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Assignment markets with the same core

F. Javier Martínez-de-Albéniz[1], Marina Núñez, Carles Rafels[2]

Adreça correspondència:

Departament de Matemàtica Econòmica, Financera i Actuarial
Facultat d'Economia i Empresa
Universitat de Barcelona
Av. Diagonal 690,
08034 BARCELONA
Telèfon 93 402 0415
Fax 93 403 48 92
e-mail: javier.martinezdealbeniz@ub.edu, mnunez@ub.edu, crafels@ub.edu

[1] Corresponding author. E-mail address: javier.martinezdealbeniz@ub.edu

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Assignment markets with the same core

Abstract: In the framework of bilateral assignment games, we study the set of matrices associated with assignment markets with the same core. We state conditions on matrix entries that ensure that the related assignment games have the same core. We prove that the set of matrices leading to the same core form a join-semilattice with a finite number of minimal elements and a unique maximum. We provide a characterization of the minimal elements. A sufficient condition under which the join-semilattice reduces to a lattice is also given.
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Resumen: En el contexto de los juegos de asignación bilaterales, estudiamos el conjunto de matrices asociadas a mercados de asignación con el mismo núcleo. Se proporcionan condiciones sobre las entradas de la matriz que aseguran que los juegos de asignación asociados tienen el mismo núcleo. Se prueba que este conjunto de matrices que dan lugar al mismo núcleo forman un semirretículo con un número finito de elementos minimales y un único máximo. Se da una caracterización de estos elementos minimales. También se proporciona una condición suficiente para obtener un retículo.

1 Introduction

Since Shapley and Shubik (1972) the bilateral assignment game has been analyzed from different points of view, with special emphasis placed on the analysis of the core. In a bilateral assignment game there is a finite set of buyers, each one demanding one unit of an indivisible good, and a finite set of sellers, each one supplying one unit of the good. From the valuations of the buyers and the reservation prices of the sellers, a non-negative matrix can be obtained that represents the joint profit that each buyer-seller pair can achieve. This market situation can be represented by a game in coalitional form where the worth of each coalition is the profit that it can obtain from an optimal matching between buyers and sellers in the corresponding submarket. Those allocations of the total profit where each coalition receives at least its worth constitute the core of the game.

It is known that the core of an assignment game is a non-empty convex and compact polyhedral endowed with a lattice structure which implies the existence of two opposite extreme allocations, one of them optimal for the buyers and the other optimal for the sellers. An analysis of the extreme core allocations of the assignment game is carried out by Balinski and Gale (1987). They show how to check, by means of the connectedness of a graph, whether a core allocation is an extreme point. Later, Hamers et al. (2002), Núñez and Rafels (2003) and Izquierdo et al. (2007) describe the set of vertices by means of three different procedures, each of them involving the definition of a payoff vector for each possible ordering on the player set.

The aim of this paper is to study the matrices that define assignment markets with the same core. The importance of this question lies in the fact that all the markets defined by these matrices have the same extreme core allocations and also the same nucleolus. The nucleolus is a single-valued solution for games in coalitional form that was introduced by Schmeidler (1969) and, for games with a non-empty core, it occupies a central position in the core. It is defined as the unique payoff that lexicographically minimizes the vector of nonincreasingly-ordered excesses. Solymosi and Raghavan (1994) give an algorithm to compute the nucleolus of an assignment game. It is known from Núñez (2004) that all assignment games with the same core also have the same nucleolus. Thus, once identified the class of matrices leading to markets with the same core, it is sufficient to compute the nucleolus of one of these markets. The same applies to the set of extreme core allocations.

In addition, when looking for assignment matrices leading to the same core we identify which valuations of the agents are essential to determine the core. It turns out that, in most markets, some agents can lower some of their valuations without modifying the core of the market. As a consequence, these agents (assume they are buyers) will not benefit, in the sense of being able to achieve a more favorable core allocation, from pretending to have lower valuations for some specific objects.

To characterize the set of assignment matrices defining markets with the same core, we first develop a necessary and sufficient condition to determine if two given square matrices lead to the same core. This condition states that by removing an arbitrarily given mixed pair formed by a buyer and a seller the optimal profit for both submarkets thus obtained must coincide. As a consequence, it turns out that the optimal profit of both entire markets also coincide. We then show that the set of matrices defining markets with the same core is a join-semilattice with respect to the usual order on the set of matrices, with one maximum element and finitely many minimal elements.

The above result is obtained in Section 4, after providing in Section 3 the aforementioned useful characterization to recognize whether two matrices lead to the same core. In Section 5 a sufficient condition on the assignment matrix is given so that its related join-semilattice is in fact a lattice. A characterization of the minimal elements of the join-semilattice is given in Section 6.

2 The assignment game: notations and preliminaries

A two-sided assignment market (M, M', A) is defined by a finite set of buyers M, a finite set of sellers M', and a nonnegative matrix $A = (a_{ij})_{(i,j)\in M\times M'}$. The number a_{ij} represents the profit obtained by the mixed-pair $(i, j) \in M \times M'$ if they trade. Let us assume there are $|M| = m$ buyers and $|M'| = m'$ sellers, and $n = m + m'$ is the cardinality of $N = M \cup M'$. If $m = m'$, the assignment market is said to be square.

A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M_0' \subseteq M'$, such that $|M_0| = |M_0'| = \min\{|M|, |M'|\}$. We write $(i,j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. The set of all matchings is denoted by $\mathcal{M}(M, M')$.

A *matching* $\mu \in \mathcal{M}(M, M')$ is *optimal* for the assignment market (M, M', A) if for all $\mu' \in \mathcal{M}(M, M')$ we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and we denote the set of optimal matchings by $\mathcal{M}_A^*(M, M')$.

Given $S \subseteq M$ and $T \subseteq M'$, we denote by $\mathcal{M}(S, T)$ and $\mathcal{M}_A^*(S, T)$ the set of matchings and optimal matchings of the submarket $\left(S, T, A_{|S\times T}\right)$.

