DOCUMENTS DE TREBALL
DE LA FACULTAT DE CIÈNCIES
ECONÒMIQUES I EMPRESARIALS

Col·lecció d'Economia

On the dimension of the core of the assignment game

Marina Núñez[1]
Carles Rafels[1]

University of Barcelona

Adreça correspondència:
Departament de Matemàtica Econòmica, Financera i Actuarial, i CREB
Facultat de Ciències Econòmiques i Empresarials
Universitat de Barcelona
Av. Diagonal 690
08034 Barcelona (Spain)
Tfn. 93 403 19 91
Fax. 93 403 48 92
e-mail: mnunez@ub.edu, crafels@ub.edu

[1]The authors thank the support of the Barcelona Economics Program of CREA. Institucional support from Ministerio de Ciencia y Tecnología and FEDER, under grant BEC 2002-00642, and of Generalitat de Catalunya, under grant SGR2001-0029, is also acknowledged.

Abstract: The set of optimal matchings in the assignment matrix allows to define a reflexive and symmetric binary relation on each side of the market, the equal-partner binary relation. The number of equivalence classes of the transitive closure of the equal-partner binary relation determines the dimension of the core of the assignment game. This result provides an easy procedure to determine the dimension of the core directly from the entries of the assignment matrix and shows that the dimension of the core is not as much determined by the number of optimal matchings as by their relative position in the assignment matrix.

Key words: assignment game, core, core dimension

JEL classification: C71, C78

Resum: El conjunt d'assignacions òptimes d'un joc d'assignació ens permet definir una relació binària, reflexiva i simètrica, en cada costat del mercat. El nombre de classes d'equivalència de la clausura transitiva d'aquesta relació determina la dimensió del core del joc. Aquest resultat ens dóna un procediment senzill per determinar la dimensió del core d'un joc d'assignació, només a partir de les entrades de la seva matriu. Això mostra que la dimensió del core d'aquests mercats depèn no tant del nombre d'assignacions òptimes que tenen sinó de la posició relativa d'aquestes assignacions òptimes dins la matriu.

1 Introduction

The assignment game (Shapley and Shubik, 1972) is a cooperative model for a two-sided market where side payments are allowed. In this market a product that comes in indivisible units is exchanged for money, and each participant either supplies or demands exactly one unit. The units need not be alike and the same unit may have different values for different participants. From these valuations, a matrix can be written, $A = (a_{ij})_{(i,j) \in M \times M'}$, which reflects the profit that can be obtained by each buyer-seller pair if they trade.

Shapley and Shubik prove that the core of the assignment game is nonempty, and has a lattice structure. It is a closed and convex polyhedral whose dimension is typically the cardinality of the short side of the market, but may be less in the presence of degeneracies, that is, special arithmetical relations among the matrix entries a_{ij}.

The aim of our paper is to characterize the dimension of this core only in terms of the assignment matrix. To do that, we must identify the aforementioned arithmetical relations among the matrix entries. In doing so we realize that the dimension decreases by the existence of arithmetical relations not only among the entries of the original assignment matrix but also of a related matrix, its buyer-seller exact representative. The buyer-seller exact representative of a given assignment market is introduced in Núñez and Rafels (2002) as the matrix with the entries as high as possible among those that define an assignment game with the same core. It is then no surprising that in the buyer-seller exact representative the number of optimal matchings might increase. In fact, we will see that the optimal matchings of this buyer-seller exact representative capture all the arithmetical relations that determine the dimension of the core. What is important is that all these relations can also be read from the relative position of the (fewer) optimal matchings of the initial market.

The dimension of the core of an assignment market gives information about how large the core is and, in some sense, how much variate are the possibilities of cooperation in the market. Its parallel in discrete models of matching markets without monetary transfers might be the computation of the number of stable matchings.

In this ordinal setting, it remains an open question to find a formula for the maximum number of stable matchings as a function of the number of agents of each type. Thompson (1981) and Balinski and Gale (1987) answer what they consider the analogous question for the cardinal case. They find an upper bound for the number of extreme core points as a function of the number of agents of each type: $\binom{2m}{m}$, where m is the size of the short side of the market. The dimension of the core is a complementary measure of how large the core is. Of course, when the number of extreme points is maximal, the dimension is also as high as it can be. But we may have a maximal core dimension with less number of extreme core points.

As far as we know, the dimension of the core of an assignment market has not been studied in depth. There is a result in a very particular case which seems to relate this problem to the number of optimal matchings: in Sotomayor (2003) it is proved that if an assignment market has a unique core allocation (a zero-dimensional core) then it must have more than one optimal matching. But we show in this paper that the dimension of the core of the assignment game is not as much related to the number of optimal matchings of the market as to the position of these matchings. We will argue at the end of the paper that for each possible dimension there is an upper bound for the number of optimal matchings but, quite surprisingly, with only two optimal matchings all intermediate core dimensions can be achieved.

To reach our results, we define an equivalence relation on the set of buyers (and another one on the set of sellers) only depending on the position of the optimal matchings, and prove that the number of equivalence classes determines the dimension of the core. In the context of assignment games, also Solymosi and Raghavan (1994) identify the dimension of some specific sets of payoffs with the number of certain equivalence classes. What validates our procedure is that our equivalence relation does not make use of the space of payoffs, that is the core, but only of the original matrix entries.

In Section 2, the basic concepts regarding the assignment model are recalled. In Section 3, and given an arbitrary assignment matrix, the equal-partner binary relation is defined on each side of the market, by means of the set of optimal matchings. Its transitive closure, which we name the chained equal-partner equivalence relation, is analyzed in depth, since it plays an essential role in the results. When the assignment matrix is doubly dominant diagonal the equal-partner relations are already transitive, as shown in Section 4. The characterization of the dimension of the core in terms of the number of equivalence classes of the chained equal-partner relation is proved in Section 5. The proof of the above result relies on the fact that our equivalence relation defines the same equivalence classes both on the original market and on its buyer-seller exact representative. Section 6 concludes.

2 Preliminaries

A *two-sided assignment market* (M, M', A) is defined by a finite set of buyers M of cardinality $|M| = m$, a disjoint finite set of sellers M' of cardinality $|M'| = m'$, and a nonnegative matrix $A = (a_{ij})_{(i,j)\in M\times M'}$ where a_{ij} represents the profit obtained by the mixed–pair $(i, j) \in M \times M'$ if they trade. A *square assignment market* (M, M', A) is one with as many buyers as sellers.

Given an assignment market (M, M', A) we look for an optimal matching between the two sides of the market. A *matching* $\mu \subseteq M \times M'$ between M and M' is a bijection from some $M_0 \subseteq M$ to some $M'_0 \subseteq M'$ such that $|M_0| = |M'_0| = \min\{|M|, |M'|\}$. We write $(i, j) \in \mu$ as well as $j = \mu(i)$ and $i = \mu^{-1}(j)$. We denote the set of matchings between M and M' by $\mathcal{M}(M, M')$. We say a buyer $i \in M$ is not assigned by μ if $(i, j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

We say a matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for the two-sided market (M, M', A) if for all $\mu' \in \mathcal{M}(M, M')$, we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and denote the set of optimal matchings by $\mathcal{M}^*_A(M, M')$. Given $S \subseteq M$ and $T \subseteq M'$, we denote by $\mathcal{M}(S, T)$ and $\mathcal{M}^*_A(S, T)$ the set of matchings and optimal matchings of the submarket $(S, T, A_{|S\times T})$ defined by the subset S of buyers, the subset T of sellers and the restriction of A to $S \times T$. If $S = \emptyset$ or $T = \emptyset$, then the only possible matching is $\mu = \emptyset$ and by convention $\sum_{(i,j)\in\emptyset} a_{ij} = 0$.

