NEW-FROM-OLD FULL DUALITIES VIA AXIOMATISATION

BRIAN A. DAVEY, JANE G. PITKETHLY, AND ROSS WILLARD

ABSTRACT. We study different full dualities based on the same finite algebra. Our main theorem gives conditions on two different alter egos of a finite algebra under which, if one yields a full duality, then the other does too. We use this theorem to obtain a better understanding of several important examples from the theory of natural dualities. We also clarify what it means for two full dualities based on the same finite algebra to be different. Throughout the paper, a fundamental role is played by the universal Horn theory of the dual categories.

1. INTRODUCTION

This paper is a contribution to our understanding of full dualities. We show how to obtain new full dualities from an existing full duality using a universal Horn axiomatisation of the dual category. This provides a systematic technique for finding full dualities that can be used to 'rediscover' several important but previously ad-hoc counterexamples from the theory of natural dualities.

In this introduction, we first motivate the theory of natural dualities within a setting appropriate to a reader with a background in category theory rather than universal algebra. We then describe the problems in which we are interested and summarise our results.

The rest of the paper is structured as follows: We give a focused introduction to full dualities in Sections 2–3. In preparation for the proof of our main theorem, we present a motivating example in Section 4, and then sketch the relevant universal Horn logic for signatures that include partial operations in Section 5. The main theorem (Theorem 8.1) is proved over Sections 6–7. Corollaries and applications of this theorem are developed in Section 8–9.

Our setting. The results presented here arise from the study of 'structural dual equivalences' (known as *natural dualities*) for certain concrete categories over SET. The categories that we wish to dualise are quasivarieties \mathcal{A} of Σ -algebras (with the usual morphisms), for some finitary functional signature Σ , with the added requirement that \mathcal{A} is generated by a finite algebra \mathbf{M} in the model-theoretic sense (meaning that \mathbf{M} is a cogenerator of \mathcal{A} and its underlying set is finite). The theory of natural dualities has its roots in the early 1980s when Davey and Werner set down the basic theory [12]. The state of the theory up to the late 1990s is presented in the text by Clark and Davey [2].

²⁰¹⁰ Mathematics Subject Classification. 08C20, 08C15, 03C07.

Key words and phrases. Natural duality, full duality, alter ego, universal Horn axiomatisation. The third author was supported by a Discovery Grant from NSERC, Canada.

The primary goal. As observed by Isbell [15], an adjoint connection between two categories is normally induced by an 'object living in both categories'. Accordingly, starting from such \mathcal{A} and \mathbf{M} as described above, the theory follows a standard recipe and seeks to place \mathcal{A} in a dual adjunction by finding another concrete category \mathfrak{Z} and an object \mathbb{M} of \mathfrak{Z} with the same underlying set as \mathbf{M} such that ' \mathbb{M} commutes with \mathbf{M} '; see Johnstone [16, VI.4]. The object \mathbb{M} is known as an alter eqo of \mathbf{M} .

Under the right conditions (see, e.g., Porst and Tholen [18, 1-C]), there exist concrete functors

$$D: \mathcal{A} \to \mathfrak{Z}^{\mathrm{op}}$$
 and $E: \mathfrak{Z} \to \mathcal{A}^{\mathrm{op}}$

represented by

 $\hom_{\mathcal{A}}(-, \mathbf{M}) \colon \mathcal{A} \to \operatorname{Set}^{\operatorname{op}} \quad \operatorname{and} \quad \hom_{\mathfrak{Z}}(-, \mathbb{M}) \colon \mathfrak{Z} \to \operatorname{Set}^{\operatorname{op}},$

respectively, which are part of a dual adjunction between \mathcal{A} and \mathfrak{Z} . Letting $\eta: \mathrm{id}_{\mathcal{A}} \to ED$ and $\varepsilon: \mathrm{id}_{\mathfrak{Z}} \to DE$ denote the units of the adjunction, we obtain a dual equivalence between the fixed subcategories

Fix
$$\eta = \{ \mathbf{A} \in \mathcal{A} \mid \eta_{\mathbf{A}} \text{ is an isomorphism} \}$$
 and
Fix $\varepsilon = \{ \mathbb{Z} \in \mathfrak{Z} \mid \varepsilon_{\mathbb{Z}} \text{ is an isomorphism} \}$

in the usual way (Lambek and Scott [17, Prop. 4.2]); see Section 2 below for details. The primary goal is to find \mathfrak{Z} and \mathbb{M} such that Fix $\eta = \mathcal{A}$.

Our choice of dual categories. A wealth of examples guide our choice for the concrete categories \mathfrak{Z} . We list just three:

- Stone duality [21] between Boolean algebras and Boolean spaces (that is, compact totally disconnected spaces, also known as Stone spaces);
- Hofmann–Mislove–Stralka duality [14] between unital semilattices and Boolean topological unital semilattices;
- Priestley duality [19] between bounded distributive lattices and Priestley spaces (that is, compact totally order-disconnected ordered spaces).