Let $N = \{1, 2, ..., n\}$ denote a finite set of players, and 2^N the set of all possible coalitions or subsets of N. A cooperative game in coalitional form is a pair (N, v), where $v : 2^N \longrightarrow \mathbb{R}$, with $v(\emptyset) = 0$, is the characteristic function which assigns to each coalition the worth it can attain.

Shapley and Shubik (1972) associate to any assignment market a cooperative game in coalitional form, with player set N and characteristic function w_A defined by A in the following way: for $S \subseteq M$ and $T \subseteq M'$, $w_A(S \cup T) = \max\left\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\right\}$. The *core* of the assignment game is always non-empty, and it is enough to impose coalitional rationality for one-player coalitions and mixed-pair coalitions:

$$Core(w_A) = \left\{ (u,v) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'} \;\middle|\; \begin{array}{l} \sum_{i\in M} u_i + \sum_{j\in M'} v_j = w_A(N), \\ u_i + v_j \geq a_{ij}, \text{ for all } (i,j) \in M \times M' \end{array} \right\}. \quad (1)$$

It follows from (1) that those pairs $(i,j) \in M \times M'$ that are assigned by any optimal matching share exactly the worth a_{ij} of their mixed-pair coalition, and unassigned players get zero.

By adding dummy players, that is, null rows or columns in the assignment matrix, we can assume from now on and without loss of generality that the number of sellers equals the number of buyers, and in this way the assignment matrix is square. It is easy to see that these dummy players get zero at any core allocation.

An assignment game $(M \cup M', w_A)$ is *buyer-seller exact* (Núñez and Rafels, 2002) if for any $(i,j) \in M \times M'$ there is a point (u,v) in its core such that $u_i + v_j = a_{ij}$.

Notice that if an assignment game is buyer-seller exact then no entry in the matrix can be raised without modifying the core. It is shown that for each assignment game $(M \cup M', w_A)$ there exists a unique buyer-seller exact assignment game $(M \cup M', w_{\overline{A}})$ such that $Core\,(w_A) = Core\,(w_{\overline{A}})$. This matrix $\overline{A}$ is said to be the buyer-seller exact representative of A and it is defined by:

$$\overline{a}_{ij} = \min_{(u,v) \in Core(w_A)} u_i + v_j. \tag{2}$$

Then, an assignment game $(M \cup M', w_A)$ is buyer-seller exact if and only if $A = \overline{A}$, and as a consequence $\overline{A} = \overline{\overline{A}}$.

Notice that if $(i,j) \in \mu$ for some $\mu \in \mathcal{M}^*_A(M, M')$ we get $a_{ij} = \overline{a}_{ij}$. Then it is obvious that $\mathcal{M}^*_A(M, M') \subseteq \mathcal{M}^*_{\overline{A}}(M, M')$.

When A is square and $\mu \in \mathcal{M}^*_A(M, M')$, the matrix entries of $\overline{A}$ can be expressed, for all $(i,j) \in M \times M'$, either by

$$\overline{a}_{ij} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} - w_A(N) + w_A(N \setminus \{\mu(i), \mu^{-1}(j)\}), \tag{3}$$

or

$$\overline{a}_{ij} = \max\{a_{ij}, \widetilde{a}_{ij}\}, \tag{4}$$

where

$$\widetilde{a}_{ij} = \max\left\{a_{i\mu(k_1)} + a_{k_1\mu(k_2)} + \ldots + a_{k_rj} - a_{k_1\mu(k_1)} - \ldots - a_{k_r\mu(k_r)}\right\}, \tag{5}$$

with $k_1, k_2, \ldots, k_r \in M \setminus \{i, \mu^{-1}(j)\}$ and all different. Notice that (3) makes use of the characteristic function, while (4) only makes use of the matrix entries.

Buyer-seller exact assignment games are also characterized (Núñez and Rafels, 2002) by a property of the assignment matrix, namely doubly dominant diagonal. Following Solymosi and Raghavan (2001), given a square assignment market (M, M', A), and $\mu \in \mathcal{M}^*_A(M, M')$ an optimal matching, the market (M, M', A) is said to have a *doubly dominant diagonal* with respect to μ if for all $(i,j) \in M \times M'$ and $k \in M$,

$$a_{ij} + a_{k\mu(k)} \geq a_{i\mu(k)} + a_{kj}. \tag{6}$$

This means that what any pair $(i,j) \in M \times M'$ gets together with an optimal assigned pair $(k, \mu(k))$ cannot be improved by rearranging the agents. This is only restrictive if $k \in M \setminus \{i, \mu^{-1}(j)\}$. Since the property of being buyer-seller exact does not depend on the optimal matching μ, but only on the core, if (M, M', A) has a doubly dominant diagonal with respect some μ, it also has with respect any other $\mu' \in \mathcal{M}^*_A(M, M')$.

3 Assignment matrices leading to markets with the same core

It is a fact that different assignment markets may have the same core. For instance, we assert that matrices A and B below have the same core, although they do not have any optimal matching in common. Let it be $M = \{1, 2, 3\}, M' = \{1', 2', 3'\}$,

$$A = \begin{pmatrix} 4 & 5 & 5 \\ 4 & 5 & 1 \\ 4 & 1 & 5 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 5 & 5 \\ 4 & 0 & 5 \\ 4 & 5 & 0 \end{pmatrix},$$

and notice that if we remove from the market a mixed pair $(i, j) \in M \times M'$, both resulting submarkets get the same optimal profit. Take for instance the mixed pair $(1, 1') \in M \times M'$ and notice that $w_A(N \setminus \{1, 1'\}) = w_B(N \setminus \{1, 1'\}) = 10$, although their optimal matchings differ.

The coincidence of the optimal profit of these submarkets, where an arbitrarily given mixed pair has been removed, is the criterion to check the core coincidence that will be stated in Theorem 3.1. To this end, the next lemma provides a new and alternative description of the core of a square assignment game.