Shapley and Shubik associate to any assignment market (M, M', A) an *assignment game* $(M \cup M', w_A)$ where the set of players is the union of the sets of buyers and sellers and the characteristic function is defined as follows. Given $S \subseteq M$ and $T \subseteq M'$, $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\}$ (notice that a coalition formed only by sellers or only by buyers has worth zero).

With this definition, they prove that the *core*, $C(w_A)$, of an arbitrary assignment game $(M \cup M', w_A)$ is nonempty and can be represented in terms of any optimal matching $\mu \in \mathcal{M}^*_A(M, M')$. Once fixed any such optimal matching, $(u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ is in the core if and only if $u_i \geq 0$ for all $i \in M$, $v_j \geq 0$ for all $j \in M'$, $u_i + v_j \geq a_{ij}$ for all $(i,j) \in M \times M'$, $u_i + v_j = a_{ij}$ if $(i,j) \in \mu$ and $u_i = 0$ if $i \in M$ is not matched by μ, while $v_j = 0$ if $j \in M'$ is not matched by μ.

Moreover, the core has a lattice structure with two special extreme points: the *buyers–optimal core allocation*, $(\overline{u}, \underline{v})$, where each buyer attains his maximum core payoff, and the *sellers–optimal core allocation*, $(\underline{u}, \overline{v})$, where each seller does.

From Demange (1982) and Leonard (1983) we know that the maximum core payoff of any player coincides with his or her marginal contribution:

$$\begin{aligned} \overline{u}_i &= w_A(M \cup M') - w_A((M \cup M') \setminus \{i\}) \text{ for all } i \in M, \text{ and} \\ \overline{v}_j &= w_A(M \cup M') - w_A((M \cup M') \setminus \{j\}) \text{ for all } j \in M'. \end{aligned} \tag{1}$$

From (1), once fixed $\mu \in \mathcal{M}^*_A(M, M')$, and taking into account that $\underline{u}_i + \overline{v}_{\mu(i)} = a_{i\mu(i)}$, since $(\underline{u}, \overline{v}) \in C(w_A)$, we get that the minimum core payoff of a buyer i who is matched by μ is

$$\underline{u}_i = a_{i\mu(i)} + w_A((M \cup M') \setminus \{\mu(i)\}) - w_A(M \cup M'), \tag{2}$$

while $\underline{u}_i = 0$ if i is not assigned by μ. Similarly the minimum core payoff of a seller j who is matched by μ is

$$\underline{v}_j = a_{\mu^{-1}(j)j} + w_A((M \cup M') \setminus \{\mu^{-1}(j)\}) - w_A(M \cup M'). \tag{3}$$

An assignment game with as many buyers as sellers is such that each agent has a null minimal core payoff if and only if it has *dominant diagonal* (Solymosi and Raghavan, 2001), that is to say, given any $\mu \in \mathcal{M}^*_A(M, M')$, for all $i_* \in M$, $a_{i_*\mu(i_*)} \geq a_{i_*j}$ for all $j \in M'$ and $a_{i_*\mu(i_*)} \geq a_{i\mu(i_*)}$ for all $i \in M$.

An agent is said to be *active* when his or her payoff in the core is not constant, while *non-active* agents are those with a constant core payoff.

Definition 1 *Let* $(M \cup M', w_A)$ *be an assignment game and let* $(\underline{u}, \overline{v})$ *and* $(\overline{u}, \underline{v})$ *be the sellers-optimal core allocation and the buyers-optimal core allocation. A buyer* $i \in M$ *is active if and only if* $\overline{u}_i > \underline{u}_i$ *, and a seller* $j \in M'$ *is active when* $\overline{v}_j > \underline{v}_j$ *.*

Throughout this paper, without loss of generality, we assume that A is square by adding null rows or columns if necessary. Notice that this does not modify the dimension of the core.

Given a set X , a *binary relation* on X is an ordered pair (X, R) where $R \subseteq X \times X$. For any $x, y \in X$, we denote $(x, y) \in R$ by $x\,R\,y$ and say that the element x is related to the element y . A binary relation (X, R) is *reflexive* if for all $x \in X$ we have $x\,R\,x$; it is *symmetric* if for all $x, y \in X$, $x\,R\,y$ implies $y\,R\,x$; and it is *transitive* if for all $x, y, z \in X$, $x\,R\,y$ and $y\,R\,z$ imply $x\,R\,z$. A binary relation on X satisfying the three above properties is named an *equivalence relation* on X . Any equivalence relation on a set induces a partition of this set by means of its equivalence classes. For all $x \in X$, the class of x is $\bar{x} = \{y \in X \mid y\,R\,x\}$.

There is a standard procedure to associate to any reflexive and symmetric binary relation R on X an equivalence relation. This procedure is known as the *transitive closure* of R which is denoted by $\bar{R}$. For all $x, y \in X$ we say $x\,\bar{R}\,y$ if there exist $k_1, k_2, \ldots, k_r$ elements in X such that $x\,R\,k_1$, $k_i\,R\,k_{i+1}$ for all $i \in \{1, 2, \ldots, r-1\}$ and $k_r\,R\,y$. The transitive closure represents the minimum that we have to add to a reflexive and symmetric relation in order to obtain an equivalence relation. Of course, for any equivalence relation R we have $\bar{R} = R$.

3 The chained equal-partner equivalence relation

We introduce two related binary relations on each side of the market (M, M', A) . First, the *equal-partner* binary relation is defined in such a way that a pair of buyers (or a pair of sellers) are related if they have the same partner by two optimal matchings. The equal-partner relations are reflexive and symmetric but might not be transitive (see matrix A_1 below). We then consider their transitive closures and name them the *chained equal-partner* equivalence relations. The chained equal-partner equivalence relation is the keystone to determine the dimension of the core of the assignment market (see Theorem 12). This is the reason why we begin studying its properties.

Definition 2 *Let* (M, M', A) *be a square assignment market. The* chained equal-partner equivalence relation $\bar{R}_A$ on the set of buyers *is the transitive closure of the relation* R_A *on* M *defined by: for* $i_1, i_2 \in M$ *,* $i_1 R_A i_2$ *if and only if there exist* $\mu_1, \mu_2 \in \mathcal{M}^*_A(M, M')$ *such that* $\mu_1(i_1) = \mu_2(i_2)$ *.*

Similarly, the chained equal-partner binary relation $\bar{R}'_A$ on the set of sellers *is the transitive closure of the relation* R'_A *on* M' *defined by: for* $j_1, j_2 \in M'$ *,* $j_1 R'_A j_2$ *if and only if* $\mu_1^{-1}(j_1) = \mu_2^{-1}(j_2)$ *for some* $\mu_1, \mu_2 \in \mathcal{M}^*_A(M, M')$ *.*

We name R_A and R'_A the *equal-partner* relations on M and M' respectively. Assuming that the assignment matrix is square and taking into account that then each optimal matching is a bijection between M and M', reflexiveness of these relations follows easily. These relations are also symmetric by definition, but may fail to be transitive. For this reason, we add to each equal-partner relation those pairs that are connected by a chain of agents on their same side of the market such that each two consecutive agents on the chain have a common partner by some pair of optimal matchings. This completion leads to the equivalence relations $\bar{R}_A$ and $\bar{R}'_A$. We denote by I_1^A, I_2^A, ..., I_r^A the equivalence classes of $\bar{R}_A$ and by J_1^A, J_2^A, ..., J_s^A the equivalence classes of $\bar{R}'_A$.

The number of equivalence classes is a relevant information for our purposes. This number does not depend as much on the number of optimal matchings as on the relative position of these optimal matchings. To argue this point, let us consider the following two assignment markets with four agents on each side:

A_1 :

	1'	2'	3'	4'
1	1	0	0	1
2	1	1	0	0
3	0	1	1	0
4	0	0	1	1

and

A_2 :

	1'	2'	3'	4'
1	1	0	0	1
2	0	1	1	0
3	0	1	1	0
4	1	0	0	1

Both markets have two optimal matchings. The market defined by matrix A_2 has one optimal matching placed on the main diagonal and the other one on the secondary diagonal. For matrix A_1 one optimal matching is also on the main diagonal while the other one is placed just below it and in the up-right corner.

