Robust distributed model predictive control of linear systems: analysis and synthesis

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Abstract

To provide robustness of distributed model predictive control (DMPC), this work proposes a robust DMPC formulation for discrete-time linear systems subject to unknown-but-bounded disturbances. Taking advantage of the structure of certain classes of distributed systems seen in applications with interagent coupling like vehicle platooning, a novel robust DMPC is formulated. The proposed approach is characterised by separable terminal costs and locally robust terminal sets, with the latter sets adaptively estimated in the online optimisation problem. A constraint tightening approach based on a set-membership approach is used to guarantee constraint satisfaction for coupled subsystems in the presence of disturbances. Under this formulation, the closed-loop system is shown to be recursively feasible and input-to-state stable. To aid in the deployment of the proposed robust DMPC, a possible synthesis method and design conditions for practical implementation are presented. Finally, simulation results with a mass-spring-damper system are provided to demonstrate the proposed robust DMPC.

Key words: Distributed Model Predictive Control; Robust Constraint Tightening; Set-membership Approach; Robust Local Terminal Sets.

1 Introduction

Model predictive control (MPC) is a powerful methodology (Maciejowski, 2002; Rawlings, Mayne, & Diehl, 2018) that has been widely considered and used in a variety of industrial applications, such as chemical processes (Ellis, Liu, & Christofides, 2017), water networks (Wang, Puig, & Cembrano, 2017) as well as building energy managements (Oldewurtel et al., 2012). As the size of the system increases, challenges in maintaining a centralised control strategy may arise due to issues including computational complexity and the mixed spatial and/or temporal scales necessitating larger communication requirements, all of which can impact on the ability to meet real-time requirements.

With the development of communication and distributed optimisation techniques, distributed MPC (DMPC) has been an active research field during the past two decades. DMPC can be also regarded as an alternative way to overcomes issues for centralised control strategy. Recent works on DMPC can be found in the literature (Conte et al., 2016; Darivianakis et al., 2020; Maestre & Negenborn, 2014; Giselsson & Rantzer,

2014). Among these DMPC methods, the corresponding DMPC optimisation problem is built in a separable structure so that it can be decomposed into sum of local optimisation problems. Therefore, distributed optimisation techniques, such as dual decomposition (Farokhi, Shames, & Johansson, 2014), alternating direction method of multipliers (ADMM) and improved/accelerated ADMM (Ghadimi et al., 2015; Mota et al., 2013; Teixeira et al., 2016), and game theoretic approaches (Barreiro-Gomez, 2019), can be implemented. Since local optimisation problems are solved in parallel, the computation time is significantly reduced, which is helpful for practical implementation.

For systems subject to disturbances, robustness is necessarily considered in an MPC setup (Mayne et al., 2005; Rawlings et al., 2018). Tube-based approach as one of popular robust MPC has been widely studied, see e.g. Alvarado et al. (2010), Limon et al. (2008), Broomhead et al. (2015), Pereira et al. (2017). The idea of tube-based MPC is to optimise the nominal system model along the prediction horizon subject to tightened constraints by means of an effective robust constraint tightening approach. In principle, robust positively invariant (RPI) sets (Stoican et al., 2011; Wang et al., 2019) are used for constraint satisfaction in order to guarantee recursive feasibility and stability of the

closed-loop systems.

In this work we consider systems subject to unknown but bounded disturbances. Previous research on robust DMPC has considered modifying the controller to guarantee recursive feasibility and closed-loop stability in the presence of disturbances. In Richards and How (2007) and Trodden and Richards (2006), a robust approach was formulated for systems with fully decoupled dynamics but subject to coupled constraints. A similar vein of work was considered for systems with coupled dynamics via states in (Conte, Zeilinger, Morari, & Jones, 2013) and via inputs in Al-Gherwi, Budman, and Elkamel (2011), and via both states and input in Grancharova and Olaru (2014). In Zhang, Wang, Ge, and Wang (2014), a robust MPC was also designed for a networked system with time delays. In these existing results, fixed terminal sets were used in the robust DMPC formulations.

Recently, Conte et al. (2016) and Darivianakis et al. (2020) consider the DMPC problem with the local terminal sets adapted online. In Conte et al. (2016), time-varying local terminal sets are defined with a relaxation term from the Lyapunov function for decentralised systems from Jokic and Lazar (2009). In Darivianakis et al. (2020), adaptive local terminal sets are defined, where the parameters of these sets are chosen as decision variables to be determined in the same optimisation problem as DMPC. In both cases, the adaptation is motivated by a desire to avoid unnecessarily small terminal sets, which can adversely impact performance. However, the class of systems considered in this work does not explicitly consider disturbances in the system model. An open challenge is not only to implement a robust constraint tightening approach for distributed systems but also to properly define time-varying local terminal sets taking into account the effects of disturbances.

The main contribution of this work is to build on the work described in the previous two paragraphs to propose a robust DMPC for discrete-time linear systems subject to unknown-but-bounded disturbances. We formulate a robust DMPC that admits a separable structure along with appropriate design rules, in order to facilitate distributed optimisation techniques for implementation. Specifically,

- We introduce *robust adaptive local terminal sets*, whose sizes are determined online;
- We prove the closed-loop control system recursively feasible and input-to-state (ISS) stable;
- We present a synthesis method for a robust DMPC controller and develop design conditions for those local terminal sets.

The paper organisation begins after the problem statement in Section 2. The robust DMPC is formulated in Section 3. The closed-loop performance analysis is discussed in Section 4. The distributed synthesis method, the design conditions for terminal constraint, as well as summary of robust DMPC algorithm are presented in Section 5. Implementation of the synthesis results and validation of the theoretical results are undertaken via simulation in Section 6. Finally, the conclusion is drawn in Section 7.

Notation. We use I to denote an identity matrix of appropriate dimension. For a matrix A, we denote $\mathrm{tr}(A)$ and $\mathrm{rank}(A)$ as the trace and the rank of A, A^{-1} and $A^\top$ as the inverse and the transpose of A, and $A \succ 0$ being the positive definiteness. For two matrices A and B, we use $\mathrm{diag}(A, B)$ to denote a block diagonal matrix. For a set of matrices A_j with $j \in N$, we denote $\mathrm{col}_{j \in \mathcal{N}}\{A_j\} := \left[A_{j_1}^\top, A_{j_2}^\top, \ldots, A_{j_N}^\top\right]^\top$, where $j_1 < j_2 < \cdots < j_N$ are the (ordered) elements of N. Besides, we define the following sets $\mathbb{S}^n := \left\{X \in \mathbb{R}^{n \times n} \mid X = X^\top\right\}$, $\mathbb{S}_{\succ 0}^n := \left\{X \in \mathbb{R}^{n \times n} \mid X = X^\top, X \succ 0\right\}$ and $\mathbb{S}_{\succeq 0}^n := \left\{X \in \mathbb{R}^{n \times n} \mid X = X^\top, X \succeq 0\right\}$. For a vector $z \in \mathbb{R}^n$ and a matrix $W \in \mathbb{S}^n$, we use $\|z\|$ to denote the 2-norm and the weighted 2-norm by W, respectively. We use $\mathrm{col}_{j \in \mathcal{N}}\{z_j\}$ to denote the column vector with elements given by the vector z_j, $\forall j \in \mathcal{N}$. We use $\mathrm{diag}(z)$ to denote a diagonal matrix with elements of its argument z. For any two sets $\mathcal{X}$ and $\mathcal{Y}$, the Minkowski sum, Pontryagin difference and Cartesian product are denoted as $\mathcal{X} \oplus \mathcal{Y} = \{x+y : x \in \mathcal{X}, y \in \mathcal{Y}\}$, $\mathcal{X} \ominus \mathcal{Y} = \{z : z+y \in \mathcal{X}, \forall y \in \mathcal{Y}\}$, $\mathcal{X} \times \mathcal{Y} = \{(x,y) : x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}$, respectively. Besides, for sets $\mathcal{X}_j$ with $\forall j \in \mathcal{N}$, we denote $\bigoplus_{j \in \mathcal{N}} \mathcal{X}_j := \mathcal{X}_{j_1} \oplus \cdots \oplus \mathcal{X}_{j_N}$ and $\bigtimes_{j \in \mathcal{N}} \mathcal{X}_j := \mathcal{X}_{j_1} \times \cdots \times \mathcal{X}_{j_N}$.

2 Problem statement

Let us consider a network of M nodes (or called *agents*). The agents exchange information according to a fixed graph defined by $\mathcal{G} := (\mathcal{M}, \mathcal{D})$, where the vertex set $\mathcal{M} := \{1, \ldots, M\}$ indicates all the agents and the edge set $\mathcal{D} \subset \mathcal{M} \times \mathcal{M}$ specifies pairs of agents that can communicate. In general, the network admits the following linear time-invariant system dynamics:

$$x(k+1) = Ax(k) + Bu(k) + w(k), \qquad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ denote the state vector and the control input vector and the discrete-time index $k \in \mathbb{N}$. $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. For each agent $i \in \mathcal{M}$, coupled dynamics via states is considered as follows:

$$x_i(k+1) = A_{\mathcal{N}_i} x_{\mathcal{N}_i}(k) + B_i u_i(k) + w_i(k), \ \forall i \in \mathcal{M}, \ (2)$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ denote the local state vector and the local control input vector, $w_i \in \mathbb{R}^{n_i}$ denotes the local disturbance vector of the i-th agent, respectively.

$A_{\mathcal{N}_i} := \text{col}_{j \in \mathcal{N}_i} \{A_{ij}\} \in \mathbb{R}^{n_i \times n_{\mathcal{N}_i}}$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$. $\mathcal{N}_i \subseteq \mathcal{M}$ is defined as the set that includes all the agents related to the agent i (i also included). Furthermore, local variables can be selected by using lifting matrices $T_i \in \{0,1\}^{n_i \times n}$, $L_i \in \{0,1\}^{m_i \times m}$, and $T_{\mathcal{N}_i} \in \{0,1\}^{n_{\mathcal{N}_i} \times n}$ such that

$$x_i = T_i x, \ u_i = L_i u, \ x_{\mathcal{N}_i} = T_{\mathcal{N}_i} x. \tag{3}$$

Assumption 1 *For the global system* (1), *the pair* (A, B) *is controllable. The closed-loop system states* $x_i(k)$ *can be measured at each time step* $k \in \mathbb{N}$.