Motivated by examples such as these, and following Clark and Krauss [5] and Davey and Werner [12], we confine our search to concrete categories \mathbf{z} of the following special kind: for some finitary signature Δ , the category \mathbf{z} consists of all Boolean spaces enriched with Δ -structure that is continuous (for operations) and closed (for relations) with respect to the topology. This restriction to such categories \mathbf{z} is severe, and makes the entire project impossible for some quasivarieties \mathcal{A} (for example, the quasivariety of implication algebras [12, pp. 148–151]).

We say that \mathcal{A} is *dualisable* (in our strict sense) if, for some (equivalently, for every [13, 20]) finite cogenerator \mathbf{M} , there exists a category \mathfrak{Z} of this kind containing an alter ego \mathbb{M} of \mathbf{M} such that Fix $\eta = \mathcal{A}$. In this setting, the requirement that ' \mathbb{M} commutes with \mathbf{M} ' becomes ' \mathbb{M} is compatible with \mathbf{M} '; see Section 2 for the formal definition.

The categories \mathfrak{Z} that we consider have concrete powers and an internal notion of 'induced structural subobject', so that if \mathbf{A} is an object in \mathcal{A} with underlying set A, then $D(\mathbf{A})$ is the induced substructure of \mathbb{M}^A with topologically closed underlying set $\hom_{\mathcal{A}}(\mathbf{A}, \mathbf{M})$. Hence, each object in Fix ε is isomorphic to a topologically closed induced substructure of a non-zero power of \mathbb{M} . We summarise this observation by writing Fix $\varepsilon \subseteq \mathsf{IS}_{c}\mathsf{P}^+(\mathbb{M})$. Following [5], we call $\mathsf{IS}_{c}\mathsf{P}^+(\mathbb{M})$ the topological quasivariety generated by \mathbb{M} . Again guided by examples like the three listed above, we identify $\mathsf{IS}_{c}\mathsf{P}^+(\mathbb{M})$ as a 'structurally simple' subcategory of \mathfrak{Z} . Being greedy, we aim to find \mathfrak{Z} and \mathbb{M} that not only satisfy Fix $\eta = \mathcal{A} = \mathsf{ISP}(\mathbf{M})$, but also satisfy Fix $\varepsilon = \mathsf{IS}_c\mathsf{P}^+(\mathbb{M})$. In this case, we deem Fix ε to be adequately understood and say that \mathcal{A} is *fully dualised* within \mathfrak{Z} via the pair (\mathbf{M}, \mathbb{M}). Since the algebra \mathbf{M} determines the quasivariety \mathcal{A} and the alter ego \mathbb{M} determines both \mathfrak{Z} and $\mathsf{IS}_c\mathsf{P}^+(\mathbb{M})$, we usually localise to the pair (\mathbf{M}, \mathbb{M}) and say simply that \mathbb{M} fully dualises \mathbf{M} . (Note that, as for dualisability, the full dualisability of \mathcal{A} is independent of the choice of the cogenerator \mathbf{M} [8].)

Partial operations cannot be avoided. A fact worth noting is that the signature of \mathfrak{Z} often must contain operations, or even partial operations, if \mathcal{A} is to be fully dualised within \mathfrak{Z} via a pair (\mathbf{M}, \mathbb{M}) . For example, every endomorphism of \mathbf{M} must be represented as a term function of \mathbb{M} (Davey, Haviar and Willard [9, Prop. 4.3(b)]). The proof is easy, so we sketch it here: Let \mathbb{X} be the smallest induced substructure of $D(\mathbf{M})$ containing id_M . Then $\mathbb{X} \in \mathsf{IS}_{\mathsf{C}}\mathsf{P}^+(\mathbb{M})$ and the underlying set of \mathbb{X} consists of all total unary term functions of \mathbb{M} . It can be shown that $u: E(\mathbb{X}) \to \mathbf{M}$ defined by $u(h) = h(\mathrm{id}_M)$ is a bijective morphism in \mathcal{A} , so is an isomorphism. Assuming \mathcal{A} is fully dualised within \mathfrak{Z} via (\mathbf{M}, \mathbb{M}) , we have $\mathbb{X} \in \mathrm{Fix} \varepsilon$ and so $\mathbb{X} \cong DE(\mathbb{X}) \cong D(\mathbf{M})$. But \mathbb{X} is an induced substructure of $D(\mathbf{M})$, so by finiteness, $\mathbb{X} = D(\mathbf{M})$. Since the underlying set of $D(\mathbf{M})$ is the set of endomorphisms of \mathbf{M} , the result follows.

More generally, there exist quasivariety-cogenerator pairs $(\mathcal{A}, \mathbf{M})$ that are fully dualisable, but only via $(\mathfrak{Z}, \mathbb{M})$ whose signature contains partial operations [9]. So we allow the signature of \mathfrak{Z} to include partial operations.

