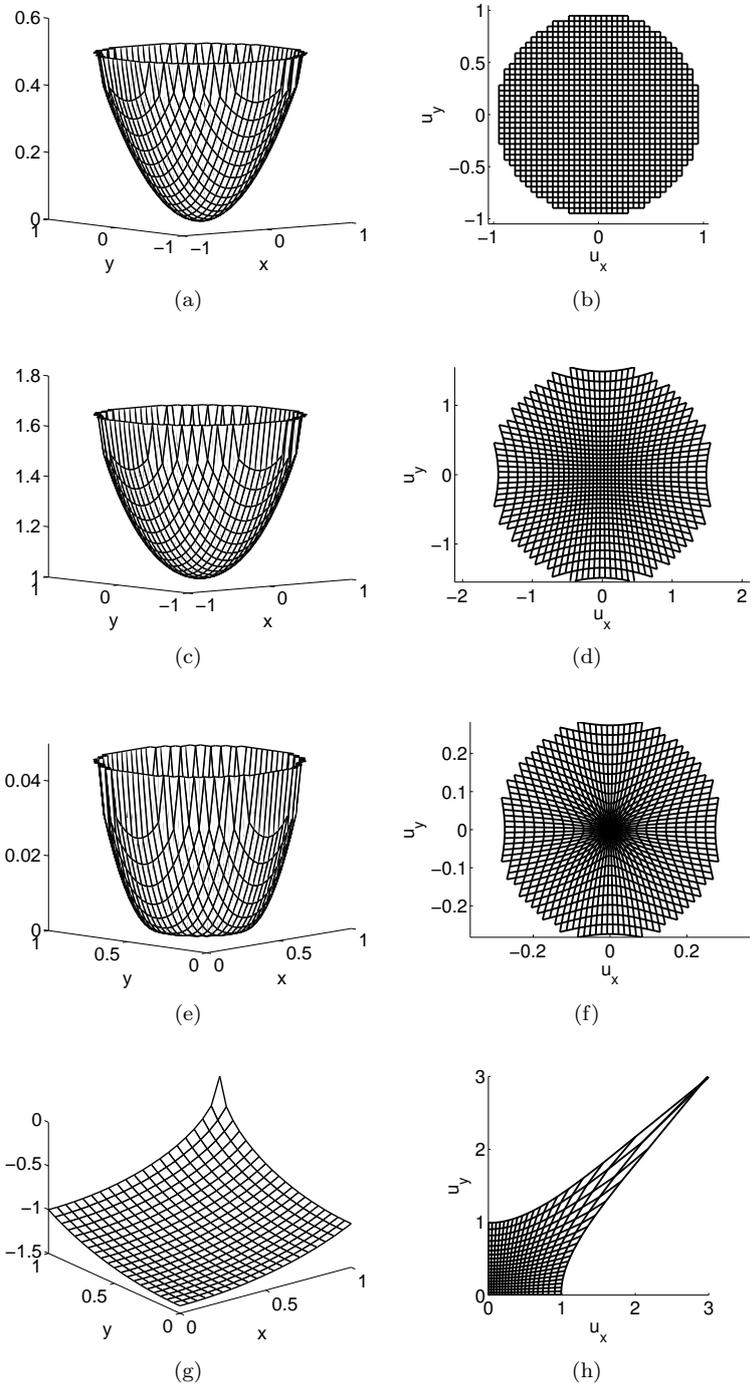


Figure 3: Solutions and mappings for the (a),(b) identity map, (c),(d) C^2 example, (e),(f) C^1 example, and (g),(h) example with blow-up.

C^2 Example (14)				
N	Newton	CPU Time (seconds)		
	Iterations	Newton	Poisson	Gauss-Seidel
31	3	0.2	0.7	2.2
45	4	0.2	1.1	4.1
63	4	0.4	1.9	15.0
89	4	1.0	4.8	57.6
127	5	2.9	9.6	236.7
181	5	9.0	23.2	1004.0
255	5	30.5	52.6	—
361	6	131.4	162.6	—

C^1 Example (15)				
N	Newton	CPU Time (seconds)		
	Iterations	Newton	Poisson	Gauss-Seidel
31	4	0.4	1.1	0.8
45	6	0.4	6.1	2.8
63	7	0.8	20.5	9.5
89	9	2.0	80.0	35.9
127	11	5.7	256.8	145.5
181	13	17.7	—	558.0
255	16	55.3	—	—
361	20	200.0	—	—