Assumption 2 *The network graph is undirected, that is, any two neighbouring agents* $i \in \mathcal{N}_j$ *and* $j \in \mathcal{N}_i$ *can communicate with information exchange in a bidirectional way.*

We are solving a finite horizon problem minimising $\sum_{k=0}^{N} \ell(x(k), u(k))$, $N > 0$, where

$$\ell(x(k), u(k)) := \|x(k)\|_Q^2 + \|u(k)\|_R^2, \tag{4}$$

with $Q \in \mathbb{S}_{\succ 0}^n$ is a positive-definite weighting matrix, and $R \in \mathbb{S}_{\succ 0}^m$ is a block-diagonal positive-definite weighting matrix. In addition, (4) can be rewritten in a separable structure

$$\begin{aligned}
\ell(x(k), u(k)) &= \sum_{i \in \mathcal{M}} \ell_i(x_{\mathcal{N}_i}(k), u_i(k)) \\
&= \sum_{i \in \mathcal{M}} \left(\|x_{\mathcal{N}_i}(k)\|_{Q_{\mathcal{N}_i}}^2 + \|u_i(k)\|_{R_i}^2 \right)
\end{aligned} \tag{5}$$

where $Q_{\mathcal{N}_i} = T_{\mathcal{N}_i} Q T_{\mathcal{N}_i}^\top \in \mathbb{S}_{\succ 0}^{n_{\mathcal{N}_i}}$ and $R_i = L_i R L_i^\top \in \mathbb{S}_{\succ 0}^{m_i}$, $\forall i \in \mathcal{M}$.

The system states and control inputs are constrained in convex sets

$$x(k) \in \mathcal{X} := \bigtimes_{i \in \mathcal{M}} \mathcal{X}_i, \ u(k) \in \mathcal{U} := \bigtimes_{i \in \mathcal{M}} \mathcal{U}_i, \tag{6}$$

for $k \in \mathbb{N}$, where $\mathcal{X}_i \subseteq \mathbb{R}^{n_i}$ and $\mathcal{U}_i \subseteq \mathbb{R}^{m_i}$, $\forall i \in \mathcal{M}$ are convex sets.

Assumption 3 *The disturbance vector* $w(k)$ *is unknown but bounded by convex sets*

$$w(k) \in \mathcal{W} := \bigtimes_{i \in \mathcal{M}} \mathcal{W}_i, \ \forall k \in \mathbb{N}. \tag{7}$$

A nominal distributed model is now introduced as follows:

$$\bar{x}_i(k+1) = A_{\mathcal{N}_i} \bar{x}_{\mathcal{N}_i}(k) + B_i \bar{u}_i(k), \ \forall i \in \mathcal{M}. \tag{8}$$

From the nominal system (8), the resulting global system can be formulated as

$$\bar{x}(k+1) = A\bar{x}(k) + B\bar{u}(k), \tag{9}$$

where $\bar{x} = \text{col}_{i \in \mathcal{M}}(\bar{x}_i)$ and $\bar{u} = \text{col}_{i \in \mathcal{M}}(\bar{u}_i)$.

For the system (1), centralised robust MPC formulation can be formulated (Mayne et al., 2005)

$$\underset{\bar{u}(0),...,\bar{u}(N-1)}{\text{minimise}} \ V_f(\bar{x}(N)) + \sum_{t=0}^{N-1} \ell(\bar{x}(t), \bar{u}(t)), \tag{10a}$$

subject to

$$\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t), \tag{10b}$$
$$\bar{x}(t) \in \bar{\mathcal{X}}, \tag{10c}$$
$$\bar{u}(t) \in \bar{\mathcal{U}}, \tag{10d}$$
$$\bar{x}(N) \in \Omega_f, \tag{10e}$$
$$\bar{x}(0) = x(k), \tag{10f}$$

where $\bar{\mathcal{X}}$ and $\bar{\mathcal{U}}$ are tightened sets for states and inputs. $V_f(\bar{x}(N))$ and Ω_f are terminal cost function and terminal set to guarantee the closed-loop convergence.

To formulate a distributed solution for (10), the following problems must be considered:

Problem 1 *How can the constraints* $\bar{\mathcal{X}}$ *and* $\bar{\mathcal{U}}$ *be tightened for distributed systems with disturbances?*

Problem 2 *How can the terminal cost* V_f *and the terminal set* Ω_f *be defined in a distributed way?*

3 Robust DMPC based on set-membership approach

In this section, we formulate the robust DMPC considering a finite prediction horizon $N > 0$. We first propose a constraint tightening approach based on set-membership approach. Besides, we also use another auxiliary terminal gain to find separable terminal cost function and local terminal sets.

3.1 Robust constraint tightening approach

Since system states are coupled in the system dynamics (2), the effect of disturbance $w_i(k)$ for agent i will be propagated into other neighbours. To solve Problem 1, we propose a robust constraint tightening approach to tighten constraints on states and inputs iteratively along the MPC prediction horizon N in terms of the global systems (1) and (9), and then project the tightened constraints into each agent.

For the system (1), there exists a state feedback controller

$$u = \kappa(x) := Kx, \qquad (11)$$

where $K \in \mathbb{R}^{m \times n}$ is a stabilising state feedback gain.

By comparing the systems (1) and (9), let us define the error $e := x - \bar{x}$, and therefore the error dynamics can be formulated with the control law in (11) as

$$e(k+1) = A_K e(k) + w(k), \ \forall k \in \mathbb{N}, \qquad (12)$$

where $A_K := A + BK$.

Set $\bar{x}(0) = x(k)$. Then, we can derive $e(0) = 0$, $e(1) = A_K e(0) + w(1) = w(1)$, and for $t \geq 1$, $e(t+1) = A_K e(t) + w(t)$.

Remark 1 *In order to clarify the different time steps, we use $t \in \{0, 1, \ldots, N\}$ to denote the MPC prediction steps while $k \in \mathbb{N}$ to denote the closed-loop simulation steps.*

The closed-loop states and inputs are constrained in the convex sets, that is, $x \in \mathcal{X}$ and $u \in \mathcal{U}$. Based on the above analysis, from (12) with $w(k) \in \mathcal{W}$ under Assumption 3, the constraints on nominal states and inputs can be tightened as follows:

$$\bar{\mathcal{X}}(t) := \mathcal{X} \ominus \mathcal{R}(t), \ t = 1, \ldots, N-1, \qquad (13a)$$
$$\bar{\mathcal{U}}(t) := \mathcal{U} \ominus K\mathcal{R}(t), \ t = 1, \ldots, N-1 \qquad (13b)$$

with $\bar{\mathcal{X}}(0) := \mathcal{X}$ and $\bar{\mathcal{U}}(0) := \mathcal{U}$, where the set $\mathcal{R}(t)$ is defined as

$$\mathcal{R}(t) := \bigoplus_{j=0}^{t-1} A_K^j \mathcal{W}, \ t \geq 1. \qquad (14)$$

From the definition of $\mathcal{R}(t)$ in (14), it holds

$$\mathcal{R}(t+1) \ominus A_K^t \mathcal{W} = \mathcal{R}(t), \ t \geq 1, \ \forall i \in \mathcal{M}. \qquad (15)$$

Remark 2 *Note it is possible to generalise the problem formulation slightly to consider bounded measurement error at time step k, so that $e(0) \neq 0$. This will incur further refinement of $\mathcal{R}(t)$ in (14), but all subsequent steps in the following theoretical results are maintained.*

Assumption 4 *There exists a K guaranteeing that the matrix A_K is Schur stable, and for the chosen K the sets $\bar{\mathcal{X}}(N)$ and $\bar{\mathcal{U}}(N)$ from (13) are non-empty.*

A synthesis method is proposed in Section 5 to find a K if Assumption 4 holds.

From the tightened constraints in (13), we can find the underlying constraints for each agent i by the projection with the lifting matrices.

$$\bar{\mathcal{X}}_{\mathcal{N}_i}(t) = T_{\mathcal{N}_i} \bar{\mathcal{X}}(t), \ \forall i \in \mathcal{M}, \qquad (16a)$$
$$\bar{\mathcal{U}}_i(t) = L_i \bar{\mathcal{U}}(t), \ \forall i \in \mathcal{M}. \qquad (16b)$$

Remark 3 *The state and input constraints can be also tightened locally for each agent if there exist local stabilising feedback control laws*

$$u_i = \kappa_i(x_{\mathcal{N}_i}) := K_{\mathcal{N}_i} x_{\mathcal{N}_i}, \ \forall i \in \mathcal{M},$$

where $K_{\mathcal{N}_i} \in \mathbb{R}^{m_i \times n_{\mathcal{N}_i}}$ is a local control gain. Since the local disturbance vector is bounded in a separable set $w_i \in \mathcal{W}_i$, we can also tighten the constraints locally to find $\bar{\mathcal{X}}_i$ and $\bar{\mathcal{U}}_i$, $\forall i \in \mathcal{M}$. If all stabilising $K_{\mathcal{N}_i}$ are found, the global feedback gain K may be determined using local gains and the lifting matrices, i.e.

$$K = \sum_{i \in \mathcal{M}} L_i^\top K_{\mathcal{N}_i} T_{\mathcal{N}_i}.$$

3.2 Separable terminal cost

We now turn our attention to Problem 2. To set up a distributed optimisation problem, since the stage cost function and constraints can be set in a distributed way as in (4) and (16), we formulate the separable terminal cost function as

$$V_f(x) = \sum_{i \in \mathcal{M}} V_{f_i}(x_i) = \sum_{i \in \mathcal{M}} x_i^\top P_{f_i} x_i, \qquad (17)$$

with $P_{f_i} \in \mathbb{S}_{\succ 0}^{n_i}$, and terminal control law can be defined as

$$u_i = \kappa_{f_i}(x_{\mathcal{N}_i}) := K_{f_i} x_{\mathcal{N}_i}, \ \forall i \in \mathcal{M}, \qquad (18)$$

where $K_{f_i} \in \mathbb{R}^{m_i \times n_{\mathcal{N}_i}}$ is a terminal control gain.

The following lemma indicates the local terminal cost may be increasing with a relaxation term but the combination of the relaxation terms across all agents leads to a strictly decreasing global terminal cost.

