

Kempe changes in degenerate graphs

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Abstract

We consider Kempe changes on the k -colorings of a graph on n vertices. If the graph is $(k - 1)$ -degenerate, then all its k -colorings are equivalent up to Kempe changes. However, the sequence between two k -colorings that arises from the proof may be exponential in the number of vertices. An intriguing open question is whether it can be turned polynomial. We prove this to be possible under the stronger assumption that the graph has treewidth at most $k - 1$. Namely, any two k -colorings are equivalent up to $O(kn^2)$ Kempe changes. We investigate other restrictions (list coloring, bounded maximum average degree, degree bounds). As a main result, we derive that given an n -vertex graph with maximum degree Δ , the Δ -colorings are all equivalent up to $O(n^2)$ Kempe changes, unless $\Delta = 3$ and some connected component is a 3-prism.

Keywords: reconfiguration, coloring, graph theory, treewidth

Given a k -colored graph, a *Kempe chain* is a connected component in the subgraph induced by two given colors. A Kempe change consists in swapping the two colors in a Kempe chain, thereby obtaining a new k -coloring of the graph. Two k -colorings of a graph are *Kempe equivalent* if one can be obtained from the other through a series of Kempe changes. This elementary operation on the k -colorings of a graph was introduced by Kempe in 1879, in an unsuccessful attempt to prove the four color theorem [Kem79].

The study of Kempe changes has a vast history, see e.g. [Moh06] for a comprehensive overview or [BBFJ19] for a recent result on general graphs. We refer the curious reader to the relevant chapter of a 2013 survey by Van Den Heuvel [vdH13]. Kempe equivalence falls within the wider setting of combinatorial reconfiguration, which [vdH13] is also an excellent introduction to. Beyond the intrinsic interest of theoretical results, the study of Kempe changes is motivated by practical applications in statistical physics and approximate counting of colorings (see e.g. [Sok00, MS09] for nice overviews). Closer to graph theory, Kempe equivalence can be studied with a goal of obtaining a random coloring by applying random walks and rapidly mixing Markov chains, see e.g. [Vig00].

Kempe changes were introduced as a mere tool, and are decisive in the proof of Vizing's edge coloring theorem [Viz64]. However, the equivalence class they define on the set of k -colorings is itself highly interesting. In which cases is there a single equivalence class? In which cases does every equivalence class contain a coloring that uses the minimum number of colors? Vizing conjectured in 1965 [Viz68] that the second scenario should be true in every line graph, no matter the choice of k .

Our main interest is the study of the reconfiguration graph $R^k(G)$, whose vertices are the k -colorings of G and in which two colorings are adjacent if and only if they differ by one Kempe change. We are interested in the following questions: in which setting is the reconfiguration

graph connected, that is, any two k -colorings are Kempe equivalent? When this is the case, can we bound the length of the shortest sequence of Kempe changes between any two colorings, i.e. the diameter of the reconfiguration graph?

A graph is said to be d -degenerate if all its (non-empty) subgraphs contain a vertex of degree at most d . Note that if G has maximum degree Δ then it is trivially Δ -degenerate, and even $(\Delta - 1)$ -degenerate if G is not regular. Las Vergnas and Meyniel [LM81] showed in 1981 that there exists a sequence of Kempe changes between any two k -colorings of a d -degenerate graph G when $k \geq d + 1$. In other words, the corresponding reconfiguration graph is connected. However, the sequence they provided may have exponential length.

Reconfiguration restricted to trivial Kempe changes – Kempe changes involving only one vertex – is another well-studied topic, known as vertex recoloring. The lemma of Las Vergnas and Meyniel echoes a result proved by Cereceda [Cer07] in the setting of vertex recoloring: all the k -colorings of a d -degenerate graph are equivalent up to trivial Kempe changes when $k \geq d + 2$. However, the sequence between two k -colorings that arises from the corresponding proof may once again be exponential in the number n of vertices. Cereceda conjectured that there exists one of length $O(n^2)$. In a breakthrough paper, Bousquet and Heinrich proved that there exists a sequence of length $O(n^{d+3})$ [BH19]. However, Cereceda’s conjecture remains open, even for $d = 2$.

Obtaining similar bounds with regular Kempe changes on d -degenerate graphs with one fewer color would have many consequences, as the lemma of Las Vergnas and Meyniel is used as a base ground in several proofs. Unfortunately, the bound of Bousquet and Heinrich [BH19] does not extend to this setting, and even a polynomial upper-bound on the number of changes would be highly interesting. With this in mind, we prove three polynomial bounds on the diameter of the reconfiguration graph in closely related settings.

Kempe equivalence of Δ colorings

Any graph G can be greedily colored with $(\Delta + 1)$ colors, where Δ is the maximum degree of G . Brooks’ theorem states that if G is not a clique or an odd cycle, then Δ colors suffice. Many different proofs of this theorem exist, see [CR14] for a collection of proofs of Brooks’ theorem using various techniques – for that matter, some of them using Kempe changes.

In the more restrictive setting of Brooks’ theorem, Mohar [Moh06] conjectured that all the k -colorings of a graph are Kempe equivalent for $k \geq \Delta$. Note that the result of Las Vergnas and Meyniel [LM81] settles the case of non-regular graphs. Feghali *et al.* [FJP17] proved that the conjecture holds for all cubic graphs but the 3-prism (see Fig. 1). The conjecture does not hold for the 3-prism, as can be seen through the lenses of *frozen* colourings. A colouring is frozen if any bichromatic subgraph is connected, hence performing any Kempe change leaves the color partition unchanged. Since the 3-prism admits two frozen 3-colorings with different color partitions, they are not Kempe equivalent. To our knowledge, this is the only argument at our disposal to prove that a reconfiguration graph is disconnected.



Figure 1: Two frozen 3-colorings of the 3-prism

Bonamy *et al.* [BBFJ19] later showed that the conjecture also holds for Δ -regular graphs with

$\Delta \geq 4$. Both paper heavily rely on the lemma of Las Vergnas and Meyniel and provide sequences that possibly have exponential length.

In Section 3, we give a polynomial upper bound on the diameter of the reconfiguration graph in this setting.

Theorem 0.1. *All the k -colorings of an n -vertex graph G with maximum degree at most $\Delta \leq k$ are equivalent up to $O(n^2)$ Kempe changes, unless $k = 3$ and G is the 3-prism.*

The main idea of the proof is to improve the result of Las Vergnas and Meyniel [LM81] by showing that there exists a sequence of Kempe changes of length $O(n^2)$ between any two k -colorings of a d -degenerate graph when $k \geq d + 1$, under the additional assumption that all the vertices but one have degree at most $d + 1$ (see Section 2).

Kempe equivalence in graphs of bounded mad

The maximum average degree of a graph G is a measure of the sparsity of G , defined as the maximum of the

$$\text{mad}(G) = \max_{\emptyset \neq H \subseteq G} \frac{2|E(H)|}{|V(G)|}.$$

For all $d \geq 1$, if a graph has mad strictly less than d , then it is $(d - 1)$ -degenerate: all its subgraph have average degree less than d , so admit a vertex of degree at most $d - 1$.

We prove that the mad of a graph, while related to its degeneracy, proves to be easier to work with in this setting.

Theorem 0.2. *Let G be a graph with $\text{mad}(G) \leq k - \varepsilon$. All the k -colorings of G are Kempe equivalent up to $O(\text{Poly}_\varepsilon(n))$ Kempe changes.*

We prove the above theorem by giving an upper bound on the number of Kempe changes when lists are involved (see Section 2), and by adapting ideas developed by Bousquet and Perarnau in [BP16] in the setting of single vertex recoloring (see Section 4).

Kempe equivalence in bounded treewidth graphs

Another way to strengthen the degeneracy assumption involves the treewidth of a graph (the treewidth is a graph parameter that measures how close a graph is from being a tree, see Section 1 for a definition). A graph of treewidth k is k -degenerate, while there are 2-degenerate graphs with arbitrarily large treewidth. Bonamy and Bousquet [BB18] confirmed Cereceda's conjecture for graphs of treewidth k . In Section 5 we extend this result to non-trivial Kempe changes with one fewer color.

Theorem 0.3. *Let tw and n be to integer. Any two k -colorings of an n -vertex graph G with treewidth tw are equivalent up to $O(\text{tw} n^2)$ Kempe changes, when $k \geq \text{tw} + 1$.*

Additionally, the proof of Theorem 0.3 is constructive and yields an algorithm to compute such a sequence in time $f(k) \cdot \text{Poly}(n)$. Given a witness that the graph has treewidth at most k , the complexity drops to $k \cdot \text{Poly}(n)$.

1 Preliminaries

A k -coloring of a graph G is a map $\alpha: V(G) \rightarrow [k]$ such that for every edge $uv \in E(G)$, $\alpha(u) \neq \alpha(v)$. Given a k -coloring α of a graph G and $c \in [k]$, we denote $K_{u,c}(\alpha, G)$ the connected component of the subgraph of G induced by the colors c and $\alpha(u)$, that contains u . With a slight

abuse of notation, we will also use $K_{u,c}(\alpha, G)$ to denote the corresponding Kempe change. We may drop the parameter G when there is no ambiguity. When describing algorithms, we will denote $\tilde{\alpha}$ (or $\tilde{\beta}$, etc) the current coloring, obtained from the original coloring α (or β , etc). We denote $\mathcal{C}^k(G)$ the set of k -colorings of G and $R^k(G)$ the reconfiguration graph, whose vertex set is $\mathcal{C}^k(G)$ and in which two colorings are adjacent if they differ by one Kempe change.

