Wilson-Schreiber Colourings of Cubic Graphs

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Abstract

An $\mathcal{S}$-colouring of a cubic graph G is an edge-colouring of G by points of a Steiner triple system $\mathcal{S}$ such that the colours of any three edges meeting at a vertex form a block of $\mathcal{S}$. In this note we present an infinite family of point-intransitive Steiner triple systems $\mathcal{S}$ such that (1) every simple cubic graph is $\mathcal{S}$-colourable and (2) no proper subsystem of $\mathcal{S}$ has the same property. Only one point-intransitive system satisfying (1) and (2) was previously known.

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1 Introduction

A *Steiner triple system* $\mathcal{S} = (V, B)$ of order v consists of a set V of v elements, called *points*, and a collection B of 3-element subsets of V, called *blocks*, such that every 2-element subset of V is contained in exactly one block. It is well known that a Steiner triple system of order v exists if and only if $v \equiv 1$ or 3 (mod 6) (see Kirkman [5]).

Given a Steiner triple system $\mathcal{S}$, an $\mathcal{S}$*-colouring* of a cubic graph G is an edge-colouring of G by points of $\mathcal{S}$ such that the colours of any three edges meeting at a vertex form a block of $\mathcal{S}$. This kind of colouring was introduced by Archdeacon [1] in 1986 and later studied by a number of authors (for example, see [2,4,6,8]). Interesting connections between Steiner colourings and several difficult conjectures, such as the cycle double cover conjecture and the Fulkerson conjecture, are discussed in [1,7,8].

One of the questions that naturally arise in this area is whether a given Steiner triple system $\mathcal{S}$ is *universal*, that is, whether every simple cubic graph admits an $\mathcal{S}$-colouring. Somewhat surprisingly, the best known geometric examples of Steiner triple systems, the projective systems $PG(n, 2)$ and the affine systems $AG(n, 3)$, include no universal member [4]. The first universal Steiner triple system (of order 381) was found by Grannell et al. [2]. Pál and Škoviera [9] improved this result by identifying a subsystem of the previous system of order 21 that is also universal. Further significant progress was made by Král' et al. [6] who proved that every non-trivial point-transitive Steiner triple system that is neither projective nor affine is universal. In particular, the smallest order of a universal system is 13. In contrast, very little is known about colourings by point-intransitive Steiner triple systems. In fact, only one universal point-intransitive system is currently known [6].

In this note we describe an infinite family of point-intransitive universal Steiner triple systems based on the Wilson-Schreiber construction [10,11]. The smallest member of the family has order 15. Infinitely many of these systems are minimally universal, that is, they do not contain a proper universal subsystem. A detailed discussion and proofs will appear in a further paper [3].

2 Wilson-Schreiber Systems and Colourings

Let A be an Abelian group of order n, written additively. We construct a Steiner triple system $\mathcal{S}$ of order $v = n + 2$ whose points are the elements of A and two additional points α and β. The construction applies only when, for every prime divisor p of n, the order of $-2 \pmod p$ is even; we call such a group *admissible*. Since v is the order of a Steiner triple system, we have $v \equiv 1$ or $3 \pmod 6$, so $n \equiv 1$ or $5 \pmod 6$, and therefore neither 2 nor 3 divides n.

Let us list all unordered triples $\langle a, b, c\rangle$ of elements of A with $a+b+c=0$ and repetitions allowed. For each triple $\langle a, b, c\rangle$ with pairwise distinct entries we include the set $\{a, b, c\}$ as a block of $\mathcal{S}$. The triples of the form $\langle a, a, -2a\rangle$ where $a \in A-0$ can be partitioned into orbits under the action of the mapping $z \mapsto -2z$, $z \in A$. Since A is admissible, the number of triples in each orbit is even. Pick one of the orbits and replace the repeated element in each triple by α and β alternately along the orbit. Process each orbit similarly, and include all sets $\{\alpha, a, -2a\}$ and $\{\beta, b, -2b\}$ obtained in this way as blocks of $\mathcal{S}$. Finally, replace the triple $\langle 0, 0, 0\rangle$ with $\{0, \alpha, \beta\}$ and include it in $\mathcal{S}$.

It is easy to see that, with the above collection of blocks, $\mathcal{S}$ is a Steiner triple system. Since there exist infinitely many primes p such that -2 has even order $\pmod p$, there are infinitely many such Wilson-Schreiber systems. Furthermore, it can be shown that the systems constructed from the prime groups $\mathbb{Z}_p$ do not contain any non-trivial proper subsystem [3].

Every Wilson-Schreiber system $\mathcal{S}$ constructed from an admissible group A of order greater than 9 is *point-intransitive*, that is, it contains two points that cannot be mapped onto each other by an automorphism of the system. This follows from the fact that the number of mitres having 0 as an apex differs from the number having $x \in A - 0$ as an apex, where a *mitre* is partial subsystem of $\mathcal{S}$ having the form $\{\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, f\}, \{c, e, g\}\}$ and the *apex* of the mitre is the point a.

Our main result is the following theorem.