Structural embeddings. The decision to allow the signature of \mathfrak{Z} to include partial operations comes at a price: the internal notion of 'induced structural subobject', as well as the corresponding notion of 'structural embedding', becomes fragile. In the absence of partial operations, structural embeddings in \mathfrak{Z} are injective morphisms that reflect the relations in the signature; they are characterised categorically in \mathfrak{Z} as concrete embeddings (monomorphisms that are initial with respect to the underlying-set functor), and also as regular monomorphisms (equalisers of morphism pairs). However, once partial operations are allowed in the signature of \mathfrak{Z} , the two categorical notions split; concrete embeddings in the new setting are injective morphisms that reflect relations and graphs of partial operations. Since each inclusion $D(\mathbf{A}) \hookrightarrow \mathbb{M}^A$ is an embedding in the latter (stronger) sense, we adopt the latter sense as the 'correct' notion of structural embedding, and take $\mathsf{IS}_{\mathsf{C}}\mathsf{P}^+(\mathbb{M})$ to mean the full subcategory of \mathfrak{Z} consisting of the objects that structurally embed (in the stronger sense) into a non-zero power of \mathbb{M} .

Problems of interest. With our setting described, we turn to the problems that interest us. Primarily, we would like to know which quasivarieties \mathcal{A} with a finite cogenerator \mathbf{M} are fully dualisable in our sense, and why. We do not address this problem here. Nevertheless, our aim here is closely related:

• For fixed \mathcal{A} and \mathbf{M} with \mathcal{A} fully dualisable, we seek to understand *all* \mathfrak{Z} and \mathbb{M} for which \mathcal{A} is fully dualised within \mathfrak{Z} via (\mathbf{M}, \mathbb{M}) .

To aid the discussion that follows, let H_{Ω} and R_{Ω} denote, respectively, the sets of all finitary partial operations and all finitary relations on the underlying set Mof \mathbf{M} that 'commute' with the structure of \mathbf{M} , and define $\Omega := H_{\Omega} \cup R_{\Omega}$. Each subset Δ of Ω may be interpreted as a signature. Let \mathfrak{Z}_{Δ} denote the category of all Boolean spaces enriched with continuous/closed structure of signature Δ , and let \mathbb{M}_{Δ} denote the obvious object in \mathfrak{Z}_{Δ} with underlying set M. Every alter ego of \mathbf{M} is of the form \mathbb{M}_{Δ} , for some $\Delta \subseteq \Omega$. Thus our question becomes:

• Given that \mathcal{A} is fully dualisable, for which signatures $\Delta \subseteq \Omega$ is \mathcal{A} fully dualised within \mathfrak{Z}_{Δ} via $(\mathbf{M}, \mathbb{M}_{\Delta})$?

Localising to the cogenerators, this question becomes:

• Given that some alter ego fully dualises \mathbf{M} , for which signatures $\Delta \subseteq \Omega$ does \mathbb{M}_{Δ} fully dualise \mathbf{M} ?

Here we address the practical problem of recognising those $\Delta \subseteq \Omega$ for which \mathbb{M}_{Δ} fully dualises **M**. To explain our approach, we first describe three helpful tools.

Tool 1. The first tool is a syntactic quasi-ordering of signatures $\Delta \subseteq \Omega$, with Ω at the top, whose corresponding equivalence relation \equiv captures a useful notion of 'syntactic equivalence'. (See Definitions 3.1–3.3 and Lemma 9.1.) It is known that:

- (a) the 'fully dualises **M**' relation on alter egos is invariant under \equiv , and
- (b) if some \mathbb{M}_{Δ} fully dualises \mathbf{M} , then \mathbb{M}_{Ω} does as well [11, 5.3].

A naive expectation is that, more generally, if \mathbb{M}_{Δ} fully dualises \mathbf{M} and $\Delta \subseteq \Delta'$, then $\mathbb{M}_{\Delta'}$ (being at least as rich as \mathbb{M}_{Δ}) should also fully dualise \mathbf{M} .

Tool 2. The second tool is 'reduction to the finite level'. Once again, fix \mathfrak{Z} and \mathbb{M} . Let \mathcal{A}_{fin} and $\mathfrak{Z}_{\text{fin}}$ denote the subcategories of \mathcal{A} and \mathfrak{Z} , respectively, consisting of the objects whose underlying sets are finite. We say that \mathcal{A} is fully dualised within \mathfrak{Z} via (\mathbf{M}, \mathbb{M}) at the finite level if $\mathcal{A}_{\text{fin}} \subseteq \text{Fix } \eta$ and $\mathsf{IS_cP^+}(\mathbb{M}) \cap \mathfrak{Z}_{\text{fin}} \subseteq \text{Fix } \varepsilon$. Localising to \mathbf{M} and \mathbb{M} , we say that \mathbb{M} fully dualises \mathbf{M} at the finite level. Informally, this means that the dual adjunction satisfies the conditions of being a full duality at the level of finite objects.

