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ABSTRACT

In this paper, a gradually deteriorating system with imperfect repair is considered. The deterioration is modeled by a stationary stochastic process. The system fails once the deterioration level exceeds a given threshold L. At failure, an imperfect repair is performed and the deterioration level is reduced to a fixed value r, say. The system can be repaired $n-1$ times and will be replaced after the nth failure. The article aims to estimate the parameters of the proposed deterioration process based on the observed failures. To this end we consider the Wiener and Gamma processes which are the most common used stochastic process models. In Wiener process, an explicit expression for the estimators is obtained. Birnbaum–Saunders approximation is extended to estimate the parameters in Gamma process. An optimal replacement policy is also discussed. Finally, a Monte-Carlo simulation is conducted to investigate the performance of estimators.

1. Introduction

For some high reliable systems, it is difficult to observe failure times. Degradation measurements contain useful information about system reliability in these situations. Degradation is the reduction in performance, reliability, and life span of systems. The deterioration level of a system is represented by a degradation process. Such degradation process can be modeled using a stochastic process. According to Lehmann [1], the stochastic-process-based approach shows great flexibility in describing the failure mechanisms caused by degradation. The most popular processes used in this area are Gamma and Wiener.

Abdel-Hameed [2] was the first to propose the Gamma process as a proper model for non-decreasing deterioration in time. A broad survey about Gamma process were performed by Çinlar et al. [3], Van Noortwijk and Klatter [4], Grall et al. [5], Nicolai et al. [6] and Van Noortwijk [7]. Wiener process is based on normally distributed increments and models a continuous deterioration with an increasing trend and a non-monotonous trajectory. Doksum and Hóyland [8], Whitmore and Schenkelberg [9] and Kahle [10] proposed the Wiener process as a proper degradation model.

We consider the situation in which failure is defined in terms of an observable characteristic. A system fails when its level of degradation reaches a specified failure threshold. For example, consider fluorescent light that its brightness decreases over time. The failure may be defined to occur when the lights luminosity reaches to a specified percent of its luminosity at 100 h of use. In growing a crack, a failure can also be defined when the length of a crack exceeds a critical level. Such failures are known as “soft” failures because the units are still working, but their performance are not

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acceptable. Thus, some preventive maintenance actions should be applied to maintain the system. Most of the efforts in this area deal with maintenance policy optimization and the issue of model parameter estimation is not largely addressed in the literature. See for example, Abdel-Hameed [11,12], Abdel-Hameed and Nakhi [13], Speijker et al. [14], Wang [15], Kallen and Van Noortwijk [16] and Liao et al. [17].

Estimation of reliability and maintainability parameters is essential in modeling repairable systems and determining maintenance policies. In order to help users for more degradation data exploitation in imperfect maintenance planning, it is important to emphasize more extensively the statistical inference of this kind of data for specific case studies. In this regard, we consider a repairable system equipped with a warning light turned on as soon as the degradation level passes L. The system undergoes an imperfect repair and its degradation reduces to a fixed value $0 \leq r < L$, say. The system can be repaired $n-1$ times and will be replaced after the nth failure. For example, a worn aircraft tire is refreshed several times using the technique of retreading and replaced after a specified numbers of landings. A toner cartridge can be refilled up to three times and eventually needs to be replaced. Steel structures are protected by an organic coating system to prevent them from corrosion. Imperfect maintenance actions such as spot repair and repainting can be done to extend the lifetime of the coating [6]. Finally in a replacement action the old coating and all corrosion is completely removed and a new coating is applied. Maintenance of the roadways is the primary way that unwanted pavement distresses are reduced or eliminated. After several reconstructions of the distresses, the road asphalt needs to be replaced.

We assume that the system deterioration can be modeled by a stationary Wiener or Gamma process, because the "virtual age" of the unit just after each imperfect repair is assumed to be unknown and then no age-dependent degradation can be assumed. This assumption leads to homogeneous degradation processes for the proposed model. Based on the observed failure times, we find an expression for the likelihood functions in presence of Wiener and Gamma processes and estimate the model parameters. In our derivations, we use the distribution of the first hitting time of the processes. It is known that the first hitting time distribution of Wiener process follows an inverse Gaussian distribution and for a Gamma process, it can be approximated by Birnbaum–Saunders (BS) distribution. We extend the Birnbaum–Saunders approximation to find a closed form of the likelihood function for Gamma process. For more details on the first hitting time, we refer to Abdel-Hameed [2], Chhikara and Folks [18], Frenk and Nicolai [19], Shu et al. [20] and Balakrishnan and Qin [21].

The paper is structured as follows. Section 2 is devoted description the maintenance policy and model assumptions. Likelihood functions and parameter estimations for two mentioned processes are derived in Section 3. In Section 4, the expected cost per time unit is derived and the optimal number of imperfect repairs which minimize the expected cost rate are determined. Next in Section 5, we conduct a simulation study. Finally, the paper ends with some conclusions and directions for future works.

2. First hitting time of Wiener and Gamma processes

First hitting times are used in a wide area of applications including medicine, environmental science, engineering, economy and sociology. Consider a repairable system with degradation process $\{X_t, t \geq 0\}$, under the following assumption:

1. The system starts working at time 0.
2. After starting, the degradation process is stochastically increasing with time, which can be modeled by Wiener or Gamma process.
3. A system failure occurs when the deterioration level exceeds the threshold L. Let τ_L be the time to the system failure (the first hitting time), that is

$$\tau_L = \inf\{t > 0 : X_t > L\}$$

4. The system is equipped with a warning light turned on as soon as a failure is detected. The repair time is also negligible.
5. After failure, an imperfect repair reduces the degradation level to $r \in [0, L)$.
6. The system can be repaired $n-1$ times and will be replaced after the nth failure.

In the sequel, the probability density function of the first hitting time of Wiener and Gamma processes is obtained which is used in our derivations.