Lemma 3.1. *Let (M, M', A) be a square assignment market. Then,*

$$Core\,(w_A) = \left\{ (u, v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \;\middle|\; \begin{array}{l} \sum_{i \in M} u_i + \sum_{j \in M'} v_j = w_A(N), \\ u_i + v_j \leq w_A(N) - w_A(N \setminus \{i, j\}), \\ \text{for all } (i, j) \in M \times M' \end{array} \right\}. \qquad (7)$$

Proof. Given (M, M', A) an assignment market, recall that the core of the assignment game has been described in (1). We denote by C the right-hand side of (7) and claim that $Core\,(w_A)$ equals C.

The inclusion of the core in C is straightforward because by coalitional rationality any allocation $(u,v) \in Core\,(w_A)$ satisfies, for all $(i,j) \in M \times M'$

$$\sum_{i' \in M \setminus \{i\}} u_{i'} + \sum_{j' \in M' \setminus \{j\}} v_{j'} \geq w_A(N \setminus \{i,j\}),$$

and by efficiency we obtain $u_i + v_j \leq w_A(N) - w_A(N \setminus \{i,j\})$.

To prove the converse inclusion, take $(u,v) \in C$, and notice that once fixed $\mu \in \mathcal{M}_A^*(M,M')$, we have $u_i + v_j \leq w_A(N) - w_A(N \setminus \{i,j\}) = a_{ij}$ for $(i,j) \in \mu$, that, together with $\sum_{i \in M} u_i + \sum_{j \in M'} v_j = w_A(N)$, leads to $u_i + v_j = a_{ij}$ for all $(i,j) \in \mu$.

Moreover for any $(i,j) \in M \times M'$ consider their assigned players[2] by any optimal matching $\mu \in \mathcal{M}_A^*(M,M')$. We have, for any $(u,v) \in C$,

$$u_{\mu^{-1}(j)} + v_{\mu(i)} \leq w_A(N) - w_A(N \setminus \{\mu^{-1}(j), \mu(i)\}),$$

and taking into account the previous remark we obtain:

$$a_{\mu^{-1}(j)j} - v_j + a_{i\mu(i)} - u_i \leq w_A(N) - w_A(N \setminus \{\mu^{-1}(j), \mu(i)\}).$$

Thus,

$$u_i + v_j \geq a_{i\mu(i)} + a_{\mu^{-1}(j)j} - w_A(N) + w_A(N \setminus \{\mu^{-1}(j), \mu(i)\}) \geq a_{ij},$$

where the last inequality comes from the superadditivity of the game and the fact that $w_A(N) - a_{i\mu(i)} - a_{\mu^{-1}(j)j} = w_A(N \setminus \{i, j, \mu(i), \mu^{-1}(j)\})$. □

The above description of the core in (7), as it is also the case of description (1), is given by the efficiency condition and one inequality constraint associated with each $(i,j) \in M \times M'$. In contrast with description (1) the inequalities in (7) are reversed. However, what is more remarkable of (7) is that each of these constraints is tight at some core allocation (see Núñez and Rafels, 2002), which is not necessarily the case in the classical description of the core.

With this expression for the core, it is quite straightforward to realize that the coincidence of the core of two assignment markets is characterized by the coincidence of the worth, in both games, of the coalitions of type $N \setminus \{i,j\}$, for each $(i,j) \in M \times M'$.

[2]Notice that there are no agents unassigned because the market is square.

Theorem 3.1. *Let* (M, M', A) *and* (M, M', B) *be two square assignment markets. Then, the following statements are equivalent:*

1. $Core\,(w_A) = Core\,(w_B)$.

2. $w_A(N \setminus \{i, j\}) = w_B(N \setminus \{i, j\})$, *for all* $(i, j) \in M \times M'$.

Proof. To prove 1. ⇒ 2., notice first that from $Core\,(w_A) = Core\,(w_B)$ we trivially have $w_A(N) = w_B(N)$.

Consider now $\overline{A}$, the buyer-seller representative of matrix A. Take $\mu \in \mathcal{M}^*_A\,(M, M')$ and recall that also $\mu \in \mathcal{M}^*_{\overline{A}}\,(M, M')$. For any $i \in M$ we have $a_{i\mu(i)} = \overline{a}_{i\mu(i)}$. Moreover, recall that $\overline{\overline{A}} = \overline{A}$.

Then, we have that for any mixed pair $(i, j) \in M \times M'$:

$$\begin{aligned} \overline{a}_{ij} &= a_{i\mu(i)} + a_{\mu^{-1}(j)j} - w_A(N) + w_A(N \setminus \{\mu(i), \mu^{-1}(j)\}), \text{ and} \\ \overline{\overline{a}}_{ij} &= \overline{a}_{i\mu(i)} + \overline{a}_{\mu^{-1}(j)j} - w_{\overline{A}}(N) + w_{\overline{A}}(N \setminus \{\mu(i), \mu^{-1}(j)\}. \end{aligned}$$

As a consequence we obtain $w_A(N \setminus \{\mu(i), \mu^{-1}(j)\}) = w_{\overline{A}}(N \setminus \{\mu(i), \mu^{-1}(j)\})$, for any $(i, j) \in M \times M'$. Notice that since μ is a bijection and i and j are arbitrary, we have $w_A(N \setminus \{i, j\}) = w_{\overline{A}}(N \setminus \{i, j\})$ for all $(i, j) \in M \times M'$.

Now, since $Core\,(w_A) = Core\,(w_B)$, from (2) we know that $\overline{A} = \overline{B}$, and then for any $(i, j) \in M \times M'$, we have $w_A(N \setminus \{i, j\}) = w_{\overline{A}}(N \setminus \{i, j\}) = w_B(N \setminus \{i, j\})$.