Lemma 1 *((Jokic & Lazar, 2009)) If there exists the functions $V_{f_i}(x_i)$, $\gamma_i(x_{\mathcal{N}_i})$ and $\ell_i(x_{\mathcal{N}_i}, K_{f_i} x_{\mathcal{N}_i})$, as well as $\mathcal{K}_\infty$ functions β_{1_i}, β_{2_i} and β_{3_i}, $\forall i \in \mathcal{M}$ such that*

$$\beta_{1_i}(\|x_i\|) \leq V_{f_i}(x_i) \leq \beta_{2_i}(\|x_i\|), \qquad (19a)$$
$$\beta_{3_i}(\|x_{\mathcal{N}_i}\|) \leq \ell_i(x_{\mathcal{N}_i}, K_{f_i} x_{\mathcal{N}_i}), \qquad (19b)$$
$$V_{f_i}(A_{\mathcal{N}_i} x_{\mathcal{N}_i} + B_i \kappa_{f_i}(x_{\mathcal{N}_i})) - V_{f_i}(x_i)$$
$$\leq -\ell_i(x_{\mathcal{N}_i}, \kappa_{f_i}(x_{\mathcal{N}_i})) + \gamma_i(x_{\mathcal{N}_i}), \qquad (19c)$$
$$\sum_{i \in \mathcal{M}} \gamma_i(x_{\mathcal{N}_i}) \leq 0, \ i \in \mathcal{M}, \qquad (19d)$$

then $V_f(x) = \sum_{i \in \mathcal{M}} V_{f_i}(x_i)$ defined in (17) is a Lyapunov function for the system (8) with $u_i = \kappa_{f_i}(x_{\mathcal{N}_i})$ defined in (18), $\forall i \in \mathcal{M}$.

3.3 Robust adaptive local terminal sets

We next discuss how to choose local terminal sets for each agent. In general, let us define the local terminal sets as

$$\Omega_{f_i}(\alpha_i) := \left\{ x_i \in \mathbb{R}^{n_i} : x_i^\top F_i x_i \le \alpha_i \right\}, \ \forall i \in \mathcal{M}, \quad (20)$$

where $F_i \in \mathbb{S}_{\succ 0}^{n_i}$ and a scalar $\alpha_i > 0$ determine the size of local terminal sets. For each agent i, since V_{f_i} satisfies the conditions (19), it can be seen that the local terminal cost might increase. Therefore, the corresponding local terminal set should be also adaptive along the terminal cost increasing.

Considering a prediction horizon of N, the constraints on states and inputs are tightened iteratively as in (16). We now define the local terminal sets $\Omega_{f_i}(\alpha_i)$ for the robust DMPC controller with updating parameters.

Definition 1 (Robust adaptive local terminal sets) *For each agent $i \in \mathcal{M}$, the set $\Omega_{f_i}(\alpha_i)$ is said to be a robust adaptive local terminal set if there exists a matrix $F_i \in \mathbb{S}_{\succ 0}^{n_i}$ and a scalar $\alpha_i > 0$ such that*

$$A_{K_{f_i}} \left\{ \bar{x}_{\mathcal{N}_i} \oplus \bar{\mathcal{E}}_{\mathcal{N}_i}(N-1) \right\} \in \Omega_{f_i}(\alpha_i), \quad (21)$$

where $\bar{\mathcal{E}}_{\mathcal{N}_i}(N-1) := T_{\mathcal{N}_i} A_K^{N-1} \mathcal{W}$ and $A_{K_{f_i}} := A_{\mathcal{N}_i} + B_i K_{f_i}$.

Remark 4 *For the selection of F_i and α_i, we can set $F_i = P_{f_i}$ as widely used in Maestre et al. (2011); Conte et al. (2016); Darivianakis et al. (2020) and update the scalar α_i online.*

3.4 Robust DMPC optimisation problem

Based on the discussions above, we now formulate the optimisation problem of the robust DMPC in the following.

$$\underset{\substack{\bar{u}_i(0), \dots, \bar{u}_i(N-1) \\ \alpha_1, \dots, \alpha_M}}{\text{minimise}} \sum_{i=1}^{M} \left(V_{f_i}(\bar{x}_i(N)) + \sum_{t=0}^{N-1} \ell_i(\bar{x}_{\mathcal{N}_i}(t), \bar{u}_i(t)) \right),$$

$$(22a)$$

subject to

$$\bar{x}_i(t+1) = A_{\mathcal{N}_i} \bar{x}_{\mathcal{N}_i}(t) + B_i \bar{u}_i(t), \quad (22b)$$
$$\bar{x}_{\mathcal{N}_i}(t) \in \bar{\mathcal{X}}_{\mathcal{N}_i}(t), \quad (22c)$$
$$\bar{u}_i(t) \in \bar{\mathcal{U}}_i(t), \quad (22d)$$
$$\bar{x}_i(N) \in \Omega_{f_i}(\alpha_i), \quad (22e)$$
$$\bar{x}_i(0) = x_i(k). \quad (22f)$$

The optimisation problem (22) can be implemented by means of alternative distribution optimisation techniques, such as dual decomposition or ADMM Farokhi et al. (2014); Boyd et al. (2011). We use the superscript $*$ to denote the variables related to the optimal solutions of (22). For instance, let us denote the feasible solutions of (22) at time step $k \in \mathbb{N}$ as follows:

$$\bar{x}_{\mathcal{N}_i}^*(0; x_i(k)), \dots, \bar{x}_{\mathcal{N}_i}^*(N; x_i(k)), \quad (23a)$$
$$\bar{u}_i^*(0; x_i(k)), \dots, \bar{u}_i^*(N-1; x_i(k)), \quad (23b)$$

and α_i^*, $\forall i \in \mathcal{M}$. Therefore, by proceeding with the receding-horizon strategy, the optimal MPC law can be chosen for the closed-loop system at the time step k as

$$\kappa_N(x_i(k)) := \bar{u}_i^*(0; x_i(k)), \ \forall i \in \mathcal{M}. \quad (24)$$

4 Properties of the closed-loop system

We now analyse the properties of the closed-loop system (2) operated by the robust DMPC controller (22).

Since the prediction model (22b) does not contain disturbances, there exists a mismatch between the predicted states and the closed-loop states. Based on the constraint tightening approach in Section 3.1, we used an auxiliary control gain K to attenuate the effect of this mismatch in closed-loop. Similar to the robust tube-based technique originally proposed in (Mayne et al., 2005), the optimal control action at time step k can be chosen as

$$u_i(k) := \bar{u}_i^*(0; x_i(k)) + L_i K(x(k) - \bar{x}^*(0; x(k))). \quad (25)$$

Remark 5 *If the local feedback control gains $K_{\mathcal{N}_i}$ are chosen for constraint tightening in Section 3.1, then the control action (25) can be adapted to be*

$$u_i(k) := \bar{u}_i^*(0; x_i(k)) + K_{\mathcal{N}_i}(x_{\mathcal{N}_i}(k) - \bar{x}_{\mathcal{N}_i}^*(0; x_i(k))),$$

where the mismatch of local states for Agent i is attenuated by local feedback gain $K_{\mathcal{N}_i}$.

With the proposed robust DMPC controller, recursive feasibility of the closed-loop system is summarised.

Theorem 1 (Recursive feasibility) *Consider that Assumptions 1-3 and the conditions of Lemma 1 hold. For any feasible initial condition $x_i(0)$, $\forall i \in \mathcal{M}$, the closed-loop system (1) with (22) is recursively feasible.*

PROOF. See Appendix A.1.

Since the closed-loop system is recursively feasible, we next consider the closed-loop stability.

Theorem 2 (ISS stability) *Consider that Assumptions 1-3 and the conditions of Lemma 1 hold. For any feasible initial condition $x_i(0)$, $\forall i \in \mathcal{M}$, the closed-loop system (1) with (5) and (22) is ISS stable.*

PROOF. See Appendix A.2.

5 Synthesis and design for robust DMPC

In this section, we propose a synthesis method to design feedback control gains and robust adaptive local terminal sets needed for implementation of the theoretical results underpinning the robust DMPC algorithm that were introduced in Section 3.

5.1 Synthesis of local feedback gains

We first present a synthesis method to find a global feedback gain K as well as local feedback gains $K_{\mathcal{N}_i}$, $\forall i \in \mathcal{M}$.

5.1.1 Synthesis of K

Consider the convex sets in the polytopic forms:

$$\mathcal{X} := \left\{ x \in \mathbb{R}^n : a_i^\top x \le d_i, i = 1, \dots, n_r \right\}, \quad (26a)$$

$$\mathcal{U} := \left\{ u \in \mathbb{R}^m : h_j^\top u \le g_j, j = 1, \dots, m_r \right\}, \quad (26b)$$

where $a_i \in \mathbb{R}^n$, $h_j \in \mathbb{R}^m$, $d_i \in \mathbb{R}$, $g_j \in \mathbb{R}$ with non-zero elements, and n_r and m_r are the number of linear constraints of $\mathcal{X}$ and $\mathcal{U}$.

We also consider the disturbance set.

$$\begin{aligned} \mathcal{W} &:= \{ w \in \mathbb{R}^n : |w| \le v \} \\ &:= \left\{ w \in \mathbb{R}^n : w^\top W w \le 1 \right\}, \end{aligned} \quad (27)$$

where $v \in \mathbb{R}^n$ with assuming non-zero elements and a diagonal matrix $W \in \mathbb{S}_{\succ 0}^n$. Besides, $\mathcal{W} = \mathcal{W}_1 \times \cdots \times \mathcal{W}_n$.

In general, the synthesis objectives for K are concluded as follows:

- For the system (1) with (11), there exists a matrix $P \in \mathbb{S}_{\succ 0}^n$ such that the set

$$\mathcal{Z} := \left\{ x \in \mathbb{R}^n : x^\top P x \le 1 \right\}, \quad (28)$$

  is a minimum RPI set, $\forall k \in \mathbb{N}$, $\forall w \in \mathcal{W}$. From the definition of $\mathcal{R}(t)$, it holds $\mathcal{R}(t) \subseteq \mathcal{Z}$, $\forall t \in \mathbb{N}$.
- The tightened constraint sets $\bar{\mathcal{X}}(N)$ and $\bar{\mathcal{U}}(N)$ are non-empty.

Based on these objectives, the following result provides a condition to find the RPI set $\mathcal{Z}$.

Theorem 3 *Given the set W defined in (27). If there exist matrices $S \in \mathbb{S}_{\succ 0}^n$, $Y \in \mathbb{R}^{m \times n}$, and two scalars $\tau_1 \ge 0$, $\tau_2 \ge 0$ such that*

$$\begin{bmatrix} \tau_2 W & I & 0 \\ \star & S & AS + BY \\ \star & \star & \frac{1}{\tau_1} S \end{bmatrix} \succeq 0, \quad (29a)$$

$$\tau_1 + \tau_2 \le 1, \quad (29b)$$

then $\mathcal{Z}$ is an RPI set, that is, $x(k+1) \in \mathcal{Z}$, for any $x(k) \in \mathcal{Z}$, $\forall w(k) \in \mathcal{W}$, $\forall k \in \mathbb{N}$. Moreover, $P = S^{-1}$ and $K = YS^{-1}$.

PROOF. See Appendix B.1.

Another objective is to make sure $\bar{\mathcal{X}}(N)$ and $\bar{\mathcal{U}}(N)$ are non-empty, which can be satisfied if $\mathcal{X} \ominus \mathcal{Z}$ and $\mathcal{U} \ominus K\mathcal{Z}$ are non-empty due to $R(t) \subseteq \mathcal{Z}$, $\forall t \in \mathbb{N}$. We give the corresponding conditions in the following theorem.

