Rainbow Matchings: existence and counting

Guillem Perarnau and Oriol Serra

October 16, 2018

Abstract

A perfect matching M in an edge–colored complete bipartite graph $K_{n,n}$ is rainbow if no pair of edges in M have the same color. We obtain asymptotic enumeration results for the number of rainbow matchings in terms of the maximum number of occurrences of a color. We also consider two natural models of random edge–colored $K_{n,n}$ and show that, if the number of colors is at least n, then there is *whp* a random matching. This in particular shows that almost every square matrix of order n in which every entry appears at most n times has a Latin transversal.

Keywords: Rainbow Matchings, Latin Transversals, Random Edge–colorings.

1 Introduction

A subgraph H of an edge–colored graph G is *rainbow* if no color appear twice in $E(H)$. The study of rainbow subgraphs has a large literature; see e.g. [1, 5, 9, 10, 8]. In this paper we deal with *rainbow perfect matchings* of edge–colored complete bipartite graphs $K_{n,n}$. These are equivalent to latin transversals in square matrices of order n, sets of n pairwise distinct entries no two in the same row nor the same column. The following is a longstanding conjecture by Ryser [15] on the existence of latin transversals in latin squares:

Conjecture 1 (Ryser) *Every latin square of odd order admits a latin transversal.*

For even size, there are some latin squares that have no rainbow matchings, such as the additive table of $\mathbb{Z}_{2n}$. Nevertheless, it was also conjectured (see e.g. [16]) that every latin square of even size has a partial latin transversal of length $n-1$.

There are different approaches to address these conjectures. For instance, Hatami and Shor [6] proved that every latin square has a partial transversal of size $n - O(\log n^2)$. Another approach was given by Erdős and Spencer [4] where they prove the following result:

Theorem 2 (Erdős, Spencer [4]) *Let A be square matrix of order n. If every entry in A appears at most $\frac{n-1}{4e}$ times, then A has a latin transversal.*

In order to get this result the authors developed the Lopsided version of the Lovász Local Lemma. The main idea of this version is to set a different dependency graph called *lopsidependency graph*. In this graph edges may no longer represent dependencies and the hypothesis of the Local Lemma are replaced by a weaker assumption.

In this paper we address two problems related to rainbow matchings in edge–colored $K_{n,n}$: asymptotic enumeration and existence in random edge–colorings. Our edge–colorings are non necessarily proper and the results apply to proper edge–colorings as well.

Previous results on enumeration of latin transversals have been obtained by McKay, McLeod and Wanless [13], where the authors give upper and lower bounds on the maximum number of transversals that a latin square can have. However there is still a large gap between these bounds. Here, we provide, under some hypothesis, upper and lower bounds for the probability that a random matching in an edge–colored $K_{n,n}$ is rainbow, that are asymptotically tight.

The bounds are obtained by techniques inspired by the framework devised by Lu and Székely [11] to obtain asymptotic enumeration results with the Local Lóvasz Lemma.

For an edge–coloring of the complete bipartite graph $K_{n,n}$, we let $\mathcal{M}$ denote the family of pairs of non incident edges that have the same color. Let M be a random matching of $K_{n,n}$. For each $(e, f) \in \mathcal{M}$, we denote by A_{ef} the event that the pair of edges e, f belongs to M. This is the set of bad events in the sense that, if none of these events occur, then the matching M is rainbow.

Therefore, given a set of bad events $A_1, \ldots, A_m$, we consider the problem of estimating the probability of the event $\cap_{i=1}^{m} \overline{A_i}$. If the bad events are mutually independent, then the number of bad events that are satisfied follows a Poisson distribution with parameter $\mu = \sum_{i=1}^{m} \Pr(A_i)$. Hence,

$$\Pr(\cap_{i=1}^{m} \overline{A_j}) = e^{-\mu}.$$

It is natural to expect a similar behaviour if the dependencies among the events are rare. This is known as the Poisson Paradigm (see e.g. [2]). Our objective is to show that

$$\Pr(\cap_{i=1}^{m} \overline{A_j}) \to e^{-\mu} \quad (n \to \infty).$$

Let X_M denote the indicator variable that a random perfect matching M is rainbow in a fixed edge–coloring of $K_{n,n}$. Let $\mathcal{M}$ denote the set of pairs of independent edges that have the same color (bad events.) Our first result is the following:

Theorem 3 *Fix an edge–coloring of $K_{n,n}$ such that no color appears more than n/k times, where $k = k(n)$. Let $\mu = |\mathcal{M}|/n(n-1)$.*

If $k \geq 12$ then there exist constants $c_1 < 1 < c_2$ depending only in k, such that

$$e^{-c_2\mu} \leq \Pr(X_M = 1) \leq e^{-c_1\mu}.$$

In particular, if $k = \omega(1)$ then

$$\Pr(X_M = 1) = e^{-(1+o(1))\mu}.$$

Moreover, if $k = \omega(n^{1/2})$ then

$$\Pr(X_M = 1) = e^{-\mu}(1 + o(1)).$$

In the proof of Theorem 3 we obtain $c_1 = 1-2/k-12/k^2$ and $c_2 = 1+16/k$. Note that the probability of having a rainbow matching only depends on the number of bad events that the given coloring defines. Perhaps surprisingly, this probability does not depend on the structure of the set of bad events in the coloring.