As for the converse implication, notice first that by the non-emptiness of the core, any element $(u, v) \in Core\,(w_A)$ satisfies

$$w_A(N \setminus \{i, j\}) \leq \sum_{i' \in M \setminus \{i\}} u_{i'} + \sum_{j' \in M' \setminus \{j\}} v_{j'} \text{ for all } (i, j) \in M \times M'.$$

By addition of these inequalities for all $(i, j) \in \mu$, where $\mu \in \mathcal{M}\,(M, M')$ is an arbitrary matching, we have

$$\sum_{(i,j) \in \mu} w_A(N \setminus \{i, j\}) \leq (m - 1) w_A(N),$$

that holds with equality for any optimal matching $\mu \in \mathcal{M}^*_A\,(M, M')$, since then $w_A(N \setminus \{i, j\}) + a_{ij} = w_A(N)$, for any $(i, j) \in \mu$. Thus,

$$\max_{\mu \in \mathcal{M}(M, M')} \sum_{(i,j) \in \mu} w_A(N \setminus \{i, j\}) = (m - 1) w_A(N).$$

Therefore, under statement 2, we have $w_A(N) = w_B(N)$. The remainder of the proof is a direct application of Lemma 3.1. $\square$

An immediate consequence of the above theorem is that for assignment games with two agents on each side, different matrices lead to different cores.

Corollary 3.1. *Let* (M, M', A) *and* (M, M', B) *be two assignment markets, with* $|M| = |M'| = 2$. *Then,* $Core\,(w_A) = Core\,(w_B)$ *if and only if* $A = B$.

Let us remark that in the proof of the above Theorem 3.1 we have seen that the coincidence of the worth of coalitions $(N \setminus \{i,j\})$ for $(i,j) \in M \times M'$, that is, $w_A(N \setminus \{i,j\}) = w_B(N \setminus \{i,j\})$, implies the coincidence of the worth of the grand coalition, $w_A(N) = w_B(N)$. Also, under the same conditions and for each $k \in N$, the coincidence of the worths of coalition $N \setminus \{k\}$ is obtained. If, for instance, $k \in M$, then we have

$$w_A(N \setminus \{k\}) = \max_{j \in M'} w_A(N \setminus \{k,j\}) = \max_{j \in M'} w_B(N \setminus \{k,j\}) = w_B(N \setminus \{k\}).$$

4 The semilattice structure of the set of matrices leading to the same core

In this section we analyze the structure of the class of square matrices that give rise to the same core in the associated assignment game.

Define $\mathbf{M}_m^+$ as the set of all matrices of m rows and m columns with non-negative entries. Notice that $\mathbf{M}_m^+$ is a lattice with the usual ordering $\leq$, that is, given $A, B \in \mathbf{M}_m^+$, $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq m$. Moreover, $A < B$ if and only if $A \leq B$ and $A \neq B$. If $A, B \in \mathbf{M}_m^+$, then the interval $[A, B]$ is the set of all matrices in between A and B, that is $[A, B] = \{C \in \mathbf{M}_m^+ \mid A \leq C \leq B\}$.

Now, for each matrix $A \in \mathbf{M}_m^+$, define

$$\langle A \rangle = \left\{ B \in \mathbf{M}_m^+ \mid Core(w_B) = Core(w_A) \right\}.$$

This is the equivalence class of all matrices such that the associated assignment game has the same core. By definition, $\langle A \rangle$ contains the buyer-seller exact representative

$\overline{A}$. Notice that if $A_1 \leq B \leq A_2$ with $A_1, A_2 \in \langle A \rangle$, then $B \in \langle A \rangle$, as can be seen from Theorem 3.1.

The next theorem provides the structure of the set of all matrices leading to the same core. It turns out that it is a finite union of lattices (intervals) with the same maximum.

Theorem 4.1. *For any square assignment market (M, M', A), with matrix $A \in \mathbf{M}_m^+$, there exists a finite number of matrices $A_1, A_2, \ldots, A_p$, in $\mathbf{M}_m^+$, such that*

$$\langle A \rangle = \bigcup_{q=1}^{p} \left[A_q, \overline{A} \,\right],$$

where $\overline{A}$ is the buyer-seller exact representative of A.

Proof. Given a matrix $A \in \mathbf{M}_m^+$, we analyze which entries can be lowered from $\overline{A} = (\overline{a}_{ij})_{(i,j)\in M\times M'}$ without modifying the core of the assignment game.

To this end, for each set K of pairs in $M \times M'$, we define matrix $\overline{A}^K$, by

$$\overline{a}_{ij}^K = \begin{cases} 0 & \text{if} \quad (i,j) \in K, \\ \overline{a}_{ij} & \text{if} \quad (i,j) \notin K. \end{cases}$$

Next, consider the set $\Lambda(A) = \left\{ K \subseteq M \times M' \mid Core(w_{\overline{A}^K}) = Core(w_A) \right\}$, and notice that $\emptyset \in \Lambda(A)$. Therefore, $\Lambda(A)$ is a non-empty finite set and as a consequence the set $\Lambda^M(A)$ of maximal elements of $(\Lambda(A), \subseteq)$ is a non-empty finite set: $\Lambda^M(A) = \{K_1, K_2, \ldots, K_p\}$. At this point we define $A_q = \overline{A}^{K_q}$ for $q \in \{1, 2, \ldots, p\}$, the matrices corresponding to these maximal elements. The inclusion $\bigcup_{q=1}^{p} \left[A_q, \overline{A} \,\right] \subseteq \langle A \rangle$ is immediate.

To prove the converse inclusion, let us take $B \in \langle A \rangle$. We know that $B \leq \overline{B} = \overline{A}$, and now define $K_B = \left\{ (i,j) \in M \times M' \mid b_{ij} < \overline{b}_{ij} = \overline{a}_{ij} \right\}$.

If $K_B = \emptyset$, then $B = \overline{B} = \overline{A}$ and we are done. In other case, we have $\overline{A}^{K_B} \leq B \leq \overline{A}$ and we claim that $Core(w_{\overline{A}^{K_B}}) = Core(w_{\overline{A}})$.