Example with blow-up (16)				
N	Newton	CPU Time (seconds)		
	Iterations	Newton	Poisson	Gauss-Seidel
31	4	0.2	0.5	0.8
45	4	0.4	1.4	5.3
63	4	0.7	2.9	19.4
89	5	1.8	8.1	74.1
127	7	5.1	17.7	293.3
181	7	12.9	51.4	1637.1
255	7	36.1	128.2	—
361	8	152.9	374.5	—

$C^{0,1}$ (Lipschitz) Example (17)				
N	Newton	CPU Time (seconds)		
	Iterations	Newton	Poisson	Gauss-Seidel
31	9	0.5	5.3	0.8
45	11	0.6	27.8	5.9
63	15	1.4	91.9	21.5
89	22	4.3	451.0	90.5
127	32	14.1	1758.2	373.9
181	30	34.6	—	1576.1
255	34	101.7	—	—
361	29	280.2	—	—

Table 1: Computation times for the Newton, Poisson, and Gauss-Seidel methods for four representative examples.

Method	Regularity of Solution		
	$C^{2,\alpha}$ (14)	$C^{1,\alpha}$ (15) and (16)	$C^{0,1}$ (17)
Gauss-Seidel	Moderate ($\sim \mathcal{O}(M^{1.8})$)	Moderate ($\sim \mathcal{O}(M^{1.9})$)	Moderate ($\sim \mathcal{O}(M^2)$)
Poisson	Fast ($\sim \mathcal{O}(M^{1.4})$)	Fast-Slow ($\sim \mathcal{O}(M^{1.4})$ -blow-up)	Slow ($\sim \mathcal{O}(M^2)$ -blow-up)
Newton	Fast ($\sim \mathcal{O}(M^{1.3})$)	Fast ($\sim \mathcal{O}(M^{1.3})$)	Fast ($\sim \mathcal{O}(M^{1.3})$)

Table 2: Order of magnitude computation time for the different solvers in terms of the regularity of solutions. Here $M = N^2$ is the total number of grid points.

402 *The C^2 solution (14)*

403 The standard finite difference schemes gives $\mathcal{O}(h^2)$ accuracy (see [28]). In this case,
404 the hybrid scheme is slightly *less* accurate, because the monotone scheme is used near the
405 boundary. On a strictly convex domain the hybrid scheme would reduce to the standard
406 discretization and achieve the same accuracy.

407 The effect diminishes as the number of grid points increases so that the number of
408 interior points using the higher order scheme dominates. Accuracy approaches $\mathcal{O}(h^2)$ as
409 the number of grid points increases. This is a definite improvement over the monotone
410 scheme, which has its accuracy limited by the stencil width.

411 *The C^1 solution (15)*

412 The accuracy is $\mathcal{O}(h)$, which is similar to the standard discretization and better
413 than the limited accuracy permitted by the monotone discretization with a fixed stencil
414 width. We also look at the error at each point (see Figure 4); it is evident that the
415 singularity around the circle is the factor that most affects the accuracy. Because of this
416 non-smoothness, there is no reason to expect our scheme to produce the $\mathcal{O}(h^2)$ accuracy
417 that is possible on C^2 solutions.

418 *The blow-up solution (16)*

419 In this case, the hybrid scheme accuracy is $\mathcal{O}(h^{1.5})$. This is better than the accuracy
420 of both the standard discretization, which was $\mathcal{O}(h^{0.5})$ [28], and the monotone scheme,
421 which is limited by the stencil width.

422 *The cone solution (17)*

423 For this singular example, the hybrid scheme is identical to the monotone scheme
424 (since the right-hand side is either 0 or very large everywhere in the domain). Con-
425 sequently, the angular resolution (stencil width) limits the accuracy of solutions. We
426 observed that the 17 point stencil reduced the error by an order of magnitude compared
427 to the 9 point stencil. This dependence on the stencil width is also evident in the surface
428 plot of error (Figure 4), which demonstrates that error is largest along directions that
429 are not captured by the stencil. Since this solution is so singular the reduced accuracy
430 is to be expected.