The Wiener process has continuous sample path and is a simple model for random accumulation of the degradation over time. This process with a linear drift, extensively was studied in degradation modeling, especially due to the existence of an analytical expression of the first hitting time distribution which allows more simple mathematical developments. A Wiener process $\{X_t, t \geq t_0\}$ with a linear drift (stationary Wiener process) can be generally described by the following model [see 22]

$$X_t = x_0 + \sigma W_{t-t_0} + \mu(t - t_0), \qquad \forall t \geq t_0, \tag{1}$$

with

t_0—beginning of the process ($t_0 \in \mathbb{R}$),

x_0—constant initial process (degradation) at time t_0 ($x_0 \in \mathbb{R}$),
μ—drift parameter ($\mu \in \mathbb{R}$),
σ—standard deviation parameter ($\sigma > 0$),
W_t—the standard Brownian motion on $[0, \infty)$.

The stationary Wiener process has independent and normally distributed random increments, i.e. for all $s < t$, $X_t - X_s$ is independent of X_s. The conditional distribution of the degradation level X_t given $X_s = y$ follows a normal distribution with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$ given by

$$f_{X_t}(x|X_s = y) = \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \exp\Big(-\frac{(x-y-\mu(t-s))^2}{2\sigma^2(t-s)}\Big).$$

Let X_t represent the deterioration accumulated by a system on $[t_0, t]$. The system is assumed to fail as soon as its degradation level exceeds a known fixed threshold L. Then, the time-to-failure of the system is given by

$$\tau_L = \inf\{t > t_0 : X_t > L\},$$

If the degradation process is modeled by a Wiener process such as Eq. (1), then it is well known that for $x_0 < L$, the lifetime τ_L follows an inverse Gaussian distribution with Lebesgue density function

$$f_{\tau_L}(t) = \frac{L - x_0}{\sqrt{2\pi\sigma^2(t-t_0)^3}} \exp\Big(-\frac{(L-x_0-\mu(t-t_0))^2}{2\sigma^2(t-t_0)}\Big) I_{\{t>t_0\}}, \tag{2}$$

where $I(.)$ denotes the indicator function. More details on the first hitting times of Wiener process is given by Chhikara and Folks [18].

There are some situations with monotonous trajectories which the Wiener process cannot cover all degradation measurements. Consequently, the Gamma process is preferred to the Wiener process. This process is applicable to model the always positive and strictly increasing degradation data and is a pure jump process with an infinite number of jumps over any finite time interval. Therefore, this process is suitable to model some non-decreasing deterioration with many tiny increments.

Let $\{X_t, t > 0\}$, be a right-continuous stochastic process with left-side limits. Then, X_t is called a homogeneous Gamma process with shape function αt ($\alpha > 0$) and scale parameter $\beta > 0$ if

1. $X_0 = 0$, almost surely.
2. The stochastic process $\{X_t, t > 0\}$, has independent increments.
3. An increment of this process is expressed as $X_t - X_s$, $t > s$, which is independent of X_s. The conditional distribution of the degradation level X_t given $X_s = y$ has a gamma distribution with mean $\alpha(t-s)\beta$ and variance $\alpha(t-s)\beta^2$. Then, we have

$$f_{X_t}(x|X_s = y) = \frac{1}{\Gamma(\alpha(t-s))\beta^{\alpha(t-s)}} (x-y)^{\alpha(t-s)-1} e^{-\frac{(x-y)}{\beta}}, \quad \forall y < x. \tag{3}$$

Let τ_L be the first passage time of the degradation process to the known constant threshold L, the distribution function of τ_L is given by

$$F_{\tau_L}(t) = P(\tau_L \le t) = P(X_t > L) = \int_L^\infty \frac{1}{\Gamma(\alpha t)\beta^{\alpha t}} x^{\alpha t - 1} e^{-\frac{x}{\beta}} dx,$$

or equivalently

$$F_{\tau_L}(t) = \frac{\Gamma(\alpha t, L/\beta)}{\Gamma(\alpha t)}, \tag{4}$$

where $\Gamma(\alpha, z) = \int_z^\infty x^{\alpha-1} e^{-x} dx$ denotes the incomplete gamma function for $z \ge 0$ and $\alpha > 0$. The density of τ_L can be also expressed as

$$f_{\tau_L}(t) = \frac{d}{dt} \frac{\Gamma(\alpha t, L/\beta)}{\Gamma(\alpha t)}. \tag{5}$$

3. Data modeling and estimation of unknown parameters

3.1. Data modeling and contributions to likelihood

We consider a repairable system that undergoes an imperfect repair when its degradation level exceeds the threshold L. The imperfect repair reduces the degradation level to constant r, where $0 \le r < L$. The system can be repaired $n-1$ times and is replaced after the nth failure. Under the Wiener and Gamma processes a part of sample path of the proposed model is presented in Fig. 1. Let τ_i, $1 \le i \le n$ denote the time that the system degradation exceeds the level L for ith times. In this section, we first find an expression for the joint density of successive hitting times $\tau_1, \tau_2, \ldots, \tau_n$, with observed values $\mathbf{t} = (t_1, \ldots, t_n)$. Then, we obtain the maximum likelihood estimator (MLE) of the unknown parameters, under the Wiener and Gamma processes.

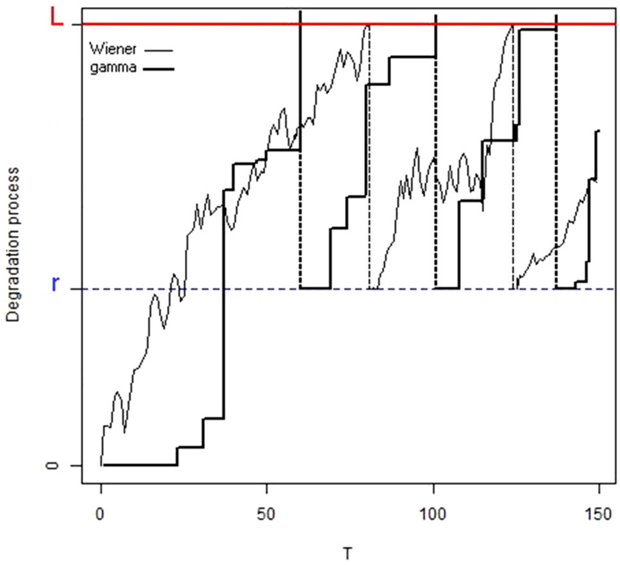

Fig. 1. A sample path of the degradation model.