Theorem 4 *Given the convex sets in (26)-(27). For the RPI set $\mathcal{Z}$ in (28), if there exist matrices $S \in \mathbb{S}_{\succ 0}^n$, $Y \in \mathbb{R}^{m \times n}$ such that*

$$\begin{bmatrix} d_i^2 & a_i^\top S \\ \star & S \end{bmatrix} \succeq 0, \ i = 1, \dots, n_r, \quad (30a)$$

$$\begin{bmatrix} g_j^2 & h_j^\top Y \\ \star & S \end{bmatrix} \succeq 0, \ j = 1, \dots, m_r, \quad (30b)$$

then the sets $\mathcal{X} \ominus \mathcal{Z}$ and $\mathcal{U} \ominus K\mathcal{Z}$ are non-empty. Moreover, $P = S^{-1}$ and $K = YS^{-1}$.

PROOF. See Appendix B.2.

As a result, the auxiliary control gain K can be synthesised via offline solving the following optimisation with the objective of finding a minimum RPI set $\mathcal{Z}$.

$$\underset{S,G,Y,\tau_2,\mu}{\text{minimise}} \ \text{trace}(S), \quad (31)$$

subject to (29)-(30), for given $\tau_1 > 0$.

Remark 6 *If we choose a structured P for the Lyapunov candidate function, i.e.*

$$V(x) = \sum_{i \in \mathcal{M}} x_{\mathcal{N}_i}^\top P_{\mathcal{N}_i} x_{\mathcal{N}_i} = \sum_{i \in \mathcal{M}} x^\top \bar{P}_i x,$$

where $P_{\mathcal{N}_i} \in \mathbb{S}^{n_i}_{\succ 0}$ and $\bar{P}_i = T_{\mathcal{N}_i}^\top P_{\mathcal{N}_i} T_{\mathcal{N}_i}$, then we may find an RPI set for $x_{\mathcal{N}_i}$

$$\mathcal{Z}_{\mathcal{N}_i} := \left\{ x_{\mathcal{N}_i} \in \mathbb{R}^{n_{\mathcal{N}_i}} : x_{\mathcal{N}_i}^\top P_{\mathcal{N}_i} x_{\mathcal{N}_i} \leq \varphi_i \right\}, \ \forall i \in \mathcal{M},$$

where $P_{\mathcal{N}_i} \in \mathbb{S}^{n_{\mathcal{N}_i}}_{\succ 0}$, $\varphi_i \geq 0$ and $\sum_{i \in \mathcal{M}} \varphi_i \leq 1$. The synthesis condition can be found in (Conte et al., 2013, (21)-(25)).

5.1.2 Synthesis of $K_{\mathcal{N}_i}$

For a non-empty disturbance set, it may become more challenging to apply a centralised constraint tightening approach as the system order increases. Instead, a distributed robust constraint tightening approach can be implemented with local feedback control gains $K_{\mathcal{N}_i}$, $\forall i \in \mathcal{M}$ so that less conservative tightening may be applied. We next present the synthesis conditions for finding these local feedback control gains.

Corollary 1 For each agent $i \in \mathcal{M}$, if there exist matrices $S_i \in \mathbb{S}^{n_i}_{\succ 0}$ with $S_{ij} = T_{\mathcal{N}_i} T_j^\top S_j T_j T_{\mathcal{N}_i}^\top$, $G_i \in \mathbb{R}^{n_{\mathcal{N}_i} \times n_{\mathcal{N}_i}}$, $Y_i \in \mathbb{R}^{m_i \times n_{\mathcal{N}_i}}$, and two scalars $\bar{\tau}_i \geq 0$, $\bar{\tau}_{ij} \geq 0$, $\forall j \in \mathcal{N}_i$ such that

$$\begin{bmatrix} \bar{\tau}_i W_i & I & 0 \\ \star & S_i & A_{\mathcal{N}_i} G_i + B_i Y_i \\ \star & \star & G_i + G_i^\top - \sum_{j \in \mathcal{N}_i} \dfrac{S_{ij}}{\bar{\tau}_{ij}} \end{bmatrix} \succeq 0, \qquad (32a)$$

$$\bar{\tau}_i + \sum_{j \in \mathcal{N}_i} \bar{\tau}_{ij} \leq 1, \qquad (32b)$$

then $\mathcal{Z}_i = \left\{ x_i \in \mathbb{R}^{n_i} : x_i^\top P_i x_i \leq 1 \right\}$ is an RPI set, that is, $x_i(k+1) \in \mathcal{Z}_i$, $\forall x_i(k) \in \mathcal{Z}_i$, $\forall w_i(k) \in \mathcal{W}_i$, $\forall k \in \mathbb{N}$. Moreover, $P_i = S_i^{-1}$ and $K_{\mathcal{N}_i} = Y_i G_i^{-1}$.

PROOF. See Appendix B.3.

Deriving from (26), the local constraints on states and inputs are considered as follows:

$$\mathcal{X}_i := \left\{ x_i \in \mathbb{R}^{n_i} : a_{il}^\top x_i \leq d_{il}, l = 1, \ldots, n_{r_i} \right\}, \quad (33a)$$
$$\mathcal{U}_i := \left\{ u_i \in \mathbb{R}^{m_i} : h_{ip}^\top u_i \leq g_{ip}, p = 1, \ldots, m_{r_i} \right\}, \quad (33b)$$

and the local disturbance set

$$\begin{aligned} \mathcal{W}_i &:= \{ w_i \in \mathbb{R}^{n_i} : |w_i| \leq v_i \} \\ &= \left\{ w_i \in \mathbb{R}^{n_i} : w_i^\top W_i w_i \leq 1 \right\}. \end{aligned} \qquad (34)$$

Corollary 2 For each agent $i \in \mathcal{M}$, if there exist matrices $S_i \in \mathbb{S}^{n_i}_{\succ 0}$ with $S_{ij} = T_{\mathcal{N}_i} T_j^\top S_j T_j T_{\mathcal{N}_i}^\top$,

$G_i \in \mathbb{R}^{n_{\mathcal{N}_i} \times n_{\mathcal{N}_i}}$, $Y_i \in \mathbb{R}^{m_i \times n_{\mathcal{N}_i}}$, and scalars $\tilde{\tau}_{ijp} \geq 0$, $\forall j \in \mathcal{N}_i$, $p = 1, \ldots, m_{r_i}$ such that

$$\begin{bmatrix} d_{il}^2 & a_{il}^\top S_i \\ \star & S_i \end{bmatrix} \succeq 0, \ l = 1, \ldots, n_{ri}, \qquad (35)$$

and

$$\begin{bmatrix} g_{ip} & h_{ip}^\top Y_i \\ \star & G_i + G_i^\top - \sum_{j \in \mathcal{N}_i} \dfrac{S_{ij}}{\tilde{\tau}_{ijp}} \end{bmatrix} \succeq 0, \ p = 1, \ldots, m_{r_i},$$
$$(36a)$$

$$\sum_{j \in \mathcal{N}_i} \tilde{\tau}_{ijp} \leq g_{ip}, \qquad (36b)$$

then the tightened constraint sets on states and inputs are non-empty. Moreover, $P_i = S_i^{-1}$ and $K_{\mathcal{N}_i} = Y_i G_i^{-1}$.

PROOF. The proof follows directly from the proofs of Theorem 4 and Corollary 1. □

5.2 Synthesis of terminal gains and costs

Considering the terminal cost function defined in (17), for each agent $i \in \mathcal{M}$, we have $V_{f_i}(x_i) = x_i^\top P_{f_i} x_i$ with $P_{f_i} \in \mathbb{S}^{n_i}_{\succ 0}$. We also consider the relaxation function as $\gamma_i(x_{\mathcal{N}_i}) = x_{\mathcal{N}_i}^\top \Gamma_i x_{\mathcal{N}_i}$ with $\Gamma_i \in \mathbb{S}^{n_{\mathcal{N}_i}}_{\succeq 0}$. The terminal gain K_{f_i} and the matrix P_{f_i} for $i \in \mathcal{M}$ can be obtained by using (Conte et al., 2016, Lemma 10).

5.3 Conditions for robust local adaptive terminal sets

To implement the optimisation problem (22), we need conditions for the terminal constraint (22e). Based on Definition 1, we give the condition for robust adaptive local terminal sets in the following theorem.

Theorem 5 Given the system (2) with K, F_i and K_{f_i}, $\forall i \in \mathcal{M}$. For each agent $i \in \mathcal{M}$, if there exist scalars α_i, and $\sigma_i \geq 0$, $\sigma_{ij} \geq 0$, $\forall j \in \mathcal{N}_i$ such that

$$\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \sigma_{ij} F_{ij} & (\alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top A_{K_{f_i}}^\top & 0 \\ \star & \alpha_i^{\frac{1}{2}} F_i^{-1} & A_{K_{f_i}} \\ \star & \star & \sigma_i E_{\mathcal{N}_i} \end{bmatrix} \succeq 0 \qquad (37a)$$

$$\sigma_i + \sum_{j \in \mathcal{N}_i} \sigma_{ij} \leq \alpha_i^{\frac{1}{2}}, \qquad (37b)$$

where $\alpha_{\mathcal{N}_i} = T_{\mathcal{N}_i} \alpha T_{\mathcal{N}_i}^\top$, $\alpha = \mathrm{diag}(\alpha_1 I_{n_1}, \ldots, \alpha_M I_{n_M})$, $F_{ij} = T_{\mathcal{N}_i} T_i^\top F_i T_i T_{\mathcal{N}_i}^\top$, and $E_{\mathcal{N}_i} = T_{\mathcal{N}_i} A_K^{N-1} W (T_{\mathcal{N}_i} A_K^{N-1})^\top$, then the condition (21) is satisfied.

PROOF. See Appendix B.4.

Let $\Omega_{f_{\mathcal{N}_i}} = \bigtimes_{j \in \mathcal{N}_i} \Omega_{f_j}(\alpha_j)$, $\forall i \in \mathcal{M}$. Any state $x_{f_{\mathcal{N}_i}} \in \Omega_{f_{\mathcal{N}_i}}$ should satisfy

$$x_{f_{\mathcal{N}_i}} \in \mathcal{X}_{\mathcal{N}_i} = \bigtimes_{j \in \mathcal{N}_i} \mathcal{X}_j, \tag{38a}$$

$$K_{f_i} x_{f_{\mathcal{N}_i}} \in \mathcal{U}_i, \tag{38b}$$

where $\mathcal{X}_{\mathcal{N}_i} := \left\{ x_{\mathcal{N}_i} \in \mathbb{R}^n : \bar{a}_{il}^\top x_{\mathcal{N}_i} \le \bar{d}_{il}, l = 1, \ldots, n_{r_{\mathcal{N}_i}} \right\}$.