The equal-partner binary relation R_{A_1} is not transitive: $1R_{A_1}2$ and $2R_{A_1}3$, but buyer 1 is not related to buyer 3 by R_{A_1}. However, transitivity holds for the relation R_{A_2}, that is $R_{A_2} = \bar{R}_{A_2}$.

Moreover, despite the fact that both matrices have the same number of optimal matchings, their corresponding chained equal-partner equivalence relations define different sets of equivalence classes. The equivalence relation $\bar{R}_{A_1}$ defined on M has only one equivalence class, $I_1^{A_1} = \{1,2,3,4\}$ (similarly, $\bar{R}'_{A_1}$ has only one equivalence class that is $J_1^{A_1} = \{1',2',3',4'\}$), while $\bar{R}_{A_2}$ defines two equivalence classes, $I_1^{A_2} = \{1,4\}$ and $I_2^{A_2} = \{2,3\}$ (and similarly $J_1^{A_2} = \{1',4'\}$ and $J_2^{A_2} = \{2',3'\}$ are the equivalence classes of $\bar{R}'_{A_2}$).

Next proposition states that, as it happens in the matrices above, each equivalence class in M is mapped onto the same equivalence class in M' by any of the possible optimal matchings.

Proposition 3 *Let (M, M', A) be a square assignment market and let $\{I_p^A\}_{p=1}^r$ and $\{J_q^A\}_{q=1}^s$ be the equivalence classes of the chained equal-partner relations $\bar{R}_A$ and $\bar{R}'_A$, respectively. Then,*

1. *$r = s$ and*
2. *for all $p \in \{1,\ldots,r\}$ there exists a unique $q \in \{1,\ldots,s\}$ such that, for all $\mu \in \mathcal{M}_A^*(M,M')$, $\mu(I_p^A) = J_q^A$.*

PROOF: Statement 1) follows from statement 2) and the fact that μ is a bijection. To prove statement 2), let us take an equivalence class I_p^A of $\bar{R}_A$ and choose any element $i_0 \in I_p^A$. Choose also an optimal matching $\mu^* \in \mathcal{M}_A^*(M,M')$ and take $q \in \{1,2,\ldots,s\}$ such that $\mu^*(i_0) \in J_q^A$. Notice that for all other $\mu \in \mathcal{M}_A^*(M,M')$, $\mu(i_0)R'_A\mu^*(i_0)$ and thus

$$\mu(i_0) \in J_q^A \text{ for all } \mu \in \mathcal{M}_A^*(M,M'). \tag{4}$$

We will first prove that $\mu(I_p^A) \subseteq J_q^A$ for all $\mu \in \mathcal{M}_A^*(M,M')$. Take $i \in I_p^A$, $i \neq i_0$ and let us see that $\mu(i) \in J_q^A$ for all $\mu \in \mathcal{M}_A^*(M,M')$. Since $i_0 \overline{R}_A i$, there exist $i_1, i_2, \ldots, i_k \in M$ such that $i_0 R_A i_1$, $i_1 R_A i_2$, …, $i_k R_A i$. From $i_0 R_A i_1$ follows that there exist $\mu_0, \mu_1 \in \mathcal{M}_A^*(M,M')$ with $\mu_0(i_0) = \mu_1(i_1)$ which implies, by (4), $\mu_1(i_1) \in J_q^A$ and consequently, since $\mu_1(i_1) R'_A \mu(i_1)$ for all $\mu \in \mathcal{M}_A^*(M,M')$, we obtain $\mu(i_1) \in J_q^A$ for all $\mu \in \mathcal{M}_A^*(M,M')$. By repeating the same argument, we iteratively obtain that $\mu(i_2), \mu(i_3), \ldots, \mu(i_k)$ and $\mu(i)$ belong to J_q^A for all $\mu \in \mathcal{M}_A^*(M,M')$.

Let us check now that $J_q^A \subseteq \mu(I_p^A)$ for all $\mu \in \mathcal{M}_A^*(M,M')$. Take $j \in J_q^A$ and any $\mu \in \mathcal{M}_A^*(M,M')$. From $\mu(i_0) \in J_q^A$ it follows that there exist $j_1, j_2, \ldots, j_k \in M'$ such that $\mu(i_0) R'_A j_1$, $j_1 R'_A j_2$,…, $j_k R'_A j$. Since $\mu(i_0) R'_A j_1$, there exist $i_1 \in M$ and $\mu_0, \mu_1 \in \mathcal{M}_A^*(M,M')$ such that $i_1 = \mu_0^{-1}(j_1) = \mu_1^{-1}(\mu(i_0))$. Then, $\mu_1(i_1) = \mu(i_0)$ which means that $i_1 R_A i_0$ and consequently $i_1 \in I_p^A$. Moreover, $\mu_0(i_1) = j_1 = \mu(\mu^{-1}(j_1))$, and this implies that $i_1 R_A \mu^{-1}(j_1)$, that is to say $\mu^{-1}(j_1) \in I_p^A$ or, equivalently, $j_1 \in \mu(I_p^A)$. By repeatedly applying the same argument we obtain $j_2, \ldots, j_k$ and j also belong to $\mu(I_p^A)$ for all $\mu \in \mathcal{M}_A^*(M,M')$. □

As a consequence of this lemma, from now on the equivalence classes will be numbered in such a way that $\mu(I_k^A) = J_k^A$, for all $k = \{1, 2, \ldots, r\}$. Notice that corresponding equivalence classes on both sides of the market have the same cardinality.

Moreover, each equivalence class of $\bar{R}_A$ and $\bar{R}'_A$ consists of either only active agents or only non-active agents (see Definition 1).

Lemma 4 *Let $(M \cup M', w_A)$ be a square assignment game and let $\{I_k^A\}_{k=1}^r$ be the equivalence classes of $\bar{R}_A$, while $\{J_k^A\}_{k=1}^r$ are the equivalence classes of $\bar{R}'_A$.*

1. *If $i_0 \in I_k^A$ is active, then all $i \in I_k^A$ are active.*

2. *If $j_0 \in J_k^A$ is active, then all $j \in J_k^A$ are active.*

Proof: Assume on the contrary that $i_0 \in I_k^A$ is active and $i_1 \in I_k^A$ is non-active. Then $i_0 \bar{R}_A i_1$. Assume without loss of generality that $i_0 R_A i_1$. Then there exist $\mu_1, \mu_2 \in \mathcal{M}_A^*(M,M')$ such that $\mu_1(i_0) = \mu_2(i_1) = j$. Since $(\overline{u}, \underline{v}) \in C(w_A)$, this means that $\overline{u}_{i_1} + \underline{v}_j = a_{i_1 j}$ and $\overline{u}_{i_0} + \underline{v}_j = a_{i_0 j}$, and if we substitute $\underline{v}_j$ from the first equation into the second one, we get $\overline{u}_{i_0} + (a_{i_1 j} - \overline{u}_{i_1}) = a_{i_0 j}$. Taking into account that $\overline{u}_{i_1} = \underline{u}_{i_1}$, since i_1 is a non-active agent, and $(\underline{u}, \overline{v}) \in C(w_A)$, we obtain

$$a_{i_0 j} = \overline{u}_{i_0} + (a_{i_1 j} - \underline{u}_{i_1}) = \overline{u}_{i_0} + \overline{v}_j \, . \tag{5}$$

Moreover, from $(\underline{u}, \overline{v}) \in C(w_A)$, we also have that $\underline{u}_{i_0} + \overline{v}_j = a_{i_0 j}$ and together with equation (5) it follows that $\underline{u}_{i_0} = \overline{u}_{i_0}$, which contradicts i_0 being active. The proof of statement 2) is similar and thus left to the reader. □

In addition to this, any optimal partner of an active agent must also be active (if $\mu(i_0)$ is non-active, then from $a_{i_0 \mu(i_0)} = \underline{u}_{i_0} + \overline{v}_{\mu(i_0)} = \underline{u}_{i_0} + \underline{v}_{\mu(i_0)}$ and $a_{i_0 \mu(i_0)} = \overline{u}_{i_0} + \underline{v}_{\mu(i_0)}$ follows i_0 is non-active). Similarly, an optimal partner of a non-active buyer is also non-active. Thus, if I_k^A is a class of active (non-active) buyers, $\mu(I_k^A)$ is a class of active (non-active) sellers, for any $\mu \in \mathcal{M}_A^*(M, M')$.