3.2. Wiener process

The Wiener process in (1) has stationary, independent and normally distributed increments. Then, for $i \geq 2$, $\tau_i|(\tau_{i-1} = t_{i-1})$ has the same distribution with the first hitting time in a Wiener process with initial value $x_0 = r$ at time $t_0 = t_{i-1}$, that the degradation level exceeds the threshold L, i.e. $\tau_i|(\tau_{i-1} = t_{i-1}) \overset{d}{=} \tau_L|X_{t_0} = r, i = 2, \ldots, n$. So from Eq. (2),

$$f_{\tau_i|(\tau_{i-1}=t_{i-1})}(t_i) = \frac{L-r}{\sqrt{2\pi\sigma^2(t_i - t_{i-1})^3}} \exp(-\frac{(L-r-\mu(t_i - t_{i-1}))^2}{2\sigma^2(t_i - t_{i-1})}), \quad i \geq 2.$$

The likelihood function can also be formulated as

$$\begin{aligned} L(\mu, \sigma|\boldsymbol{t}) &= f_{\tau_1,\tau_2,\ldots,\tau_n}(t_1, t_2, \ldots, t_n) \\ &= f_{\tau_1}(t_1) f_{\tau_2}(t_2|\tau_1 = t_1) \cdots f_{\tau_n}(t_n|\tau_{n-1} = t_{n-1}) \\ &= \frac{L}{\sqrt{2\pi\sigma^2 t_1^3}} \exp(-\frac{(L-\mu t_1)^2}{2\sigma^2 t_1}) \\ &\times \prod_{i=2}^{n} \frac{L-r}{\sqrt{2\pi\sigma^2(t_i - t_{i-1})^3}} \exp(-\frac{(L-r-\mu(t_i - t_{i-1}))^2}{2\sigma^2(t_i - t_{i-1})}), \qquad 0 < t_1 < \cdots < t_n. \end{aligned}$$

In the case of k systems we have:

$$\begin{aligned} L(\mu, \sigma|\tilde{\boldsymbol{t}}) &= \prod_{j=1}^{k} \frac{L}{\sqrt{2\pi\sigma^2 t_{j,1}^3}} \exp(-\frac{(L-\mu t_{j,1})^2}{2\sigma^2 t_{j,1}}) \\ &\times \prod_{j=1}^{k} \prod_{i=2}^{m_j+1} \frac{L-r}{\sqrt{2\pi\sigma^2(t_{j,i} - t_{j,i-1})^3}} \exp(-\frac{(L-r-\mu(t_{j,i} - t_{j,i-1}))^2}{2\sigma^2(t_{j,i} - t_{j,i-1})}), \end{aligned}$$

where $\tilde{\boldsymbol{t}} = (\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots, \boldsymbol{t_k})$, $\mathbf{t}_i$ is the observed first hitting times of ith system and m_j is the number of performed imperfect repairs in the jth system. Then, the log-likelihood function can be represented as

$$\begin{aligned} \ell \propto &- \frac{1}{2} k \log \sigma^2 - \sum_{j=1}^{k} \frac{(L-\mu t_{j,1})^2}{2\sigma^2 t_{j,1}} - \frac{1}{2} \sum_{j=1}^{k} m_j \log(\sigma^2) \\ &- \sum_{j=1}^{k} \sum_{i=2}^{m_j+1} \frac{(L-r-\mu(t_{j,i} - t_{j,i-1}))^2}{2\sigma^2(t_{j,i} - t_{j,i-1})}. \end{aligned}$$

The MLEs of unknown parameters can be obtained by taking the partial derivatives of the log-likelihood function with respect to the parameters as

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma^2}\sum_{j=1}^{k}(L-\mu t_{j,1}) + \sum_{j=1}^{k}\sum_{i=2}^{m_j+1}\frac{L-r-\mu(t_{j,i}-t_{j,i-1})}{\sigma^2},$$

so,

$$\frac{\partial \ell}{\partial \mu} = 0 \Rightarrow \hat{\mu} = \frac{kL+(L-r)\sum_{j=1}^{k} m_j}{\sum_{j=1}^{k} t_{j,1} + \sum_{j=1}^{k}\sum_{i=2}^{m_j+1}(t_{j,i}-t_{j,i-1})} = \frac{kL+(L-r)\sum_{j=1}^{k} m_j}{\sum_{j=1}^{k} t_{j,m_j+1}}.$$

Also

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{1}{2}\frac{k+\sum_{j=1}^{k} m_j}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{j=1}^{k}\frac{(L-\mu t_{j,1})^2}{t_{j,1}} + \frac{1}{2\sigma^4}\sum_{j=1}^{k}\sum_{i=2}^{m_j+1}\frac{(L-r-\mu(t_{j,i}-t_{j,i-1}))^2}{t_{j,i}-t_{j,i-1}}.$$

Then,

$$\frac{\partial \ell}{\partial \sigma^2} = 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{k+\sum_{j=1}^{k} m_j}\Big(\sum_{j=1}^{k}\frac{(L-\hat{\mu} t_{j,1})^2}{t_{j,1}} + \sum_{j=1}^{k}\sum_{i=2}^{m_j+1}\frac{(L-r-\hat{\mu}(t_{j,i}-t_{j,i-1}))^2}{t_{j,i}-t_{j,i-1}}\Big).$$

3.3. Gamma process

Suppose $\tau_1, \tau_2, \ldots, \tau_n$ are the successive of first passage times of threshold L after n repairs. Recall that τ_1 is the first time that the process with initial value $x_0 = 0$ passes L. Then, τ_i; $2 \le i \le n$ denotes the next passage times when $x_0 = r$. Since in a homogeneous Gamma process, the increments are independent and gamma distributed, so, by considering the characteristics of increments, the conditional distribution of τ_i given $\tau_{i-1} = t_{i-1}$ is the same as the first time that a shifted Gamma process with initial value r exceeds the threshold L, i.e.