Theorem 6 *For each agent $i \in \mathcal{M}$, if there exist scalars $\phi_{ijl} \ge 0$, $\forall j \in \mathcal{N}_i$, $l = 1, \ldots, n_{r_{\mathcal{N}_i}}$, and $\psi_{ijp} \ge 0$, $\forall j \in \mathcal{N}_i$, $p = 1, \ldots, m_{r_i}$ such that*

$$\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \phi_{ijl} F_{ij} & (\alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top \bar{a}_{il} \\ \star & \bar{d}_{il} \end{bmatrix} \succeq 0, \tag{39a}$$

$$\sum_{j \in \mathcal{N}_i} \phi_{ijl} \le \bar{d}_{il}, \ l = 1, \ldots, n_{r_{\mathcal{N}_i}}, \tag{39b}$$

and

$$\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \psi_{ijp} F_{ij} & (\alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top K_{f_i}^\top h_{ip} \\ \star & g_{ip} \end{bmatrix} \succeq 0, \tag{40a}$$

$$\sum_{j \in \mathcal{N}_i} \psi_{ijp} \le g_{ip}, \ p = 1, \ldots, m_{r_i}, \tag{40b}$$

then the conditions in (38) *are satisfied.*

PROOF. See Appendix B.5.

Moreover, by using the Schur complement, the constraint (22e) can be rewritten as

$$\begin{bmatrix} \alpha_i^{\frac{1}{2}} & \bar{x}_i(N)^\top \\ \star & \alpha_i^{\frac{1}{2}} F_i^{-1} \end{bmatrix} \succeq 0, \forall i \in \mathcal{M}. \tag{41}$$

To this end, the conditions for the constraint (22e) in (22) can be summarised as follows:

$$\bar{x}_i(N) \in \Omega_{f_i}(\alpha_i) \Leftrightarrow (37), (39), (40), (41),$$

for given F_i, K_{f_i}. The decision variables are $\alpha_i^{\frac{1}{2}}$, σ_i, σ_{ij}, ϕ_{ijl} and ψ_{ijl}. Besides, $\alpha_{\mathcal{N}_i}^{\frac{1}{2}} = T_{\mathcal{N}_i} \alpha T_{\mathcal{N}_i}^\top$ and $\alpha^{\frac{1}{2}} = \text{diag}(\alpha_1^{\frac{1}{2}} I_{n_1}, \ldots, \alpha_M^{\frac{1}{2}} I_{n_M})$.

5.4 Summary of robust DMPC algorithm

The proposed robust DMPC algorithm has both offline (synthesis) and online (optimisation) components. These are summarised in Algorithm 1 and 2, respectively.

Algorithm 1 Offline synthesis of $K/K_{\mathcal{N}_i}$, K_{f_i} and P_{f_i}

1: Solve the optimisation problem (31) to obtain K or alternatively use conditions in Corollary 1-2 to obtain $K_{\mathcal{N}_i}$.
2: **for** $t = 0, \ldots, N - 1$ **do**
3:     Compute the set $\mathcal{R}(t)$ based on (14).
4:     Compute the tightened constraint sets $\bar{\mathcal{X}}(t)$ and $\bar{\mathcal{U}}(t)$ based on (13).
5:     Project the global constraint sets into local ones $\bar{\mathcal{X}}_{\mathcal{N}_i}(t)$ and $\bar{\mathcal{U}}_i(t)$ based on (16).
6: **end for**
7: Obtain K_{f_i} and P_{f_i} for each agent $i \in \mathcal{M}$.

Algorithm 2 Online robust DMPC

1: Choose and fix $F_i = P_{f_i}$ and K_{f_i} from the offline synthesis for each agent $i \in \mathcal{M}$.
2: **while** $k \ge 0$ **do**
3:     Each agent $i \in \mathcal{M}$ measures its local current state $x_i(k)$.
4:     Solve optimisation problem (22) with conditions (37), (39), (40), (41) for terminal constraints by distributed optimisation, where agents $\mathcal{N}_i$ iteratively communicate.
5:     Apply $u_i(k) = \kappa_N(x_i(k))$ as in (24).
6: **end while**

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6 Simulation results

In this section, we use a mass-spring-damper system to demonstrate the proposed robust DMPC. Let us consider the system as shown in Fig. 1.
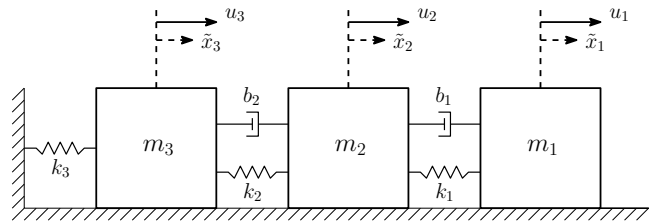

Fig. 1. The chain of three masses, connected by springs and dampers.

Let the vectors of states and inputs be selected as $x = \left[\tilde{x}_1, \dot{\tilde{x}}_1, \tilde{x}_2, \dot{\tilde{x}}_2, \tilde{x}_3, \dot{\tilde{x}}_3 \right]^\top \in \mathbb{R}^6$ and $u = [u_1, u_2, u_3]^\top \in \mathbb{R}^3$. The continuous-time state-space model with additive

disturbances can be formulated as follows:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{k_1}{m_1} & \frac{b_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{m_2} & \frac{b_1}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{b_1+b_2}{m_2} & \frac{k_2}{m_2} & \frac{b_2}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_2}{m_3} & \frac{b_2}{m_3} & -\frac{k_2+k_3}{m_3} & -\frac{b_2}{m_3} \end{bmatrix} x$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} u + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

where the parameters of masses, spring constants and damping coefficient are chosen asymmetrically with $k_1 > k_2 > k_3$ and $b_1 > b_2$ from $m_1, m_2, m_3 \in [5, 10]\,\text{kg}$, $k_1, k_2, k_3 \in [0.8, 1.2]\,\text{N} \cdot \text{m}$ and $b_1, b_2 \in [0.8, 2]\,\text{kg/s}$, respectively. The resulting discrete-time LTI system in the form of (1) can be obtained by using the Euler discretisation method to the above continuous-time state-space model with the sampling time $T_s = 0.1$s. Each mass $i \in \mathcal{M} := \{1, 2, 3\}$ with the external force u_i can be considered as one agent. Asymmetric constraints are imposed as

$$\mathcal{X}_1 = \left\{ x_1 \in \mathbb{R}^2 : [-10, -10]^\top \le x_1 \le [10, 10]^\top \right\},$$
$$\mathcal{X}_2 = \left\{ x_2 \in \mathbb{R}^2 : [-2, -3]^\top \le x_2 \le [2, 3]^\top \right\},$$
$$\mathcal{X}_3 = \left\{ x_3 \in \mathbb{R}^2 : [-3, -5]^\top \le x_3 \le [3, 5]^\top \right\},$$
$$\mathcal{U} = \left\{ u \in \mathbb{R}^3 : |u_1| \le 10, |u_2| \le 1.5, |u_3| \le 5 \right\},$$

and the disturbances are unknown but bounded in given sets

$$\mathcal{W}_1 = \left\{ w_1 \in \mathbb{R}^2 : [-0.15, -0.3] \le w_1 \le [0.15, 0.3]^\top \right\},$$
$$\mathcal{W}_2 = \left\{ w_2 \in \mathbb{R}^2 : [-0.05, -0.1] \le w_2 \le [0.05, 0.1]^\top \right\},$$
$$\mathcal{W}_3 = \left\{ w_3 \in \mathbb{R}^2 : [-0.05, -0.1] \le w_3 \le [0.05, 0.1]^\top \right\}.$$

Table 1
Parameters for robust DMPC.

	$Q_{\mathcal{N}_i}$	R_i
Agent 1	diag(10, 10)	0.1
Agent 2	diag(1, 1)	0.01
Agent 3	diag(2.5, 2.5)	0.05

The weighting matrices for the stage cost functions are given in Table 1.

For comparison, in the closed-loop simulations, we have implemented the nominal DMPC proposed in (Darivianakis et al., 2020) with this uncertain system. The initial condition $x(0) = [-5, -3, 1.2, 1, -1, -2]^\top$ is
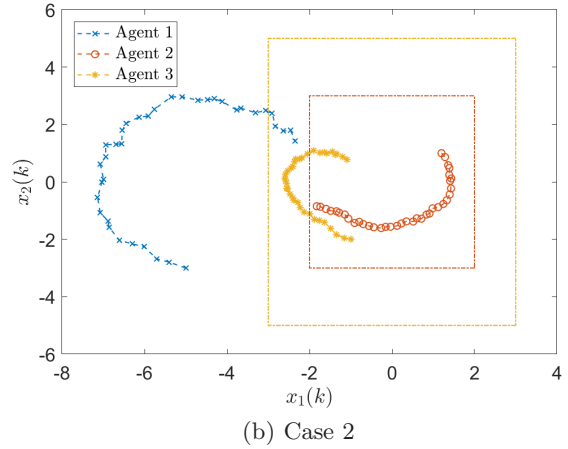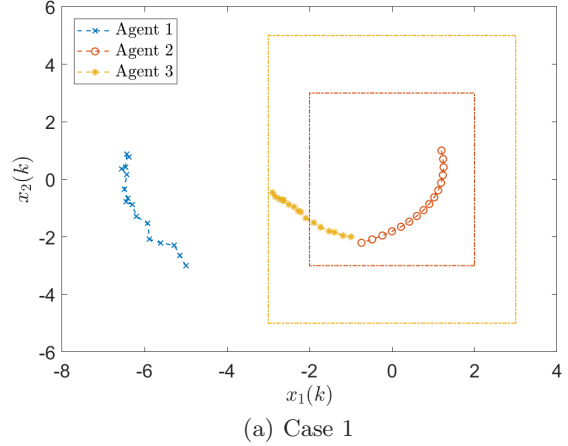

(a) Case 1

(b) Case 2

Fig. 2. State trajectories under nominal DMPC with noise sequences leading to infeasibility.
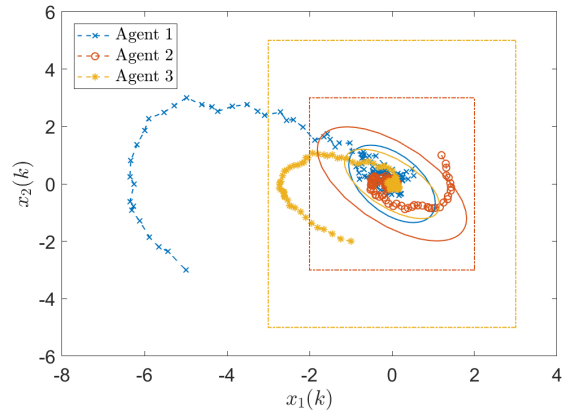

Fig. 3. State trajectories under robust DMPC, and local terminal sets.

given and the prediction horizon is considered as $N = 5$. With a number of realisations of the disturbance sequences sampled from $\mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{W}_3$, two sequences leading to infeasible closed-loop results were identified

Table 2
Synthesis results for robust DMPC.