The results in Theorem 3 require the condition $k \geq 12$, which is one unit more than the one given by Erdős and Spencer [4] for the existence of rainbow matchings. This prompts us to analyze the existence of rainbow matchings in *random* edge–colorings of $K_{n,n}$ in the more general setting when $k \geq 1$ (we can not use less than n colors.) For the existence of rainbow matchings in random edge–colorings of $K_{n,n}$ we restrict ourselves to colorings with a fixed number $s = kn$ of colors. We define two natural random models that fit with this condition.

In the Uniform random model, URM, each edge gets one of the s colors independently and uniformly at random. In this model, every possible edge coloring with at most s colors appears with the the same probability. In the Regular random model, RRM, we choose an edge coloring uniformly at random among all the equitable edge colorings, where each color class has prescribed size $\frac{n}{k}$. Although they have the same expected behaviour, both models are interesting. A result analogous to the one in Theorem 3 can be proven for these two models.

Theorem 4 *Let c be a random edge coloring of $K_{n,n}$ in the model* URM *with $s \geq n$ colors. Then,*

$$\Pr(X_M = 1) = e^{-c(k)\mu}$$

where $\mu \sim \frac{n^2}{2s}$ and

$$c(k) = 2k\left(1 - (k-1)\log\left(\frac{k}{k-1}\right)\right)$$

Let c be a random edge-coloring of $K_{n,n}$ in the RRM *model with $s \geq n$ colors. Then*

$$\Pr(X_M = 1) = e^{-(c(k)+o(1))\mu}$$

Observe that both models lead to similar results. In particular, if $k = \omega(1)$

$$\Pr(X_M = 1) = e^{-(1+o(1))\mu}$$

The RRM behaves as expected since, as we have observed, just the number of bad events is relevant, and in this case it is approximately $\frac{n^3}{2k}$.

Since the colorings are random, we have a stronger concentration of the rainbow matching probability than in the case of fixed colorings. By using the random model URM we show that *with high probability* (*whp*, meaning with probability tending to one as $n \to \infty$), for any constant $k \geq 1$, every random coloring has a rainbow matching.

Theorem 5 *Every random edge–coloring of $K_{n,n}$ in the* URM *with $s \geq n$ colors has whp a rainbow matching.*

To prove the Theorem 5 we use the second moment method on the random variable that counts the number of rainbow matchings in the URM model. Observe that the same result can be proven using the same idea for the RRM model.

The paper is organized as follows. In Section 2 we provide a proof for Theorem 3. The random coloring models are defined in Section 3, where we also prove Theorem 4. In the Subsection 4 we display a prove for Theorem 5. Finally on Section 5 we discuss about open problems about rainbow matchings that arise from the paper.

2 Asymptotic enumeration

In this section we prove Theorem 3. When $k = \omega(1)$ for $n \to \infty$, it gives an asymptotically tight estimation of the probability that a random matching is rainbow. For constant k the theorem provides exponential upper and lower bounds for this probability.

2.1 Lower bound

One of the standard tools to give a lower bound for $\Pr(\cap_{i=1}^{m}\overline{A_j})$ is the Local Lemma. In particular, as it is shown in [4], it is convenient in our current setting to use the Lopsided version of it.

Given a set of events $A_1, \ldots, A_m$, a graph H with vertex set $V(H) = \{1, \ldots, m\}$ is a *lopsidependency graph* for the events if, for each i and each subset $S \subseteq \{j \mid ij \notin E(H),\, j \neq i\}$, we have

$$\Pr(A_i \mid \cap_{j\in S}\overline{A_j}) \leq \Pr(A_i).$$

Following Lu and Szekely [11], we adopt the more explanatory term *negative dependency graph* for this notion. We next recall the statement of the Lóvasz Local Lemma we will use. It includes an intermediate step, that appears in its proof, which will also be used later on.