Since $B \in \langle A \rangle$, by Theorem 3.1 we have that

$$w_B(N \setminus \{i^*, j^*\}) = w_{\overline{A}}(N \setminus \{i^*, j^*\}), \text{ for any } (i^*, j^*) \in M \times M'. \tag{8}$$

Then, for any $(i^*, j^*) \in M \times M'$ we assert that no $(i,j) \in K_B$ belongs to any optimal matching of the submarket $\left(M \setminus \{i^*\}, M' \setminus \{j^*\}, B_{|M\setminus\{i^*\}\times M'\setminus\{j^*\}}\right)$. Otherwise, if there exist $\mu \in \mathcal{M}^*_B\left(M \setminus \{i^*\}, M' \setminus \{j^*\}\right)$ and $(i,j) \in K_B \cap \mu$, then we will have

$$w_B(N \setminus \{i^*, j^*\}) = \sum_{(i',j')\in\mu} b_{i'j'} < \sum_{(i',j')\in\mu} \overline{a}_{i'j'} \leq w_{\overline{A}}(N \setminus \{i^*, j^*\}),$$

in contradiction with (8).

As a consequence, we have $w_{\overline{A}^{K_B}}(N \setminus \{i^*, j^*\}) = w_{\overline{A}}(N \setminus \{i^*, j^*\})$, and by Theorem 3.1 we prove our claim that $Core(w_{\overline{A}^{K_B}}) = Core(w_{\overline{A}})$. To sum up, $B \in \left[\overline{A}^{K_B}, \overline{A}\,\right]$ with $\overline{A}^{K_B} \in \langle A \rangle$.

If $K_B \in \Lambda^M(A)$ we are done. Otherwise, there exists $K \in \Lambda(A)$, maximal with $K_B \subseteq K$, and thus, from $\overline{A}^{K} \leq \overline{A}^{K_B} \leq B \leq \overline{A}$, we get $B \in \left[\overline{A}^{K}, \overline{A}\,\right]$. □

Theorem 4.1 shows that $(\langle A \rangle, \leq)$ is a join-semilattice with a finite number of minimal elements, and one maximal element, its maximum, $\overline{A}$. This maximum can be computed by means of (3).

Theorem 3.1 and Theorem 4.1 suggest a procedure to find all the minimal matrices in $(\langle A \rangle, \leq)$ under the assumption that the given matrix A is buyer-seller exact. One must identify those matrix entries that are essential to preserve the worth of coalitions $N \setminus \{i,j\}$ for all $(i,j) \in M \times M'$. Those entries cannot be lowered, nor can decrease the null entries. But it may happen that a submarket $\left(M \setminus \{i\}, M' \setminus \{j\}, A_{|M\setminus\{i\}\times M'\setminus\{j\}}\right)$ have several optimal matchings and then we have different choices. In this way we will obtain the different minimal elements of $\langle A \rangle$. For instance, consider the following example.

Example 4.1. *Let A be a buyer-seller exact matrix*

$$A = \begin{pmatrix} \mathbf{5} & \mathbf{6} & 5 \\ \mathbf{2} & \mathbf{3} & 2 \\ \mathbf{0} & 1 & \mathbf{1} \end{pmatrix}$$

with $M = \{1,2,3\}$, and $M' = \{1',2',3'\}$. The elements that are zero or strictly necessary to preserve $w_A(N \setminus \{i,j\})$ for $(i,j) \in M \times M'$ are in boldface. Notice that, since in any of those submarkets at least one optimal matching has to be preserved, *we cannot lower simultaneously entries a_{13} and a_{23}. Thus, in this case the minimal matrices in $\langle A\rangle$ are:*

$$A_1 = \begin{pmatrix} 5 & 6 & 0 \\ 2 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 5 & 6 & 5 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This example illustrates that $\langle A\rangle$ is not in general a lattice. The next section provides a condition that is sufficient to ensure that $\langle A\rangle$ is a lattice.

5 A sufficient condition to obtain a lattice

In this section we look for conditions on a matrix A that guarantee that $\langle A\rangle$ has only one minimal element. In this case the join-semilattice is in fact a lattice.

To this end we focus on assignment matrices with a unique optimal matching and we determine the unique minimal matrix in a constructive way. This is the generic case, since small perturbations can destroy the multiplicity of optimal matchings.

Theorem 5.1. *Let it be (M, M', A) a square assignment market. If $A \in \mathbf{M}_m^+$ has a unique optimal matching, then $\langle A\rangle$ has a unique minimal element.*

Proof. Making use of Theorem 6 in Núñez and Rafels (2008), it can be deduced that if A has a unique optimal matching, its buyer-seller exact representative $\overline{A}$ has also a unique optimal matching, the same one. We also have $\langle A\rangle = \langle\overline{A}\rangle$, and therefore we can assume without loss of generality that matrix A is buyer-seller exact, that is $A = \overline{A}$. Let us identify a minimal element of $\langle A\rangle$ and prove its uniqueness.

Let μ be the unique optimal matching of A and consider the following set of pairs of agents:

$$I = \left\{ (i,j) \in M \times M' \;\middle|\; \begin{array}{l} \text{there exists } k \in M \setminus \{i,\ \mu^{-1}(j)\} \\ \text{with } a_{ij} + a_{k\mu(k)} = a_{i\mu(k)} + a_{kj} \end{array} \right\}.$$

We claim that matrix B defined by

$$b_{ij} = \begin{cases} 0 & \text{if} \quad (i,j) \in I, \\ a_{ij} & \text{if} \quad (i,j) \notin I. \end{cases}$$

is the unique minimal element in $\langle A \rangle$.

We first check that $B \in \langle A \rangle$ by proving that $\overline{B} = A$. Notice now that I does not include any assigned pair by μ, since otherwise there would be another optimal matching for (M, M', A). Then, $b_{ij} = a_{ij}$ for all $(i,j) \in \mu$ and, as a consequence, A and B have the same optimal matching μ, and the same optimal profit for the grand coalition. Since $B \leq A$, and $w_A(N) = w_B(N)$, we know that $Core\,(w_A) \subseteq Core\,(w_B) = Core\,(w_{\overline{B}})$. Moreover, since A is buyer-seller exact, for any $(i,j) \in M \times M'$ there is a point $(u,v) \in Core\,(w_A)$ such that $u_i + v_j = a_{ij}$. Then $a_{ij} \geq \overline{b}_{ij}$, for all $(i,j) \in M \times M'$ and therefore $\overline{B} \leq A$.

