b-chromatic number of cacti

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Abstract

A b-colouring of a graph G is a proper colouring of G such that each colour contains a vertex that is adjacent to all other colours and the b-chromatic number $\chi_b(G)$ is the maximum number of colours used in a b-colouring of G. If $m(G)$ is the largest integer k such that G has at least k vertices with degree at least $k-1$, then we know that $\chi_b(G) \leq m(G)$. Irving and Manlove [1] prove that, if T is a tree, then the b-chromatic number of T is at least $m(T) - 1$. In this paper, we prove that, if G is a connected cactus and $m(G) \geq 7$, then the b-chromatic number of G is at least $m(G) - 1$.

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1 Introduction

Let G be a simple graph. A *proper coloring of* G is an assignment of colors to the vertices of G such that no two adjacent vertices have the same color. In a proper coloring, the color class of a color c is the set of vertices of G colored with color c. The *chromatic number of* G is the minimum integer $\chi(G)$ such that G has a proper coloring with $\chi(G)$ colors. Suppose that we have a proper coloring of G and a color class C such that every vertex in C is not adjacent to at least one other color class. We can change the color of the vertices in C, obtaining a proper coloring that uses less colors than before. However, as one can expect, we cannot apply this heuristic iteratively until we reach the chromatic number of G, since the coloring problem is $\mathcal{NP}$-hard. It was this idea that made Irving and Manlove introduce the notion of b-coloring in [1]. Intuitively, a b-coloring is a proper coloring that cannot be improved by the described heuristic and the b-chromatic number $\chi_b(G)$ of G measures the worst possible such coloring. More formally:

Definition 1.1 A vertex u in color class C is said to be a b-vertex if u has at least one neighbor in each color class other than C.

Definition 1.2 A b-coloring of a graph G is a proper coloring of G such that each color class contains at least one b-vertex.

Definition 1.3 The b-chromatic number of G is the largest integer k such that G has a b-coloring with k colors. We denote it by $\chi_b(G)$.

Naturally, we have that a proper coloring of G with $\chi(G)$ colors is a b-coloring of G, since it cannot be improved. So, $\chi(G) \leq \chi_b(G)$. For an upper bound, note that if G has a b-coloring with k colors, then G has at least k vertices with degree at least $k-1$ (the b-vertices). So, if $m(G)$ is the largest integer such that G has at least $m(G)$ vertices with degree at least $m(G)-1$, we know that G cannot have a b-coloring with more than $m(G)$ colors, i.e., $\chi_b(G) \leq m(G)$. This upper bound was introduced by Irving and Manlove in [1], where they also showed that the difference between $\chi_b(G)$ and $m(G)$ can be arbitrarily large for a general graph and that it is at most one for trees. In addition, the problem was proved to be $\mathcal{NP}$-hard [1], even when restricted to bipartite graphs [2].

We say that G is a cactus if G does not contain two cycles that share an edge. In this article, we prove that, if G is a connected cactus and $m(G) \geq 7$, then the difference between $\chi_b(G)$ and $m(G)$ is at most one and we can obtain $\chi_b(G)$ in polynomial time.

We say that G is *m-defective* if $m(G) = m$ and $\chi_b(G) < m$. Additionally, G is *minimal m-defective* if G is m-defective and any proper subgraph of G is not m-defective.

The general idea for our main result is as follows. We present the class of pivoted cacti and show that every pivoted cactus G is $m(G)$-defective. Then, we show that $\chi_b(G) = m(G)-1$, for every pivoted cactus G. Finally, we prove that a minimal $m(G)$-defective cactus is pivoted. To do so, first we describe the structure of minimal $m(G)$-defective cacti. We say that a subgraph H of G is a *b-kernel of* G if $m(H) = m(G)$ and $\chi_b(H) = m(G)$ implies $\chi_b(G) = m(G)$. If G is not a pivoted cactus, we find a special b-kernel H_G of G and we either show how to b-color H_G with $m(G)$ colors or show that H_G contains no minimal $m(G)$-defective cactus as a subgraph. In any case, this implies that $\chi_b(H_G) = m(G)$ and, by the definition of a b-kernel, $\chi_b(G) = m(G)$. This proves that if G is not a pivoted cactus, then it is not $m(G)$-defective. More details on this general idea are provided in the remainder of this paper.

2 Pivoted Cacti

In this section, we define the class of pivoted cacti and show that if G is a pivoted cactus, then $\chi_b(G) = m(G)-1$. This implies that every pivoted cactus G is $m(G)$-defective.

Let $G = (V, E)$ be a connected cactus. We say that a vertex v in V is a *dense vertex* if $d(v) \geq m(G) - 1$. Let $M(G)$ denote the set of dense vertices of G.

Let V' be any subset of $m(G)$ vertices of $M(G)$. Let $u \in V \setminus V'$ and $v \in V'$. If u and v are adjacent or have a common neighbor w in V' with $d(w) = m(G) - 1$, then we say that *v is reachable from u within V'*. Note that if $u \in V \setminus V'$ reaches every vertex of V', then there is no b-colouring of G with V' as the set of b-vertices. Observe that it also happens if we have a pair of vertices $u, v \in V \setminus V'$ that reaches every vertex of V' but one, say w: we could colour u or v with the colour of w, but the other one cannot be coloured without repeating some colour in the neighborhood of a vertex of V' with degree $m(G)-1$. So, we say that *V' encircles vertex u in $V \setminus V'$* if every vertex v in V' is reachable from u within V'. Below, we describe the situations where we have a pair that prevent V' from being a set of b-vertices of some b-colouring of G. We say that *V' encircles the pair $x, y \in V \setminus V'$* if it does not encircle x or y and if one of the following occurs:

(E1) There are $V'' \subset V'$ and $u, v \in V''$ such that $|V''| = m(G) - 1$, $\langle x, u, y, v \rangle$

is a cycle and:

(a) $d(u) = d(v) = m(G) - 1$ and every $w \in V'' \setminus \{u, v\}$ is adjacent to u or v; or

(b) $d(u) = m(G) - 1$ and every $w \in V'' \setminus \{u, v\}$ is adjacent to u; or

(c) $d(u) = m(G)$, $d(v) = m(G) - 1$ and every $w \in V' \setminus \{u, v\}$ is adjacent to u or v; or

(d) $d(u) = m(G)$ and every $w \in V' \setminus \{u, v\}$ is adjacent to u.