We denote $N(u)$ and $N[u]$ the open and closed neighborhoods of a vertex u , respectively. Given an ordering $v_1 \prec \dots \prec v_n$ of the vertices of G , we denote $N^+(v_i) = \{v_j \in N(v_i) \mid j > i\}$ and $N^-(v_i) = \{v_j \in N(v_i) \mid j < i\}$.

A graph G is *d-degenerate* if all its (non-empty) subgraphs contain a vertex of degree at most d . This is equivalent to admitting a *d-degeneracy sequence*: an ordering of the vertices such that for all $v \in V$, $N^+(v)$ is of size at most d . A d -degenerate graph is trivially $(d+1)$ -colorable, by doing a decreasing induction on the vertices. We will extensively use the lemma of Las Vergnas and Meyniel:

Lemma 1.1 ([LM81]). *All k -colorings of a d -degenerate graph are Kempe equivalent when $k \geq d+1$.*

For completeness, we include a proof of it.

Proof. We proceed by induction on the number of vertices. Let G be a d -degenerate graph on n vertices and $k \geq d+1$. Since G is d -degenerate, there exists a vertex v of degree at most d . Let $G' = G \setminus \{v\}$. Let α and β be two k -colorings of G . We claim that we can apply a series of Kempe changes on α in G so as to obtain a coloring γ whose restriction to G' is equal to $\beta|_{G'}$. Assuming this claim, if $\gamma(v) \neq \beta(v)$, then neither $\gamma(v)$ nor $\beta(v)$ appears in $\gamma(N(v))$ since $\gamma|_{G'} = \beta|_{G'}$. As a result, performing the trivial Kempe change $K_{v,\beta(v)}(\gamma, G)$ yields β .

It remains only to prove the above claim. By the induction hypothesis, all k -colorings of G' are Kempe equivalent, so there exists a sequence S of Kempe changes in G' leading from $\alpha|_{G'}$ to $\beta|_{G'}$. Our goal is to extend S to a series of Kempe changes in G . Each Kempe change $K_{u,c}(\tilde{\alpha}, G')$ in S is either applied directly or preceded by an appropriate trivial Kempe change, as follows. If $\tilde{\alpha}(v) \notin \{c, \tilde{\alpha}(u)\}$ or no neighbor of v belongs to the corresponding chain, we directly apply the same change in G . Otherwise, the issue is that adding v to G' might result in merging two connected bichromatic components of G' . Assume from now on without loss of generality that $\tilde{\alpha}(v) = \tilde{\alpha}(u)$.

If $K_{u,c}(\tilde{\alpha}, G)$ and $K_{u,c}(\tilde{\alpha}, G')$ agree except on v (in particular, if v has at most one neighbor of color c), we also apply directly the same Kempe change. Indeed, the change will impact the color of v in G but the restriction to G' behaves as expected. Therefore, we can assume that v has at least two neighbors colored c . Since v has degree at most d and $k \geq d+1$, there exists a color c_v that is not used in $N[v]$. After the trivial Kempe change $K_{v,c_v}(\tilde{\alpha}, G)$, the vertex v is colored c_v . We can now apply the desired Kempe change.

This concludes the proof of the claim, hence of the lemma. ■

A graph H is *chordal* if every induced cycle is a triangle. Equivalently, there is an ordering of the vertices such that $N^+[v]$ is a clique for any vertex v . As a consequence, the chromatic number $\chi(H)$ of a chordal graph H is equal to the size $\omega(H)$ of a largest clique in H . Note that we also have $\omega(H) = d+1$ where d is the degeneracy of H .

Proposition 1.2 ([BHI⁺20]). *Given an n -vertex chordal graph G and an integer p , any two p -colorings of G are equivalent up to at most n Kempe changes.*

The proof presented in [BHI⁺20] for Proposition 1.2 is constructive and the corresponding algorithm runs in linear time.

The *treewidth* of a graph measures how much a graph “looks like” a tree. Out of the many equivalent definitions of treewidth, we use the following: a graph G has treewidth at most k if

there exists a chordal graph H such that G is a (not necessarily induced) subgraph of H , with $\omega(H) = \chi(H) \leq k + 1$. For example, if G is a tree, we note that G is chordal and 2-colorable. By taking $H = G$, we derive that G has treewidth at most 1. If G has treewidth at most k , there is a chordal graph H with $\omega(H) = k + 1$ that admits G as a subgraph. Therefore, H and thus G are k -degenerate. The converse does not hold, as rectangular grids have degeneracy 2 yet unbounded treewidth.

2 Kempe recoloring with list assignments

Let G be a graph and $L : V(G) \rightarrow \mathcal{P}(\mathbb{N})$ be a list assignment. A coloring α of G is an L -coloring if $\alpha(u) \in L(u)$ for every vertex u . Kempe changes can be defined as before, although the resulting coloring may not be an L -coloring. Possible obstructions will be defined later on.

We strengthen the lemma of Las Vergnas and Meyniel under two different additional assumption:

Proposition 2.1. *Let G be a graph and let $v_1 \prec \dots \prec v_n$ be an ordering of $V(G)$.*

1. *If the ordering yields a $(d - 1)$ -degeneracy sequence and $\deg(v_i) \leq d$ for every $i < n$, then any two k -colorings of G are Kempe equivalent up to $O(n^2)$ Kempe changes, for $k \geq d$. More precisely, the color of each vertex is changed a linear number of times.*
2. *Given a list assignment L of G , if $|L(v_i)| \geq \deg(v_i) + 1$ for all $i < n$ then any two L -colorings of G are equivalent up to $O(n)$ Kempe changes.*

Let α be a L -coloring of G , v a vertex and a color $c \in L(v) \setminus \alpha(N^+(v))$. Let $K = K_{v,c}(\alpha, G)$. We say that a vertex $u \in K \setminus \{v\}$ is *blocking* the Kempe change if c or $\alpha(v)$ does not belong to $L(u)$. If a Kempe chain admits at least one blocking vertex, then we say it is *not feasible*. To further analyze how possible obstructions can appear, we introduce the following definitions. We say that a vertex $u \in K \setminus \{v\}$ is

- *branching* if u has degree at least 3 in the Kempe chain K ,
- *problematic* if u has at least two greater neighbors in the Kempe chain K .

We say that u is a *bad vertex* for K if it is either branching, problematic or blocking K . It is a *first bad vertex* if there exists a path from v to u in K that contains no other bad vertex than u . Note that a first bad vertex is necessarily smaller than v .

We use the same algorithm to prove both parts of Proposition 2.1.

2.1 Recoloring $(d - 1)$ -degenerate graphs with an additional assumption on the degree

Assume that G is $(d - 1)$ -degenerate, with corresponding degeneracy sequence $v_1 \prec \dots \prec v_n$, and that all the vertices but possibly v_n have degree at most d . To prove the first point of Proposition 2.1, we prove the following lemma:

Lemma 2.2. *Let α be a k -coloring of G , let $1 \leq j \leq n$ and $1 \leq c \leq k$ such that $c \notin \alpha(N^+(v_j))$. Algorithm 1 applied on (α, v_j, c) with list assignment $L(v_i) = \{1, \dots, k\}$ for every i returns a k -coloring β such that $\beta(v_j) = c$ and $\forall \ell > j, \beta(v_\ell) = \alpha(v_\ell)$. Furthermore, the smaller vertices are recolored at most $|N^-(v_j)|$ times, v_j is recolored at most once, and the bigger vertices are not recolored.*

Algorithm 1:

Input : An L -coloring α of G , a vertex v , a color $c \in L(v) \setminus \alpha(N^+[v])$.
Output: An L -coloring β of G which agrees with α on $\{w \mid w \succ v\}$, with $\beta(v) = c$.
Let $\beta = \alpha$;
1 **while** $\exists u \in K_{v,c}(\beta, G) \setminus \{v\}$ *that is bad* **do**
2 Let u be the greatest first bad vertex;
3 **if** *there exists a color c' in $L(u) \setminus \beta(N[u])$* **then**
4 // In particular, this is the case if u is blocking or branching.
 Let β be the result of the trivial Kempe change $K_{u,c'}(\beta, G)$;
 else
 Let c' be a color in $L(u) \setminus \beta(N^+[u])$;
 // The set is non-empty as u is problematic.
 Let β be the result of Algorithm 1 on input (β, u, c') ;
 end
 Perform $K_{v,c}(\beta, G)$ in β ;
 Return β ;

Proof. We proceed by induction on j . If $j = 1$, then the algorithm does not enter the while loop, and recolors v_j with a trivial Kempe change. Assume now that $1 < j < n$. Figure 2 shows an example of the execution of Algorithm 1. Since L is constant, there is no blocking vertex.

Consider a k -coloring γ of G that agrees with α on all vertices greater or equal than v_j . Assume that $K_{v_j,c}(\gamma, G) \setminus \{v_j\}$ contains a bad vertex, in other words the condition of the while loop is satisfied for γ . Let u be the greatest bad vertex as in the line 2. For every neighbor $w \in N^-(v_j)$ such that $\gamma(w) = c$ we define

$$C_\gamma(w) = K_{w,\gamma(v_j)}(\gamma, G \setminus \{v_j\}) \cup \{v_j\}.$$

We have $K_{v_j,c}(\gamma, G) = \bigcup_{w \in N^-(v_j) \mid \gamma(w)=c} C_\gamma(w)$. A neighbor $w \in N^-(v_j)$ is a *safe neighbor* in γ if $w \succ u$ and either $\gamma(w) \neq c$ or $\gamma(w) = c$ and $C_\gamma(w)$ is a decreasing path ending at a vertex greater than u . A neighbor $w \in N^-(v_j)$ is *unsafe* if it is not safe.