From the lemma above, we have the same number of classes of active (non-active) agents on each side of the market. We delay until Corollary 11 in Section 5 the proof of the fact that the non-active agents on each side of the market form a unique equivalence class. This is obtained there as a straightforward consequence of Proposition 8.

The partition in equivalence classes has some consequences on the structure of the core. We see now that the payoffs to two buyers on the same class have a constant difference in all core allocation. A similar statement could be done for the sellers.

Lemma 5 *Let* $(M \cup M', w_A)$ *be a square assignment game and let* $\{I_k^A\}_{k=1}^r$ *be the equivalence classes of* $\bar{R}_A$ *.*

1. *If* $i_1,\, i_2 \in I_k^A$ *for some* $k \in \{1, 2, \ldots, r\}$ *, then* $u_{i_1} - u_{i_2}$ *is constant in* $C(w_A)$ *.*
2. *If* $i_1,\, i_2 \in I_k^A$ *for some* $k \in \{1, 2, \ldots, r\}$ *, then, for all* $\mu \in \mathcal{M}_A^*(M, M')$ *,* $u_{i_1} + v_{\mu(i_2)}$ *is constant in* $C(w_A)$ *.*

Proof: 1) First, if $i_1 R_A i_2$, there exist $\mu_1, \mu_2 \in \mathcal{M}_A^*(M, M')$ such that $\mu_1(i_1) = \mu_2(i_2)$. This implies that for all $(u, v) \in C(w_A)$, $u_{i_1} + v_{\mu_1(i_1)} = a_{i_1 \mu_1(i_1)}$ and $u_{i_2} + v_{\mu_2(i_2)} = a_{i_2 \mu_2(i_2)}$ and, as a consequence, $u_{i_1} - u_{i_2} = a_{i_1 \mu_1(i_1)} - a_{i_2 \mu_2(i_2)}$. Secondly, if $i_1 \bar{R}_A i_2$, there exist $k_1, k_2, \ldots, k_r \in M$ such that $i_1 R_A k_1$, $k_1 R_A k_2$, …, $k_r R_A i_2$.

Then we can write $u_{i_1} - u_{i_2} = (u_{i_1} - u_{k_1}) + (u_{k_1} - u_{k_2}) + \cdots + (u_{k_r} - u_{i_2})$ and since each of these differences is constant in $C(w_A)$ we obtain that $u_{i_1} - u_{i_2}$ is constant in $C(w_A)$.

2) By part 1), $u_{i_1} - u_{i_2}$ is constant in $C(w_A)$. Then, $u_{i_1} + v_{\mu(i_2)} = u_{i_2} + v_{\mu(i_2)} + (u_{i_1} - u_{i_2}) = a_{i_2\mu(i_2)} + (u_{i_1} - u_{i_2})$ is also constant in $C(w_A)$. □

The above lemma shows that the equivalence relations $\bar{R}_A$ and $\bar{R}'_A$ imply some connection between the core payoffs of some agents and thus it is not surprising that these equivalence classes turn out to determine the dimension of the core. In fact, the converse of statement (1) in the above lemma also holds, that is, if two buyers have a constant difference of payoffs in all core allocation, then they belong to the same class. But this will be seen in Corollary 10 as a consequence of a main result of Section 5 that states that the equivalence classes remain invariant among all the assignment markets with the same core. Before reaching this point, we must analyze those assignment markets where the equal partner relation is already transitive, with no need of considering its transitive closure $\bar{R}_A$.

4 The doubly dominant diagonal case

An assignment game $(M \cup M', w_A)$ with as many buyers as sellers has *doubly dominant diagonal* (Solymosi and Raghavan, 2001) if and only if, for any $\mu \in \mathcal{M}^*_A(M, M')$, $a_{ij} + a_{k\mu(k)} \geq a_{i\mu(k)} + a_{kj}$ for all $i, j, k \in M$ [2]. This definition does not depend on the chosen optimal matching $\mu \in \mathcal{M}^*_A(M, M')$. In Núñez and Rafels (2002) it is proved that, for a square assignment game, having doubly dominant diagonal is equivalent to being *buyer-seller exact*, that is to say, to satisfying that each matrix entry is attained in some core allocation: for all $i \in M$ and all $j \in M'$ there exists $(u, v) \in C(w_A)$ such that $u_i + v_j = a_{ij}$. Thus, if a square assignment game has doubly dominant diagonal, no matrix entry can be raised without modifying the core of the game.

[2] The names of dominant diagonal and doubly dominant diagonal make more sense if the optimal matching is placed on the diagonal of the assignment matrix. But since these two properties of the matrix characterize properties of the core of the game, and the core does not depend on the selected optimal matching, they can be stated in terms of any optimal matching.

We are now interested in some additional properties of the core of an assignment game with doubly dominant diagonal in relation with the defined equivalence classes. The next technical lemma shows that in a buyer-seller exact assignment game, or an assignment game with doubly dominant diagonal, if one core constraint is tight at all core allocations, then the corresponding mixed-pair coalition must be paired in some optimal matching. In fact, all mixed-pair $(i,j) \in I_k^A \times J_k^A$ belongs to some optimal matching.

Lemma 6 *Let $(M \cup M', w_A)$ be a square assignment game with doubly dominant diagonal, and let $\{I_k^A\}_{k=1}^r$ and $\{J_k^A\}_{k=1}^r$ be the equivalence classes of $\bar{R}_A$ and $\bar{R}'_A$ respectively. Then the following statements are equivalent:*

1. *$(i,j) \in I_k^A \times J_k^A$ for some $k \in \{1,2,\ldots,r\}$.*
2. *$u_i + v_j = a_{ij}$ for all $(u,v) \in C(w_A)$.*
3. *There exists $\mu \in \mathcal{M}_A^*(M,M')$ such that $(i,j) \in \mu$.*

Proof: The implication $3) \Rightarrow 1)$ follows easily since if $j = \mu(i)$ and i belongs to some class I_k^A, then, by Lemma 3, $\mu(i)$ belongs to $\mu(I_k^A) = J_k^A$. As for $1) \Rightarrow 2)$, if $(i,j) \in I_k^A \times J_k^A$ we have that, once fixed $\mu \in \mathcal{M}_A^*(M,M')$, $j = \mu(i')$ for some $i' \in I_k^A$. Then, by Lemma 5, $u_i + v_{\mu(i')} = u_i + v_j$ is constant in $C(w_A)$. Now, since $(M \cup M', w_A)$ is buyer-seller exact, $u_i + v_j = a_{ij}$ for all $(u,v) \in C(w_A)$.