$$\begin{aligned} F_{\tau_i}(t_i|\tau_{i-1} = t_{i-1}) &= P(\tau_i \le t_i|\tau_{i-1} = t_{i-1}) \\ &= P(X_{t_i - t_{i-1}} + r > L) \\ &= \frac{\Gamma(\alpha(t_i - t_{i-1}), (L-r)/\beta)}{\Gamma(\alpha(t_i - t_{i-1}))}, \qquad \text{for } i = 2, \ldots, n. \end{aligned}$$

Then, the joint density function of $\tau_1, \tau_2, \ldots, \tau_n$ can be derived from Eqs. (3) and (5) as

$$\begin{aligned} f_{\tau_1,\tau_2,\ldots,\tau_n}(t_1, t_2, \ldots, t_n) &= f_{\tau_1}(t_1) f_{\tau_2}(t_2|\tau_1 = t_1)\cdots f_{\tau_n}(t_n|\tau_{n-1} = t_{n-1}) \\ &= \frac{d}{dt_1}\frac{\Gamma(\alpha t_1, L/\beta)}{\Gamma(\alpha t_1)}\prod_{i=2}^{n}\frac{d}{dt_i}\frac{\Gamma(\alpha(t_i - t_{i-1}), (L-r)/\beta)}{\Gamma(\alpha(t_i - t_{i-1}))}. \end{aligned} \tag{6}$$

As it can be seen, finding an explicit expression for the likelihood function from Eq. (6) is difficult and leads to complex derivations. Therefore, we conduct to use some useful approximation like 'Birnbaum–Saunders' and follow the same approach in Birnbaum and Saunders [23] to find MLEs of the parameters. Park and Padgett [24] approximated (4) based on Birnbaum–Saunders (BS) distribution as follows

$$F_{\tau_L}(t) \simeq \Phi\Big(\frac{1}{\alpha^*}(\sqrt{\frac{t}{\beta^*}} - \sqrt{\frac{\beta^*}{t}})\Big), \tag{7}$$

where $\alpha^* = \sqrt{\frac{\beta}{L}}$, $\beta^* = \frac{L}{\alpha\beta}$ and $\Phi(\cdot)$ is the cdf of the standard normal distribution.

In the sequel, for more convenience, the following equations are utilized

$$\xi(x) = x^{\frac{1}{2}} - x^{-\frac{1}{2}}, \tag{8}$$

$$\xi^2(x) = x - x^{-1} - 2, \tag{9}$$

$$\xi'(x) = \frac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{2x} = \frac{1}{2\xi(x)}(1 - \frac{1}{x^2}), \tag{10}$$

$$\frac{x\xi''(x)}{\xi'(x)} = -1 + \frac{1}{2}\frac{x-1}{x+1} = -\frac{1}{2} - \frac{1}{x+1}. \tag{11}$$

Now Eq. (7) can be rewritten as

$$F_{\tau_L}(t) \simeq \Phi(\alpha^{*-1}\xi(\frac{t}{\beta^*})), \qquad t>0,\ \alpha>0,\ \beta>0.$$

Then the pdf of τ_L is

$$f_{\tau_L}(t) \simeq \frac{1}{\alpha^*\beta^*}\xi'(\frac{t}{\beta^*})\phi(\alpha^{*-1}\xi(\frac{t}{\beta^*})), \tag{12}$$

which is shown by $f_{BS}(t;\alpha^*,\beta^*)$, and $\phi(\cdot)$ is the pdf of the standard normal distribution.

Now the likelihood function can be approximated by substituting (12) in (6) as

$$\begin{aligned} L(\breve{\alpha},\breve{\beta}|\boldsymbol{t}) &\simeq f_{BS}(t_1;\alpha^*,\beta^*)\prod_{i=2}^{n} f_{BS}(t_i-t_{i-1};\breve{\alpha},\breve{\beta}) \\ &= \sqrt{\frac{L-r}{L}}\frac{1}{\breve{\alpha}\breve{\beta}}\phi\Big((\sqrt{\frac{L-r}{L}}\breve{\alpha})^{-1}\xi(\frac{t_1(L-r)}{\breve{\beta}L})\Big)\xi'(\frac{t_1(L-r)}{\breve{\beta}L}) \\ &\times\prod_{i=2}^{n}\frac{1}{\breve{\alpha}\breve{\beta}}\phi\Big(\breve{\alpha}^{-1}\xi(\frac{t_i-t_{i-1}}{\breve{\beta}})\Big)\xi'(\frac{t_i-t_{i-1}}{\breve{\beta}}), \end{aligned} \tag{13}$$

where $\boldsymbol{t}=(t_1,t_2,\ldots,t_n)$, $\breve{\alpha}=\sqrt{\frac{\beta}{L-r}}=\sqrt{\frac{L}{L-r}}\alpha^*$, $\breve{\beta}=\frac{L-r}{\alpha\beta}=\frac{L-r}{L}\beta^*$ and $0<r<L$.

The log-likelihood function is given by

$$\begin{aligned} \ell = \log(L(\breve{\alpha},\breve{\beta}|\boldsymbol{t})) &\propto -n\log(\breve{\alpha}\breve{\beta}) - \frac{1}{2}\frac{L}{L-r}\breve{\alpha}^{-2}\xi^2(\frac{t_1^*}{\breve{\beta}}) - \frac{1}{2}\sum_{i=2}^{n}\breve{\alpha}^{-2}\xi^2(\frac{t_i-t_{i-1}}{\breve{\beta}}) \\ &+ \log(\xi'(\frac{t_1^*}{\breve{\beta}})) + \sum_{i=2}^{n}\log(\xi'(\frac{t_i-t_{i-1}}{\breve{\beta}})), \end{aligned} \tag{14}$$

where $t_1^*=\frac{L-r}{L}t_1$. By taking derivative with respect to $\breve{\alpha}$ of Eq. (14) and equating to zero we have

$$-\breve{\alpha}^3\frac{\partial\ell}{\partial\breve{\alpha}} = n\breve{\alpha}^2 - \frac{L}{L-r}\xi^2(\frac{t_1^*}{\breve{\beta}}) - \sum_{i=2}^{n}\xi^2(\frac{t_i-t_{i-1}}{\breve{\beta}}). \tag{15}$$