A reasonable intuition is that, while the full dualisability of a quasivariety \mathcal{A} is determined at the infinite level, if \mathcal{A} is fully dualisable, then the problem of determining which signatures $\Delta \subseteq \Omega$ give rise to fully dualising alter egos should be determined at the finite level. In particular, a second naive expectation is that, if some alter ego fully dualises \mathbf{M} and \mathbb{M}_{Δ} fully dualises \mathbf{M} at the finite level, then \mathbb{M}_{Δ} should fully dualise \mathbf{M} .

It turns out that both naive expectations stated above are falsified by examples (see [11, 5.1] and [9, Thm 1]), which explains in part the delicateness of the problem.

Tool 3. The third tool, which will help us to overcome these issues, is the logic of universal Horn sentences in finitary signatures with partial operations. An easy observation is that, if two alter egos \mathbb{M}_1 and \mathbb{M}_2 both fully dualise \mathbf{M} , then the categories $\mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_1)$ and $\mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_2)$ are equivalent, as they are both dually equivalent to \mathcal{A} . We will show that, conversely, in certain situations we can translate a full duality from a known fully dualising alter ego \mathbb{M}_1 to another alter ego \mathbb{M}_2 by defining a concrete isomorphism between the categories $\mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_1)$ and $\mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_2)$. (In particular, we can define such an isomorphism whenever both \mathbb{M}_1 and \mathbb{M}_2 fully dualise \mathbf{M} .) The logic of universal Horn sentences is used both to articulate the assumptions that make this work and to define the isomorphism.

We outline the relevant universal Horn logic in Sections 2 and 5. For a more detailed introduction to universal Horn logic as it applies to the axiomatisation of dual categories, see Clark, Davey, Haviar, Pitkethly and Talukder [3].

Our results. We assume that we are given an alter ego \mathbb{M}_1 that fully dualises a finite algebra \mathbf{M} at the finite level. Our main theorem (Theorem 8.1) gives necessary and sufficient conditions, organised into three families, under which an alter ego \mathbb{M}_2 also fully dualises \mathbf{M} at the finite level. These conditions are a fragment of those in the known characterisation (Lemma 3.7), and depend on both \mathbf{M} and \mathbb{M}_1 . In particular, one family of conditions is constructed from a basis for the universal Horn theory of \mathbb{M}_1 .

Self-contained corollaries. While the statement of the main theorem is rather technical, we use the theorem to obtain a series of self-contained corollaries. One corollary gives a new and very natural condition under which every finite-level full duality lifts to the infinite level. First, we say that an alter ego \mathbb{M} is *standard* if the category $\mathsf{IS_cP^+}(\mathbb{M})$ consists of all the structures in \mathfrak{Z} that satisfy the universal Horn theory of \mathbb{M} ; see Definition 2.2. Here is the corollary (Theorem 8.3):

Assume that some standard alter ego fully dualises \mathbf{M} . If an alter ego \mathbb{M} fully dualises \mathbf{M} at the finite level, then \mathbb{M} fully dualises \mathbf{M} and is also standard.

It follows from this corollary that, for any quasi-primal algebra \mathbf{M} , every finite-level full duality based on \mathbf{M} lifts to a full duality (Example 8.4).

Other corollaries include a new characterisation of the alter egos that yield a finite-level full duality (Theorem 8.5) and a new constructive description of the smallest alter ego that yields a finite-level full duality (Theorem 8.6). We also obtain the known characterisation (Davey, Pitkethly and Willard [11, 5.3]) of how the structure on an alter ego can be enriched without destroying a full duality (Theorem 8.7).

Seminal counterexamples explained. We use our main theorem to elucidate two important counterexamples in the theory of natural dualities:

- The first example of a finite-level full duality that is not equivalent to \mathbb{M}_{Ω} (Davey, Haviar and Willard [9]). This example is based on the threeelement bounded lattice **3** and the alter ego \mathfrak{Z}_h defined in Section 4.
- The first example of a full duality that is not equivalent to \mathbb{M}_{Ω} (Clark, Davey and Willard [4]). This example, which solved a 27-year-old problem from [12], is based on a four-element quasi-primal algebra \mathbf{Q} and the alter ego \mathbb{Q}_0 defined in Example 8.8.

We give a general algorithm (Algorithm 8.9) that, given (i) an alter ego \mathbb{M}_1 that fully dualises **M** at the finite level (typically, but not necessarily, equivalent to \mathbb{M}_{Ω}), and (ii) a finite basis for the universal Horn theory of \mathbb{M}_1 , produces the smallest alter ego (up to \equiv) that fully dualises **M** at the finite level. This algorithm can be applied to obtain the two examples listed above; see Example 8.10.