Lemma 6 (LLLL) *Let $\{A_1, \ldots, A_m\}$ be events and let $H = (V, E)$ be a graph on $\{1, \ldots, m\}$ such that, for each i and each $S \subseteq \{j \mid ij \notin E(H),\, j \neq i\}$,*

$$\Pr(A_i | \cap_{j\in S} \overline{A_j}) \leq P(A_i).$$

Let $x_1, \ldots, x_m \in (0, 1)$. If, for each i,

$$\Pr(A_i) \leq x_i \prod_{ij\in E(H)} (1 - x_j), \tag{1}$$

then, for each $T \subset [m]$ we have

$$\Pr(A_i | \cap_{j\in T} \overline{A_j}) \leq x_i. \tag{2}$$

In particular, for each $S \subset [m]$ disjoint from T we have

$$\Pr(\cap_{i\in S}\overline{A_i} | \cap_{j\in T} \overline{A_j}) \geq \prod_{i\in S}(1 - x_i), \tag{3}$$

and

$$\Pr(\cap_{j\in[m]}\overline{A_j}) \geq \prod_{j\in[m]} (1 - x_j). \tag{4}$$

Recall that $\mathcal{M}$ denotes the family of pairs of independent edges that have the same color and, for each such pair $\{e, f\} \in \mathcal{M}$, we denote by $A_{e,f}$ the event that the pair belongs to a perfect random matching M. We identify $\mathcal{M}$ with this set of events. We consider the following dependency graph:

Definition 7 *The rainbow dependency graph H has the family $\mathcal{M}$ as vertex set. Two elements in $\mathcal{M}$ are adjacent in H whenever the corresponding pairs of edges share some end vertex in $K_{n,n}$.*

It is shown in Erdős and Spencer [4] that the graph H defined above is a negative dependency graph. The following lower bound can be obtained in a similar way to Lu and Szekely [11, Lemma 2]. Recall that we consider edge–colorings of $K_{n,n}$ in which each color appears at most n/k times.

Lemma 8 *With the above notations, if $k \geq 12$ then*

$$\Pr(\cap_{\{e,f\}\in\mathcal{M}}\overline{A_{e,f}}) \geq e^{-(1+16/k)\mu},$$

where $\mu = \sum_{\{e,f\}\in\mathcal{M}} \Pr(A_{e,f})$.

In particular, if $k = k(n) = \omega(\sqrt{n})$, then

$$\Pr(\cap_{\{e,f\}\in\mathcal{M}}\overline{A_{e,f}}) \geq (1 + o(1))e^{-\mu}.$$

Proof Set $\mathcal{M} = \{A_1, \ldots, A_m\}$. The size of $\mathcal{M}$ depends on the configuration of the colors in $E(K_{n,n})$. In the worst case all the colors appear repeated in exactly n/k disjoint edges. Thus,

$$|\mathcal{M}| \leq kn\binom{n/k}{2} \sim \frac{n^3}{2k}$$

Since we are taking a random perfect matching, $p = \Pr(A_i) = \frac{1}{n(n-1)}$ for each i. Then

$$\mu = \frac{|\mathcal{M}|}{n(n-1)} = \frac{1}{2}\left(1 + \frac{1}{n-1}\right)\left(\frac{n}{k} - 1\right) \leq \frac{n}{2k} \tag{5}$$

Set $t=4/k$. Since $n \geq k \geq 12$ we have $t \leq 1/3$ and $p \leq 1/35$. It can be checked that, for $4/n < t < 7/50$ and $0<p<1/35$, we have

$$pe^{(1+4t)t} < 1 - e^{-(1+4t)p}.$$

Choose x_i in the interval $(pe^{(1+4t)t}, 1-e^{-(1+4t)p})$. For each $1 \leq i \leq m$ we have

$$\Pr(A_i) = p < x_i e^{-(1+4t)t} < x_i \prod_{ij \in E(H)} e^{-(1+4t)\Pr(A_j)} < x_i \prod_{ij \in E(H)} (1-x_j). \tag{6}$$

Thus, by Lemma 6,

$$\Pr(\cap_{A_i \in \mathcal{M}} \overline{A_i}) \geq \prod_{i=1}^{m} (1-x_i) \geq e^{-(1+16/k)\mu}.$$

This proves the first part of the Lemma. In particular, since $\mu \leq n/2k$,

$$\Pr(\cap_{A_i \in \mathcal{M}} \overline{A_i}) \geq e^{-\mu}\left(1-\frac{16\mu}{k}\right) \geq e^{-\mu}\left(1-\frac{8n}{k^2}\right).$$

so that, if $k = k(n) = \omega(\sqrt{n})$, then

$$\Pr(\cap_{A_i \in \mathcal{M}} \overline{A_i}) \geq e^{-\mu}(1+o(1)).$$

□

2.2 Upper bound

Lu and Szekely [11] propose a new enumeration tool using the Local Lemma. Their objective is to find an upper bound for the non occurrence of rare events comparable with the Janson inequality. In order to adapt the Local Lemma, they define a new type of parametrized dependency graph: the *ε-near dependency graph*.