After substitution (9) in (15), we have

$$n\breve{\alpha}^2 = \frac{1}{\breve{\beta}}\Big(\frac{L}{L-r}t_1^* + \sum_{i=2}^{n}(t_i-t_{i-1})\Big) + \breve{\beta}\Big(\frac{L}{L-r}\frac{1}{t_1^*} + \sum_{i=2}^{n}\frac{1}{t_i-t_{i-1}}\Big) - 2(\frac{L}{L-r}+n-1),$$

Or equivalently

$$\begin{aligned} \frac{n\breve{\alpha}^2}{\frac{L}{L-r}+n-1} &= \frac{1}{\breve{\beta}}\Big(\frac{\frac{L}{L-r}t_1^* + \sum_{i=2}^{n}(t_i-t_{i-1})}{\frac{L}{L-r}+n-1}\Big) + \breve{\beta}\Big(\frac{\frac{L}{L-r}\frac{1}{t_1^*} + \sum_{i=2}^{n}\frac{1}{t_i-t_{i-1}}}{\frac{L}{L-r}+n-1}\Big) - 2, \\ &= \frac{s}{\breve{\beta}} + \frac{\breve{\beta}}{h} - 2, \end{aligned} \tag{16}$$

where s is the weighted mean of $t_1^*, t_2-t_1, \ldots, t_n-t_{n-1}$ and $\frac{1}{h}$ is the weighted mean of $\frac{1}{t_1^*}, \frac{1}{t_2-t_1}, \ldots, \frac{1}{t_n-t_{n-1}}$ with correspond weights $\frac{L}{L-r}, 1, \ldots, 1$, respectively.

Taking derivative of Eq. (14) with respect to $\breve{\beta}$ and using Eqs. (8) to (11), we conduct to

$$\begin{aligned} \frac{\partial\ell}{\partial\breve{\beta}} &= -\frac{n}{\breve{\beta}} + \frac{1}{\breve{\alpha}^2\breve{\beta}}\Big(\frac{L}{L-r}\frac{t_1^*}{\breve{\beta}}\xi(\frac{t_1^*}{\breve{\beta}})\xi'(\frac{t_1^*}{\breve{\beta}}) + \sum_{i=2}^{n}\frac{t_i-t_{i-1}}{\breve{\beta}}\xi(\frac{t_i-t_{i-1}}{\breve{\beta}})\xi'(\frac{t_i-t_{i-1}}{\breve{\beta}})\Big) \\ &-\frac{1}{\breve{\beta}^2}\Big(t_1^*\frac{\xi''(\frac{t_1^*}{\breve{\beta}})}{\xi'(\frac{t_1^*}{\breve{\beta}})} + \sum_{i=2}^{n}(t_i-t_{i-1})\frac{\xi''(\frac{t_i-t_{i-1}}{\breve{\beta}})}{\xi'(\frac{t_i-t_{i-1}}{\breve{\beta}})}\Big), \\ &= -\frac{n}{2\breve{\beta}} + \frac{1}{2\breve{\alpha}^2\breve{\beta}}\Big(\frac{1}{\breve{\beta}}(\frac{L}{L-r}t_1^* + \sum_{i=2}^{n}(t_i-t_{i-1})) \\ &-\breve{\beta}(\frac{L}{L-r}\frac{1}{t_1^*} + \sum_{i=2}^{n}\frac{1}{t_i-t_{i-1}})\Big) + \frac{1}{t_1^*+\breve{\beta}} + \sum_{i=2}^{n}\frac{1}{t_i-t_{i-1}+\breve{\beta}}, \end{aligned} \tag{17}$$

so

$$2\breve{\alpha}^2\breve{\beta}\frac{\partial \ell}{\partial \breve{\beta}} = -n\breve{\alpha}^2 + \frac{As}{\breve{\beta}} - \frac{A\breve{\beta}}{h} + \frac{2n\breve{\alpha}^2\breve{\beta}}{K(\breve{\beta})}, \tag{18}$$

where $A = \frac{L}{L-r} + n - 1$ and $K(\breve{\beta}) = n\left(\frac{1}{t_1^*+\breve{\beta}} + \sum_{i=2}^{n}\frac{1}{t_i-t_{i-1}+\breve{\beta}}\right)^{-1}$. Therefore, Eq. (18) cause to

$$\frac{n\breve{\alpha}^2}{A} = \frac{s}{\breve{\beta}} - \frac{\breve{\beta}}{h} + \frac{2n\breve{\alpha}^2\breve{\beta}/A}{K(\breve{\beta})}.$$

Thus, from Eq. (16) we have

$$\frac{\breve{\beta}}{h} = 1 + \frac{n\breve{\alpha}^2\breve{\beta}/A}{K(\breve{\beta})}. \tag{19}$$

Substituting (16) in (19) implies

$$\frac{\breve{\beta}}{h} = 1 + \frac{s + \frac{\breve{\beta}^2}{h} - 2\breve{\beta}}{K(\breve{\beta})}.$$

The maximum likelihood estimate of $\widehat{\breve{\beta}}$ is the solution of $g(\breve{\beta}) = 0$, where

$$g(\breve{\beta}) = \breve{\beta}^2 - \breve{\beta}(2h + K(\breve{\beta})) + h(s + K(\breve{\beta})).$$

In the next theorem the properties of $\widehat{\breve{\beta}}$ such as existence, uniqueness and being MLE of $\breve{\beta}$ are given. The proof is provided in the Appendix.

Theorem 1. *Let $\tau_1, \tau_2, \ldots, \tau_n$ denote independent random variables with joint distribution in Eq. (13). Then*

(a) $g(\breve{\beta}) = 0$ results a unique positive solution, denoted by $\widehat{\breve{\beta}}$ and $h < \widehat{\breve{\beta}} < s$.
(b) $\widehat{\breve{\beta}}$ is the MLE of $\breve{\beta}$.
(c) The MLE $\widehat{\breve{\alpha}}$ of $\breve{\alpha}$ is given by

$$\widehat{\breve{\alpha}} = \left(\frac{A}{n}\left(\frac{s}{\widehat{\breve{\beta}}} + \frac{\widehat{\breve{\beta}}}{h} - 2\right)\right)^{\frac{1}{2}},$$

where $A = \frac{L}{L-r} + n - 1$.