Different full dualities. In the final section of the paper, we clarify what it means for two full dualities based on the same finite algebra \mathbf{M} to be 'different'. We show that the concept of 'structural embedding' is not categorical in the concrete dual category Fix $\varepsilon = \mathsf{IS}_{c}\mathsf{P}^+(\mathbb{M})$. (This is in contrast to the fact that the concept *is* categorical in the larger category \mathfrak{Z} .) More precisely, if \mathbb{M}_1 and \mathbb{M}_2 are two alter egos, both of which fully dualise \mathbf{M} , then the two dual categories $\mathsf{IS}_{c}\mathsf{P}^+(\mathbb{M}_1)$ and $\mathsf{IS}_{c}\mathsf{P}^+(\mathbb{M}_2)$ are necessarily isomorphic as concrete categories (Lemma 9.2), but the isomorphism cannot preserve and reflect structural embeddings unless the signatures of \mathbb{M}_1 and \mathbb{M}_2 are equivalent (Lemma 9.1).

2. Preliminaries: Full dualities

In this section, we formalise many of the concepts discussed more informally in the introduction. For a comprehensive introduction to the theory of natural dualities, see the Clark–Davey text [2].

Fix a finite algebra $\mathbf{M} = \langle M; \Sigma \rangle$ and consider the quasivariety $\mathcal{A} := \mathsf{ISP}(\mathbf{M})$, that is, the class of all isomorphic copies of subalgebras of arbitrary powers of \mathbf{M} . Our conventions are that \mathcal{A} never contains the empty algebra and that \mathcal{A} always contains the one-element algebras (via the zero power); for other consistent conventions, see [11].

- Let r be an n-ary relation on M, for some $n \ge 0$. Then r is said to be compatible with **M** if it forms a subalgebra **r** of **M**ⁿ.
- Let h be an n-ary partial operation on M, for some $n \ge 0$. Then h is said to be *compatible with* **M** if the (n + 1)-ary relation

$$graph(h) := \{ (\vec{a}, h(\vec{a})) \mid \vec{a} \in \operatorname{dom}(h) \}$$

is compatible with \mathbf{M} , or equivalently, if the *n*-ary relation $r := \operatorname{dom}(h)$ is compatible with \mathbf{M} and $h : \mathbf{r} \to \mathbf{M}$ is a homomorphism.

An alter ego of **M** is a topological structure $\mathbb{M} = \langle M; H, R, \mathcal{T} \rangle$ with the same underlying set as **M**, where

- *H* is a set of partial operations that are compatible with **M**,
- R is a set of relations that are compatible with **M**, and
- \mathcal{T} is the discrete topology on M.

It is common to add a set G of total operations to the signature of \mathbb{M} , but to simplify the notation, we include total operations in H.

An alter ego \mathbb{M} is the starting point for creating a potential dual category \mathfrak{X} for the quasivariety $\mathcal{A} = \mathsf{ISP}(\mathbf{M})$. First, we form the category \mathfrak{Z} whose objects are the *Boolean structures* of signature (H, R). (That is, each member of \mathfrak{Z} is a topological structure with a Boolean topology and with continuous partial operations on closed domains and with closed relations.) The morphisms of \mathfrak{Z} are continuous structurepreserving maps. The potential dual category \mathfrak{X} of \mathcal{A} will be a full subcategory of \mathfrak{Z} .

We require the usual concept of induced substructure and the concept of structural embedding:

- For X, Y ∈ Z, we say that X is an *induced substructure* of Y if X ⊆ Y, the topology on X is the induced subspace topology from Y, the relations in R^X are the restrictions of those in R^Y, and the domains and graphs of the partial operations in H^X are the restrictions of those in H^Y.
- We define a morphism in \mathfrak{Z} to be a *structural embedding* if
 - (a) it is a homeomorphism from its domain to its range considered as an induced subspace of its codomain, and
 - (b) it preserves and reflects the relations in the signature and the domains and graphs of partial operations in the signature.

The alter ego \mathbb{M} of \mathbb{M} induces a pair of contravariant hom-functors $D: \mathcal{A} \to \mathfrak{Z}$ and $E: \mathfrak{Z} \to \mathcal{A}$, and a pair of natural transformations $\eta: \mathrm{id}_{\mathcal{A}} \to ED$ and $\varepsilon: \mathrm{id}_{\mathfrak{Z}} \to DE$.