Let $A_1, \ldots, A_m$ a set of events. A graph H with vertex set $\{A_1, \ldots, A_m\}$ is an ε-near-positive dependency graph (ε-NDG) if,

i) if $A_i \sim A_j$, then $\Pr(A_i \cap A_j) = 0$.

ii) for any set $S \subseteq \{j : A_j \nsim A_i\}$ it holds $\Pr(A_i \mid \cap_{j \in S} \overline{A_j}) \geq (1-\varepsilon)\Pr(A_i)$.

Condition $i)$ implies that only incompatible events can be connected. Condition $ii)$ says that this set of non connected events can not shrink the probability of A_i too much.

Theorem 9 (Lu and Szekely [11]) *Let $A_1, \ldots, A_m$ be events with an ε–near–positive dependency graph H. Then we have,*

$$\Pr(\cap \overline{A_i}) \leq \prod_i (1-(1-\varepsilon)\Pr(A_i)).$$

Observe that this upper bound gives an exponential upper bound of the form $e^{(1-\varepsilon)\mu}$.

Lu and Szekely [11] show also that an ε–near–positive dependency H can be constructed using a family of matchings $\mathcal{M}$. Unfortunately the conditions of [11, Theorem 4] which would provide the upper bound in our case do not apply to our family $\mathcal{M}$ of matchings. We give instead a direct proof for the upper bound which is inspired by their approach.

Lemma 10 *The graph H is an ε–near-positive dependency graph with $\varepsilon = 1-e^{-(2/k+32/k^2)}$.*

Proof Set $\mathcal{M} = \{A_1, \ldots, A_m\}$. The graph H clearly satisfies condition i) in the definition of ε-NDG. For condition ii) we want to show that, for each i and each $T \subseteq \{j \mid ij \notin E(H), j \neq i\}$, we have the inequality

$$\Pr(A_i|B) \geq (1-\epsilon)\Pr(A_i),$$

where $B = \cap_{j\in T}\overline{A_j}$. This is equivalent to show

$$\Pr(B|A_i) \geq (1-\epsilon)\Pr(B).$$

Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be the vertices of the two stable sets of $K_{n,n}$. We may assume that A_i consists of the two edges $a_{n-1}b_{n-1}, a_nb_n$. Then $\{A_j : j \in T\}$ consists of a set of 2–matchings in $K_{n,n} - \{a_{n-1}, a_n, b_{n-1}, b_n\}$. This is the complete bipartite graph $K_{n',n'}$, $n' = n-2$, with an edge–coloring in which each color appears at most n'/k' times, where $k' = k(1+2/(n-2))$. Let us call B' the event B viewed in $K_{n'n'}$ (dashes in notation indicate changing the probability space from random matchings in $K_{n,n}$ to random matchings in $K_{n'n'}$), so that

$$\Pr(B|A_i) = \Pr(B'). \tag{7}$$

Let $C_{r,s}$ denote the 2–matching $a_{n-1}b_r, a_nb_s$, where $r \neq s$. Define $T_{r,s} \subset T$ in such a way that $\{A_j : j \in T_{r,s}\}$ are the 2–matchings in $\{A_j : j \in T\}$ which meet none of the two vertices b_r, b_s. Set $B_{r,s} = \cap_{j\in T_{r,s}}\overline{A_j}$. Let us show that

$$\Pr(B) = \frac{1}{n(n-1)}\sum_{r\neq s}\Pr(B'_{r,s}), \tag{8}$$

where, as before, $B'_{r,s}$ denotes the event $B_{r,s}$ in the probability space of random matchings in $K_{n',n'}$.

We have

$$\Pr(B) = \sum_{r\neq s}\Pr(B\cap C_{r,s}) = \sum_{r\neq s}\Pr(B_{r,s}\cap C_{r,s}),$$

Note that, since none of the matchings involved in B meets vertices in $\{a_{n-1}, a_n, b_{n-1}, b_n\}$, we have, for all r, s, $r \neq s$,

$$\Pr(B_{r,s}|C_{r,s}) = \Pr(B_{r,s}|C_{n-1,n}).$$

Moreover, observe that $\Pr(B_{r,s}|C_{n-1,n}) = \Pr(B'_{r,s})$. Therefore

$$\Pr(B) = \sum_{r\neq s}\Pr(B_{r,s}|C_{r,s})\Pr(C_{r,s}) = \frac{1}{n(n-1)}\sum_{r\neq s}\Pr(B_{r,s}|C_{n-1,n}) = \frac{1}{n(n-1)}\sum_{r\neq s}\Pr(B'_{r,s}),$$

giving equality (8).