4. Replacement policy

One of the purposes of reliability analysis is quantifying the probability by any attempt to measure it involving probabilistic and statistical methods. Reliability analysis incorporates activities to identify potential failure modes and mechanisms, to make reliability predictions, and to quantify risks for critical components to optimize the life cycle costs for a product [25].

However, a model of reliability estimation based on the lifetime analysis, should be developed to optimize the maintenance dispatches, replacement schedules, availability, reliability, etc. In this section, we determine the optimal number of preventive maintenance before replacement.

Suppose that the system can be repaired $n-1$ times and will be replaced by a new one at the nth failure, and C_1 and C_2 are related to the cost of each preventive imperfect repair and scheduled corrective replacement, respectively. Let T_i denote the time to the next failure of a system after it has been repaired $i-1$ times, $i \geq 1$. Then, $\mu_n = E(\sum_{i=1}^{n} T_i) = E(\tau_n)$ is the mean time to system replacement. The expected cost per unit of time for an infinite time span is given by

$$C(n) = \frac{(n-1)C_1 + C_2}{\mu_n},$$

[26]. Now, to find a value of n^* which minimizes $C(n)$, we solve the inequalities

$$C(n+1) \geq C(n) \quad \text{and} \quad C(n) < C(n-1),$$

which imply

$$L(n) \geq \frac{C_2}{C_1} \quad \text{and} \quad L(n-1) < \frac{C_2}{C_1}, \tag{20}$$

Table 1
Optimal number n^*.

$\frac{C_2}{C_1}$ \ c	1.2	1.5	1.8	2.1
5	4	3	2	2
10	7	4	3	3
15	8	5	4	3
20	9	5	4	3

where,

$$L(n) = \frac{\mu_{n+1}}{\mu_{n+1} - \mu_n} - n = \begin{cases} \frac{E(\sum_{i=1}^{n+1} T_i)}{E(T_{n+1})} - n & n = 1, 2, 3, \ldots, \\ 0 & n = 0. \end{cases} \tag{21}$$

When the system is not deteriorated after each successive imperfect repair, as has been assumed heretofore, let $\mu_L = E(T_1) = E(\tau_1)$ and $\mu_{L-r} = E(T_i) = E(\tau_i - \tau_{i-1})$, $i = 2, 3, \ldots$. Then, $L(n) = \frac{\mu_L}{\mu_{L-r}}$ and the optimal n value, say n^*, is given by:

$$n^* = \begin{cases} 1 & \text{if } \frac{\mu_L}{\mu_{L-r}} > \frac{C_2}{C_1}, \\ \forall n & \text{if } \frac{\mu_L}{\mu_{L-r}} = \frac{C_2}{C_1}, \\ \infty & \text{if } \frac{\mu_L}{\mu_{L-r}} < \frac{C_2}{C_1}. \end{cases}$$

In the real situations, it is more logical to assume that after each successive imperfect repair, the system is deteriorated and mean time to next failure decreases, i.e., $E(T_i) \geq E(T_{i+1})$, $i = 1, \ldots, n-1$. To take into account this fact, we consider an especial case in Gamma process with scale parameter β, such that after $(i-1)$th repair, the shape parameter of Gamma process changes to $\alpha_i = c^{i-1}\alpha$, $i = 1, 2, 3, \ldots, n$, where constant $c > 1$ is known. Note that the parameter estimation in this case can also be obtained by following the same approach in Section 3.3.

Now, by using Birnbaum–Saunders approximation, we find an expression for μ_n. If T has a Birnbaum–Saunders distribution with pdf (12), then $E(T) = \beta^*(1 + \frac{\alpha^{*2}}{2})$, more details is given by Kundu et al. [27]. Since $T_i \sim BS(\alpha_i^*, \beta_i^*)$, where $\alpha_1^* = \alpha^*$, $\beta_1^* = \beta^*$, $\alpha_i^* = \breve{\alpha}$ and $\beta_i^* = \frac{\breve{\beta}}{c^{i-1}}$; $i = 2, \ldots, n$, then $E(T_1) = \frac{L}{\alpha\beta}(1 + \frac{\beta}{2L}) = a$ and

$$E(T_i) = \frac{L-r}{\alpha\beta}(1 + \frac{\beta}{2(L-r)})\frac{1}{c^{i-1}}, \quad i = 2, \ldots, n.$$

We thus obtain

$$\mu_n = a + b\sum_{i=2}^{n} \frac{1}{c^{i-1}} = a + b\frac{\frac{1}{c} - (\frac{1}{c})^n}{1 - \frac{1}{c}} = a^* - \frac{b^*}{c^{n-1}},$$

where $a^* = a + \frac{b}{c-1}$, $b = \frac{L-r}{\alpha\beta}(1 + \frac{\beta}{2(L-r)})$ and $b^* = \frac{b}{c-1}$.

Hence, Eq. (21) reduces to

$$L(n) = \begin{cases} \frac{a^*}{b}c^n - n - \frac{1}{c-1} & n = 1, 2, 3, \ldots, \\ 0 & n = 0. \end{cases}$$

It is evident that $L(n+1) - L(n) = \frac{a^*}{b}(c^{n+1} - c^n) - 1 > 0$, and $L(n) \to \infty$ as $n \to \infty$. Thus, there exists a unique and finite $n^* > 1$ satisfying inequalities in (20) if $L(1) = \frac{ac}{b} < \frac{C_2}{C_1}$, which is equivalent to $\frac{(2L+\beta)c}{2(L-r)+\beta} < \frac{C_2}{C_1}$. Note that the values of $L(n)$ are independent of α, and decrease with β. Table 1 gives the optimal number of imperfect repair for $r = 5$, $L = 10$, $\beta = 0.5$ and different values of $\frac{C_2}{C_1}$ and c. This shows that we can make more repair on the system when $\frac{C_2}{C_1}$ is larger and c is smaller.