C^2 Example (14)			
N	Maximum Error		
	Standard	Monotone	Hybrid
31	7.14×10^{-5}	89.09×10^{-5}	24.45×10^{-5}
45	3.39×10^{-5}	60.50×10^{-5}	15.29×10^{-5}
63	1.73×10^{-5}	50.88×10^{-5}	9.06×10^{-5}
89	0.87×10^{-5}	47.51×10^{-5}	5.32×10^{-5}
127	0.43×10^{-5}	45.53×10^{-5}	3.02×10^{-5}
181	0.21×10^{-5}	44.65×10^{-5}	1.61×10^{-5}
255	0.11×10^{-5}	44.22×10^{-5}	0.87×10^{-5}
361	0.05×10^{-5}	44.00×10^{-5}	0.46×10^{-5}

C^1 Example (15)			
N	Maximum Error		
	Standard	Monotone	Hybrid
31	2.6×10^{-4}	17.5×10^{-4}	12.2×10^{-4}
45	1.8×10^{-4}	11.6×10^{-4}	5.9×10^{-4}
63	1.5×10^{-4}	9.8×10^{-4}	4.2×10^{-4}
89	0.9×10^{-4}	8.4×10^{-4}	2.6×10^{-4}
127	0.6×10^{-4}	7.9×10^{-4}	2.0×10^{-4}
181	0.4×10^{-4}	7.4×10^{-4}	1.2×10^{-4}
255	—	7.2×10^{-4}	1.0×10^{-4}
361	—	7.0×10^{-4}	0.7×10^{-4}

Example with blow-up (16)			
N	Maximum Error		
	Standard	Monotone	Hybrid
31	17.15×10^{-3}	1.74×10^{-3}	1.74×10^{-3}
45	14.59×10^{-3}	0.98×10^{-3}	0.98×10^{-3}
63	12.53×10^{-3}	0.59×10^{-3}	0.59×10^{-3}
89	10.67×10^{-3}	0.37×10^{-3}	0.35×10^{-3}
127	9.00×10^{-3}	0.35×10^{-3}	0.20×10^{-3}
181	7.59×10^{-3}	0.34×10^{-3}	0.12×10^{-3}
255	6.42×10^{-3}	0.33×10^{-3}	0.07×10^{-3}
361	5.41×10^{-3}	0.33×10^{-3}	0.04×10^{-3}

$C^{0,1}$ (Lipschitz) Example (17)			
N	Maximum Error		
	Standard	Monotone	Hybrid
31	10×10^{-3}	3×10^{-3}	3×10^{-3}
45	8×10^{-3}	3×10^{-3}	3×10^{-3}
63	6×10^{-3}	3×10^{-3}	3×10^{-3}
89	4×10^{-3}	4×10^{-3}	4×10^{-3}
127	3×10^{-3}	4×10^{-3}	4×10^{-3}
181	2×10^{-3}	4×10^{-3}	4×10^{-3}
255	—	4×10^{-3}	4×10^{-3}
361	—	4×10^{-3}	4×10^{-3}

Table 3: Accuracy for the standard, monotone, and hybrid discretizations for four representative examples.

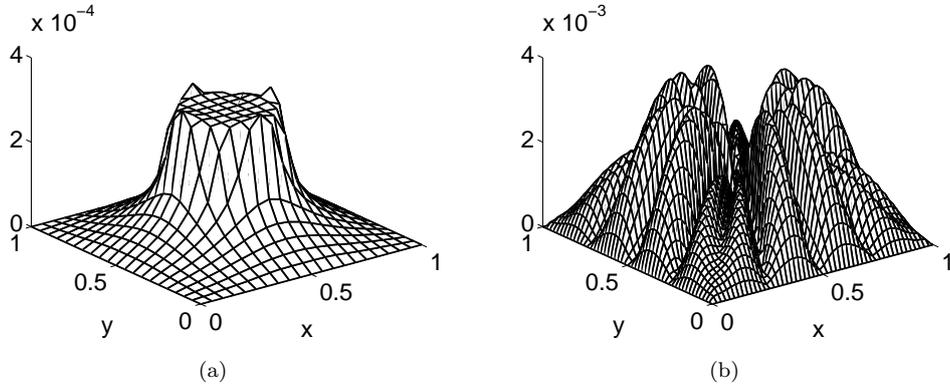


Figure 4: Surface plots of error using the hybrid scheme for the (a) C^1 example and (b) cone example.

431 **9. Computational results in three dimensions.**

432 In this section, we demonstrate the speed and accuracy of the hybrid Newton's method
 433 for three dimensional problems. These computations are performed on an $N \times N \times N$
 434 grid on the square $[0, 1]^3$. The monotone scheme used a 19 point stencil.

435 The size of the computation was restricted by the available memory, not by solution
 436 time (the computations were performed on a recent model laptop).

437 The solution methods of [28] were restricted to the two-dimensional Monge-Ampère
 438 equation, so we are no longer able to compare solution times to Newton's method for
 439 these examples.