It only remains to prove $2) \Rightarrow 3)$. Assume $u_i + v_j = a_{ij}$ for all $(u,v) \in C(w_A)$ and take some $\mu \in \mathcal{M}_A^*(M,M')$. If $(i,j) \in \mu$ we are done. If $(i,j) \notin \mu$, consider then $\mu(i) \in M'$ and $\mu^{-1}(j) \in M$. Then,

$$u_i + v_{\mu(i)} + u_{\mu^{-1}(j)} + v_j = a_{i\mu(i)} + a_{\mu^{-1}(j)j}\,, \quad \text{for all } (u,v) \in C(w_A)$$

and together with $u_i + v_j = a_{ij}$ for all $(u,v) \in C(w_A)$ it follows that $u_{\mu^{-1}(j)} + v_{\mu(i)} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} - a_{ij}$ for all $(u,v) \in C(w_A)$. This means that $u_{\mu^{-1}(j)} + v_{\mu(i)}$ is also constant in $C(w_A)$, but, since A is buyer-seller exact, we know that there exists $(u',v') \in C(w_A)$ such that $u'_{\mu^{-1}(j)} + v'_{\mu(i)} = a_{\mu^{-1}(j)\mu(i)}$. As a consequence, $a_{i\mu(i)} + a_{\mu^{-1}(j)j} - a_{ij} = a_{\mu^{-1}(j)\mu(i)}$.

We can now define a new matching

$$\mu' = \{(i,j),(\mu^{-1}(j),\mu(i))\} \cup \{(k,\mu(k)) \mid \text{ for all } k \in M,\, k \neq i,\, k \neq \mu^{-1}(j)\}$$

and prove it is optimal for A, since

$$\begin{aligned}\sum_{k\in M} a_{k\mu(k)} &= a_{i\mu(i)} + a_{\mu^{-1}(j)j} + \sum_{k\in M\setminus\{i,\mu^{-1}(j)\}} a_{k\mu(k)} \\ &= a_{ij} + a_{\mu^{-1}(j)\mu(i)} + \sum_{k\in M\setminus\{i,\mu^{-1}(j)\}} a_{k\mu(k)} = \sum_{k\in M} a_{k\mu'(k)}\,.\end{aligned}$$

Since $(i,j) \in \mu'$, the proof is finished. □

A consequence of the above lemma is that, when the assignment game has doubly dominant diagonal, the equal-partner relation R_A is already transitive, and there is no need of the transitive closure.

Theorem 7 *Let $(M \cup M', w_A)$ be a square assignment game. If A has doubly dominant diagonal, then the equal-partner binary relation is transitive, that is, $\bar{R}_A = R_A$.*

PROOF: Let us see that under the assumption that A is doubly dominant diagonal, if $i_1, i_2 \in M$ satisfy $i_1 \bar{R}_A i_2$, then $i_1 R_A i_2$. From $i_1 \bar{R}_A i_2$ follows that i_1, i_2 belong to the same equivalence class I_k^A, for some $k \in \{1,2,\ldots,r\}$. Take some $\mu \in \mathcal{M}_A^*(M,M')$, and consider $j_1 = \mu(i_1)$. Now, (i_2, j_1) belongs to $I_k^A \times J_k^A$, and then, from Lemma 6, we know there exists $\mu' \in \mathcal{M}_A^*(M,M')$ such that $(i_2,j_1) \in \mu'$. Thus, $\mu(i_1) = j_1 = \mu'(i_2)$, which means $i_1 R_A i_2$. □

To have a doubly dominant diagonal is a sufficient condition for the transitivity of R_A, but it is not necessary. Consider the matrix

A:

	1'	2'	3'
1	1	1	0
2	0	1	1
3	1	0	1

and notice that R_A is transitive although A does not have a doubly dominant diagonal, since $a_{13} + a_{22} = 1 < a_{12} + a_{23} = 2$.

It is proved in Núñez and Rafels (2002) that for all assignment game $(M \cup M', w_A)$ there exists a unique buyer-seller exact assignment game with the same core. This is denoted by $(M \cup M', w_{A^r})$, it is named the *buyer-seller exact representative* of the initial game, and it is defined by

$$a^r_{ij} = \max_{(u,v)\in C(w_A)} u_i + v_j \,, \text{ for all } (i,j) \in M \times M' \,. \tag{6}$$

Moreover, any matching that is optimal for A is also optimal for A^r, but A^r may have more optimal matchings than A. It follows that $(M \cup M', w_A)$ is buyer-seller exact if and only if $A = A^r$.

Since $C(w_A) = C(w_{A^r})$, both markets have the same core dimension and thus the buyer-seller exact representative will be crucial to determine the dimension of $C(w_A)$.

Once fixed any optimal matching $\mu \in \mathcal{M}^*_A(M, M')$, and being $|M| = |M'|$, the entries in matrix A^r can be obtained in two different ways. The first one (see p. 430 in Núñez and Rafels, 2002) makes use of the characteristic function of the assignment game and it is useful to be applied to numerical examples:

$$a^r_{ij} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} + w_A(M \cup M' \setminus \{\mu^{-1}(j), \mu(i)\}) - w_A(M \cup M') \,. \tag{7}$$

The second way to obtain the buyer-seller exact representative A^r, which can also be found in the paper cited above, is of a more theoretical nature and only makes use of the matrix entries: $a^r_{ij} = \max\{a_{ij}, \tilde{a}_{ij}\}$, where, for all $(i,j) \in M \times M'$,

$$\tilde{a}_{ij} = \max_{\substack{k_1,k_2,\ldots,k_r \in M\setminus\{i,j\} \\ different}} \{a_{i\mu(k_1)} + a_{k_1\mu(k_2)} + \cdots + a_{k_r j} - (a_{k_1\mu(k_1)} + \cdots + a_{k_r\mu(k_r)})\} \,. \tag{8}$$

This means that each mixed-pair coalition $\{i,j\}$ evaluates what it could achieve by cooperating with some optimally matched pairs on the basis that these pairs will be paid what they obtain in the fixed optimal matching, and then takes the maximum between this worth and a_{ij}. Expression (8) will be useful in the proof of Proposition 8.

Since the buyer-seller exact representative A^r is doubly dominant diagonal, by Theorem 7 its equal-partner binary relation is already transitive, $\bar{R}_{A^r} = R_{A^r}$. Also, by Lemma 6 it is easy to identify by means of its equivalence classes which core constraints are tight at all core allocations and, as a consequence, the dimension of $C(w_{A^r})$. In the next section we see that also the chained equal-partner equivalence relation $\bar{R}_A$ of the initial assignment game determines the dimension of its core.

5 The dimension of the core

As announced before, the dimension of the core of the assignment market will be determined by the number of equivalence classes of the chained equal-partner relation. We have seen in the previous section that, for the buyer-seller exact representative A^r, the equal-partner relation is already transitive. However, to use the equal-partner relation of A^r to characterize the dimension of $C(w_A)$ we first need to prove that both games not only have the same core but also the same set of equivalence classes.

Proposition 8 *Let* $(M \cup M', w_A)$ *and* $(M \cup M', w_B)$ *be two square assignment games. If* $C(w_A) = C(w_B)$, *then* $\bar{R}_A = \bar{R}_B$ *and* $\bar{R}'_A = \bar{R}'_B$.

PROOF: From expression (6) follows that two assignment games with the same core have the same buyer-seller exact representative. Thus, it is enough to prove that, if $(M \cup M', w_{A^r})$ is the buyer-seller exact representative of $(M \cup M', w_A)$, then for all $i_1, i_2 \in M$, $i_1 \bar{R}_A i_2$ if and only if $i_1 \bar{R}_{A^r} i_2$; and similarly, for all $j_1, j_2 \in M'$, $j_1 \bar{R}'_A j_2$ if and only if $j_1 \bar{R}'_{A^r} j_2$. We only prove the first statement above since the second one is proved analogously. The "only if" part is straightforward since any optimal matching for A is also optimal for A^r. Let us then prove the "if" part.

Assume $i_1, i_2 \in M$ with $i_1 \bar{R}_{A^r} i_2$ and not $i_1 \bar{R}_A i_2$. This means that i_1 and i_2 belong to two different equivalence classes of M by the relation $\bar{R}_A$. By Lemma 5, given $\mu \in \mathcal{M}^*_A(M, M')$, and taking into account that $\mathcal{M}^*_A(M, M') \subseteq \mathcal{M}^*_{A^r}(M, M')$, we get that $u_{i_1} + v_{\mu(i_2)}$ is constant in $C(w_{A^r})$.