We have to prove that also $\overline{B} \geq A$. For all $(i,j) \notin I$ we obtain trivially that $a_{ij} = b_{ij} \leq \overline{b}_{ij}$. If $I = \emptyset$, we are done. Otherwise we take any $(i,j) \in I$ and we claim that there exist $k_1, k_2, ..., k_p \in M \setminus \{i,\ \mu^{-1}(j)\}$ for some $p \geq 1$, and all different such that

$$a_{ij} + a_{k_1\mu(k_1)} + a_{k_2\mu(k_2)} + ... + a_{k_p\mu(k_p)} = a_{i\mu(k_1)} + a_{k_1\mu(k_2)} + a_{k_2\mu(k_3)} + ... + a_{k_pj}, \quad (9)$$

where $(i, \mu(k_1)), (k_1, \mu(k_2)), (k_2, \mu(k_3)), ..., (k_p, j)$ do not belong to I.

If we prove the claim we are done, because $a_{i\mu(k_1)} = b_{i\mu(k_1)}, a_{k_1\mu(k_2)} = b_{k_1\mu(k_2)}, \ldots, a_{k_pj} = b_{k_pj}$, and then

$$a_{ij} = b_{i\mu(k_1)} + b_{k_1\mu(k_2)} + \ldots + b_{k_pj} - b_{k_1\mu(k_1)} - b_{k_2\mu(k_2)} - \ldots - b_{k_p\mu(k_p)} \leq \overline{b}_{ij},$$

by (4).

To prove the claim in (9), notice first that since $(i,j) \in I$, there exists $k \in M \setminus \{i, \mu^{-1}(j)\}$ such that

$$a_{ij} + a_{k\mu(k)} = a_{i\mu(k)} + a_{kj}. \quad (10)$$

If $(i, \mu(k))$ and (k,j) do not belong to I we are done. If this is not the case, assume without loss of generality that $(i, \mu(k)) \in I$. Then

$$a_{i\mu(k)} + a_{\widetilde{k}\mu(\widetilde{k})} = a_{i\mu(\widetilde{k})} + a_{\widetilde{k}\mu(k)}, \quad (11)$$

for some $\widetilde{k} \in M \setminus \{i,\ k\}$.

By addition of (10) and (11), we get

$$a_{ij} + a_{\widetilde{k}\mu\left(\widetilde{k}\right)} + a_{k\mu(k)} = a_{i\mu\left(\widetilde{k}\right)} + a_{\widetilde{k}\mu(k)} + a_{kj}. \tag{12}$$

Let us write $k_1 = \widetilde{k}$ and $k_2 = k$, and see that $k_1, k_2 \in M \setminus \{i,\ \mu^{-1}(j)\}$ and are different. If not, the only possibility is that $\mu(k_1) = j$, but in that case (12) reduces to

$$a_{k_1\mu(k_1)} + a_{k_2\mu(k_2)} = a_{k_1\mu(k_2)} + a_{k_2\mu(k_1)},$$

in contradiction with the uniqueness of the optimal matching of (M, M', A).

Assume by iteration that we have reached a step such that:

$$a_{ij} + a_{k_1\mu(k_1)} + a_{k_2\mu(k_2)} + ... + a_{k_q\mu(k_q)} = a_{i\mu(k_1)} + a_{k_1\mu(k_2)} + a_{k_2\mu(k_3)} + ... + a_{k_qj}, \tag{13}$$

for some $q \geq 1$ and $k_1, k_2, ..., k_q \in M \setminus \{i,\ \mu^{-1}(j)\}$ all different.

Then, if all pairs $(i, \mu(k_1)), (k_1, \mu(k_2)), (k_2, \mu(k_3)), ..., (k_q, j)$ do not belong to I we are done. If this is not the case, write $i = k_0$ and $k_{q+1} = \mu^{-1}(j)$ and assume that for some $l \in \{0, 1, 2, \ldots, q+1\}$ we have $(k_l, \mu(k_{l+1})) \in I$. Then,

$$a_{k_l\mu(k_{l+1})} + a_{k'\mu(k')} = a_{k_l\mu(k')} + a_{k'\mu(k_{l+1})}, \tag{14}$$

for some $k' \in M \setminus \{k_l, k_{l+1}\}$.

Notice that there is a point $(u, v) \in Core(w_A)$ such that $u_i + v_j = a_{ij}$, since $(M \cup M', w_A)$ is a buyer-seller exact game, and also $u_{k_s} + v_{\mu(k_s)} = a_{k_s\mu(k_s)}$ for $s \in \{1, 2, \ldots, q\}$, since μ is an optimal matching for A. Thus (13) is equivalent to

$$u_{k_0} + v_{\mu(k_{q+1})} + \sum_{s=1}^{q} \left(u_{k_s} + v_{\mu(k_s)}\right) = \sum_{s=0}^{q} a_{k_s\mu(k_{s+1})},$$

and since (u, v) belongs to $Core(w_A)$ we obtain that

$$u_{k_s} + v_{\mu(k_{s+1})} = a_{k_s\mu(k_{s+1})} \text{ for } s \in \{0, 1, \ldots, q\}. \tag{15}$$

In particular, $u_{k_l} + v_{\mu(k_{l+1})} = a_{k_l\mu(k_{l+1})}$, that together with (14) gives

$$\begin{array}{rcl} u_{k_l} + v_{\mu(k')} & = & a_{k_l\mu(k')} \\ u_{k'} + v_{\mu(k_{l+1})} & = & a_{k'\mu(k_{l+1})}. \end{array} \tag{16}$$

If $k' = k_t$ for some $t \in \{0, 1, 2, \ldots, q+1\}$ (let us assume that $t < l$, since in the case that $t > l+1$, the argument is similar), then (15) and (16) lead to

$$a_{k_t\mu(k_{l+1})} + \sum_{s=t+1}^{l} a_{k_s\mu(k_s)} = u_{k_t} + v_{\mu(k_{l+1})} + \sum_{s=t+1}^{l} (u_{k_s} + v_{\mu(k_s)}) = \sum_{s=t}^{l} a_{k_s\mu(k_{s+1})}. \quad (17)$$

Then, adding up (14) and (17), and simplifying, we obtain

$$\sum_{s=t}^{l} a_{k_s\mu(k_s)} = \sum_{s=t}^{l-1} a_{k_s\mu(k_{s+1})} + a_{k_l\mu(k_t)}.$$

This equality gives rise to a different optimal matching, in contradiction with the assumption.