The hom-functors D and E are given on objects by

 $D(\mathbf{A}) :=$ the induced substructure of \mathbb{M}^A with underlying set $\hom_{\mathcal{A}}(\mathbf{A}, \mathbf{M})$

 $E(\mathbb{Z}) :=$ the subalgebra of $\mathbf{M}^{\mathbb{Z}}$ with underlying set hom_{\mathfrak{Z}}(\mathbb{Z}, \mathbb{M})

for all $\mathbf{A} \in \mathcal{A}$ and $\mathbb{Z} \in \mathfrak{Z}$. The compatibility between \mathbf{M} and \mathbb{M} guarantees that these hom-functors are well defined. The natural transformations η and ε are given by evaluation: for all $\mathbf{A} \in \mathcal{A}$, the homomorphism $\eta_{\mathbf{A}} : \mathbf{A} \to ED(\mathbf{A})$ is defined by

 $\eta_{\mathbf{A}}(a)(x) := x(a), \text{ for all } a \in A \text{ and } x \in \hom_{\mathcal{A}}(\mathbf{A}, \mathbf{M}),$

and, for all $\mathbb{Z} \in \mathfrak{Z}$, the morphism $\varepsilon_{\mathbb{Z}} \colon \mathbb{Z} \to DE(\mathbb{Z})$ is defined by

$$\varepsilon_{\mathbb{Z}}(z)(u) := u(z), \text{ for all } z \in Z \text{ and } u \in \hom_{\mathfrak{Z}}(\mathbb{Z}, \mathbb{M}).$$

It is easily seen that $\eta_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ is an embedding, for all $\mathbf{A} \in \mathcal{A}$, since $\mathcal{A} = \mathsf{ISP}(\mathbf{M})$.

We form the topological quasivariety $\mathfrak{X} := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M})$ consisting of all isomorphic copies of topologically closed induced substructures of non-zero powers of \mathbb{M} . Our conventions are that \mathfrak{X} contains the empty structure if and only if H contains no nullary operations, and that \mathfrak{X} contains a one-element structure if and only if it appears as an induced substructure of \mathbb{M} . It is easily seen that $\varepsilon_{\mathbb{X}} \colon \mathbb{X} \to DE(\mathbb{X})$ is a structural embedding, for all $\mathbb{X} \in \mathfrak{X}$.

The basic concepts, localised to the pair (\mathbf{M}, \mathbb{M}) , are defined as follows:

- (1) \mathbb{M} dualises \mathbf{M} [at the finite level] if the embedding $\eta_{\mathbf{A}} : \mathbf{A} \to ED(\mathbf{A})$ is an isomorphism, for each [finite] algebra $\mathbf{A} \in \mathcal{A}$.
- (2) \mathbb{M} fully dualises \mathbf{M} [at the finite level] if, in addition to (1), the structural embedding $\varepsilon_{\mathbb{X}} \colon \mathbb{X} \to DE(\mathbb{X})$ is an isomorphism, for each [finite] structure $\mathbb{X} \in \mathfrak{X}$.
- (3) \mathbb{M} strongly dualises \mathbf{M} [at the finite level] if, in addition to (1) and (2), the alter ego \mathbb{M} is injective with respect to structural embeddings among the [finite] structures in \mathfrak{X} .

Using the Fix notation of the introduction:

- If \mathbb{M} dualises \mathbf{M} , then Fix $\eta = \mathcal{A}$, and hence \mathcal{A} is dually equivalent to a full subcategory of \mathfrak{X} .
- If M fully dualises M, then Fix $\eta = \mathcal{A}$ and Fix $\varepsilon = \mathfrak{X}$, and hence \mathcal{A} is dually equivalent to \mathfrak{X} .

The following basic lemma will allow us to create new full dualities from old ones. There are two versions of this lemma: the phrases in square brackets can be either included or deleted.

New-from-old Lemma 2.1. Let \mathbb{M}_1 and \mathbb{M}_2 be alter egos of a finite algebra \mathbf{M} . For $i \in \{1, 2\}$, define $\mathfrak{X}_i := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_i)$. Assume that \mathbb{M}_1 fully dualises \mathbf{M} [at the finite level]. Then \mathbb{M}_2 also fully dualises \mathbf{M} [at the finite level] provided the following two conditions hold:

- (1) \mathbb{M}_2 dualises **M** [at the finite level];
- (2) for each [finite] structure X in X₂, there is a structure X' in X₁ on the same underlying set as X such that hom_{X₂}(X, M₂) = hom_{X₁}(X', M₁).

Proof. Define $\mathcal{A} := \mathsf{ISP}(\mathbf{M})$ and, for $i \in \{1, 2\}$, let $D_i : \mathcal{A} \to \mathfrak{X}_i$ and $E_i : \mathfrak{X}_i \to \mathcal{A}$ be the hom-functors induced by \mathbf{M} and \mathbb{M}_i . Assume that (1) and (2) hold. Let \mathbb{X} be a [finite] structure in \mathfrak{X}_2 . We just need to show that $\varepsilon_{\mathbb{X}} : \mathbb{X} \to D_2 E_2(\mathbb{X})$ is

surjective, that is, we need to show that every homomorphism $u: E_2(\mathbb{X}) \to \mathbf{M}$ is given by evaluation.