Claim 1. *An iteration of the while loop on γ results in a coloring γ' with more safe neighbors. Moreover, either $K_{v_j,c}(\gamma', G)$ has no bad vertex or the greatest bad vertex u' is smaller than the greatest bad vertex u for $K_{v_j,c}(\gamma, G)$.*

Proof. The Kempe changes in the while loops operate on vertices smaller or equal than u . Hence safe neighbors in γ remain safe in γ' if $u' \prec u$. We only need to show that $u' \prec u$ and that some unsafe neighbor becomes safe.

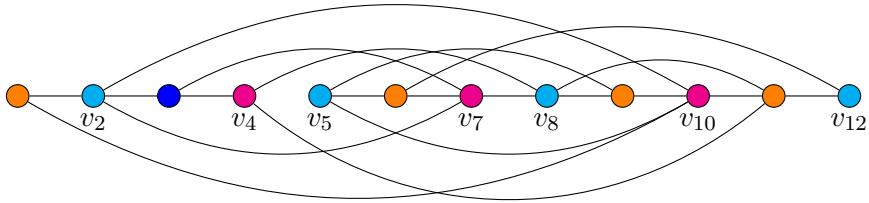
We first prove that $u' \prec u$. In either case of the if condition, only the vertices smaller or equal than u are modified. Furthermore u is not bad anymore in γ' . Hence either γ' has no bad vertex or the greatest first one is smaller than u .

We now prove that some unsafe neighbor becomes safe. Let $w \in N^-(v_j)$ be on a shortest path P from v_j to u in $K_{v_j,c}(\gamma, G)$. Then w is an unsafe neighbor with $\gamma(w) = c$. We prove that w is safe in γ' .

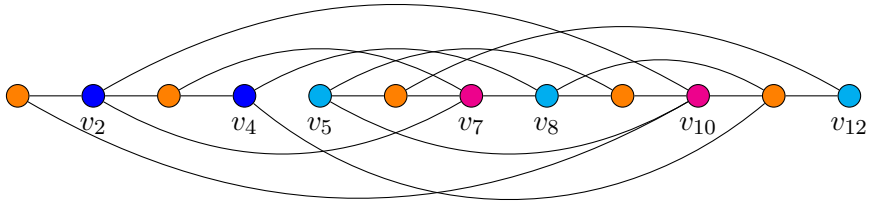
If $u = w$ then w is safe in γ' since it is recolored with a color c' different from $\{c, \gamma(v_j)\}$. If $u \neq w$, then at the end of the while loop the vertex u is recolored and $C_{\gamma'}(w) = P \setminus \{u\}$. In particular, since $u' \prec u$, w is safe in γ' . ■

Apply Claim 1 to α , then to the coloring obtained after each new iteration of the while loop (if any). If the resulting coloring has only safe neighbors, then K is a generalized star, rooted at v_j , in which each branch is decreasing. In such a scenario, the while condition is not satisfied. The Kempe change K colors v_j with c and does not affect the colors of the vertices bigger than v_j , as desired.

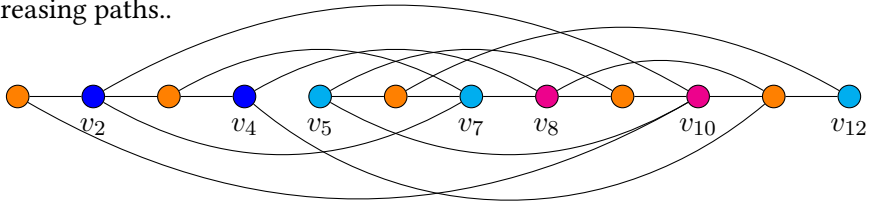
Since the number of safe neighbors increases at each step, the number of iterations is at most the number of unsafe neighbors in α , which is at most $|N^-(v_j)|$. Note that bigger vertices are not recolored. It remains to argue more carefully that smaller vertices are not recolored too many times. We observe that in each iteration of the while loop, a smaller vertex is recolored at most once. This is trivial if u satisfies the if condition on line 3. If it does not, we observe that u has at most one bad neighbor, and apply the induction hypothesis to obtain that every vertex smaller than u is recolored at most once. The conclusion follows. ■



Initial coloring α . The aim is to recolor v_8 in pink. The bad neighbors are v_4 and v_7 . $K(v_4)$ consists only of v_4 and v_8 , and is a decreasing path. $K(v_7)$ consists of $v_2, v_5, v_7, v_8, v_{10}$. No vertex is branching, but v_2 is problematic. All colors appear in $N[v_2]$, so we apply induction on v_2 with color orange.



$K_{v_8, \text{pink}}(\tilde{\beta})$ is now a generalized star, centered at v_8 , whose branches are decreasing paths..



The vertex v_8 has been recolored pink, after a total of 2 Kempe changes.

Figure 2: Example of recoloring on a 3-degenerate graph, following the steps of the proof of Lemma 2.2.

2.2 List recoloring

We now prove the second point of Proposition 2.1. Let L be a list assignment of G such that $|L(v_i)| \geq \deg(v_i) + 1$ for all $i < n$. We prove the following claim:

Claim 2. Let α be a L -coloring of G , let $1 \leq j \leq n$ and $c \in L(v) \setminus \alpha(N^+(v))$. Algorithm 1 applied on (α, v_j, c) returns an L -coloring β such that $\beta(v_j) = c$ and $\forall \ell > j, \beta(v_\ell) = \alpha(v_\ell)$. Furthermore, Algorithm 1 performs at most $|N^-(v_j)| + 1$ Kempe changes and each vertex is recolored at most once.

Proof. First note that the condition line 3 is always satisfied. Indeed, because u is different from v_n we have $|L(u)| \geq \deg(u) + 1$ and

- if u is problematic, $|\beta(N[u])| = 1 + |\beta(N(u))| \leq \deg(u)$,
- if u is blocking, $|\beta(N[u])| \leq \deg(u) + 1$ but at least one of c and $\alpha(v_j)$ belongs to $\beta(N(u))$ but not to $L(u)$,
- if u is branching, $|\beta(N[u])| = 1 + |\beta(N(u))| \leq \deg(u) - 1$.

Now the bound on the number of Kempe changes follow from the exact same analysis as in the previous proof. ■

2.3 Proof of Proposition 2.1

Proof of Proposition 2.1. Let α and β be two colorings satisfying one of the two settings. Let $1 \leq j \leq n$ be the largest such that $\alpha(v_j) \neq \beta(v_j)$. We note $v = v_j$ and proceed by induction on j . Denote $L'(v) = L(v) \setminus \alpha(N^+(v))$, where $L(v) = \{1, \dots, k\}$ in the first setting. Note that $L'(v)$ is not empty. For c in $L'(v)$, denote $p_c = |\alpha^{-1}(c) \cap N^-(v)| + |\beta^{-1}(c) \cap N^-(v)|$. If $p_c \geq 2$ for all $c \in L'(v)$, we have $2 \deg^-(v) \geq 2|L'(v)|$. However, $|L'(v)| \geq |L(v)| - \deg^+(v) > \deg^-(v)$, hence $2 \deg^-(v) > 2 \deg^-(v)$, a contradiction.

Let $c \in L'(v)$ be such that $p_c \leq 1$. By applying Algorithm 1 on v for color c in α (resp. β), we obtain a coloring α' (resp. β'). We have $\alpha'(w) = \alpha(w) = \beta(w) = \beta'(w)$ for all $w \succ v$ and $\alpha'(v) = c = \beta'(v)$, hence $\alpha'(w) = \beta'(w)$ for all $w \succ v$. Furthermore, at most $O(n)$ Kempe changes are performed in the first setting, and at most $O(1)$ in the second. We obtain the claimed bounds on the length of the sequence of Kempe changes. ■

3 Recoloring with graphs of bounded degree

We recall Theorem 0.1:

Theorem 0.1. All the k -colorings of an n -vertex graph G with maximum degree at most $\Delta \leq k$ are equivalent up to $O(n^2)$ Kempe changes, unless $k = 3$ and G is the 3-prism.

To prove Theorem 0.1, we adapt the proof [BBFJ19, FJP17] by handling separate cases depending on whether G is 3-connected. The problem is then reduced to $(\Delta(G) - 1)$ -degenerate graphs, with all vertices but possibly one of degree at most $\Delta(G)$. We proved in Section 2 that in that case, $R^k(G)$ has diameter $O(n^2)$ whenever $k \geq \Delta(G)$ (see Proposition 2.1).