(E2) There are $V'' \subseteq V'$ and $u, v, w \in V''$ such that $|V''| \geq m(G) - 1$, $\langle x, u, v, y, w \rangle$ is a cycle, $d(u) = d(v) = m(G) - 1$, every $w' \in V'' \setminus \{u, v, w\}$ is adjacent to w, and

(a) $V'' = V'$ and $d(w) = m(G)$; or

(b) $V'' \subset V'$ and $d(w) = m(G) - 1$.

Let V' be a subset of $m(G)$ vertices of $M(G)$. We say that V' is a *good set* if it does not encircle any vertex or pair of vertices and every $u \in V \setminus V'$ with degree at least $m(G)$ is either adjacent to some vertex in V' with degree $m(G) - 1$ or is within a path between two vertices of V' of length at most three, whose internal vertices are not in V'. If G does not have a good set, we say that G is a *pivoted cactus*.

The following lemma shows the possible number of encircled vertices or encircled pairs.

Lemma 2.1 *Let V' be any set of $m(G)$ dense vertices. If $m(G) \geq 7$, then V' encircles at most two vertices, or at most one pair of vertices.*

By using the structural properties presented in Lemma 2.1, we can prove that if $|M(G)| = m(G)$ and situation E1 or E2 occurs (in which case, we know that $\chi_b(G) < m(G)$), then we can b-color G with $m(G) - 1$ colors. However, the following lemma shows us that there is a situation where G has more than $m(G)$ dense vertices and still cannot be b-colored with $m(G)$ colors.

Lemma 2.2 *Let G be a connected cactus with $|M(G)| > m(G)$, $m(G) \geq 7$, and let V' be a set of $m(G) + 1$ dense vertices of G containing all vertices with degree greater than $m(G) - 1$. Then, G does not have a good set if and only if $V' = M(G)$ and there are vertices $u, v \in V'$ and $w \in V \setminus V'$ such that $d(u) = d(v) = m(G) - 1$, $\{u, v, w\}$ forms a triangle in G and every vertex in V' is adjacent to u or to v.*

Again, we can use the structural properties presented in Lemma 2.2 to prove the following lemma by giving a b-coloring of G with $m(G) - 1$ colors.

Lemma 2.3 *If G is a pivoted connected cactus, then $\chi_b(G) = m(G) - 1$.*

3 Non pivoted Cacti

In this section, we prove that if a cactus G has a good set, then it can be b-coloured with $m(G)$ colours, giving us the desired result. To do so, we first describe the structure of minimal m-defective cacti.

Theorem 3.1 *If G is a minimal $m(G)$-defective cactus and $m(G) \geq 7$, then $|M(G)| = m(G)$ and $d(v) = m(G) - 1$, for all $v \in M(G)$. Furthermore, if $u \in V(G) \setminus M(G)$, then any neighbor of u is a dense vertex of G.*

Theorem 3.2 *If G is a minimal $m(G)$-defective cactus with $m(G) \geq 7$, then either $M(G)$ encircles a vertex or $M(G)$ encircles a pair of vertices as in E1.a or E2.b.*

To link minimal m-defective cacti with non-pivoted cacti, note that if G has no minimal $m(G)$-defective subgraph, then G is not defective. This implies that $\chi_b(G) = m(G)$. We also use the following lemma.

Lemma 3.3 *If G is a non-pivoted cactus and V' is a good set of G, then $G[V' \cup N(V')]$ is a b-kernel of G.*

Now, let G be a non-pivoted cactus and let V' be a good set of G. Consider the b-kernel $H_G = G[V' \cup N(V')]$ of G. If H_G contains no minimal $m(G)$-defective subgraph, then H_G is not $m(G)$-defective itself and we have $\chi_b(H) = m(G)$. Since H_G is a b-kernel, then $\chi_b(G) = m(G)$. On the other hand, if H_G has a minimal $m(G)$-defective subgraph, then using the structural properties given by Theorem 3.2 and the fact that V' does not encircle a vertex or pair of vertices in H_G, it is possible to construct a b-colouring of G with $m(G)$ colours. So, the following theorem is true:

Theorem 3.4 *If G is a non-pivoted cactus with $m(G) \geq 7$, then $\chi_b(G) = m(G)$.*

4 Closing Comments

Our main result is the following theorem:

Theorem 4.1 *If G is a connected cactus with $m(G) \geq 7$, then $m(G) - 1 \leq \chi_b(G) \leq m(G)$.*

In addition, we found an algorithm that gives an optimum b-coloring of a pivoted cactus and an algorithm to find a good set of a non-pivoted cactus. We also have an algorithm to extend a maximum b-coloring of the b-kernel H_G

into a maximum b-coloring of G. However, we could not find an algorithm that optimally colors H_G and the proof that $\chi_b(H_G) = m(G)$ is not constructive. Thus, we can decide in polynomial time the value of $\chi_b(G)$, but we cannot always provide such coloring.

References

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