By (2), we have $\hom_{\mathfrak{X}_2}(\mathbb{X}, \mathbb{M}_2) = \hom_{\mathfrak{X}_1}(\mathbb{X}', \mathbb{M}_1)$. Thus $E_2(\mathbb{X}) = E_1(\mathbb{X}')$ in \mathcal{A} . As \mathbb{M}_1 fully dualises \mathbf{M} [at the finite level], each homomorphism $u: E_1(\mathbb{X}') \to \mathbf{M}$ is given by evaluation. It follows at once that each homomorphism $u: E_2(\mathbb{X}) \to \mathbf{M}$ is given by evaluation. \Box

We close this section with a brief discussion of universal Horn sentences and their role in attempts to axiomatise dual categories.

Fix a signature (H, R) of finitary partial-operation and relation symbols. We define a *universal Horn sentence* (*uH-sentence*, for short) in the language of (H, R) to be a first-order sentence of the form

$$\forall \vec{v} \left[\left(\&_{i=1}^{\nu} \alpha_i(\vec{v}) \right) \to \gamma(\vec{v}) \right],$$

for some $\nu \ge 0$, where each $\alpha_i(\vec{v})$ is an atomic formula and $\gamma(\vec{v})$ is either an atomic formula or \perp . (Note that, if $\nu = 0$, then we have a sentence of the form $\forall \vec{v} \gamma(\vec{v})$.)

Definition 2.2 ([3]). Let $\mathbb{M} = \langle M; H, R, \mathcal{T} \rangle$ be an alter ego of a finite algebra \mathbb{M} , and let \mathfrak{Z} be the associated category of Boolean structures of signature (H, R). The potential dual category $\mathfrak{X} = \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M})$ is always contained in the category \mathcal{Y} of all Boolean models of the uH-theory of \mathbb{M} . That is,

$$\mathfrak{X} \subseteq \mathfrak{Y} := ig \{ \mathbb{Y} \in \mathfrak{Z} \mid \mathbb{Y} \models \mathrm{Th}_{\mathrm{uH}}(\mathbb{M}) ig \},$$

where $Th_{uH}(\mathbb{M})$ denotes the set of all uH-sentences true in \mathbb{M} .

If the two categories \mathfrak{X} and \mathfrak{Y} are equal, then we say that the alter ego \mathbb{M} is *standard*. For example, the discrete semilattice $\mathbb{S} = \langle \{0, 1\}; \lor, \mathfrak{T} \rangle$ (from Hofmann–Mislove–Stralka duality) is standard [14], but the discrete chain $2 = \langle \{0, 1\}; \leqslant, \mathfrak{T} \rangle$ (from Priestley duality) is not standard [22].

Note that we always have $\mathfrak{X}_{\text{fin}} = \mathfrak{Y}_{\text{fin}}$. That is, the finite structures in \mathfrak{X} are precisely the finite Boolean models of the uH-theory of \mathbb{M} ; see [3, pp. 861–862].

3. Preliminaries: Comparing Alter Egos

This section gives the more specific background theory that we require. We start by defining the 'structural reduct' quasi-order on the alter egos of a finite algebra \mathbf{M} ; see [10, 11]. This is the natural generalisation from algebras to structures of the 'term reduct' quasi-order.

Definition 3.1 ([2]). Given any alter ego $\mathbb{M} = \langle M; H, R, \mathcal{T} \rangle$ of \mathbf{M} , we define $\operatorname{Clo_{ep}}(\mathbb{M})$ to be the *enriched partial clone* on M generated by H, that is, the smallest set of non-empty partial operations on M that contains H and the projections, $\pi_i \colon M^n \to M$ for all $n \ge 1$ and $i \le n$, and is closed under composition (when the composite has non-empty domain). This corresponds to the usual definition of partial clone, except that we exclude empty domains and we enrich the partial clone by allowing nullary operations.

Definition 3.2 ([11]). Let $\mathbb{M} = \langle M; H, R, \mathcal{T} \rangle$ be an alter ego of **M** and let $k, n \ge 0$.

• We shall call a conjunction of atomic formulæ $\Psi(\vec{v}) = [\psi_1(\vec{v}) \& \cdots \& \psi_k(\vec{v})]$ a conjunct-atomic formula. • We say that a non-empty *n*-ary relation *r* on *M* is *conjunct-atomic definable* from \mathbb{M} if it is described in \mathbb{M} by an *n*-variable conjunct-atomic formula $\Psi(\vec{v})$ in the language of \mathbb{M} , that is, if

$$r = \{ (a_1, \ldots, a_n) \in M^n \mid \Psi(a_1, \ldots, a_n) \text{ is true in } \mathbb{M} \}.$$

• We define Rel_{ca}(M) to be the set of all relations on M that are conjunctatomic definable from M.