3.1 Naive bounds for Mohar's conjecture

Let G be a graph of maximum degree Δ other than the 3-prism and let $k \geq \Delta$. Bonamy *et al.* [BBFJ19] and Feghali *et al.* [FJP17] proved that $\mathcal{C}^k(G)$ forms a single Kempe class. If $k > \Delta$ or if G is not regular, Proposition 2.1 states that $\text{diam}(R^k(G)) = O(n^2)$. Assume that $k = \Delta$ and that G is regular. Let u be a vertex of G . Given any Δ -coloring of G , there are at least two neighbors of u that are colored alike. Denote G_{v+w} the graph where two non-adjacent neighbors

v and w of u are identified and $\mathcal{C}_{v,w}^\Delta(G)$ the set of k -colorings of G in which v and w are colored alike. We have

$$\mathcal{C}^\Delta(G) = \bigcup_{\substack{v,w \in N(u) \\ vw \notin E(G)}} \mathcal{C}_{v,w}^\Delta(G).$$

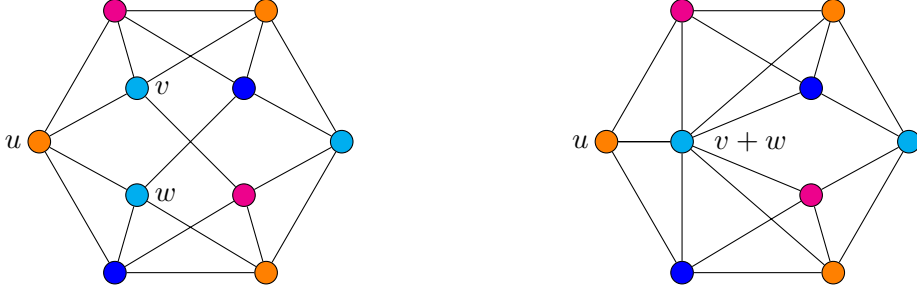


Figure 3: The graph G_{v+w} (right) obtained from the 4-regular graph G (left) is 3-degenerate with all its vertices but $v + w$ of degree less than 4

There is a one-to-one correspondence between the colorings of G_{v+w} and the colorings of G in which v and w are colored alike. Performing a Kempe change in G_{v+w} corresponds to performing one or two Kempe changes in G , to maintain identical colors on v and w . As a result,

$$\text{diam}(R^\Delta(G)) \leq 2 \sum_{\substack{v,w \in N(u) \\ vw \notin E(G)}} \text{diam}(R^\Delta(G_{v+w})). \quad (1)$$

Note that G_{v+w} is $(\Delta - 1)$ -degenerate and all its vertices but the identification of v and w are of degree less than Δ (see Figure 3). By Proposition 2.1, $R^\Delta(G_{v+w})$ has diameter at most $O(n^2)$. Together with Equation (1), this proves that $\text{diam}(R^\Delta(G)) = O(\Delta^2 n^2)$. As a result, we obtain the following theorem :

Theorem 3.1. *Let G be a graph of maximum degree Δ , different from the 3-prism and $k \geq \Delta$. Then $R^k(G)$ has diameter $O(\Delta^2 n^2)$.*

To obtain Theorem 0.1, one needs to improve this bound to $O(n^2)$. As explained before, we only need to consider the case $k = \Delta$. Without loss of generality, we may only consider the case $k \geq 4$, by including the Δ^2 obtained for $k = 3$ inside the O . The proof is very similar to the one developed in [BBFJ19], where we simply replace the use of the Las Vergnas and Meyniel's lemma [LM81] with Proposition 2.1 and count the number of Kempe changes performed (see Appendix A for an adaptation of this proof).

4 Recoloring graphs of bounded maximum average degree

A t -layering of a graph G is an ordered partition $V = V_1 \sqcup \dots \sqcup V_t$ of its vertices into t subsets. We call *layers* the atoms V_i of the partition. Given a t -layering, we denote $G_i = G[\bigcup_{j \geq i} V_j]$ for $1 \leq i \leq t$ and we define the *level* $\ell(v)$ of any vertex v as the index i of the subset V_i it belongs to. We say that a t -layering has *degree* k if v has degree at most k in $G_{\ell(v)}$ for all $v \in V$.

We recall Theorem 4.1:

Theorem 4.1. *Let G be a graph with $\text{mad}(G) \leq k - \varepsilon$. All the k -colorings of G are Kempe equivalent up to $O(\text{Poly}_\varepsilon(n))$ Kempe changes.*

The proof of Theorem 4.1 can be decomposed in two main steps. First, proving that if G has mad less than $k - \epsilon$, then it admits a t -layering of degree $k - 1$ with t being logarithmic in n . This is achieved by Proposition 4.2, see [BP16]. Second, proving that if α_i and β_i are two colorings of $G[V_i]$ that differ by only one Kempe change, then any extension α of α_i to G differs by at most $O(\text{Poly}_\epsilon(n))$ kempe changes from an extension β of β_i to G , see Proposition 4.3 applied with t as above. Finally, by the list coloring version of Proposition 2.1, each layer $G[V_i]$ can be recolored one by one with $O(|V_i|)$ Kempe changes, starting from $G[V_t]$ down to $G[V_1]$.

Proposition 4.2 ([BP16]). *For every $k \geq 1$ and every $\epsilon > 0$, there exists a constant $C = C(k, \epsilon) > 0$ such that every graph G on n vertices that satisfies $\text{mad}(G) \leq k - \epsilon$ admits a $(C \log_k n)$ -layering of degree $k - 1$.*

Consider a graph G and a t -layering of degree $k - 1$ of it. Consider an arbitrary total order \prec on the vertices that satisfies:

$$\forall i < j, \forall (u, v) \in V_i \times V_j, u \prec v.$$

Note that every vertex has at most $k - 1$ greater neighbors. We say a sequence S_1 of vertices is *lexicographically smaller* than another sequence S_2 if

- S_2 is empty and S_1 is not, or
- the first vertex of S_1 is smaller than the first vertex of S_2 , or
- $S_1 = x \oplus S'_1$ and $S_2 = x \oplus S'_2$ for some vertex x , and S'_1 is lexicographically smaller than S'_2 .

We then denote $S_1 \prec_{lex} S_2$. Note that in particular, the empty sequence is the biggest element for this order. A sequence of vertices $S = (v_1, \dots, v_p)$ is said to be *level-decreasing* if the sequence of levels $(\ell(v_1), \dots, \ell(v_p))$ is decreasing. We will say that two colorings *agree* on a set of vertices X if their restrictions to X are equal.

Given a coloring α , a vertex v and a color c , we say in this subsection that a vertex u is *problematic* for the pair (v, c) if there exists a level-decreasing path of vertices in $K_{v,c}(\alpha, G)$ going from v to u , such that u has at least two neighbors in $K_{v,c}(\alpha, G_{\ell(v)})$. Note that if the pair (v, c) has no problematic vertices and c is not used in $\alpha(N_{G_{\ell(v)}}[v])$, then the coloring β resulting from the Kempe change $K_{v,c}(\alpha, G)$ agrees with α on $V(G_{\ell(v)}) \setminus \{v\}$ and has $\beta(u) = c$.

Proposition 4.3. *Let α be a k -coloring of G and K a Kempe chain in G_i with $\alpha|_{G_i}$. Let γ be the coloring of G_i obtained from $\alpha|_{G_i}$ by performing the Kempe change on K in G_i .*

There exists a k -coloring β of G within $n^2 \cdot (2k)^t$ Kempe changes of α such that $\beta|_{G_i} = \gamma$.

4.1 Freeing one color at a single vertex

Lemma 4.4. *Let α be a k -coloring of G and v be a vertex of G . For any color c , Algorithm 2 on input $(\alpha, (v), c)$ yields a k -coloring β of G within $n(2(k - 1))^t$ Kempe changes of α , such that α agrees with β on $G_{\ell(v)}$, and the pair (v, c) admits no problematic vertices for β .*

Proof of correctness of Algorithm 2. We prove correctness of the algorithm by induction on $\ell(v)$. If $\ell(v) = 1$, then $V(G_{\ell(v)}) = V$, so (v, c) does not admit any problematic vertex. For $\ell(v) > 1$, at each iteration of the while loop, line 1 is possible since u has at least two neighbors in $G_{\ell(u)}$ that are colored identically. By induction hypothesis, the Kempe change line 2 does not modify the color of the vertices in $G_{\ell(u)} \setminus \{u\}$. Thus, $\ell(v)$ decreases at each iteration of the loop and at

Algorithm 2:

Input : A k -coloring α of G , a level-decreasing sequence of vertices S , a color c .

Output: A k -coloring β of G which agrees with α on $V(G_{\ell(v)})$ where v is the last vertex of S . Moreover, the pair (v, c) admits no problematic vertices with respect to β .

Let v be the last vertex of S ;

Let $\beta = \alpha$;

while the pair (v, c) admits problematic vertices with respect to the coloring β **do**

1 | Let u be the largest problematic vertex for (v, c) with respect to β ;

1 | Let c_u be a color in $[k] \setminus \beta(N_{G_{\ell(u)}}[u])$;

1 | Let β be the result of Algorithm 2 on input $(\beta, S \oplus u, c_u)$;

2 | Perform $K_{u, c_u}(\beta, G)$ in β ;

end

Return β ;

most n calls are generated by the current call. ■

To conclude the proof of Lemma 4.4, we need to bound the number of Kempe changes performed by Algorithm 2.

Given a call C of Algorithm 2, we will denote S_C the sequence provided in input, and u_C its last vertex.