Now, since A^r is buyer seller exact, this means that there exists $(u, v) \in C(w_{A^r})$ such that $u_{i_1} + v_{\mu(i_2)} = a^r_{i_1\mu(i_2)}$. But we have just seen that $u_{i_1} + v_{\mu(i_2)}$ is constant in $C(w_{A^r})$ and thus, $u_{i_1} + v_{\mu(i_2)} = a^r_{i_1\mu(i_2)}$ for all $(u, v) \in C(w_{A^r})$. Recall (8) for the expression of $a^r_{i_1\mu(i_2)}$ in terms of the assignment matrix.

Case 1: If $a^r_{i_1\mu(i_2)} = a_{i_1\mu(i_2)}$, then from $u_{i_1} + v_{\mu(i_1)} + u_{i_2} + v_{\mu(i_2)} = a_{i_1\mu(i_1)} + a_{i_2\mu(i_2)}$ we obtain

$$u_{i_2} + v_{\mu(i_1)} = a_{i_1\mu(i_1)} + a_{i_2\mu(i_2)} - a_{i_1\mu(i_2)}\,, \tag{9}$$

which shows that $u_{i_2} + v_{\mu(i_1)}$ is also constant for all core allocation of $(M \cup M', w_{A^r})$. Thus, for all $(u,v) \in C(w_{A^r})$,

$$u_{i_2} + v_{\mu(i_1)} = a^r_{i_2\mu(i_1)}\,. \tag{10}$$

By (8) it may happen that $a^r_{i_2\mu(i_1)} = a_{i_2\mu(i_1)}$ and then, by substitution in (9) $a_{i_1\mu(i_1)} + a_{i_2\mu(i_2)} = a_{i_1\mu(i_2)} + a_{i_2\mu(i_1)}$. This means that $\mu' = \{(i_1, \mu(i_2)), (i_2, \mu(i_1))\} \cup \{(i, \mu(i)) \mid i \in M \setminus \{i_1, i_2\}\}$ is also optimal for A and since $\mu(i_1) = \mu'(i_2)$ we get $i_1 R_A i_2$, in contradiction with the assumption.

If $a^r_{i_2\mu(i_1)} \neq a_{i_2\mu(i_1)}$, again by (8) we know that there exist $k_1, k_2, \ldots, k_r \in M \setminus \{i_1, i_2\}$ and different such that $a^r_{i_2\mu(i_1)} = a_{i_2\mu(k_1)} + a_{k_1\mu(k_2)} + \cdots + a_{k_r\mu(i_1)} - a_{k_1\mu(k_1)} - a_{k_2\mu(k_2)} - \cdots - a_{k_r\mu(k_r)}$. By substitution in (10), and taking into account (9), we get

$$\begin{aligned} &a_{i_2\mu(k_1)} + a_{k_1\mu(k_2)} + \cdots + a_{k_r\mu(i_1)} + a_{i_1\mu(i_2)} = \\ &\qquad a_{i_1\mu(i_1)} + a_{i_2\mu(i_2)} + a_{k_1\mu(k_1)} + \cdots + a_{k_r\mu(k_r)}\,. \end{aligned} \tag{11}$$

As a consequence, the matching

$$\begin{aligned} \mu' = \{&(i_2, \mu(k_1)), (k_1, \mu(k_2)), (k_2, \mu(k_3)), \ldots, (k_r, \mu(i_1)), (i_1, \mu(i_2))\} \\ &\cup \{(i, \mu(i)) \mid i \in M \setminus \{i_1, i_2, k_1, k_2, \ldots, k_r\}\} \end{aligned}$$

is also optimal for A.

Now, since $\mu'(i_1) = \mu(i_2)$, we get $i_1 \bar{R}_A i_2$, in contradiction with the assumption.

Case 2: If $a^r_{i_1\mu(i_2)} \neq a_{i_1\mu(i_2)}$, then by (8) we have that for all $(u,v) \in C(w_{A^r})$

$$u_{i_1} + v_{\mu(i_2)} = a^r_{i_1\mu(i_2)} = a_{i_1\mu(k_1)} + a_{k_1\mu(k_2)} + \cdots + a_{k_r\mu(i_2)} - a_{k_1\mu(k_1)} - a_{k_2\mu(k_2)} - \cdots - a_{k_r\mu(k_r)}$$

for some $k_1, k_2, \ldots, k_r \in M \setminus \{i_1, i_2\}$ and different.

As a consequence, for all $(u,v) \in C(w_A)$,

$$u_{i_1} + v_{\mu(i_2)} = a_{i_1\mu(k_1)} + \sum_{j=1}^{r-1} a_{k_j\mu(k_{j+1})} + a_{k_r\mu(i_2)} - \left(\sum_{j=1}^{r} (u_{k_j} + v_{\mu(k_j)}) \right)$$

and then,

$$u_{i_1} + v_{\mu(i_2)} + \sum_{j=1}^{r} (u_{k_j} + v_{\mu(k_j)}) = a_{i_1 \mu(k_1)} + \sum_{j=1}^{r-1} a_{k_j \mu(k_{j+1})} + a_{k_r \mu(i_2)} \, .$$

Since $u_{i_1} + v_{\mu(k_1)} \geq a_{i_1\mu(k_1)}$, $u_{k_r} + v_{\mu(i_2)} \geq a_{k_r\mu(i_2)}$ and $u_{k_j} + v_{\mu(k_{j+1})} \geq a_{k_j\mu(k_{j+1})}$ for all $j \in \{1, 2, \ldots, r-1\}$, we obtain that all these core constraints are tight at all core allocations. To unify notation let us write $(\tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_r, \tilde{k}_{r+1}) = (i_1, k_1, \ldots, k_r, i_2)$. Then, since $i_1 = \tilde{k}_0$ and $i_2 = \tilde{k}_{r+1}$ belong to different equivalence classes by the relation $\bar{R}_A$, there must be two consecutive buyers in this chain $(\tilde{k}_0, \tilde{k}_1, \ldots, \tilde{k}_r, \tilde{k}_{r+1})$ whose equivalence class by $\bar{R}_A$ differ. Let us say these are $\tilde{k}_j$ and $\tilde{k}_{j+1}$, for some $j \in \{0, 1, \ldots, r\}$. Moreover, $u_{\tilde{k}_j} + v_{\mu(\tilde{k}_{j+1})} = a_{\tilde{k}_j\mu(\tilde{k}_{j+1})}$ for all $(u, v) \in C(w_A)$, which means that $a^r_{\tilde{k}_j\tilde{k}_{j+1}} = a_{\tilde{k}_j\tilde{k}_{j+1}}$. This implies, by Lemma 6, that $\tilde{k}_j$ and $\tilde{k}_{j+1}$ belong to the same class by the relation $\bar{R}_{A^r}$, that is, $\tilde{k}_j \bar{R}_{A^r} \tilde{k}_{j+1}$. We are thus on the assumptions of Case 1, that is, $\tilde{k}_j$ and $\tilde{k}_{j+1}$ play the same role as i_1 and i_2 in Case 1, and thus we reach $\tilde{k}_j \bar{R}_A \tilde{k}_{j+1}$, in contradiction with the assumption. □

By the above lemma, for all assignment games with the same core the equivalence relation $\bar{R}_A$, and consequently the partition in equivalence classes, is the same. Thus, whenever we want to analyze some core property of these markets in terms of this equivalence relation, we can restrict to their buyer-seller exact representatives. For this representative, the chained equal-partner relation coincides with the equal-partner relation, which is already transitive. This is shown in the next corollary.