From inequality (6) we know that $x'_j = 1-e^{-(1+16/k')p'}$ fulfills the hypothesis (1) of the Local Lemma. We can now use the intermediate inequality (3) of the Lemma with $S = T\setminus T_{r,s}$ to obtain

$$\Pr(B'_{r,s}) = \frac{\Pr(B')}{\Pr(\cap_{j\in S}\overline{A_j})} \leq \Pr(B')\prod_{j\in S}(1-x'_j)^{-1}. \tag{9}$$

By combining (7) with (8) and (9) we get

$$\Pr(B|A_i) \geq \Pr(B)\prod_{j\in S}(1-x'_j). \tag{10}$$

Recall that $S = T\setminus T_{r,s}$ is the set of 2-matchings in $\mathcal{M}'$ that are incident to b_r or b_s. The size of this set can be bounded independently from r and s by

$$|S| \leq 2n'\left(\frac{n'}{k'}-1\right) \leq 2\frac{n^2}{k}$$

With our choice of $x'_j = 1 - e^{-(1+16/k')p'} \leq 1 - e^{-(1+16/k)p}$ (where $p = 1/n(n-1)$) we have

$$\prod_{j\in S}(1 - x'_j) \geq e^{-(1+16/k)p|S|} \geq e^{-(2/k+32/k^2)}.$$

Therefore, by (10),

$$\varepsilon = 1 - e^{-(2/k+32/k^2)},$$

satisfies the conclusion of the Lemma. □

Now we are able to prove Theorem 3.

Proof of Theorem 3 Set $\mathcal{M} = \{A_1, \dots, A_m\}$. By Lemma 10, the graph H is an ε–near–positive dependency graph with $\varepsilon = 1 - e^{-(2/k+32/k^2)}$. It follows from Theorem 9 that the probability of having a rainbow matching is upper bounded by

$$\Pr(\cap_{i\in[m]}\overline{A_i}) \leq \prod_{i\in[m]} (1 - (1-\varepsilon)\Pr(A_i)) \leq e^{-(1-\epsilon)\mu}.$$

By plugging in our value of ε and by using $e^{-(2/k+32/k^2)} \geq 1 - \frac{2}{k} - \frac{32}{k^2}$ we obtain

$$\Pr(\cap_{i\in[m]}\overline{A_i}) \leq e^{-(1-2/k-12/k^2)\mu}.$$

Combining this upper bound with the lower bound obtained in Lemma 8 we obtain

$$\exp\left\{-\left(1+\frac{16}{k}\right)\mu\right\} \leq Pr(\cap\overline{A_i}) \leq \exp\left\{-\left(1-\frac{2}{k}-\frac{12}{k^2}\right)\mu\right\}.$$

This proves the first part of the Theorem.

In particular, since $\mu \leq n/2k$, if $k = \omega(n^{1/2})$ we get

$$\Pr(\cap_{i\in[m]}\overline{A_i}) \leq e^{-\mu}(1 + o(1)),$$

which matches the lower bound obtained in Lemma 8, thus proving the second part of the Theorem. □

3 Random colorings

In this section we will analyze the existence of rainbow matchings when the edge coloring of $K_{n,n}$ is given at random.

Recall that, in the uniform random model URM, each edge of $K_{n,n}$ is given a color uniformly and independently chosen from a set C with s colors, i.e. every possible coloring with at most s colors appears with the same probability.

In the regular random model RRM a coloring is chosen uniformly at random among all colorings of $E(K_{n,n})$ with equitable color classes of size n^2/s. In order to construct a coloring in the RRM we use a complete bipartite graph $H = (A, B)$, where A contains s blocks, each of size n^2/s, representing the colors and B is the set of edges of $K_{n,n}$. Every perfect matching in H gives an equitable coloring of $E(K_{n,n})$. Moreover, every equitable coloring of $E(K_{n,n})$ corresponds to the same number of perfect matchings. Therefore, by selecting a random perfect matching in H with the uniform distribution, all equitable colorings have the same probability.

We established these two models since they simulate the worst situation in all the possible colorings admitted in Theorem 3: the probability for a matching of being rainbow only depends on the size of $|\mathcal{M}|$, and this set has its largest cardinality when there are few colors with a maximum number of occurrences. This means that we have $s = nk$ colors with n/k occurrences each. Observe that in both models the expected size of each color class is also n/k, and in this sense, they are contiguous to the hypothesis of Theorem 3. One can draw an analogy between the URM and the Erdős-Rényi model $G(n, p)$ for random graphs, and also between the RRM and the regular random graph $G(n, d)$.