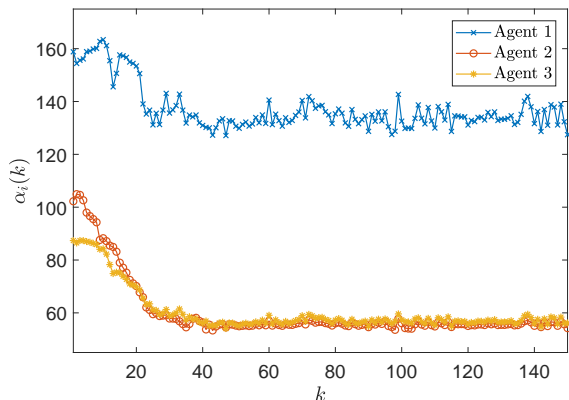|	$K_{\mathcal{N}_i}$	K_{f_i}	P_{f_i}
Agent 1	$\begin{bmatrix} -0.29 & -0.88 & -0.67 & -0.82 \end{bmatrix}$	$\begin{bmatrix} -7.68 & -11.99 & -0.26 & -0.33 \end{bmatrix}$	$\begin{bmatrix} 151.92 & 57.76 \\ 57.76 & 92.08 \end{bmatrix}$
Agent 2	$\begin{bmatrix} -0.57 & -0.70 & -0.29 & -0.83 & -0.70 & -0.28 \end{bmatrix}$	$\begin{bmatrix} -0.29 & -0.38 & -12.39 & -19.07 & -0.11 & -0.17 \end{bmatrix}$	$\begin{bmatrix} 25.95 & 14.23 \\ 14.23 & 21.53 \end{bmatrix}$
Agent 3	$\begin{bmatrix} -0.41 & -1.21 & -1.01 & -3.17 \end{bmatrix}$	$\begin{bmatrix} -0.127 & -0.19 & -9.69 & -15.30 \end{bmatrix}$	$\begin{bmatrix} 68.39 & 40.33 \\ 40.33 & 62.28 \end{bmatrix}$

Fig. 4. The optimal $\alpha_i(k)$.

and denoted 'Case 1' and 'Case 2'. The closed-loop state trajectories of three agents are shown in Fig. 2.

The same disturbance sequences are now applied to the closed-loop system with the proposed robust DMPC controller. Offline implementation of Algorithm 1, leads to the local feedback gains and terminal cost matrices along with the terminal feedback gains shown in Table 2. The online implementation of Algorithm 2 then allows the closed-loop trajectories to be determined for the same disturbance sequences that led to infeasibility with the nominal controller. As illustrated in Fig. 3, the feasibility of the closed-loop system is now retained, thereby validating the results of Theorem 1. Similarly, the convergence of the system towards the origin demonstrates that the desired ISS property is achieved, thereby validating Theorem 2. This stability is further illustrated by plotting the trajectories of the α_i parameters in Fig. 4, leading to the ellipsoidal terminal sets shown in Fig. 3.

7 Conclusions

In this paper, we have proposed a robust DMPC formulation for discrete-time LTI systems subject to disturbances. The closed-loop system with the proposed robust DMPC controller has been proved to be recursively feasible and ISS stable in the presence of unknown-but-bounded disturbances. We have presented algorithms for the offline (synthesis) problem as well as the online (optimisation) problem that have provable guarantees. As a future direction, the proposed robust DMPC formulation can be extended into tracking DMPC as well as economic DMPC.

Acknowledgements

A   Proofs in Section 4

A.1   Proof of Theorem 1

The feasible solutions of (22) at any time step $k \geq 0$ are denoted as in (23). The control action at time step k is chosen to be $u_i(k) = \kappa_N(x_i(k))$ as in (24). Due to the constraint (22f), the control action $u_i(k) = \kappa_N(x_i(k))$ also satisfies (25).

According to the constraints (22c)-(22d), the corresponding closed-loop state $x_i(k)$ and $u_i(k)$ are feasible at time step k, that is, $x_i(k) \in \mathcal{X}_i$ and $u_i(k) \in \mathcal{U}_i$ when $t = 0$. After applying the optimal control action (24), we can obtain $x_i(k+1)$ from the system (1).

Referring to (25), with $x_i(k+1)$, $\forall i \in \mathcal{M}$ and feasible solution at time step k, we define a sequence of shifted nominal control inputs

$$\begin{aligned} \bar{u}_i(t; x_i(k+1)) := & \bar{u}_i^*(t+1; x_i(k)) \\ & + L_i K(\bar{x}(t; x(k+1)) - \bar{x}^*(t+1; x(k))), \end{aligned} \quad \text{(A.1)}$$

for $t = 0, \ldots, N-2$, $\forall i \in \mathcal{M}$, and when $t = N-1$,

$$\bar{u}_i(N-1; x_i(k+1)) := K_{f_i} \bar{x}_{\mathcal{N}_i}(N-1; x_i(k+1)). \quad \text{(A.2)}$$

We can also define the shifted error for the global system states as

$$\bar{e}(t) := \bar{x}(t; x(k+1)) - \bar{x}^*(t+1; x(k)), \qquad (A.3)$$

with

$$\begin{aligned}
\bar{e}(0) &= \bar{x}(0; x(k+1)) - \bar{x}^*(1; x(k)) \\
&= x(k+1) - \operatorname{col}_{i \in \mathcal{M}} \{A_{\mathcal{N}_i} x_{\mathcal{N}_i}(k) + B_i \kappa_N(x_i(k))\} \\
&= w(k) \in \mathcal{W}.
\end{aligned}$$

With (A.1), the shifted error dynamics can be described as

$$\begin{aligned}
\bar{e}(t+1) &= \operatorname{col}_{i \in \mathcal{M}} \{(A_{\mathcal{N}_i} T_{\mathcal{N}_i} + B_i U_i K)\, \bar{e}(t)\} \\
&= A_K \bar{e}(t).
\end{aligned} \qquad (A.4)$$

From $\bar{e}(0)$ and (A.4), we denote shifted error sets $\bar{e}(t) \in \bar{\mathcal{E}}(0) = \mathcal{W}$, and $\bar{e}(t) \in \bar{\mathcal{E}}(t) = A_K^t \mathcal{W}$, $t \geq 1$, $\forall i \in \mathcal{M}$.

We now check the feasibility of all the constraints in (22) at time step $k+1$, $\forall i \in \mathcal{M}$.

- State constraint (22c): for $t = 0, 1, \ldots, N-1$,

$$\begin{aligned}
\bar{x}_{\mathcal{N}_i}(t; x_i(k+1)) &\in \bar{x}_{\mathcal{N}_i}^*(t+1; x_i(k)) \oplus T_{\mathcal{N}_i} \bar{\mathcal{E}}(t) \\
&\subseteq \bar{\mathcal{X}}_{\mathcal{N}_i}(t+1) \oplus T_{\mathcal{N}_i} \bar{\mathcal{E}}(t) \\
&= T_{\mathcal{N}_i} \mathcal{X} \ominus T_{\mathcal{N}_i} \{\mathcal{R}(t+1) \ominus \bar{\mathcal{E}}(t)\} \\
&= T_{\mathcal{N}} \mathcal{X}_i \ominus T_{\mathcal{N}_i} \mathcal{R}(t) \\
&= \bar{\mathcal{X}}_{\mathcal{N}_i}(t).
\end{aligned}$$

- Input constraint (22d): for $t = 0, 1, \ldots, N-1$,

$$\begin{aligned}
\bar{u}_i(t; x_i(k+1)) &\in \bar{u}_i^*(t+1; x_i(k)) \oplus L_i K \bar{\mathcal{E}}(t) \\
&\subseteq \bar{\mathcal{U}}_i(t+1) \oplus L_i K \bar{\mathcal{E}}(t) \\
&= L_i \mathcal{U}_i \ominus L_i K \{\mathcal{R}(t+1) \ominus \bar{\mathcal{E}}(t)\} \\
&= L_i \mathcal{U}_i \ominus L_i K \mathcal{R}(t) \\
&= \bar{\mathcal{U}}_i(t).
\end{aligned}$$

- Terminal constraint (22e): Since

$$\begin{aligned}
\bar{x}_i(N-1; x_i(k+1)) &\in \bar{x}_i^*(N; x_i(k)) \oplus T_i \bar{\mathcal{E}}(N-1) \\
&\subseteq \Omega_{f_i}(\alpha_i) \oplus T_i \bar{\mathcal{E}}(N-1),
\end{aligned}$$

  and then by using the terminal control law (A.2), when $t = N$, it holds $\bar{x}_i(N; x_i(k+1)) \in \Omega_{f_i}(\alpha_i)$ by the condition (21) in Definition 1.
- Initial condition constraint (22f): $\bar{x}_i(0; x_i(k+1)) = x_i(k+1)$.