5. Simulation study

We assess the performance of our proposed imperfect repair model by using Monte-Carlo simulated degradation data of the Wiener and Gamma processes. We repeat the simulation 100,000 times for $r = 5$ and $L = 10$, and choose

Table 2
Biases and MSEs of $(\hat{\mu}, \hat{\sigma}^2)$ in Wiener process for different numbers of imperfect repair.

n	Bias		MSE	
	$\hat{\mu}$	$\hat{\sigma}^2$	$\hat{\mu}$	$\hat{\sigma}^2$
10	0.03533	−0.20407	0.19195	0.87908
20	0.01948	−0.10703	0.10408	0.47092
40	0.01004	−0.04518	0.043834	0.19619

Table 3
Biases and MSEs of $(\hat{\alpha}, \hat{\beta})$ in Gamma process for $r = 5$ and $L = 10$.

n	Bias		MSE		The percentage of removed samples
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	
10	0.7072 (0.7164)	0.7289 (0.5223)	2.7229 (2.7340)	21.5493 (13.5591)	0.57
20	0.2345 (0.2372)	1.0505 (0.9799)	0.7057 (0.7055)	15.8289 (13.6123)	0.24
40	−0.0181 (−0.0180)	1.3016 (1.2967)	0.2766 (0.2765)	11.2253 (11.0542)	0.02

Table 4
Biases and MSEs of $(\hat{\alpha}, \hat{\beta})$ in Gamma process for $r = 30$ and $L = 60$.

n	Bias		MSE	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
10	0.5506	−0.0694	1.8858	2.8301
20	0.1989	0.0764	0.4839	1.6010
40	0.0439	0.1577	0.1797	0.8797

$n = 10$ (small sample size), $n = 20$ (moderate sample size) and $n = 40$ (large sample size) as the number of preventive imperfect repairs. Table 2 shows the bias and MSE of $(\hat{\mu}, \hat{\sigma}^2)$ in Wiener process with parameters $\mu = 5$ and $\sigma = 1.5$. We see that as the number of samples increases MSE values decrease. However, the behavior of MLEs in Gamma process is different. In this process after carrying out simulations with $\alpha = 1.5$ and $\beta = 3$, we found some small values in the simulated interarrival times between failures. These values cause to a large estimation of $\widehat{\breve{\alpha}}$ in BS distribution and finally a large value for MSE($\hat{\beta}$). For example, for $n = 10$, we examined the simulated samples and found that the largest value of $(\hat{\beta} - \beta)^2$=19.765 is related to the simulated interarrival times between failures 0.0119, 1.9973, 1.9416, 2.1320, 2.5541, 1.2168, 4.5238, 1.8299, 2.7956, 3.0053, that contains the small value 0.0119.

After removing the samples that caused large values of $\widehat{\breve{\alpha}}$ ($\widehat{\breve{\alpha}^2} \geq 6 = 10\breve{\alpha}^2$), we can improve the estimated values of $\hat{\beta}$. For $n = 10$, about 0.6 percent of the samples are removed. More details are given in Table 3. In Table 3, the parentheses show the modified values of Biases and MSEs.

For a given deterioration process, as the difference between the initial point r and the failure threshold L increases, it takes much longer to cross L. Therefore, the trajectories are longer and the number of crossing times decreases. Since the estimation is based on the crossing time data, the quality of estimates should be substantially impacted by $L - r$. Table 4 presents biases and MSEs of the estimators in Gamma process for $L - r = 30$. The plot density functions of the first hitting times in Gamma process with $\alpha = 1.5$ and $\beta = 3$ are also displayed for $L - r = 5$ and $L - r = 30$ in Fig. 2. The results show that after increasing the difference $L - r$, the percent of small samples decreases and then the estimators perform better.

6. Conclusions and future extensions

In this work, the Wiener and Gamma degradation models with preventive imperfect repair were analyzed. For a gradually deteriorating system, a simple maintenance model proposed, where the system is imperfectly repaired, as soon as, the degradation level passes a predetermined level L. After performing the maintenance action, the degradation level is reduced to a prespecific value r. The system can be repaired $n - 1$ times and will be replaced after the nth failure. If one could predict, with some measure of confidence, the next failure of the system, then it might help to make the maintenance services more effective at a lower cost. We obtained an expression for the likelihood function of the proposed model, regarding Wiener and Gamma processes, and used the MLE method to estimate the model parameters. These results help us to find a prediction of the next failure and carry out preventive maintenance activities at predetermined intervals of time. In the sequel, for characterizing the structure of the optimal repair and replacement policy a maintenance cost rate is developed. Finally, in order to have numerical examples, we used Monte-Carlo simulation method.

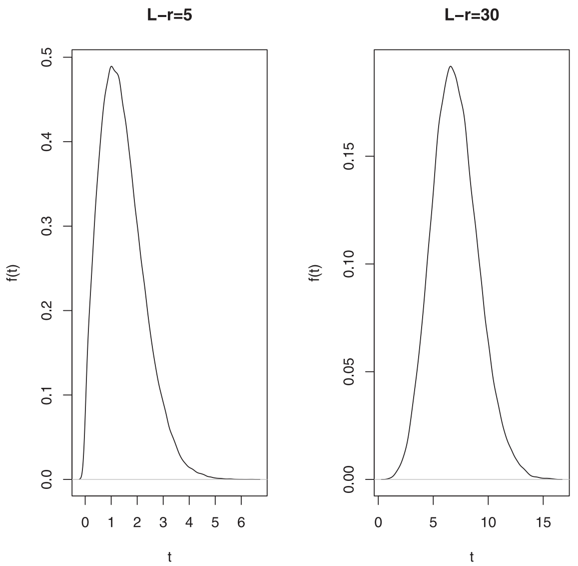

Fig. 2. Plot density functions of the first hitting times in Gamma process.

In this paper we assumed that the effect of each maintenance action is a fixed value. However, in practice depending on the operating environment, or on the use by each customer, this reduction may be different after each repair action. Further, the uncertainty in the inspections could also be taken into account to make the model more realistic.

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Appendix

Proof of Theorem 1.