Proof of Theorem 4 In the URM it is easy to compute the probability of having the rainbow property for a matching. Let M be a random perfect matching, and $K_{n,n}$ provided with a random edge coloring with $s \geq n$ colors. For the indicator variable X_M that M is rainbow we have:

$$\begin{aligned} \Pr(X_M = 1) &= \frac{s}{s} \cdot \frac{s-1}{s} \cdot \frac{s-2}{s} \cdot \ldots \cdot \frac{s-(n-1)}{s} \\ &= \prod_{i=0}^{n-1} \left(1 - \frac{i}{s}\right). \end{aligned} \tag{11}$$

For $s = n$ we can get directly from (11)

$$\Pr(X_M = 1) = \frac{n!}{n^n} \sim e^{-2\mu}.$$

Assume $s > n$. We have, for $0 < x < 1$,

$$(1 - x) = \exp\{\log(1 - x)\} \tag{12}$$

Therefore,

$$\begin{aligned} \Pr(X_M = 1) &= \prod_{i=1}^{n-1} \exp\left\{\log\left(1 - \frac{i}{s}\right)\right\} \\ &= \exp\left\{\sum_{i=1}^{n-1} \log\left(1 - \frac{i}{s}\right)\right\} \\ &\sim \exp\left\{\int_0^n \log\left(1 - \frac{x}{s}\right) dx\right\}. \end{aligned}$$

We use

$$\int_0^t \log(1 - x)\, dx = (t - 1)\log(1 - t) - t$$

By writing $k = s/n$

$$\begin{aligned} \Pr(X_M = 1) &= \exp\left\{\left((k-1)\log\left(\frac{k}{k-1}\right) - 1\right) n\right\} \\ &\sim \exp\left\{-2k\left(1 - (k-1)\log\left(\frac{k}{k-1}\right)\right)\mu\right\}. \end{aligned}$$

since $\mu \sim \frac{n}{2k}$.

It must be stressed that this result is consistent with the ones in Theorem 3. When $k = 1$ we have $\Pr(X_M = 1) = e^{-2\mu}$, while $\lim_{k\to\infty} \Pr(X_M = 1) = e^{-\mu}$. Observe that, in this case, $\mathbb{E}(|\mathcal{M}|)$ is not exactly the same as for a given coloring. This is due to the variance on the number of occurrences of each color, but does not have a significant importance.

To study the property that a random selected matching is rainbow in the RRM we express the equitable edge colorings through permutations $\sigma \in Sym(n^2)$. Then, probability for a matching M of being rainbow is,

$$
\begin{aligned}
\Pr(X_M = 1) &= \frac{n^2}{n^2} \cdot \frac{n^2 - \frac{n^2}{s}}{n^2-1} \cdot \frac{n^2 - 2\frac{n^2}{s}}{n^2-2} \cdot \ldots \cdot \frac{n^2 - (n-1)\frac{n^2}{s}}{n^2-(n-1)} \\
&= \prod_{i=0}^{n-1} \left(1 - \frac{i(n^2-s)}{s(n^2-i)}\right) \\
&= \exp\left\{\sum_{i=0}^{n-1} \log\left(1 - \frac{i(n^2-s)}{s(n^2-i)}\right)\right\} \quad \text{(by (12))} \\
&\sim \exp\left\{\int_0^n \log\left(1 - \frac{x(n^2-s)}{s(n^2-x)}\right) dx\right\}.
\end{aligned}
$$

If $s = n$ we have

$$
\int_0^n \log\left(1 - \frac{x(n-1)}{(n^2-x)}\right) dx = -n(n-1)\log\left(\frac{n}{n-1}\right),
$$

which, by using the Taylor expansion of the logarithm, gives

$$
\Pr(X_M = 1) = e^{-(1+o(1))2\mu}
$$

In the case where $s > n$, and using $k = s/n$, we have

$$
\begin{aligned}
\int_0^n \log\left(1 - \frac{x(n^2-s)}{s(n^2-x)}\right) dx &= \left((k-1)\log\left(\frac{k}{k-1}\right) - (n-k)\log\left(\frac{n}{n-1}\right)\right) n \\
&= \left((k-1)\log\left(\frac{k}{k-1}\right) - 1 + o(1)\right) n.
\end{aligned}
$$

Hence

$$
\Pr(X_M = 1) = \exp\left\{-2k\left(1 - (k-1)\log\left(\frac{k}{k-1}\right) + o(1)\right)\mu\right\}.
$$

□

Note that, for both models of random edge colorings, the probability that a fixed perfect matching is rainbow the same (up to a $o(1)$ term). Thus, in spite of being different models, the probability of having a rainbow matching is similar.

In general it is not true that $\Pr(X_M = 1) = e^{-(1+o(1))\mu}$ but, if $k = \omega(1)$, then $\Pr(X_M = 1) \to e^{-\mu}$ for both models since

$$
2k\left(1 - (k-1)\log\left(\frac{k}{k-1}\right)\right) = 1 + O\left(\frac{1}{k}\right)
$$

This is natural since, when k is large, the number of bad events decreases and the model behaves like in the case they were independent.