Thus, the optimisation problem (22) is also feasible at time step $k+1$, $\forall k \geq 0$. $\square$

A.2 Proof of Theorem 2

Denote the optimal cost of (22) at any time step $k \geq 0$ as $V_N^*(x(k)) := \sum_{i \in \mathcal{M}} V_{N_i}^*(x_i(k))$. Since $V_N^*(x(k))$ is positive-definite and continuous in a neighbourhood of the coordinate origin, there exist $\mathcal{K}$ functions β_1 and β_2 such that $\beta_1(\|x(k)\|) \leq V_N^*(x(k)) \leq \beta_2(\|x(k)\|)$. Then, at time step $k+1$, let us define

$$\begin{aligned}
\Delta V_N :=& V_N(x(k+1)) - V_N^*(x(k)) \\
=& \sum_{i \in \mathcal{M}} \left(V_{N_i}(x_i(k+1)) - V_{N_i}^*(x_i(k))\right),
\end{aligned}$$

where for $i \in \mathcal{M}$,

$$\begin{aligned}
& V_{N_i}(x_i(k+1)) - V_{N_i}^*(x_i(k)) \\
=& V_{f_i}(\bar{x}_i(N; x_i(k+1))) - V_{f_i}^*(\bar{x}_i(N; x_i(k))) \\
& + \sum_{t=0}^{N-1} \left(\ell_i(\bar{x}_{\mathcal{N}_i}(t; x_i(k+1)), \bar{u}_i(t; x_i(k+1)))\right) \\
& - \sum_{t=0}^{N-1} \left(\ell_i(\bar{x}_{\mathcal{N}_i}^*(t; x_i(k)), \bar{u}_i^*(t; x_i(k)))\right) \\
=& \|\bar{x}_i(N; x_i(k+1))\|_{P_{f_i}}^2 - \|\bar{x}_i(N; x_i(k))\|_{P_{f_i}}^2 \\
& + \sum_{t=0}^{N-1} \left(\|\bar{x}_{\mathcal{N}_i}(t; x_i(k+1))\|_{Q_{\mathcal{N}_i}}^2 + \|\bar{u}_i(t; x_i(k+1))\|_{R_i}^2\right) \\
& - \sum_{t=0}^{N-1} \left(\left\|\bar{x}_{\mathcal{N}_i}^*(t; x_i(k))\right\|_{Q_{\mathcal{N}_i}}^2 + \|\bar{u}_i^*(t; x_i(k))\|_{R_i}^2\right).
\end{aligned}$$

For $t = 0, 1, \ldots, N-2$, from (A.3)-(A.4), we have

$$\begin{aligned}
\bar{x}_{\mathcal{N}_i}(t; x_i(k+1)) - \bar{x}_{\mathcal{N}_i}^*(t+1; x_i(k)) &= T_{\mathcal{N}_i} \bar{e}(t) \\
&= T_{\mathcal{N}_i} A_K^t w(k).
\end{aligned}$$

Then, it follows

$$\begin{aligned}
& \|\bar{x}_{\mathcal{N}_i}(t; x_i(k+1))\|_{Q_{\mathcal{N}_i}}^2 - \left\|\bar{x}_{\mathcal{N}_i}^*(t+1; x_i(k))\right\|_{Q_{\mathcal{N}_i}}^2 \\
=& \|A_K^t w(k)\|_{\bar{Q}_i}^2 + 2(T_{\mathcal{N}_i}^\top Q_{\mathcal{N}_i} \bar{x}_{\mathcal{N}_i}^*(t+1; x_i(k)))^\top A_K^t w(k) \\
\leq& \|A_K^t w(k)\|_{\bar{Q}_i}^2 + 2\|T_{\mathcal{N}_i}^\top Q_{\mathcal{N}_i} \bar{x}_{\mathcal{N}_i}^*(t+1; x_i(k))\| \|A_K^t w(k)\| \\
\leq& \|A_K^t w(k)\|_{\bar{Q}_i}^2 + 2c_{i,1}\|A_K^t w(k)\|,
\end{aligned}$$

where $\bar{Q}_i = T_{\mathcal{N}_i}^\top Q_{\mathcal{N}_i} T_{\mathcal{N}_i}$, and $c_{i,1}$ is an upper bound of $\|T_{\mathcal{N}_i}^\top Q_{\mathcal{N}_i} \bar{x}_{\mathcal{N}_i}\|$ for given matrices $T_{\mathcal{N}_i}$ and $Q_{\mathcal{N}_i}$, $\forall \bar{x}_{\mathcal{N}_i} \in T_{\mathcal{N}_i} \mathcal{X}$.

Similarly, due to (A.1), we have

$$\begin{aligned}
& \bar{u}_i(t; x_i(k+1)) - \bar{u}_i^*(t+1; x_i(k)) \\
=& L_i K \bar{e}(t) = L_i K A_K^t w(k).
\end{aligned}$$

Then, it follows

$$\|\bar{u}_i(t; x_i(k+1))\|^2_{\bar{R}_i} - \|\bar{u}_i^*(t+1; x_i(k))\|^2_{\bar{R}_i}$$
$$\leq \|A_K^t w(k)\|^2_{\bar{R}_i} + 2c_{i,2}\|A_K^t w(k)\|,$$

where $\bar{R}_i = K^\top L_i^\top R_i L_i K$, and $c_{i,2}$ is an upper bound of $\|K^\top L_i^\top R_i \bar{u}_i\|$ for given matrices K, L_i and R_i, $\forall u_i \in L_i \mathcal{U}$.

When $t = N - 1$, we can obtain $\bar{x}_{\mathcal{N}_i}(N-1; x_i(k+1)) - \bar{x}_{\mathcal{N}_i}^*(N; x_i(k)) = T_{\mathcal{N}_i} A_K^{N-1} w(k)$. Besides, based on (A.2), we can derive

$$\|\bar{x}_{\mathcal{N}_i}(N-1; x_i(k+1))\|^2_{Q_{\mathcal{N}_i}} + \|\bar{u}_i(N-1; x_i(k+1))\|^2_{\bar{R}_i}$$
$$= \|\bar{x}_{\mathcal{N}_i}(N-1; x_i(k+1))\|^2_{Q_{\mathcal{N}_i}+K_{f_i}^\top R_i K_{f_i}}$$
$$\leq \|\bar{x}_{\mathcal{N}_i}^*(N; x_i(k))\|^2_{Q_{\mathcal{N}_i}+K_{f_i}^\top R_i K_{f_i}}$$
$$+ \|A_K^{N-1} w(k)\|^2_{\bar{Q}_{f_i}} + 2c_{i,3}\|A_K^{N-1} w(k)\|,$$

where $\bar{Q}_{f_i} = T_{\mathcal{N}_i}^\top \left(Q_{\mathcal{N}_i} + K_{f_i}^\top R_i K_{f_i}\right) T_{\mathcal{N}_i}$, and $c_{i,3}$ is an upper bound of $\|T_{\mathcal{N}_i}^\top (Q_{\mathcal{N}_i} + K_{f_i}^\top R_i K_{f_i})\bar{x}_{\mathcal{N}_i}\|$, for given matrices $T_{\mathcal{N}_i}$, $Q_{\mathcal{N}_i}$, R_i and K_{f_i}, $\forall \bar{x}_{\mathcal{N}_i} \in T_{\mathcal{N}_i}\mathcal{X}$.

Based on a shifted control input from (A.2) at the prediction step N, we know $\bar{x}_i^*(N+1; x_i(k)) = A_{K_{f_i}} \bar{x}_{\mathcal{N}_i}^*(N; x_i(k))$, which gives

$$\|\bar{x}_i^*(N+1; x_i(k))\|^2_{P_{f_i}} = \|\bar{x}_{\mathcal{N}_i}^*(N; x_i(k))\|^2_{A_{K_{f_i}}^\top P_{f_i} A_{K_{f_i}}}.$$

Also based on shifted control inputs from (A.1) and (A.2) at the prediction step N, it also comes

$$\bar{x}_i(N; x_i(k+1)) - \bar{x}_i^*(N+1; x_i(k)) = A_{K_{f_i}} T_{\mathcal{N}_i} A_K^{N-1} w(k).$$

Therefore, we have

$$\|\bar{x}_i(N; x_i(k+1))\|^2_{P_{f_i}}$$
$$\leq \|\bar{x}_{\mathcal{N}_i}^*(N; x_i(k))\|^2_{A_{K_{f_i}}^\top P_{f_i} A_{K_{f_i}}} + \|A_K^{N-1} w(k)\|^2_{\bar{P}_{f_i}}$$
$$+ 2c_{i,4}\|A_K^{N-1} w(k)\|,$$

where $\bar{P}_{f_i} = T_{\mathcal{N}_i}^\top A_{K_{f_i}}^\top P_{f_i} A_{K_{f_i}} T_{\mathcal{N}_i}$, and $c_{i,4}$ is an upper bound of $\|T_{\mathcal{N}_i}^\top A_{K_{f_i}}^\top P_{f_i} x_i\|$ for given matrices K_{f_i} and P_{f_i}.

From the condition (19c), we have

$$\|\bar{x}_{\mathcal{N}_i}^*(N; x_i(k))\|^2_{A_{K_{f_i}}^\top P_{f_i} A_{K_{f_i}}} - \|\bar{x}_i^*(N; x_i(k))\|^2_{P_{f_i}}$$
$$\leq -\|\bar{x}_{\mathcal{N}_i}^*(N; x_i(k))\|^2_{Q_{\mathcal{N}_i}+K_{f_i}^\top R_i K_{f_i}} + \gamma_i(\bar{x}_{\mathcal{N}_i}^*(N; x_i(k))).$$

As a result, we thus obtain

$$V_{N_i}(x_i(k+1)) - V_{N_i}^*(x_i(k))$$
$$\leq \gamma_i(\bar{x}_{\mathcal{N}_i}^*(N; x_i(k))) + \lambda_i(\|w_i(k)\|)$$
$$- \|\bar{x}_{\mathcal{N}_i}^*(0; x_i(k))\|^2_{Q_{\mathcal{N}_i}} - \|\bar{u}_i^*(0; x_i(k))\|^2_{\bar{R}_i}$$

where

$$\lambda_i(\|w(k)\|) = \sum_{t=0}^{N-2} \left(\|A_K^t w(k)\|^2_{\bar{Q}_i} + 2c_{i,1}\|A_K^t w(k)\|\right.$$
$$+ \|A_K^t w(k)\|^2_{\bar{R}_i} + 2c_{i,2}\|A_K^t w(k)\|\right)$$
$$+ \|A_K^{N-1} w(k)\|^2_{\bar{Q}_{f_i}} + 2c_{i,3}\|A_K^{N-1} w(k)\|$$
$$+ \|A_K^{N-1} w(k)\|^2_{\bar{P}_{f_i}} + 2c_{i,4}\|A_K^{N-1} w(k)\|,$$

By optimality, we know $V_{N_i}^*(x_i(k+1) \leq V_{N_i}(x_i(k+1)$. Then, by proceed with sum, we can obtain with (19d)

$$V_N^*(x(k+1)) - V_N^*(x(k))$$
$$\leq \sum_{i \in \mathcal{M}} \left(-\|\bar{x}_{\mathcal{N}_i}^*(0; x_i(k))\|^2_{Q_{\mathcal{N}_i}} + \lambda_i(\|w(k)\|)\right),$$

which is an ISS-Lyapunov function as stated in [Definition 7](Limon et al., 2009). Thus, the closed-loop system is ISS stable. □

B Proofs in Section 5

B.1 Proof of Theorem 3

The condition of the RPI set can be written as $x(k+1) \in \mathcal{Z}$, $\forall x(k) \in \mathcal{Z}$ and $\forall w \in \mathcal{W}$. By using the S-procedure (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994, Chapter 2.6.3), the above condition is satisfied if there exist $\tau_1 \geq 0$, $\tau_2 \geq 0$ such that

$$\begin{bmatrix} -A_K^\top P A_K & -A_K^\top P & 0 \\ -P A_K & -P & 0 \\ 0 & 0 & 1 \end{bmatrix} - \tau_1 \begin{bmatrix} -P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$- \tau_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & -W & 0 \\ 0 & 0 & 1 \end{bmatrix} \succeq 0$$