Observation 4.5. *If Algorithm 2 is called on (α, S, c) where S is a level-decreasing sequence, and makes some recursive call C , then the sequence S_C is longer than S and is of the form $S \oplus u_C$. Moreover, S_C is also a level-decreasing sequence by construction.*

Claim 3 (analog to [BP16]). *If a call D is initiated after a call C , then $S_D \prec_{lex} S_C$.*

Proof. If D is called by C , then by Observation 4.5, S_D is of the form $S_C \oplus u_C$ thus $S_D \prec_{lex} S_C$. By applying this argument inductively, we have also have $S_D \prec_{lex} S_C$ if the call D is generated by C .

Now, assume that D is not generated by C . Denote I the initial call of Algorithm 2 and recall that the recursive calls generated by I have a natural tree structure, rooted at I . There exists a unique sequence $\mathcal{S}_C = C_1, C_2, \dots, C_{t_1}$ such that C_i calls C_{i+1} for each $i < t_1$, with $C_1 = I$ and $C_{t_1} = C$; and a unique sequence $\mathcal{S}_D = D_1, D_2, \dots, D_{t_2}$ such that D_i calls D_{i+1} for each $i < t_2$, with $D_1 = I$ and $D_{t_2} = D$. Denote B the last common ancestor of C and D . Since neither C nor D is generated by the other, B is distinct from C and D and thus is not the last element of the sequences \mathcal{S}_C and \mathcal{S}_D . We have :

- B_C and B_D are both called by B and B_C is initiated before B_D ,
- $B_C = C$ or B_C generates C ,
- $B_D = D$ or B_D generates D .

It follows from Observation 4.5 that $S_{B_C} \prec_{lex} S_C$, so we just need to show that $S_D \prec_{lex} S_{B_C}$. By Observation 4.5 applied inductively, S_D can be written as $S_{B_D} \oplus T_D$ with $S_{B_D} = S_B \oplus u_{B_D}$. Furthermore, S_{B_C} can be written as $S_B \oplus u_{B_C}$. Since B_C is called before B_D , we have $u_{B_D} \prec u_{B_C}$, so $S_D = S_B \oplus u_{B_D} \oplus T_D \prec_{lex} S_B \oplus u_{B_C} = S_{B_C}$. ■

Claim 4 ([BP16]). *Given that the t -layering $V = V_1 \sqcup \dots \sqcup V_t$ has degree at most $k-1$, the number of level-decreasing paths between two vertices u and w in different levels is at most $(k-1)^{i-1}$ where $i = |\ell(u) - \ell(w)|$.*

Lemma 4.6. *The number of Kempe changes performed by Algorithm 2 on input $(\alpha, (v), c)$ is bounded by $n(2(k-1))^t$.*

Proof. Observe that the sequences of vertices considered in the recursive calls are subsequences of level-decreasing paths. Thus, each level-decreasing path between v and w has at most $2^{\ell(v)-\ell(w)-1}$ subsequences that contain v and w . As a result, each vertex w with level $\ell(w)$ less than $\ell(v)$ is the origin of at most $2^{\ell(v)-\ell(w)-1}(k-1)^{\ell(v)-\ell(w)-1}$ Kempe changes. ■

4.2 Performing a Kempe change

Algorithm 3:

Input : A k -coloring α of G with $k \geq \ell + 1$, a level i , a Kempe chain K of $G[V_i]$.
Output: A k -coloring β of G , whose restriction to $G[V_i]$ is the coloring resulting from the Kempe change K in $\alpha|_{G[V_i]}$.
Let c_1, c_2 be the colors involved in K ;
Let $\beta = \alpha$;
while there exists a vertex $v \in K$ such that (v, c_1) or (v, c_2) admits problematic vertices in β **do**
 Among all problematic vertices for some (v, c_1) or (v, c_2) with $v \in K$, let u be the largest one;
 Let $v \in K$ and $j \in \{1, 2\}$ be such that u is problematic for (v, c_j) in β ;
 Update β with the result of Algorithm 2 on input $(\beta, (v), c_j)$.
end
Perform the Kempe change K on β ;
Return β ;

Proof of Proposition 4.3. The coloring β is obtained by applying Algorithm 3. The correctness of Algorithm 3 follows from the correctness of Algorithm 2, which holds by Lemma 4.4. After each iteration of the loop the largest possible vertex that is problematic for a pair of vertex and a color of K decreases. Therefore, the loop is executed at most n times and at the end, the Kempe chain K is a proper Kempe chain of G . Combined with the cost analysis of Algorithm 2 by Lemma 4.6, this proves that Algorithm 3 performs at most $n^2(2(k-1))^i + 1$ Kempe changes and is correct. ■

4.3 Combining the arguments

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $V_1 \sqcup \dots \sqcup V_t$ be a t -layering of G of degree $(k-1)$, with $t = C \log_k n$ (see Proposition 4.2).

We claim that G can be recolored layer by layer, starting from $G[V_t]$, with a polynomial number of Kempe changes. We prove this by decreasing induction on the level of the layer. Let $1 \leq i \leq t$, let α and β be two k -colorings of G and assume that α and β agree on all vertices of level more than i . For $v \in V_i$, let $L(v) = [k] \setminus \alpha(N(v) \cap G_{i+1})$. We have for all $v \in V_i$, $|L(v)| \geq \deg_{G[V_i]}(v) + 1$, so by applying the second case of Proposition 2.1, there exist a sequence S of Kempe changes in $G[V_i]$ of size $O(|V_i|)$ leading from $\alpha|_{V_i}$ to $\beta|_{V_i}$. Moreover, each of the Kempe change in S is a proper Kempe change in G_i . By Proposition 4.3, each of these Kempe changes can be performed in G , after $O(\text{Poly}_\varepsilon(n))$ Kempe changes affecting the vertices of level less than i . ■

5 Recoloring bounded treewidth graphs

Let G be an n -vertex graph of treewidth tw . Let H be a chordal graph such that G is a (not necessarily induced) subgraph of H , with $\omega(H) = \chi(H) \leq \text{tw} + 1$ and $V(H) = V(G)$. Computing H is equivalent to computing a so-called tree decomposition of G , which can be done in time $f(\text{tw}) \cdot n$ [Bod96].

Since G and H are defined on the same vertex set, there may be confusion when discussing neighbourhoods and other notions. When useful, we write G or H in index to specify. There is an ordering $v_1 \prec \dots \prec v_n$ of the vertices of G such that for all $v \in V(G)$, $N_H^+[v]$ induces a clique in H . The ordering can be computed from H in $O(n)$ using Lex-BFS [JRTL75].

The core of the proof lies in Proposition 5.1: for $k \geq \text{tw} + 1$, any k -coloring of G is equivalent up to $O(\text{tw} \cdot n^2)$ Kempe changes to a k -coloring of G that yields a k -coloring of H .

Proof of Theorem 0.3 assuming Proposition 5.1. Let α and β be two k -colorings of G . By Proposition 5.1, there exists a k -coloring α' (resp. β') that is equivalent to α (resp. β) up to $O(\text{tw} \cdot n^2)$ Kempe changes. Additionally, both α' and β' yield k -colorings of H . Since H is chordal, by Proposition 1.2, there exists a sequence of at most n Kempe changes in H from α' to β' . Each of these Kempe changes in H can be simulated by at most n Kempe changes in G , which results in a sequence of length $O(\text{tw} \cdot n^2)$ between α and β . ■

Proposition 5.1. *Given any k -coloring α of G with $k \geq \text{tw} + 1$, there exists a k -coloring α' of G that is equivalent to α up to $O(\text{tw} \cdot n^2)$ Kempe changes and such that $\alpha'(u) \neq \alpha'(v)$ for all $uv \in E(H)$. The algorithm 4 computes β and a sequence of Kempe changes leading to it.*

To prove Proposition 5.1 and obtain a k -coloring of H , we gradually “add” to G the edges in $E(H) \setminus E(G)$. To add an edge, we first reach a k -coloring where the extremities have distinct colors, then propagate any later Kempe change involving one extremity to the other extremity. We formalize this process through Algorithm 4. Let $v_1w_1 \prec \dots \prec v_qw_q$ be the edges in $E(H) \setminus E(G)$ in the lexicographic order, where $v_i \prec w_i$ for every i .

We will prove the following three claims. Note that Proposition 5.1 follows from Claims 1 and 2, while Claim 3 simply guarantees that the proof of Theorem 0.3 is indeed constructive.

Claim 5 (1). *Algorithm 4 outputs a k -coloring α' of G that is Kempe equivalent to α and such that $\alpha'(u) \neq \alpha'(v)$ for all $uv \in E(H)$.*

Claim 6 (2). *Algorithm 4 performs $O(\text{tw} \cdot n^2)$ Kempe changes in G to obtain α' from α .*

Claim 7 (3). *Algorithm 4 runs in $O(\text{tw} \cdot n^4)$ time.*

In Algorithm 4, the variable \tilde{G} keeps track of how close we are to a k -coloring of H . Before the computations start, $\tilde{G} = G$. When the algorithm terminates, $\tilde{G} = H$. At every step, G is a subgraph of \tilde{G} . To refer to \tilde{G} or $\tilde{\alpha}$ at some step of the algorithm, we may say the *current* graph or *current* coloring. The Kempe changes that we discuss are performed in \tilde{G} . Consequently, the corresponding set of vertices might be disconnected in G , and every Kempe change in \tilde{G} may correspond to between 1 and n Kempe changes in G .

Proof of Claim 1. By construction, at every step $\tilde{\alpha}$ is Kempe equivalent to α . We prove the following loop invariant: at every step, $\tilde{\alpha}$ is a k -coloring of \tilde{G} . Since $\tilde{G} = H$ at the end of the algorithm, proving the loop invariant will yield the desired conclusion.