(a) Note that $g(0) = h(s + K(0)) > 0$. At first we check $g(x) \to -\infty$ as $x \to \infty$. So, pay attention that $\frac{K(x)}{x} \xrightarrow{x\to\infty} 1$, and also $x - K(x) \xrightarrow{x\to\infty} -\frac{1}{n}(t_1^* + \sum_{i=2}^{n}(t_i - t_{i-1}))$, which can be calculated by applying L'Hôpital's rule as follows

$$\lim_{x\to\infty}(x - K(x)) = \lim_{x\to\infty} \frac{\frac{x}{t_1^*+x} + \sum_{i=2}^{n} \frac{x}{t_i-t_{i-1}+x} - n}{\frac{1}{t_1^*+x} + \sum_{i=2}^{n} \frac{1}{t_i-t_{i-1}+x}}$$
$$= \lim_{x\to\infty} \frac{\frac{t_1^*}{(t_1^*+x)^2} + \sum_{i=2}^{n} \frac{t_i-t_{i-1}}{(t_i-t_{i-1}+x)^2}}{-\frac{1}{(t_1^*+x)^2} - \sum_{i=2}^{n} \frac{1}{(t_i-t_{i-1}+x)^2}}$$
$$= \lim_{x\to\infty} \frac{\frac{t_1^*}{(\frac{t_1}{x}^*+1)^2} + \sum_{i=2}^{n} \frac{t_i-t_{i-1}}{(\frac{t_i-t_{i-1}}{x}+1)^2}}{-\frac{1}{(\frac{t_1}{x}^*+1)^2} - \sum_{i=2}^{n} \frac{1}{(\frac{t_i-t_{i-1}}{x}+1)^2}}$$
$$= -\frac{1}{n}(t_1^* + \sum_{i=2}^{n}(t_i - t_{i-1})).$$

Then

$$\frac{g(x)}{x} \xrightarrow{x\to\infty} x - K(x) + h\frac{K(x)}{x} - 2h + \frac{hs}{x},$$

so

$$\frac{g(x)}{x} \xrightarrow{x\to\infty} -(\frac{1}{n}(t_1^* + \sum_{i=2}^{n}(t_i - t_{i-1})) + h) < 0.$$

But

$$g'(x) = (x-h)\Big(1 - K'(x)\Big) + x - h - K(x),$$

and

$$K'(x) = K^2(x)\,\frac{1}{n}\Big(\frac{1}{(t_1^* + x)^2} + \sum_{i=2}^{n} \frac{1}{(t_i - t_{i-1} + x)^2}\Big),$$

and since $(E|X|^{\nu})^{\frac{1}{\nu}}$ is nondecreasing in ν for any random variable X, we have $K'(x) > 1$. Thus, $x - K(x)$ is decreasing, so $x - K(x) < K(0) = h$ for $x \geq 0$ and consequently $g'(x) < 0$ for $x > h$. Therefore, based on monotonous characteristic of $g(x)$, it was shown that $\widehat{\breve{\beta}}$ is unique. After some algebraic calculations, $g(h)$ and $g(s)$ are obtained as follows

$$g(h) = h(s-h), \qquad g(s) = (s-h)(s - K(s)).$$

Since $s > h$, therefore $g(h) > 0$ and by the mentioned argument, there is a unique solution for $g(x) = 0$ as $\widehat{\breve{\beta}}$, which $\widehat{\breve{\beta}} > h$. But $g(s) < 0$ iff

$$\frac{1}{s} > \frac{1}{K(s)} \quad \text{iff} \quad 1 > \frac{1}{n}\Big(\frac{s}{t_1^* + s} + \sum_{i=2}^{n} \frac{s}{t_i - t_{i-1} + s}\Big) = 1 - \frac{1}{n}\Big(\frac{t_1^*}{t_1^* + s} + \sum_{i=2}^{n} \frac{t_i - t_{i-1}}{t_i - t_{i-1} + s}\Big). \tag{22}$$

Therefore, the unique solution $\widehat{\breve{\beta}}$ is such that

$$s > \widehat{\breve{\beta}} > h$$

(b) After proving the existence and uniqueness of $\widehat{\breve{\beta}}$, in the second step we check that it is indeed the MLE of $\breve{\beta}$. By substituting Eq. (16) for $\breve{\alpha}^2$ in Eq. (17) we have

$$\begin{aligned}
\frac{1}{n}\frac{\partial \ell}{\partial \breve{\beta}} &= -\frac{1}{2\breve{\beta}} + \frac{1}{2A\breve{\beta}(\frac{s}{\breve{\beta}} + \frac{\breve{\beta}}{h} - 2)}\Big(\frac{As}{\breve{\beta}} - \frac{A\breve{\beta}}{h}\Big) + \frac{1}{K(\breve{\beta})} \\
&= \frac{-2\frac{A\breve{\beta}}{h} + 2A}{2A\breve{\beta}(\frac{s}{\breve{\beta}} + \frac{\breve{\beta}}{h} - 2)} + \frac{1}{K(\breve{\beta})} \\
&= \frac{h - \breve{\beta}}{hs + \breve{\beta}^2 - 2h\breve{\beta}} + \frac{1}{K(\breve{\beta})}.
\end{aligned}$$

Now, to be sure that $\breve{\beta}$ is the MLE, it is sufficient to note here that $(\frac{\partial \ell}{\partial \breve{\beta}})|_{\breve{\beta}=h} = \frac{1}{K(h)} > 0$ and $(\frac{\partial \ell}{\partial \breve{\beta}})|_{\breve{\beta}=s} = -\frac{1}{s} + \frac{1}{K(s)} < 0$ by Eq. (22).

(c) By substituting $\widehat{\breve{\beta}}$ into Eq. (16) and define

$$\widehat{\breve{\alpha}} = \Big(\frac{A}{n}\big(\frac{s}{\widehat{\breve{\beta}}} + \frac{\widehat{\breve{\beta}}}{h} - 2\big)\Big)^{\frac{1}{2}},$$

it is easy to check that $\frac{\partial \ell}{\partial \breve{\alpha}} > 0$ for $0 < \breve{\alpha} < \widehat{\breve{\alpha}}$ and $\frac{\partial \ell}{\partial \breve{\alpha}} < 0$ for $\breve{\alpha} > \widehat{\breve{\alpha}}$, so $\widehat{\breve{\alpha}}$ is the MLE of $\breve{\alpha}$.

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