Corollary 9 *Let $(M \cup M', w_A)$ be an assignment game with as many buyers as sellers. Then $\bar{R}_A = R_{A^r}$, where $(M \cup M', w_{A^r})$ is the buyer-seller exact representative.*

Proof: Take an arbitrary assignment game $(M \cup M', w_A)$, not necessarily with doubly dominant diagonal. We have $\bar{R}_A = \bar{R}_{A^r} = R_{A^r}$, where the first equality follows from Proposition 8 and the second one from Theorem 7. □

A second consequence is that the statements of Lemma 5 are in fact characterizations of the chained equal-partner equivalence relation.

Corollary 10 *Let $(M \cup M', w_A)$ be a square assignment game and let $\bar{R}_A$ be its chained equal-partner relation on M. If $i_1, i_2 \in M$, then the following statements are equivalent.*

1. $i_1 \bar{R}_A i_2$.

2. $u_{i_1} - u_{i_2}$ *is constant in* $C(w_A)$.

3. $u_{i_1} + v_{\mu(i_2)}$ *is constant in* $C(w_A)$ *for all* $\mu \in \mathcal{M}^*_A(M, M')$.

And similarly for the relation $\bar{R}'_A$ *on* M' .

PROOF: The implication $(1) \Rightarrow (2)$ is part (1) of Lemma 5. As for $(2) \Rightarrow (3)$, notice that if $u_{i_1} - u_{i_2}$ is constant in $C(w_A)$ then, for all $\mu \in \mathcal{M}^*_A(M, M')$, $u_{i_1} - (a_{i_1\mu(i_2)} - v_{\mu(i_2)})$ is also constant in $C(w_A)$, which implies (3).

To see $(3) \Rightarrow (1)$ recall first that $\mathcal{M}^*_A(M, M') \subseteq \mathcal{M}^*_{A^r}(M, M')$. Observe now that if, for some $\mu \in \mathcal{M}^*_A(M, M')$, $u_{i_1}+v_{\mu(i_2)}$ is constant in $C(w_A)$, then $u_{i_1}+v_{\mu(i_2)}$ is also constant in $C(w_{A^r})$. Since A^r is buyer-seller exact, we have that $u_{i_1}+v_{\mu(i_2)} = a^r_{i_1\mu(i_2)}$ for all $(u, v) \in C(w_{A^r})$. Now, by Lemma 6, there exists $\mu' \in \mathcal{M}^*_{A^r}(M, M')$ such that $(i_1, \mu(i_2)) \in \mu'$. This means that $\mu'(i_1) = \mu(i_2)$ and thus $i_1 R_{A^r} i_2$. Finally, by Proposition 8, we get $i_1 \bar{R}_A i_2$. □

A consequence of the above corollary is that all non-active agents on the same side of the market belong to the same equivalence class. This happens because every non-active agent has a constant payoff in the core.

Corollary 11 *Let* $(M \cup M', w_A)$ *be a square assignment game. All non-active buyers (sellers) belong to the same equivalence class of the chained equal-partner relation* $\bar{R}_A$ *(* $\bar{R}'_A$ *).*

Now we know that there is at most one equivalence class formed by non-active agents on each side of the market, we denote by I^A_0 and J^A_0 the class of non-active buyers and sellers respectively. Notice that I^A_0 and J^A_0 might be empty.

Finally, next theorem states that the dimension of the core equals the number of equivalence classes formed by active agents on one side of the market. Recall that the dimension of the core as a convex polytope C is the dimension of the minimal affine variety, $Aff(C)$, in which it is contained. See Rockafellar (1970) for the definitions and properties of convex sets.

Theorem 12 *Let* $(M \cup M', w_A)$ *be a square assignment game and let the equivalence classes of* $\bar{R}_A$ *be* I_0^A, $I_1^A, \ldots, I_r^A$ *, where* I_0^A *is the (possibly empty) class of non-active buyers. Then,* $\dim C(w_A) = r$ *.*

PROOF: By Proposition 8, we assume without loss of generality that A is buyer-seller exact. We will prove first that $\dim C(w_A) \leq r$. Take any optimal matching μ. For all $(u,v) \in C(w_A)$ and all $(i,j) \in \mu$ we have that $u_i + v_j = a_{ij}$. Thus $\dim C(w_A) = \dim C_M(w_A) \leq m$, where $C_M(w_A)$ is the projection of $C(w_A)$ to the space of payoffs to the buyers. Moreover the payoff in the core of any non-active agent is constant; this is, for all $i \in I_0^A$, u_i is constant for all $(u,v) \in C(w_A)$. This means that $\dim C(w_A) \leq |I_1^A \cup I_2^A \cup \cdots \cup I_r^A|$, where recall that $|I_1^A \cup I_2^A \cup \cdots \cup I_r^A|$ stands for the cardinality of the set $I_1^A \cup I_2^A \cup \cdots \cup I_r^A$.

By Lemma 5, for all equivalence class I_k^A, $k \in \{1, 2, \ldots, r\}$, and any two agents in this class, $i_1, i_2 \in I_k^A$, it holds that $u_{i_1} - u_{i_2}$ is constant for any $(u,v) \in C(w_A)$. This implies $\dim C(w_A) \leq r$.

To prove the converse inequality, take (u,v) in the relative interior of $C(w_A)$. Recall that $x \in Aff(C)$ belongs to the relative interior of C, $ri(C)$, if and only if there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \cap Aff(C) \subset C$, where $B(x,\varepsilon)$ is the ball centered at x with radius $\varepsilon > 0$. Recall also that the relative interior of a non-empty convex set is also non-empty. Then, since all assignment game has a non-empty core, $ri(C(w_A)) \neq \emptyset$.

Since we are assuming that A is buyer-seller exact, by Lemma 6 we know that a core constraint $u_i + v_j \geq a_{ij}$ is tight at all core allocations if and only if $(i,j) \in I_k^A \times J_k^A$, for some $k \in \{1, 2, \ldots, r\}$. Then, (u,v) in the relative interior of $C(w_A)$ means that $u_i > 0$ for all $i \in M \setminus I_0^A$, $v_j > 0$ for all $j \in M' \setminus J_0^A$ and $u_i + v_j > a_{ij}$ for all $(i,j) \notin \bigcup_{k=1}^r (I_k^A \times J_k^A)$.

Now, for all $k \in \{1, 2, \ldots, r\}$ take $\varepsilon^k > 0$ small enough such that the point (u^k, v^k) belongs to the core, where

$$\begin{aligned} u_i^k &= u_i + \varepsilon^k && \forall i \in I_k^A , \\ v_j^k &= v_j - \varepsilon^k && \forall j \in J_k^A , \\ u_i^k &= u_i && \forall i \in M \setminus I_k^A , \\ v_j^k &= v_j && \forall j' \in M' \setminus J_k^A . \end{aligned}$$

If we denote by $conv\{u, u^1, u^2, \ldots, u^r\}$ the convex hull of the points u, u^1, $u^2, \ldots, u^r$ of $\mathbb{R}^m$, we have that $C_M(w_A) \supseteq conv\{u, u^1, u^2, \ldots, u^r\}$ and thus $\dim C_M(w_A) \geq \dim conv\{u, u^1, u^2, \ldots, u^r\}$. For all $k \in \{1, 2, \ldots, r\}$, let w^k be the vector in $\mathbb{R}^m$ with origin in u and extreme point in u^k, that is $w_i^k = \varepsilon^k$ for all $i \in I_k^A$ and $w_i^k = 0$ otherwise. Then, $\dim conv\{u, u^1, u^2, \ldots, u^r\}$ is the rank of the set of vectors $\{w^1, w^2, \ldots, w^r\}$ and this is r. □

As an application of the above theorem, consider the two markets with matrices A_1 and A_2 introduced in page 8. In both markets the buyers-optimal core allocation is $(\overline{u}, \underline{v}) = (1, 1, 1, 1; 0, 0, 0, 0)$ and the sellers-optimal core allocation is $(\underline{u}, \overline{v}) = (0, 0, 0, 0; 1, 1, 1, 1)$, as can be easily obtained from (1), (2) and (3). Thus, all agents are active. The assignment game $(M \cup M', w_{A_1})$ has only one equivalence class (by the chained equal-partner relation) on each side of the market. As a consequence it has a one-dimensional core. On the other side, the chained equal-partner relation on any side of the market defined by A_2 has two equivalence classes. Then, the core of the corresponding assignment game is two-dimensional.