The invariant holds at the beginning of the algorithm, when $\tilde{G} = G$.

Assume that at the beginning of the j -th iteration of the loop 1, $\tilde{\alpha}$ is a proper coloring of \tilde{G} . All the Kempe changes in the loop are performed in \tilde{G} , so we only need to prove that at the end of the iteration, $\tilde{\alpha}(v_j) \neq \tilde{\alpha}(w_j)$.

This follows from the validity of comments 4 and 5. The latter is a direct consequence of

Algorithm 4: Going from α to α'

Input : G a graph of treewidth tw , a k -coloring α of G with $k \geq \text{tw} + 1$ and $v_1w_1 \prec \dots \prec v_qw_q$ such that $G + \{v_1w_1, \dots, v_qw_q\}$ is a $\text{tw} + 1$ -colorable chordal graph H

Output: A k -coloring $\tilde{\alpha}$ of H , that is Kempe equivalent to α

Let $\tilde{G} \leftarrow G$ and $\tilde{\alpha} \leftarrow \alpha$;

```
1 for  $j$  from 1 to  $q$  do
  if  $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$  then
    Let  $c \in [k] \setminus \tilde{\alpha}(N_H^+[v_j])$ ;
    // Possible because  $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$  and  $|N_H^+[v_j]| \leq \text{tw} \leq k - 1$ .
    Let  $U = N_{\tilde{G}}^-(v_j) \cap N_{\tilde{G}}^-(w_j) = \{u_1 \prec \dots \prec u_p\}$ ;
  2 for  $i = p$  down to 1 do
    if  $\tilde{\alpha}(u_i) = c$  then
      Let  $c_i \in [k] \setminus \tilde{\alpha}(N_{\tilde{G}}^+[u_i])$ ;
      // Possible because  $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$  and  $|N_{\tilde{G}}^+[u_i]| \leq \text{tw} \leq k - 1$ .
      3  $\tilde{\alpha} \leftarrow K_{u_i, c_i}(\tilde{\alpha}, \tilde{G})$ ;
    end
    4 // Now  $c \notin \tilde{\alpha}(\{x \in N_{\tilde{G}}(v_j) \mid x \geq u_i\})$ 
  end
  5 // Now  $c \notin \tilde{\alpha}(N_{\tilde{G}}(v_j))$ 
  6  $\tilde{\alpha} \leftarrow K_{v_j, c}(\tilde{\alpha}, \tilde{G})$ ;
end
 $\tilde{G} \leftarrow \tilde{G} \cup \{v_jw_j\}$ ;
end
```

the former, so we focus on arguing that after the step i of the inner loop 2, we have $c \notin \tilde{\alpha}(\{x \in N_{\tilde{G}}(v_j) \mid x \geq u_i\})$. The key observation is that at the i -th step of the inner loop 2, the Kempe changes performed at line 3 involve only vertices smaller than u_i . We prove by induction the stronger statement the Kempe chain T involved in the Kempe change $K_{u_i, c_i}(\tilde{\alpha}, \tilde{G})$ is a tree rooted at u_i in which all the nodes are smaller than their father.

- $c_i \notin \tilde{\alpha}(N_{\tilde{G}}^+(u_i))$ so all the vertices at distance 1 in T from u_i are smaller than u_i .
- Let x at distance $d + 1$ from u_i in T . Let y be a neighbour of x at distance d from u_i . Assume by contradiction that $x \succ y$. By induction hypothesis, there is a unique neighbour z of y at distance $d - 1$ from u_i , with $y \prec z$. Both z and x are in $N_H^+(y)$ and since H is chordal, this implies $zx \in E(H)$. We have $z \prec u_i \prec v_j$ so $zx \prec v_jw_j$ and $zx \in E(\tilde{G})$. In particular, x is at distance d from u_i in T , which raises a contradiction and proves $x \prec y$. Assume by contradiction that x is adjacent to two vertices y, z at distance d from u_i in T . Then y and z are identically colored so $yz \notin E(\tilde{G})$. Moreover $y, z \in N_H^+(x)$ and H is chordal, hence yz is an edge of H . Since $y, z \prec v_j$, we have $yz \prec v_jw_j$. Thus, yz belongs to \tilde{G} , raising a contradiction.

As a result, the Kempe change of line 6 does not recolor any vertex larger than u_i with color c , and the comment 5 is true. Therefore at the beginning of line 6, $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$ and $c \notin \alpha(N(v_j))$. At the end of line 6, v_j and w_j are colored differently. ■

Proof of Claim 2. We now prove that the number of Kempe changes in G performed by the algorithm is $O(\text{tw} \cdot n^2)$.

We first prove that for each vertex x , there exists at most one step j of the loop 1 for which $v_j = x$ and we enter the conditional statement $\tilde{\alpha}(v_j) = \tilde{\alpha}(w_j)$. Indeed, the first time we enter the conditional statement, the vertex x is recolored with a color c not in $N_H^+(x)$ at line 6. Note that all the edges xy with $y \succ x$ are consecutive in the ordering of $E(H) \setminus E(G)$. Therefore, once the vertex x is recolored, all the remaining edges xy are handled without Kempe change, as the conditional statement is not satisfied. This implies directly that line 6 is executed at most n times.

Now, we bound the number of times x plays the role of u_i in the Kempe change at line 3. For each step j of loop 1 for which it happens, we have $v_j, w_j \in N_H^+(x)$. Since $|N_H^+(x)| \leq \text{tw}$ and each v_j is involved at most once by the above argument, we obtain that x plays this role at most tw times.

Consequently the overall number of Kempe changes performed in \tilde{G} by the algorithm is $O(\text{tw} \cdot n)$. Performing a Kempe change in \tilde{G} is equivalent to performing a Kempe change in all the connected component of G of the Kempe chain of \tilde{G} . Therefore, the number of Kempe changes performed in G by the algorithm is $O(\text{tw} \cdot n^2)$. ■

Proof of Claim 3. In total, the loop 2 is executed at most once for every pair of vertices in $N_H^+(u)$ for each $u \in V(G)$, that is $O(\text{tw}^2 n)$ times. However, we also take into account the number of Kempe changes that need to be performed. By Claim 2, only $O(\text{tw} \cdot n^2)$ Kempe changes are performed in G . As a result, the total complexity of the algorithm is $O(\text{tw}^2 n + \text{tw} \cdot n^4) = O(\text{tw} \cdot n^4)$ (performing Kempe changes in a naive way in G). ■

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A Proof of Theorem 0.1

This section is an adaptation of the proof provided in [BBFJ19] for $k \geq 4$. The proof handles separately the cases of graphs not 3-connected, 3-connected graphs with diameter at most 2 and 3-connected graphs of diameter at least 3.

A.1 G not 3-connected

We first need a few lemmas.

Lemma A.1 (Adapted from Lemma 3.2 in [LM81]). *Let G_1 and G_2 be two $(k - 1)$ -degenerate graphs of maximum degree k , such that $S = G_1 \cap G_2$ is a complete graph. Let α be a k -coloring of G and $\beta_{|G_1}$ be a k -coloring of G_1 . There exists a k -coloring γ of G within $O(|S|n^2)$ Kempe changes of α such that $\gamma_{|G_1} = \beta_{|G_1}$ and $\gamma_{|G_2}$ is equal to $\alpha_{|G_2}$ up to a permutation of colors.*

Proof. By Proposition 2.1, there exists a series of Kempe changes in G_1 , leading from $\alpha_{|G_1}$ to $\beta_{|G_1}$, such that the color of each vertex of G_1 is modified at most $O(n_1)$ times. We will adapt this sequence to G to ensure that at all times, the current coloring $\tilde{\alpha}$ differs from α on G_2 only by a permutation of colors. For each Kempe change $K_{u,c}(\tilde{\alpha}, G_1)$ in the sequence, if $K_{u,c}(\tilde{\alpha}, G) \cap S = \emptyset$, then simply perform $K_{u,c}(\tilde{\alpha}, G)$. If $K_{u,c}(\tilde{\alpha}, G) \cap S \neq \emptyset$, then swap the colors $\tilde{\alpha}(u)$ and c in each Kempe chain of G_2 in addition to performing $K_{u,c}(\tilde{\alpha}, G)$.