Observe that for the two random models we obtain the exact asymptotic value of the probability, while bounds provided by Theorem 3 (when the size $|\mathcal{M}|$ of the set of bad events is maximum) are not sharp, although consistent with the values for the random models. Since the result proven for fixed colorings does only depend on the size of $\mathcal{M}$, the probability for the random model RRM should be exactly the same.

4 Existence of rainbow matchings

The aim of this Section is to prove that *whp* there exists a rainbow matching for a given random coloring of $E(K_{n,n})$ with $s \geq n$ colors. We only consider the URM, but the results can be adapted to the RRM. The number of rainbow matchings is counted by $X = \sum X_M$, which, according to Theorem 4, has expected value

$$\mathbb{E}(X) = n! \Pr(X_M = 1) \sim n! \exp\left\{ -2k \left(1 - (k-1)\log\left(\frac{k}{k-1}\right)\right)\mu \right\}. \tag{13}$$

In order to have a rainbow matching we just need that $X \neq 0$.

Given two perfect matchings M and N, the events that they are rainbow are positively correlated,

$$\Pr(X_M = 1 \mid X_N = 1) \geq \Pr(X_M = 1). \tag{14}$$

Proof of Theorem 5 To show that there exists some rainbow matching *whp* we will use the second moment method. Let X be a random variable with expected value μ and variance σ^2. Then, the Chebyshev inequality asserts that

$$\Pr(|X - \mu| > \alpha\sigma) \leq \frac{1}{\alpha^2} \tag{15}$$

In particular if $\alpha = \frac{\mu}{\sigma}$,

$$\Pr(X = 0) \leq \Pr(|X - \mu| > \mu) \leq \frac{\sigma^2}{\mu^2} \tag{16}$$

Observe that $X = 0$ is equivalent to the non existence of any rainbow matching. Therefore, we need to compute $\sigma^2(X)$ and show that it is asymptotically smaller than $\mathbb{E}(X)^2$. Note that

$$\mathbb{E}(X^2) = \sum_{M,N} \mathbb{E}(X_M X_N)$$

Let M and N two fixed matchings, then

$$\mathbb{E}(X_M X_N) = \Pr(X_M)\Pr(X_{N\setminus M})$$

Given a fixed matching M and a fixed intersection size t, we claim there are at most $e^{-1}\binom{n}{t}(n-t)!$ matchings N, such that $|M \cap N| = t$. There are $\binom{n}{t}$ ways of choosing which edges will be shared and once this edges have been fixed, at most $e^{-1}(n-t)!$ ways of completing the matching. Suppose that $\tau = \sigma_N \mid_{N\setminus M} \in \mathcal{S}_{n-t}$ is the permutation for extending the matching in the disjoint part. Since N has intersection exactly t with M, not any permutation is valid. We can assume *wlog* that M is given by $\sigma_M = Id$, and therefore τ must be a derangement. Classical results state that the proportion of derangements in permutations of any length is at most e^{-1}. This concludes our claim.

Hence,

$$\mathbb{E}(X^2) = e^{-1} n! \sum_{t=0}^{n} \binom{n}{t} (n-t)! \Pr(X_M)\Pr(X_{N\setminus M})$$

Since $\sigma^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2$,

$$\begin{aligned} \frac{\sigma^2(X)}{\mathbb{E}(X)^2} &= \frac{e^{-1} n! \sum_{t=0}^{n} \binom{n}{t}(n-t)! \Pr(X_M)\Pr(X_{N\setminus M})}{(n!\Pr(X_M))^2} - 1 \\ &= e^{-1} \sum_{t=0}^{n} \frac{1}{t!} \frac{\Pr(X_{N\setminus M})}{\Pr(X_M)} - 1 \end{aligned}$$

Given X_M, we know that the edges of $M \cap N$ are rainbow. In the remaining $n-t$ edges to color, we must avoid the t colors that appear in $M \cap N$

$$\Pr(X_N \mid X_M) = \prod_{i=t}^{n-1} \left(1 - \frac{i}{s}\right)$$

Then,

$$f(s) = \sum_{t=0}^{n} \frac{1}{t!} \frac{\Pr(X_{N \setminus M})}{\Pr(X_M)} \sim \sum_{t=0}^{n} \frac{1}{t!} e^{(1+O(1/k))\frac{t^2}{2s}} \sim \sum_{t=0}^{\infty} \frac{1}{t!} e^{(1+O(1/k))\frac{t^2}{2s}}$$

If the number of colors $s = \omega(1)$, then $f(s) \to e$. Observe that $s \geq n$. Otherwise, $\Pr(X_M) = 0$ in the Equation (11).