Thus, we can guarantee $k' \notin \{i, k_1, k_2, ..., k_q, \mu^{-1}(j)\}$, and by addition of (13) and (14) we obtain an equality like (13) with one more term on each side, and all agents involved different. Since the market is finite, we can continue until we reach a step with an equality like in (13) with $(k_l, \mu(k_{l+1})) \notin I$ for all $l \in \{0, 1, 2, \ldots, q\}$, and this proves our claim.

Now we have to prove that B is minimal in $\langle A \rangle$ and unique with this minimality property, and therefore $[B, A] = \langle A \rangle$. Assume on the contrary that there is another matrix $C \in \langle A \rangle$ and minimal, and such that $c_{i'j'} < a_{i'j'}$ for some $(i', j') \notin I$. In this case all matrices in $[C, A]$ also belong to $\langle A \rangle$. Because $(i', j') \notin I$, we have that for all $k \in M \setminus \{i',\ \mu^{-1}(j')\}$, $a_{i'j'} + a_{k\mu(k)} > a_{i'\mu(k)} + a_{kj'}$. Then, let $\varepsilon > 0$ be such that

$$a_{i'j'} - \varepsilon > a_{i'\mu(k)} + a_{kj'} - a_{k\mu(k)}$$

for all $k \in M \setminus \{i', \mu^{-1}(j')\}$, and also $a_{i'j'} - \varepsilon > c_{i'j'}$. Now define matrix $\widehat{C}$ as

$$\widehat{c}_{ij} = \begin{cases} a_{i'j'} - \varepsilon & \text{if} \quad (i, j) = (i', j'), \\ a_{ij} & \text{otherwise.} \end{cases}$$

Obviously $\widehat{C} \in [C, A] \subseteq \langle A \rangle$, $\widehat{C} \neq A$ and $\widehat{C}$ is a buyer-seller exact matrix using (6), in contradiction with the fact that A is the buyer-seller exact representative of $\langle A \rangle$. □

Notice that in the proof of Theorem 5.1 we show how to find the minimal element in $\langle A \rangle$ for matrices A with a unique optimal matching: first compute its buyer-seller representative $\overline{A}$, and then find the set I to obtain matrix $\overline{A}^I$, which is the minimal element.

A natural question following this result is whether the converse statement is true. That is, if whenever the semilattice is a lattice, we have the uniqueness of the optimal matching. The answer is negative, as the following example shows.

Example 5.1. *The matrix*

$$\begin{pmatrix} 3 & 4 & 0 \\ 4 & 5 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

has two optimal matchings, but the same argument as in Example 4.1 shows that the corresponding semilattice has only one minimal element that is

$$\begin{pmatrix} 3 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

6 A characterization of minimality

In the previous section we have seen how to find the unique minimal matrix in the set $\langle A \rangle$ whenever A has a unique optimal matching. In the general case, when multiple minimal elements may exist, it is necessary to identify which are the entries that must be modified to find these minimal elements.

The basic idea is that if a matrix B is not minimal in $\langle A \rangle$ then some matrix entries can be lowered to zero without modifying the core. The reason is that the corresponding core constraint is already implied by other core inequalities. This is captured by the following definition.

Definition 6.1. *Let it be (M, M', A) a square assignment market and $\mu \in \mathcal{M}^*_A(M, M')$. We say $(i,j) \in M \times M'$ is* core-redundant with respect to μ *if and only if $(i,j) \notin \mu$ and there exist $r \geq 1$ and $k_1, k_2, ..., k_r \in M \setminus \{i, \mu^{-1}(j)\}$ all different such that*

$$0 \;<\; a_{ij} \;\leq\; a_{i\mu(k_1)} + a_{k_1\mu(k_2)} + a_{k_2\mu(k_3)} + ... + a_{k_r j} - \textstyle\sum_{l=1}^{r} a_{k_l\mu(k_l)}. \qquad (18)$$

We denote by $R^A(\mu)$ the set of all pairs that are core-redundant with respect to the optimal matching μ.

Notice that if $(i,j) \in R^A(\mu)$, any payoff vector $(u,v) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ that satisfies the core constraints related to the matrix entries of the right-hand side of (18) also satisfies the core constraint related to a_{ij}. Notice also that if $a_{ij} = 0$ for some $(i,j) \in M \times M'$, the related core constraint is already guaranteed by the non-negativity of the payoffs.

It could seem that if the core-redundant pairs coincide for all optimal matchings, we could set their entries to zero, but in Example 4.1, although there are two optimal matchings, namely μ_1 and μ_2, and they have the same core-redundant pairs,

$$R^A(\mu_1) = R^A(\mu_2) = \{(1,3'),(2,3'),(3,2')\},$$

we cannot lower simultaneously entries a_{13} and a_{23}.

Now we turn to a lemma that shows that one arbitrary core-redundant pair can be lowered to zero without modifying the core.

Lemma 6.1. *Let it be (M,M',A) a square assignment market with $A \in \mathbf{M}_m^+$ and $\mu \in \mathcal{M}_A^*(M,M')$. If $(i^*,j^*) \in R^A(\mu)$, then $B \in \mathbf{M}_m^+$ defined by*

$$b_{ij} = \begin{cases} 0 & \text{if} \quad (i,j) = (i^*,j^*), \\ a_{ij} & \quad \text{otherwise}\ , \end{cases}$$

satisfies $B \in \langle A \rangle$.