The latter case occurs at most $O(|S|n_1)$ times, each time resulting in $O(n_2)$ Kempe changes in G_2 . The sequence we obtain has length $O(n_1^2 + |S|n_1n_2) = O(|S|n^2)$. ■

Corollary A.2 (Adapted from Lemma 3.3 in [LM81] and in [BBFJ19]). *Let G_1 and G_2 be two $(k - 1)$ -degenerate graphs of maximum degree k , such that $S = G_1 \cap G_2$ is a complete graph. Let $G = G_1 \cup G_2$, any two k -colorings of G are equivalent up to $O(|S|n^2)$.*

Proof. Let α and β be two k -colorings of G . By Lemma A.1 applied to α and $\beta_{|G_1}$, there exists γ within $O(|S|n^2)$ Kempe changes of α , such that $\gamma_{|G_1} = \beta_{|G_1}$ and $\gamma_{|G_2}$ is equal to $\alpha_{|G_2}$ up to a permutation of colors. By Lemma A.1 applied to γ and $\beta_{|G_2}$, there exists δ within $O(|S|n^2)$ Kempe changes of γ , such that $\delta_{|G_2} = \beta_{|G_2}$ and $\delta_{|G_1}$ is equal to $\gamma_{|G_1} = \beta_{|G_1}$ up to a permutation of colors σ . Note that if a color c is used in S , then $c = \sigma(c)$ and $\delta^{-1}(c) = \beta^{-1}(c)$, even on G_1 . By iteratively swapping the other colors with their image by σ in each Kempe chain of G_2 , one obtains the coloring β within at most $O(kn)$ Kempe changes. ■

Proposition A.3 (Adapted from Proposition 3.1 in [BBFJ19]). *Let $k \geq 3$ and G be a k -regular graph that is neither 3-connected, nor a clique or the 3-prism. Any two k -colorings of G are equivalent up to $O(n^2)$ Kempe changes.*

Proof. Let S be a separator of minimal size, $|S| = 1$ or $|S| = 2$. Consider G_1 and G_2 such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = S$. Two cases can occur:

- S is a complete graph, then the result stems from Corollary A.2.
- S is composed of two non-adjacent vertices u and v . If both u and v have only one neighbor in G_1 , consider w the unique neighbor of u in G_1 . Note that w is not adjacent to v , otherwise, $\{w\}$ would be a separator of G . Since G is k -regular with $k \geq 3$, w has at least two neighbors in G_1 . As a result, we can assume that u or v has at least two neighbors in G_1 (respectively G_2). We prove the two following claims.

Claim 8. *There is a sequence Kempe changes of length at most $O(n^2)$ between any two k -colorings α and β of G , such that $\alpha(u) \neq \alpha(v)$ and $\beta(u) \neq \beta(v)$.*

Let G'_1 (respectively G'_2 and G') the graphs obtained from G_1 (respectively G_2 and G), by adding an edge between u and v . The graph G'_1 (respectively G'_2) has maximum degree k and is $(k - 1)$ -degenerate because either u or v has at most $k - 2$ neighbors in G_1 (respectively G_2). By Corollary A.2, $R^k(G')$ has diameter $O(n^2)$. As a result, $R^k(G)$ has also diameter $O(n^2)$.

Claim 9. [BBFJ19] *Given any k -coloring α of G , such that $\alpha(u) = \alpha(v)$, there exists a k -coloring β within at most 3 Kempe changes from α such that $\alpha'(u) \neq \alpha'(v)$.*

Assume that $\alpha(u) = \alpha(v) = c$. If there exists a color c' that is unused in the closed neighborhood of u or v , say u , then one obtains the desired property after the trivial Kempe change $K_{u,c'}(\alpha)$. Henceforth, assume that both u and v have a neighbor of each color. Two cases may occur:

- u or v has at least two neighbors in both G_1 and G_2 . Note that this case can only happen if $k \geq 4$. By symmetry, assume that u has at least two neighbors in each of G_1 and G_2 . By assumption, each color but one is used exactly once in the neighborhood of u . As a result, there exists $c_1 \notin \alpha(N[u] \cap G_1)$ and $c_2 \notin \alpha(N[u] \cap G_2)$. The (c_1, c_2) -Kempe chain containing the neighbor of u colored c_2 is fully contained in G_1 . After performing it, the color c_2 does not appear any more in the neighborhood of u and one can conclude in one trivial Kempe change.
- u has only one neighbor u' in G_1 and v only one neighbor v' in G_2 (or the other way around). If u' and v' are colored differently, the Kempe chain $K_{u',\alpha(v')}(\alpha)$ is contains neither u , v nor v' . Therefore, after performing the corresponding Kempe change, u' and v' are both colored c' . Consider a third color c'' , u has no neighbor colored c'' in G_1 and v has no neighbor colored c'' in G_2 , so $K_{u,c''}(\tilde{\alpha})$ does not contain v , and after performing it, u and v are colored differently.

■

A.2 G 3-connected

In this subsection, G is a 3-connected k -regular graph of diameter. Given a vertex $u \in V(G)$, we say that a pair of vertices (t_1, t_2) is an eligible pair of u if t_1 and t_2 are two non-adjacent neighbors of u and denote the set of such pairs $P(u) = N(u)^2 \setminus E$.

Lemma A.4 (Adapted from Lemma 4.3 in [BBFJ19]). *Assume that there exists two vertices u and x and an eligible pair (t_1, t_2) of u such that for each eligible pair (w_1, w_2) of x , there exist a k -coloring that colors t_1 and t_2 alike and w_1 and w_2 alike. That is :*

$$\forall (w_1, w_2) \in P(x), \mathcal{C}_{w_1, w_2}^k(G) \cap \mathcal{C}_{t_1, t_2}^k(G) \neq \emptyset.$$

Then $\mathcal{C}^k(G)$ forms a single Kempe class and $R^k(G)$ has diameter $O(n^2)$.

Proof. Let α and β be two colorings of G . Since G is k -regular, there exists an eligible pair (w_1, w_2) (respectively (w_3, w_4)) of x that is colored identically by α (respectively β). By assumption, there exists a k -coloring $\alpha' \in \mathcal{C}_{w_1, w_2}^k(G) \cap \mathcal{C}_{t_1, t_2}^k(G)$ and $\beta' \in \mathcal{C}_{w_3, w_4}^k(G) \cap \mathcal{C}_{t_1, t_2}^k(G)$. By applying Proposition 2.1 on $G_{w_1+w_2}$ (respectively $G_{w_3+w_4}$), there exists a sequence of $O(n^2)$ Kempe changes between α and α' (respectively β and β'). Finally by applying Proposition 2.1 in $G_{t_1+t_2}$, we get that α' is Kempe equivalent to β' up to $O(n^2)$ Kempe changes. As a result, α and β are equivalent up to $O(n^2)$ Kempe changes. ■

Lemma A.5 (Adapted from Lemma 4.4 in [BBFJ19]). *If G admits a vertex cut S of size 3 such that one of the components of $G \setminus S$ is isomorphic to K_k . Then $R^k(G)$ has diameter at most $O(n^2)$.*

Proof. Let C be the aforementioned component. Since G is k -regular, each vertex of C has exactly one neighbor in S and all the others in C , namely, S weakly dominates $V(C)$. $|S| = 3$ and $d \geq 4$, hence at least one vertex u of S is adjacent to two or more vertices of C . Let w_1 be a neighbor of u in C . Note that $S \setminus \{u\}$ is not a vertex cut of G , otherwise G would not be 3-connected, therefore there exists a neighbor w_2 of u that does not belong to C . By Proposition 2.1, $R^k(G_{w_1+w_2})$ has diameter $O(n^2)$ and so does $R^k_{w_1, w_2}(G)$. Thus, if we prove that from any coloring α of G , one can reach a coloring where w_1 and w_2 are colored alike with a bounded number of moves, then we obtain $\text{diam}(R^k(G)) = O(n^2)$.

Let α be a k -coloring of G such that $\alpha(w_1) \neq \alpha(w_2)$. Since C is a clique on d vertices, one of them is colored $\alpha(w_2)$, say w_3 . Two cases may occur:

- If w_3 is adjacent to u , then it is not adjacent to any other vertex of S and $\{w_1, w_3\}$ forms a Kempe chain. After performing the corresponding Kempe change, w_1 and w_2 are colored alike.
- Otherwise, let v be the neighbor of w_3 in S . From what precedes, u has at least two neighbors in C so one of them, say w_4 , is colored differently from v (it is possible that $w_4 = w_1$). Then $\{w_3, w_4\}$ forms a Kempe chain and after performing the corresponding move, either we had $w_4 = w_1$ thus w_1 and w_2 are now colored alike, or w_4 and w_1 were distinct and we are now in the first case.

This proves that any coloring of G is within at most 2 Kempe changes of a coloring in which w_1 and w_2 are colored alike, thus $\text{diam}(R^k(G)) = O(n^2)$ by applying Proposition 2.1. ■

Lemma A.6 (Adapted from Lemma 4.5 in [BBFJ19]). *Let u, v be two vertices of G and (w_1, w_2) be an eligible pair in $P(v)$ such that neither w_1 nor w_2 is adjacent to u . Assume that there exists an eligible pair (t_1, t_2) in $P(u)$ such that there is no k -coloring of G that colors w_1 and w_2 alike and t_1 and t_2 alike. Then G contains a subgraph weakly dominated by both $\{t_1, t_2\}$ and $\{w_1, w_2\}$, that is isomorphic K_{k-1} .*

Proof. The proof of Lemma 4.5 presented in [BBFJ19], results in the same outcome by assuming the stronger hypothesis that $C^k(G)$ does not form a single Kempe class, but it only uses the fact that there is no k -coloring of G that colors w_1 and w_2 alike and t_1 and t_2 alike, therefore we can use it directly. ■

Proposition A.7 (Adapted from Lemma 4.6 in [BBFJ19]). *Assume that there are two non-adjacent vertices u, v of G and an eligible pair (w_1, w_2) in $P(v)$ such that neither w_1 nor w_2 is adjacent to u – note that this is in particular the case in graphs of diameter at least 3. Then $C^k(G)$ forms a single Kempe class and $R^k(G)$ has diameter $O(n^2)$.*

Proof. If for all eligible pair (t_1, t_2) in $P(u)$, $C^k_{t_1, t_2}(G) \cap C^k_{w_1, w_2}(G) \neq \emptyset$, then the results holds by Lemma A.4. Otherwise, there exist an eligible pair (t_1, t_2) in $P(u)$ such that $C^k_{w_1, w_2}(G) \cap C^k_{t_1, t_2}(G) = \emptyset$. By Lemma A.6, G contains a subgraph C isomorphic to K_{k-1} and weakly dominated by both $\{t_1, t_2\}$ and $\{w_1, w_2\}$. Each vertex of C is adjacent to the $k - 2$ others and to one vertex of $\{t_1, t_2\}$ and one of $\{w_1, w_2\}$. The clique C does not contain u nor v since they are adjacent to both $\{t_1, t_2\}$ and $\{w_1, w_2\}$.