Theorem 2.1 *Let $\mathcal{S}$ be a Wilson-Schreiber system obtained from an admissible Abelian group of order greater than 9. Then $\mathcal{S}$ is universal.*

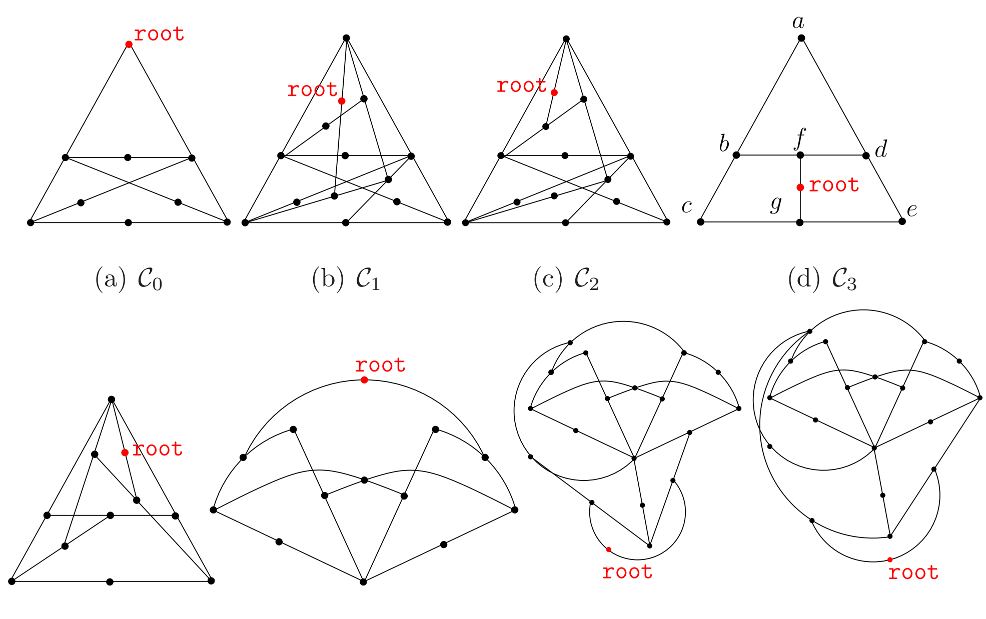
To show that $\mathcal{S}$ is universal we employ a sufficient condition based on the existence of certain substructures in $\mathcal{S}$. A *rooted configuration* is a configuration $\mathcal{C}$ of points and 3-element blocks with one distinguished point, the *root*. A *rooted homomorphism* of $\mathcal{C}$ into $\mathcal{S}$ is a homomorphism $\mathcal{C} \to \mathcal{S}$ such that the root of $\mathcal{C}$ is mapped to a given point of $\mathcal{S}$.

The following result, based on ideas from [4] and [6], will be proved in [3].

Theorem 2.2 *Let P be a set of points of a Steiner triple system $\mathcal{S}$. Suppose that for every configuration $\mathcal{C}_i \in \mathbf{U} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_7\}$ (see* Fig. 1*) and for every point $y \in P$ there exists a rooted homomorphism $\mathcal{C}_i \to \mathcal{S}$ taking the points of $\mathcal{C}_i$ to P and the root to y. Then $\mathcal{S}$ is universal.*

Sketch of proof of Theorem 2.1. First observe that every non-trivial subgroup of an admissible group is admissible, and that the Wilson-Schreiber system constructed from a subgroup is a subsystem of that constructed from the whole group. By the classification of finite Abelian groups, and since $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_{11}$ are not admissible groups, it suffices to prove the result for all admissible cyclic groups of prime order $p \geq 13$ as well as for three individual groups $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_7$, and $\mathbb{Z}_7 \times \mathbb{Z}_7$.

Let $p \geq 13$ be a prime, and let $\mathcal{S}$ be a Wilson-Schreiber system based on $\mathbb{Z}_p$. To apply Theorem 2.2, we take $P = (\mathbb{Z}_p - 0) \cup \{\alpha, \beta\}$ and define $\mathcal{D}$ to be the partial subsystem of $\mathcal{S}$ induced by the points from P. For each $\mathcal{C}_i$ we construct two particular rooted homomorphisms $\mathcal{C}_i \to \mathcal{D}$, one taking the root to α or β and the other taking the root to some element of $\mathbb{Z}_p - 0$. All other rooted homomorphisms $\mathcal{C}_i \to \mathcal{D}$ required by Theorem 2.2 are then obtained from these two by applying automorphisms of $\mathcal{S}$. As an example, in Table 1 we display the two homomorphisms for the configuration $\mathcal{C}_3$. The remaining configurations, as well as the three small groups will be dealt with in [3].



(a) $\mathcal{C}_0$ (b) $\mathcal{C}_1$ (c) $\mathcal{C}_2$ (d) $\mathcal{C}_3$

(e) $\mathcal{C}_4$ (f) $\mathcal{C}_5$ (g) $\mathcal{C}_6$ (h) $\mathcal{C}_7$

Fig. 1. Set of configurations **U** from Theorem 2.2

group	`root`	a	b	c	d	e	f	g
$\mathbb{Z}_p$, $p \geq 13$	-2	1	-3	2	-4	3	7	-5
	β	1	2	-3	-6	5	4	-2

Table 1

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