Proof. First notice that $w_A(N) = w_B(N)$, because $(i^*,j^*) \notin \mu$. Moreover, $B \leq A$, and therefore $Core(w_A) \subseteq Core(w_B)$. Conversely, let it be $(u,v) \in Core(w_B)$. For all $(i,j) \neq (i^*,j^*)$ it holds $u_i + v_j \geq b_{ij} = a_{ij}$. Since $(i^*,j^*) \in R^A(\mu)$, there exist $r \geq 1$ and $k_1, k_2, ..., k_r \in M \setminus \{i^*, \mu^{-1}(j^*)\}$ all different such that

$$\begin{aligned} u_{i^*} + v_{j^*} \quad &= \quad u_{i^*} + v_{j^*} + \textstyle\sum_{l=1}^r (u_{k_l} + v_{\mu(k_l)}) - \sum_{l=1}^r a_{k_l\mu(k_l)} \quad \geq \\ & \quad a_{i^*\mu(k_1)} + \textstyle\sum_{l=1}^{r-1} a_{k_l\mu(k_{l+1})} + a_{k_rj^*} - \sum_{l=1}^r a_{k_l\mu(k_l)} \quad \geq \quad a_{i^*j^*}, \end{aligned}$$

where the first inequality follows from $(u,v) \in Core(w_B)$ and the last one from $(i^*,j^*) \in R^A(\mu)$. Then $Core(w_B) \subseteq Core(w_A)$. □

From the above lemma we obtain that if there exists some $(i^*,j^*) \in R^A(\mu)$, for some $\mu \in \mathcal{M}_A^*(M,M')$, matrix A cannot be minimal in $\langle A \rangle$, since there exists $B \in \langle A \rangle$ with $B < A$. The converse implication also holds, as the next theorem states.

Theorem 6.1. *Let it be* (M, M', A) *a square assignment market with* $A \in \mathbf{M}_m^+$. *Then* $B \in \langle A \rangle$ *is minimal if and only if* $R^B(\mu) = \emptyset$ *for all* $\mu \in \mathcal{M}_B^* (M, M')$.

Proof. Assume that $B \in \langle A \rangle$ is not minimal and $R^B(\mu) = \emptyset$ for all $\mu \in \mathcal{M}_B^* (M, M')$. In this case there exists $\widetilde{C} \in \langle A \rangle$ with $\widetilde{C} < B$, and therefore there exists $(i^*, j^*) \in M \times M'$ such that $\widetilde{c}_{i^*j^*} < b_{i^*j^*}$. Notice that $w_{\widetilde{C}}(N) = w_B(N)$ and for any $\mu \in \mathcal{M}_{\widetilde{C}}^* (M, M')$, it also holds $\mu \in \mathcal{M}_B^* (M, M')$. As a consequence $(i^*, j^*) \notin \mu$, for any $\mu \in \mathcal{M}_{\widetilde{C}}^* (M, M')$.

Take then $\mu \in \mathcal{M}_{\widetilde{C}}^* (M, M')$. Since $b_{i^*j^*} > \widetilde{c}_{i^*j^*} \geq 0$ and $(i^*, j^*) \notin \mu$, the assumption that $(i^*, j^*) \notin R^B(\mu)$ implies that for all $r \geq 1$ and $k_1, k_2, ..., k_r \in M \setminus \{i^*, \mu^{-1}(j^*)\}$ all different it holds that

$$b_{i^*j^*} > b_{i^*\mu(k_1)} + \sum_{l=1}^{r-1} b_{k_l\mu(k_{l+1})} + b_{k_rj^*} - \sum_{l=1}^{r} b_{k_l\mu(k_l)}. \tag{19}$$

Let $\varepsilon > 0$ be such that $b_{i^*j^*} - \varepsilon > \widetilde{c}_{i^*j^*}$ and also

$$b_{i^*j^*} - \varepsilon > b_{i^*\mu(k_1)} + \sum_{l=1}^{r-1} b_{k_l\mu(k_{l+1})} + b_{k_rj^*} - \sum_{l=1}^{r} b_{k_l\mu(k_l)},$$

for all $k_1, k_2, ..., k_r \in M \setminus \{i^*, \mu^{-1}(j^*)\}$ and different.

Now define matrix $C \in \mathbf{M}_m^+$ by:

$$c_{ij} = \begin{cases} b_{ij} - \varepsilon & \text{if} \quad (i,j) = (i^*, j^*), \\ b_{ij} & \qquad \text{otherwise.} \end{cases}$$

The above considerations, making use of (4), imply that $\overline{c}_{i^*j^*} = c_{i^*j^*}$. Thus there exists $(u, v) \in Core(w_C)$ with $u_{i^*} + v_{j^*} = c_{i^*j^*} < b_{i^*j^*}$, a contradiction with the fact that $Core(w_C) = Core(w_B)$.

The converse implication is immediate from Lemma 6.1. □

By means of the above theorem we can identify when a given matrix $A \in \mathbf{M}_m^+$ is minimal in $\langle A \rangle$. This is illustrated in the following example.

Example 6.1. *Let us consider* $M = \{1, 2, 3, 4\}, M' = \{1', 2', 3', 4'\}$ *and the matrix*

$$A = \begin{pmatrix} 10 & 1 & 4 & 0 \\ 1 & 10 & 1 & 1 \\ 2 & 0 & 4 & 3 \\ 1 & 3 & 3 & 4 \end{pmatrix},$$

where the only optimal matching μ is placed on the main diagonal.

The reader can check that $(1, 2') \in R^A(\mu)$, since $1 = a_{12} < a_{13} + a_{34} + a_{42} - a_{33} - a_{44} = 2$, and thus, A is not minimal in $\langle A \rangle$.

We can use (3) to compute the buyer-seller exact representative of the class and apply twice Lemma 6.1 to matrix A or the procedure in Theorem 5.1 to obtain the minimal element in $\langle A \rangle$. Then, all the matrices

$$\begin{pmatrix} 10 & \alpha & 4 & \beta \\ 1 & 10 & 1 & 1 \\ 2 & \gamma & 4 & 3 \\ \delta & 3 & 3 & 4 \end{pmatrix}$$

for $\alpha \in [0, 2], \beta \in [0, 3], \gamma \in [0, 2], \delta \in [0, 1]$, define the same market with the same core, kernel and nucleolus.

In general, there may exist several minimal elements of $\langle A \rangle$ below a given matrix A. We would obtain them by iterated application of Lemma 6.1.

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