At least three vertices of $\{t_1, t_2, w_1, w_2\}$ are adjacent to at least one vertex of C , otherwise G is not 3-connected. If exactly one vertex, say t_1 , is not adjacent to a vertex in C , then $\{u, w_1, w_2\}$

is a cut set between the clique $C \cup \{t_2\}$ and the rest of the graph, so one can apply Lemma A.5. We will show that this is the only possible case, thereby completing the proof.

Assume towards contradiction that all the vertices of $\{t_1, t_2, w_1, w_2\}$ are adjacent to at least one vertex of C . Assume, without loss of generality that w_1 has at least as many neighbors in C as w_2 . Without loss of generality, assume that t_1 and w_1 have a common neighbor x in C . Then (x, v) is an eligible pair of w_1 , such that neither w_1 , nor x , nor v is adjacent to u . If for all eligible pair (t_3, t_4) in $P(u)$, $\mathcal{C}_{t_3, t_4}^k(G) \cap \mathcal{C}_{x, v}^k(G) \neq \emptyset$, we can once again conclude by Lemma A.4.

Otherwise, there exists an eligible pair (t_3, t_4) of u such that $\mathcal{C}_{x, v}^k(G) \cap \mathcal{C}_{t_3, t_4}^k(G) = \emptyset$. By Lemma A.6, there exists a $(k - 1)$ -clique C' in G weakly dominated by both $\{t_3, t_4\}$ and $\{x, v\}$. With the same reasoning, each of the vertices $\{t_3, t_4, x, v\}$ is adjacent to at least one vertex of C' .

Assume towards contradiction that neither t_1 nor t_2 belong to C' . Then $N(x) = C \setminus \{x\} \cup \{t_1, w_1\}$ and at least one neighbor of x belongs to C . By assumption, $t_1 \notin C'$ and w_1 neither since it is adjacent to both x and v . So there is a vertex $y \neq x \in C \cup C'$ and $(k - 2)$ other neighbors of y belong to C' . As none of $\{t_1, t_2, w_1, x\}$ belong to C' , we have $C' = C \setminus \{x\} \cup \{w_2\}$ so w_2 is adjacent to all the vertices of $C \setminus \{x\}$. By assumption, w_1 has at least as many neighbors in C as w_2 so C must be a clique on 2 vertices, and $k = 3$ which is a contradiction and proves that, exactly one of t_1 and t_2 belong to C' (t_1 and t_2 are non-adjacent by definition).

For $i \in \{1, 2\}$, if t_i belongs to C' , then it has $k - 2$ neighbors in C' and two other neighbors. By definition, t_i is adjacent to u and to one of $\{x, v\}$. But neither of t_3 and t_4 belongs to $C' \cup \{u, x, v\}$ which contradicts the definition of C' and proves that $\{t_1, t_2, w_1, w_2\}$ cannot all have at least one neighbor in C . This concludes the proof. ■

We denote $N^2(u)$ the *second neighborhood* of a vertex u , that is the set of vertices at distance exactly two from u .

Proposition A.8. *Let G be a 3-connected k -regular graph of diameter at most 2, with $k \geq 4$. Then $\mathcal{C}^k(G)$ forms a single Kempe class and $\text{diam}(R^k(G)) = O(n^2)$.*

Proof. If G is of diameter less than 2, the result is obvious, thus we can assume that $\text{diam}(G) = 2$. If there exists two non-adjacent vertices u, v in G and an eligible pair (w_1, w_2) in $P(v)$, such that neither w_1 nor w_2 is adjacent to u , then the result holds by Proposition A.7. Otherwise, the second neighborhood of any vertex can contain no path on three vertices, thus is a collection of disjoint cliques.

Assume that the second neighborhood of a vertex v consists of at least two disjoint cliques C_1 and C_2 . Let x and y be two vertices of C_1 and C_2 respectively. If x is adjacent to a neighbor z of v that is not adjacent to y , then $\{x, y, z\}$ induce a path in the second neighborhood of v , and one can conclude by Proposition A.7. As a result, we can assume that $N(x) \cap N(v) = N(y) \cap N(v)$. By repeating this argument for all of pair of vertices in $C_1 \times C_2$, we obtain that all the vertices in $C_1 \cup C_2$ have the same set of neighbors in $N(v)$. Given a coloring α of G , if x and y have distinct colors, say 1 and 2, the $\{1, 2\}$ Kempe chain containing x cannot contain any vertices of C_2 , thus, after performing it, x and y are colored identically. Since they have at least one common neighbor, we can conclude by Proposition 2.1.

Therefore, we can assume without loss of generality that the second neighborhood of each vertex consist in a single clique. Let $v \in V$ and let α and β be two colorings of G . Denote by C the second neighborhood of v . Up to a Kempe change, one can assume that $\alpha(v) = \beta(v) = 1$.

If the color 1 is not used by α in C , then let x be the vertex of C colored 1 in β (or any vertex in C if the color is also not used by β in C). The only vertex of C in the Kempe chain $K_{x,1}(\alpha, G)$ is x , and as no vertices of $N(v)$ can be colored 1, $K_{x,1}(\alpha, G)$ is a trivial Kempe change. We can apply it to color identically x and v with color 1. If no vertex of C is colored 1 by β , we can recolor x with 1 using the same argument, without changing the color of v . By Proposition 2.1,

the two colorings obtained are equivalent up to at most $O(n^2)$ Kempe changes.

Therefore, one can assume without loss of generality that there exists two vertices u, w in C such that $\alpha(u) = \beta(w) = 1$. Once again if $u = w$, we can conclude by Proposition 2.1, so one can assume that $u \neq w$. We will require the following definition: a vertex x is said to be *locked* in γ if all the colors are present in $N(x)$, making a trivial Kempe change on x impossible.

Claim 10. *If any of the vertices in $N[u] \setminus \{w\}$ is not locked or if the color $\alpha(w)$ does not appear twice in the neighborhood of u , then $R^k(G)$ is connected and of diameter $O(n^2)$.*

If u is not locked, then a trivial Kempe change on u removes the color 1 from C and we can conclude with the aforementioned argument. So one can assume that u is locked. If $v \notin K_{u, \alpha(w)}(\alpha, G)$, then performing the corresponding Kempe change yields a coloring in which both w and v are colored 1 and since this is also the case in β , one can conclude with Proposition 2.1. Assume that $v \in K_{u, \alpha(w)}(\alpha, G)$, as all the vertices are adjacent to u or v , w cannot be adjacent to any vertex colored 1 other than u , so there must be a vertex y adjacent to both v and u colored $\alpha(w)$. To sum up, all the colors except color 1 are present in the neighborhood of u , and the color $\alpha(w)$ is used by both w and y .

If y is not locked, one can perform a trivial Kempe change on y , after which $v \notin K_{u, \alpha(w)}(\tilde{\alpha}, G)$ and we are back in the previous situation. If any vertex of $N(u) \setminus \{w, y\}$ is not locked, after a trivial Kempe change, u is not locked anymore.

This proves the claim. From now on, we assume that v belongs to $K_{u, \alpha(w)}(\alpha, G)$, that the vertices of $N[u] \setminus \{w\}$ are locked in α , and that the color $\alpha(w)$. We make the symmetric assumptions for β . To conclude this proof, we discuss the two following cases :

- If $|C| \geq 3$, let $z \in C \setminus \{u, w\}$. Each vertex in G is adjacent to u or v , so u is the only neighbor of w colored 1 in α . Likewise, w is the only neighbor of u colored 1 in β . By assumption, z is the only neighbor of u colored $\alpha(z)$ in α , so $K_{z, 1}(\alpha, G) = \{z, u\}$. The same argument applied in β gives us $K_{z, 1}(\beta, G) = \{z, w\}$. By performing each of these Kempe changes, we obtain two colorings that both color z and v with color 1 and we can conclude with Proposition 2.1.
- Otherwise, if $|C| = 2$, G contains v , its k neighbors and u and w . The vertices u and w are adjacent to one another and are each adjacent to $k - 1$ neighbors of v . So $S = N(u) \cap N(w)$ contains at least $k - 2$ vertices. By assumption, there exists a vertex z in $N(u) \cap N(v)$ that is colored $\alpha(w)$ by α and thus is not adjacent to w . Likewise, there exists a vertex z' in $N(w) \cap N(v)$ that is colored $\beta(u)$ by β and thus is not adjacent to u . As a result, S contains exactly $k - 2$ vertices. By assumption, all the vertices of S are locked in α and since both v and u are colored 1 in α , the vertices of S each have exactly one neighbor of each other color. But the vertices of S are adjacent to w , so they cannot be adjacent to z which is colored alike. So z has at most three neighbors : v, w and z' , contradicting $k \geq 4$. As a result, the assumptions of Claim 10 are always met when $|C| = 2$.

■