Some more comments relating the dimensionality of the core with the position of the optimal matchings on the assignment matrix are given in the next section.

6 Some concluding remarks

Some other straightforward consequences follow from the partitioned form of the assignment matrix given by the equivalence classes. For instance, once fixed a set of agents $M \cup M'$, for all $t \in \{0, 1, \ldots, m\}$, there exists a matrix A such that $\dim C(w_A) = t$. That is, all possible dimensions are effectively achieved. To see that, you only have to built a $m \times m$ matrix with t non-zero diagonal blocks, each diagonal block being a constant and positive matrix and all blocks out of the diagonal being null.

Moreover, it may well happen that a submarket has a core with a dimension that is higher than the dimension of the core of the initial assignment market. The reason is that in general the submarket does not preserve the optimal matchings of the initial market and thus the number of equivalence classes may increase.

We may also ask if there is any relationship between the number of optimal matchings and the dimension of the core. As a first answer we obtain that an assignment market with all agents active is full dimensioned, that is $\dim C(w_A) = m$, if and only if there is only one optimal matching. This happens because, if only one optimal matching exists, then each equivalence class is a singleton. Moreover, if the optimal matching is not unique, then at least two agents will be related by the equal-partner relation, and then the number of equivalence classes will be less than m, since at least one class will contain more than one agent.

As the dimension of the core decreases, its relationship with the number of optimal matchings is not so tight. One could think that as the dimension diminishes, the number of optimal matchings increases. But this is not exactly the case.

With only two optimal matchings, and assuming again that all agents are active, the dimension of the core may take any value from 1 to $m-1$. To obtain an assignment game with a core of dimension $k \in \{1, 2, \ldots, m-1\}$ you only have to define a matrix with these two optimal matchings: $\mu = \{(i,i) \mid i \in M\}$ and $\mu' = \{(i,i) \mid 1 \leq i \leq k-1\} \cup \{(i, i+1) \mid k \leq i \leq m-1\} \cup \{(m,k)\}$. Then the equivalence classes on the set of buyers are $I_r^A = \{r\}$ for all $r \in \{1, 2, \ldots, k-1\}$ and $I_k^A = \{k, k+1, \ldots, m\}$.

Thus, there is not a lower bound to the number of optimal matchings depending on the dimension of the core. However, it is true that, as the dimension decreases the upper bound for the number of optimal matchings increases.

To argue this assertion, notice that if the matrix has doubly dominant diagonal, and once assumed that the agents on the same equivalence class are consecutive, all matching contained in the diagonal blocks is optimal. This is to say, all matching $\mu \in \mathcal{M}_{A^r}(M,M')$ such that for all $(i,j) \in \mu$ there exists $k \in \{1,2,\ldots,r\}$ with $(i,j) \in I_k^A \times J_k^A$ is an optimal matching for A^r. Thus, the number of equivalence classes of an assignment market $(M \cup M', w_A)$, together with the cardinality of these classes, determines the number of optimal matchings of the buyer-seller exact representative A^r, which is $\prod_{k=1}^{r} |I_k^A|!$. But the number of optimal matchings of the initial market $(M \cup M', w_A)$ may be less than that.

Thus, for a given structure of the r equivalence classes, $\prod_{k=1}^{r} |I_k^A|!$ is an upper bound for the number of optimal matchings. Among all the markets with m agents on each side and r equivalence classes, the maximum possible number of optimal matchings is $(m-r+1)!$, which is attained at a doubly dominant matrix with $r-1$ singleton equivalence classes and another one with $m-r+1$ agents. Thus, when all agents are active, an attainable upper bound for the number of optimal matchings when the core is r-dimensional is $(m-r+1)!$

Also, from a given structure of the equivalence classes, a lower bound for the number of optimal matchings of $(M \cup M', w_A)$ can be deduced: for each equivalence class I_k^A not reduced to a singleton, the submatrix $A_{|I_k^A \times J_k^A}$ must have at least two optimal matchings. As a consequence, the result in Sotomayor (2003) is easily obtained: if the core of a market with at least two agents on each side reduces to only one point, then the optimal matching cannot be unique. The reason is that if the core is zero-dimensional then all the agents on each side of the market belong to the same equivalence class (the one formed by non-active agents), and this cannot be achieved with only one optimal matching.

Finally, from the description of the equivalence classes of the transitive closure of the equal-partner relation we can recognize when the kernel of an assignment game coincides with the core. The kernel is another set-solution concept for coalitional games with transferable utility which was introduced by Davis and Maschler (1965). Taking into account that the kernel of an assignment game is included in the core (Driessen, 1998), we know by Granot and Granot (1992) that the kernel of an assignment game coincides with the core if and only if there does not exist an agent with non-constant payoff in the core that is matched with the same partner in all optimal matching. We can now say that this happens if and only if no equivalence class formed by active agents is a singleton.

Corollary 13 *Let* $(M \cup M', w_A)$ *be an assignment game with as many buyers as sellers, and let* $\{I_k^A\}_{k=0}^r$ *and* $\{J_k^A\}_{k=0}^r$ *be its equivalence classes by the relations* $\bar{R}_A$ *and* $\bar{R}'_A$ *, where* I_0^A *and* J_0^A *are the possibly empty classes of non-active agents. The kernel coincides with the core if and only if* $|I_k^A| > 1$ *for all* $k \in \{1, 2, \ldots, r\}$ *.*

References

[1] Balinski, M.L., Gale, D., 1987. On the core of the assignment game. In: *Functional Analysis, Optimization and Matematical Economics*. Oxford University Press, New York, 274–289.

[2] Davis, M., Maschler, M., 1965. The kernel of a cooperative game. Naval Research Logistics Quarterly, 12, 223–259.

[3] Demange, G., 1982. Strategyproofness in the Assignment Market Game. Laboratorie d'Econometrie de l'Ecole Politechnique, Paris. Mimeo.

[4] Driessen, T.S.H., 1998. A note on the inclusion of the kernel in the core of the bilateral assignment game. International Journal of Game Theory, 27, 301–303.

[5] Granot, D., Granot, F., 1992. On some network flow games. Mathematics of Operations Research, 17, 792–841.

[6] Leonard, H.B., 1983. Elicitation of Honest Preferences for the Assignment of Individuals to Positions. Journal of Political Economy, 91, 461–479.

[7] Núñez, M., Rafels, C., 2002. Buyer–seller exactness in the assignment game. International Journal of Game Theory, 31, 423–436.

[8] Rockafellar, R.T., 1970. Convex analysis. Princeton University Press, Princeton, N.J.

[9] Shapley, L.S., Shubik, M., 1972. The Assignment Game I: The Core. International Journal of Game Theory, 1, 111–130.

[10] Solymosi T., Raghavan T.E.S., 1994. An algorithm for finding the nucleolus of assignment games. International Journal of Game Theory, 23, 119–143.

[11] Solymosi, T., Raghavan, T.E.S., 2001. Assignment games with stable core. International Journal of Game Theory, 30, 177–185.

[12] Sotomayor, M., 2003. Some further remark on the core of the assignment game. Mathematical Social Sciences, 46, 261–265.

[13] Thompson, G.L., 1981. Auctions and Market Games. In *Essays in Game Theory and Mathematical Economics in honor of Oskar Morgenstern,* Bibliographisches Institute AG, Zürich, 181–196.