The above condition is equivalent to (29b) and

$$\begin{bmatrix} \tau_1 P & 0 \\ 0 & \tau_2 W \end{bmatrix} - \begin{bmatrix} A_K^\top \\ I \end{bmatrix} P \begin{bmatrix} A_K & I \end{bmatrix} \succeq 0.$$

By using the Schur complement to the above condition, we can obtain

$$
\begin{bmatrix} \tau_1 P & 0 & A_K^\top \\ 0 & \tau_2 W & I \\ A_K & I & P^{-1} \end{bmatrix} \succeq 0.
$$

Again, by using the Schur complement, we have

$$
\begin{bmatrix} \tau_2 W & I \\ I & P^{-1} \end{bmatrix} - \begin{bmatrix} 0 \\ A_K \end{bmatrix} (\tau_1 P)^{-1} \begin{bmatrix} 0 & A_K^\top \end{bmatrix} \succeq 0.
$$

By using the Schur complement to the above condition with setting $Y = KP^{-1}$ and $S = P^{-1}$, we thus obtain (29a). $\square$

B.2 Proof of Theorem 4

First, let us discuss the condition to guarantee $\mathcal{X} \ominus \mathcal{Z}$. By using the results in (Boyd & Vandenberghe, 2004, Chapter 8.4.2), the condition to guarantee $\mathcal{Z} \subseteq \mathcal{X}$ can be expressed as

$$
\left\| P^{-\frac{1}{2}} a_j \right\| \le d_j, \ j = 1, \ldots, n_r.
$$

which can be rewritten as

$$
d_j^2 - a_j^\top P^{-1} a_j \ge 0, \ j = 1, \ldots, n_r.
$$

By using the Schur complement to the above condition with setting $Y = KP^{-1}$ and $S = P^{-1}$, we can obtain (30a).

Then, similarly, considering $\mathcal{U}$ defined in (26b) and the control gain K, we have $h_j^\top K x \le g_j, \ j = 1, \ldots, m_r$. The condition to guarantee non-empty $\mathcal{U} \ominus K\mathcal{Z}$ can be expressed as

$$
g_j^2 - h_j^\top P^{-1} h_j \ge 0, \ j = 1, \ldots, m_r.
$$

By using the Schur complement to the above condition, we thus obtain (30b). $\square$

B.3 Proof of Corollary 1

Similar to the proof of Theorem 3, for each agent i, the condition $x_i(k+1) \in \mathcal{Z}_i, \ \forall x_i(k) \in \mathcal{Z}_i$ and $\forall w_i \in \mathcal{W}_i$ is

satisfied if there exist two scalars $\bar{\tau}_i \ge 0, \bar{\tau}_{ij} \ge 0, \forall j \in \mathcal{N}_i$ such that (32b) and

$$
\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \bar{\tau}_{ij} P_{ij} & 0 \\ 0 & \bar{\tau}_i W_i \end{bmatrix} - \begin{bmatrix} A_{K_{\mathcal{N}_i}}^\top \\ I \end{bmatrix} P_i \begin{bmatrix} A_{K_{\mathcal{N}_i}} & I \end{bmatrix} \succeq 0,
$$

with $A_{K_{\mathcal{N}_i}} := A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i}$. By using the Schur complement twice, the above condition is equivalent to

$$
\begin{bmatrix} \bar{\tau}_i W_i & I \\ I & P_i^{-1} \end{bmatrix} - \begin{bmatrix} 0 \\ A_{K_{\mathcal{N}_i}} \end{bmatrix} \left(\sum_{j \in \mathcal{N}_i} \bar{\tau}_{ij} P_{ij} \right)^{-1} \begin{bmatrix} 0 & A_{K_{\mathcal{N}_i}}^\top \end{bmatrix} \succeq 0.
$$

Pre-multiplying and post-multiplying (32a) by $[I \ -\Theta_i]$ and $[I \ -\Theta_i]^\top$ with $\Theta_i^T = \begin{bmatrix} 0 & A_{K_{\mathcal{N}_i}}^\top \end{bmatrix}$ can obtain the above condition with setting $S_i = P_i^{-1}$ and $Y_i = K_{\mathcal{N}_i} G_i$. $\square$

B.4 Proof of Theorem 5

From the condition (21), we have

$$
A_{K_{f_i}}(x_{\mathcal{N}_i} + e_{\mathcal{N}_i}) \in \Omega_{f_i}(\alpha_i), \forall x_j \in \Omega_{f_j}(\alpha_j), \forall j \in \mathcal{N}_i,
$$

and $\forall e_{\mathcal{N}_i} \in \bar{\mathcal{E}}_{\mathcal{N}_i}(N-1)$, which is equivalent to

$$
(x_{\mathcal{N}_i} + e_{\mathcal{N}_i})^\top A_{K_{f_i}}^\top F_i A_{K_{f_i}}(x_{\mathcal{N}_i} + e_{\mathcal{N}_i}) \le \alpha_i,
$$
$$
\text{for all } x_j^\top F_i x_j \le \alpha_j, \forall j \in \mathcal{N}_i, \text{ and } e_{\mathcal{N}_i}^\top E_{\mathcal{N}_i} e_{\mathcal{N}_i} \le 1.
$$

Refer to (Darivianakis et al., 2020, Proposition 2), we set $x_i = \alpha_i^{\frac{1}{2}} s_i$, $x_{\mathcal{N}_i} = \alpha_{\mathcal{N}_i}^{\frac{1}{2}} s_{\mathcal{N}_i}$ and $e_{\mathcal{N}_i} = \alpha_{\mathcal{N}_i}^{\frac{1}{2}} z_{\mathcal{N}_i}$.

The above condition is equivalent to

$$
(s_{\mathcal{N}_i} + z_{\mathcal{N}_i})^\top (A_{K_{f_i}} \alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top F_i A_{K_{f_i}} \alpha_{\mathcal{N}_i}^{\frac{1}{2}}(s_{\mathcal{N}_i} + z_{\mathcal{N}_i}) \le \alpha_i,
$$
$$
\text{for all } s_{\mathcal{N}_i}^\top F_{ij} s_{\mathcal{N}_i} \le 1, \forall j \in \mathcal{N}_i,
$$
$$
\text{and } z_{\mathcal{N}_i}^\top (\alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top E_{\mathcal{N}_i} \alpha_{\mathcal{N}_i}^{\frac{1}{2}} z_{\mathcal{N}_i} \le 1.
$$

By using the S-procedure (Boyd et al., 1994, Chapter 2.6.3), the above condition is satisfied if there exist scalars $\sigma_i \ge 0, \sigma_{ij} \ge 0, \forall j \in \mathcal{N}_i, \forall i \in \mathcal{M}$ such that (37b) and

$$
\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \sigma_{ij} F_{ij} & 0 \\ 0 & (\alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top \sigma_i E_{\mathcal{N}_i} \alpha_{\mathcal{N}_i}^{\frac{1}{2}} \end{bmatrix}
$$
$$
- \begin{bmatrix} (A_{K_{f_i}} \alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top \\ (A_{K_{f_i}} \alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top \end{bmatrix} \alpha_i^{-\frac{1}{2}} F_i \begin{bmatrix} A_{K_{f_i}} \alpha_{\mathcal{N}_i}^{\frac{1}{2}} & A_{K_{f_i}} \alpha_{\mathcal{N}_i}^{\frac{1}{2}} \end{bmatrix} \succeq 0.
$$

By using the Schur complement and arranging its rows and columns, we can obtain

$$\begin{bmatrix} \sum_{j \in \mathcal{N}_i} \sigma_{ij} F_{ij} & (\alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top A_{K_{f_i}}^\top & 0 \\ \star & \alpha_i^{\frac{1}{2}} F_i^{-1} & A_{K_{f_i}} \alpha_{\mathcal{N}_i}^{\frac{1}{2}} \\ \star & \star & (\alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top \sigma_i E_{\mathcal{N}_i} \alpha_{\mathcal{N}_i}^{\frac{1}{2}} \end{bmatrix} \succeq 0.$$

Again, by using the Schur complement twice, we thus obtain (37a). □

B.5 Proof of Theorem 6

Also based on the results in (Boyd & Vandenberghe, 2004, Chapter 8.4.2), the condition for $\Omega_{f_i}(\alpha_i)$ satisfying $\mathcal{X}_{\mathcal{N}_i} = \bigtimes_{j \in \mathcal{N}_i} \mathcal{X}_j$ can be formulated as

$$\left\| \bar{a}_{il}^\top x_{\mathcal{N}_i} \right\| \le \bar{d}_{il}, \text{ for all } x_j^\top F_i x_j \le \alpha_j, \forall j \in \mathcal{N}_i,$$

for each $l = 1, \ldots, n_{r_{\mathcal{N}_i}}$.

Set $x_i = \alpha_i^{\frac{1}{2}} s_i$, $x_{\mathcal{N}_i} = \alpha_{\mathcal{N}_i}^{\frac{1}{2}} s_{\mathcal{N}_i}$. The above condition is equivalent to

$$s_{\mathcal{N}_i}^\top (\alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top \bar{a}_{il} \bar{d}_{il}^{-1} \bar{a}_{il}^\top \alpha_{\mathcal{N}_i}^{\frac{1}{2}} s_{\mathcal{N}_i} \le \bar{d}_{il},$$
$$\text{for all } s_{\mathcal{N}_i}^\top F_{ij} s_{\mathcal{N}_i} \le 1, \forall j \in \mathcal{N}_i,$$

which is satisfied if there exist scalars $\phi_{ijl} \ge 0$, $\forall j \in \mathcal{N}_i$ such that (39b) and

$$\sum_{j \in \mathcal{N}_i} \phi_{ijl} F_{ij} - (\alpha_{\mathcal{N}_i}^{\frac{1}{2}})^\top \bar{a}_{il} \bar{d}_{il}^{-1} \bar{a}_{il}^\top \alpha_{\mathcal{N}_i}^{\frac{1}{2}} \succeq 0.$$

By using the Schur complement to the above condition, we thus obtain (39a).

On the other hand, following the same procedure, we have

$$x_{\mathcal{N}_i}^\top K_{f_i}^\top h_{ip} g_{ip}^{-1} h_{ip}^\top K_{f_i} x_{\mathcal{N}_i} \le g_{ip},$$
$$\text{for all } x_j^\top F_i x_j \le \alpha_j, \forall j \in \mathcal{N}_i,$$

for each $p = 1, \ldots, m_{r_i}$, is satisfied, if the conditions in (40) hold. □

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