Hence $\frac{\sigma^2}{\mu^2} \to 0$ and the theorem holds. □

Actually,

$$f(s) = \frac{1}{s} + O(s^{-2})$$

and Equation (16) also provides an upper bound estimation for the probability p that a random coloring has no rainbow matchings of the type

$$p \leq (1 + o(1))\frac{1}{n}$$

Observe that the proportion of Latin squares among the set of square matrices with n symbols is of the order of e^{-n^2} (see e.g. [17]), so that this estimation falls short to prove an asymptotic version of the original conjecture of Ryser.

5 Open Problems

On the ennumeration of Rainbow matchings, it would be interesting to prove exact upper and lower bounds for the case where the number of occurences of each color is at most k, with constant k. Theorem 3 provides exponential upper and lower bounds as long as $k \geq 12$, but both are asymptotically equal if and only if $k = \omega(1)$.

A related problem is to improve the lower bound $k \geq 4e$ given by Erdős and Spencer [4, Theorem 2] for the existence of rainbow matchings.

On the other hand, another really interesting problem is to prove that almost all latin squares have a latin transversal, i.e. the asymptotic version of the Ryser conjecture. We have stablished a probabilistic way to approach the problem. Unfortunately, as far as we know, there are no random models for latin squares. Some results on generating random latin squares can be found in [14, 7]. Nevertheless, there are some almost sure results on Latin squares (see e.g.[12, 3]).

References

[1] Noga Alon, Tao Jiang, Zevi Miller, and Dan Pritikin, *Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints*, Random Structures Algorithms **23** (2003), no. 4, 409–433. MR 2016871 (2004i:05106)

[2] Noga Alon and Joel H. Spencer, *The probabilistic method*, third ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., Hoboken, NJ, 2008, With an appendix on the life and work of Paul Erdős. MR 2437651 (2009j:60004)

[3] Nicholas J. Cavenagh, Catherine Greenhill, and Ian M. Wanless, *The cycle structure of two rows in a random Latin square*, Random Structures Algorithms **33** (2008), no. 3, 286–309. MR 2446483 (2009e:05053)

[4] Paul Erdős and Joel Spencer, *Lopsided Lovász local lemma and Latin transversals*, Discrete Appl. Math. **30** (1991), no. 2-3, 151–154, ARIDAM III (New Brunswick, NJ, 1988). MR 1095369 (92c:05160)

[5] Alan Frieze and Michael Krivelevich, *On rainbow trees and cycles*, Electron. J. Combin. **15** (2008), no. 1, Research paper 59, 9. MR 2398851 (2008m:05106)

[6] Pooya Hatami and Peter W. Shor, *A lower bound for the length of a partial transversal in a Latin square*, J. Combin. Theory Ser. A **115** (2008), no. 7, 1103–1113. MR 2450332 (2009h:05039)

[7] Mark T. Jacobson and Peter Matthews, *Generating uniformly distributed random Latin squares*, J. Combin. Des. **4** (1996), no. 6, 405–437. MR 1410617 (98b:05021)

[8] Svante Janson and Nicholas Wormald, *Rainbow Hamilton cycles in random regular graphs*, Random Structures Algorithms **30** (2007), no. 1-2, 35–49. MR 2283220 (2008i:05171)

[9] Mikio Kano and Xueliang Li, *Monochromatic and heterochromatic subgraphs in edge-colored graphs—a survey*, Graphs Combin. **24** (2008), no. 4, 237–263. MR 2438857 (2009h:05083)

[10] Timothy D. LeSaulnier, Christopher Stocker, Paul S. Wenger, and Douglas B. West, *Rainbow matching in edge-colored graphs*, Electron. J. Combin. **17** (2010), no. 1, Note 26, 5,. MR 2651735

[11] L. Lu and L. A. Szekely, *A new asymptotic enumeration technique: the Lovasz Local Lemma*, ArXiv e-prints (2009).

[12] B. D. McKay and I. M. Wanless, *Most Latin squares have many subsquares*, J. Combin. Theory Ser. A **86** (1999), no. 2, 322–347. MR 1685535 (2000c:05031)

[13] Brendan D. McKay, Jeanette C. McLeod, and Ian M. Wanless, *The number of transversals in a Latin square*, Des. Codes Cryptogr. **40** (2006), no. 3, 269–284. MR 2251320 (2007f:05030)

[14] Brendan D. McKay and Nicholas C. Wormald, *Uniform generation of random Latin rectangles*, J. Combin. Math. Combin. Comput. **9** (1991), 179–186. MR 1111853 (92b:05013)

[15] H.J. Ryser, *Neuere probleme der kombinatorik*, Vorträgeüber Kombinatorik, Oberwolfach (1967).

[16] S. K. Stein, *Transversals of Latin squares and their generalizations*, Pacific J. Math. **59** (1975), no. 2, 567–575. MR 0387083 (52 #7930)

[17] J. H. van Lint and R. M. Wilson, *A course in combinatorics*, second ed., Cambridge University Press, Cambridge, 2001. MR 1871828 (2002i:05001)