440 As before, we provide specific results for three representative examples of varying
 441 regularity. In this section we use the notation

$$\mathbf{x} = (x, y, z)$$

442 and let $\mathbf{x}_0 = (.5, .5, .5)$ be the centre of the domain.

443 The first example is the C^2 solution given by

$$u(\mathbf{x}) = \exp\left(\frac{|\mathbf{x}|^2}{2}\right), \quad f(\mathbf{x}) = (1 + |\mathbf{x}|^2) \exp\left(\frac{3}{2}|\mathbf{x}|^2\right). \quad (18)$$

444 The second example is the C^1 solution given by

$$u(\mathbf{x}) = \frac{1}{2} ((|\mathbf{x} - \mathbf{x}_0| - 0.2)^+)^2, \quad (19)$$

445

$$f(\mathbf{x}) = \begin{cases} 1 - \frac{0.4}{|\mathbf{x} - \mathbf{x}_0|} + \frac{0.04}{|\mathbf{x} - \mathbf{x}_0|^2}, & |\mathbf{x} - \mathbf{x}_0| > 0.2 \\ 0 & \text{otherwise.} \end{cases}$$

446 The third example is the surface of a ball, which is differentiable in the interior of the
 447 domain, but has an unbounded gradient at the boundary.

$$u(\mathbf{x}) = -\sqrt{3 - |\mathbf{x}|^2}, \quad f(\mathbf{x}) = 3(3 - |\mathbf{x}|^2)^{-5/2}. \quad (20)$$

448 As indicated by the results in Table 4, the hybrid Newton’s method continues to
 449 perform well in three dimensions. (The fact that the solver required only one iteration
 450 for Example (19) was simply an artifact—for larger problems sizes more iterations were
 451 required.

C^2 Example (18)			
N	Max Error	Iterations	CPU Time (s)
7	0.0151	2	0.04
11	0.0140	3	0.10
15	0.0129	5	0.71
21	0.0121	6	6.72
31	0.0111	5	86.63

C^1 Example (19)			
N	Max Error	Iterations	CPU Time (s)
7	0.0034	1	0.02
11	0.0022	1	0.09
15	0.0016	1	0.22
21	0.0009	1	1.03
31	0.0005	1	17.12

Example with Blow-up (20)			
N	Max Error	Iterations	CPU Time (s)
7	9.6×10^{-3}	1	0.03
11	5.2×10^{-3}	3	0.11
15	4.6×10^{-3}	3	0.48
21	4.0×10^{-3}	6	7.42
31	2.9×10^{-3}	8	138.74

Table 4: Maximum error and computation time for the hybrid Newton’s method on three representative examples.

452 10. Conclusions

453 The purpose of this work was to build a fast, accurate finite difference solver for the
 454 elliptic Monge-Ampère equation.

455 A hybrid finite difference discretization was used which selects between an accurate
 456 standard finite difference discretization and a stable (provably convergent) monotone
 457 discretization. The choice of discretization was based on known regularity results which
 458 depended on the boundary data, g , the right hand side function f , and strict convexity of
 459 the domain. Wherever the requirements on the data are not met, the hybrid discretization
 460 chooses the monotone discretization.

461 The discretized equations were solved by Newton’s method, which is fast, $\mathcal{O}(M^{1.3})$,
 462 where M is the number of data points, independent of the regularity of the solution. The
 463 implementation of Newton’s method was significantly (orders of magnitude) faster than

464 the two other methods used for comparison. The hybrid discretization was shown to be
465 necessary for stability of Newton's method: an example with a mildly singular solution
466 showed that the standard discretization leads to instabilities.

467 The hybrid discretization was introduced to improve the accuracy of the monotone
468 discretization on regular solutions. This expected improvement was achieved. On regular
469 solutions the hybrid solver was (asymptotically) as accurate as the standard finite dif-
470 ference discretization. For one moderately singular example the hybrid solver was more
471 accurate than standard finite differences by $\mathcal{O}(h)$.

472 The discretization and solution method used was not restricted to two dimensions.
473 This allowed for the solution of three dimensional problems on moderate sized grids, with
474 the problem size limited by computer memory, not solution time.

475 In summary, the solver presented used a novel discretization in general dimensions,
476 accompanied by a fast solution method. The resulting solver is a significant improvement
477 over existing methods for the solution of possibly singular solutions of the elliptic Monge-
478 Ampère equation, in terms of solution time, stability, and accuracy.

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