Definition 3.3 ([10]). Let $\mathbb{M}_1 = \langle M; H_1, R_1, T \rangle$ and $\mathbb{M}_2 = \langle M; H_2, R_2, T \rangle$ be alter egos of **M**. Then we say that \mathbb{M}_1 is a *structural reduct* of \mathbb{M}_2 if

- (a) each partial operation in H_1 has an extension in $\text{Clo}_{ep}(\mathbb{M}_2)$, and
- (b) each relation in $R_1 \cup \text{dom}(H_1)$ belongs to $\text{Rel}_{ca}(\mathbb{M}_2)$.

We say that \mathbb{M}_1 and \mathbb{M}_2 are *structurally equivalent* if each is a structural reduct of the other.

Under the 'structural reduct' quasi-order, the alter egos of **M** form a doubly algebraic lattice $\mathcal{A}_{\mathbf{M}}$; see [11, 2.6]. The top element of this lattice is represented by the *top alter ego* of **M**, which we denote by $\mathbb{M}_{\Omega} = \langle M; H_{\Omega}, R_{\Omega}, \mathcal{T} \rangle$, where

- H_{Ω} is the set of all partial operations that are compatible with **M**, and
- R_{Ω} is the set of all relations that are compatible with **M**.

With the help of the following definitions and lemmas, we will be able to describe how the various flavours of duality occur within the lattice $\mathcal{A}_{\mathbf{M}}$; see Facts 3.9.

Definition 3.4. Let r be an n-ary relation compatible with \mathbf{M} , for some $n \ge 0$, and let **r** be the subalgebra of \mathbf{M}^n with r as underlying set.

- We say that the relation r is *hom-minimal* if every homomorphism from **r** to **M** is a projection (see [10]).
- We say that \mathbb{M} is operationally rich at r if every compatible partial operation on \mathbb{M} with domain r has an extension in $\operatorname{Clo}_{ep}(\mathbb{M})$.

Duality Lemma 3.5 ([11, 4.1]). Let \mathbb{M} be an alter ego of a finite algebra \mathbf{M} . Then \mathbb{M} dualises \mathbf{M} at the finite level if and only if every hom-minimal relation on \mathbf{M} belongs to $\operatorname{Rel}_{ca}(\mathbb{M})$.

Remark 3.6. If a finite algebra **M** has an alter ego that yields a duality, then every finite-level duality based on **M** lifts to the infinite level; see [11, p. 19]. Note that the same is not true in general for full duality [9]; see Lemma 4.4 and Remark 4.5.

We shall use the description of finite-level full duality provided by the following lemma. In fact, our main theorem will allow us to give a more refined version of this lemma (see Theorem 8.5).

Full Duality Lemma 3.7 ([11, 4.3]). Let \mathbb{M} be an alter ego of a finite algebra \mathbb{M} . Then \mathbb{M} fully dualises \mathbb{M} at the finite level if and only if

- (a) every hom-minimal relation on **M** belongs to $\operatorname{Rel}_{ca}(\mathbb{M})$, and
- (b) \mathbb{M} is operationally rich at each relation in $\operatorname{Rel}_{\operatorname{ca}}(\mathbb{M})$.

Remark 3.8. Since $M^n \in \operatorname{Rel}_{ca}(\mathbb{M})$, for all $n \ge 0$, it follows from (b) above that every compatible total operation (that is, every homomorphism $g: \mathbb{M}^n \to \mathbb{M}$) belongs to $\operatorname{Clo}_{ep}(\mathbb{M})$. In particular, every element of M that forms a one-element subalgebra of \mathbb{M} must be the value of a nullary operation in $\operatorname{Clo}_{ep}(\mathbb{M})$.

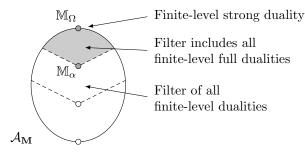


FIGURE 1. The lattice of alter egos of a finite algebra M

Facts 3.9. The following facts about the lattice of alter egos $\mathcal{A}_{\mathbf{M}}$ are proved in [11]; see Figure 1.

- (1) By the Duality Lemma 3.5, the alter egos that dualise \mathbf{M} at the finite level form a principal filter of $\mathcal{A}_{\mathbf{M}}$.
- (2) It follows from the Full Duality Lemma 3.7 that, under the 'structural reduct' quasi-order, there is a smallest alter ego \mathbb{M}_{α} that fully dualises **M** at the finite level; see [11, 4.4].
- (3) The alter egos that fully dualise \mathbf{M} at the finite level form a complete sublattice $\mathcal{F}_{\mathbf{M}}$ of $\mathcal{A}_{\mathbf{M}}$ and those that fully dualise \mathbf{M} form an up-set of $\mathcal{F}_{\mathbf{M}}$; see [11, 5.5].
- (4) An alter ego strongly dualises **M** at the finite level if and only if it is structurally equivalent to the top alter ego \mathbb{M}_{Ω} , and so there is essentially only one candidate for a strong duality; see [11, 4.6].

4. MOTIVATING EXAMPLE

In this section, we illustrate the general idea behind the proof of our New-fromold Theorem 8.1 using the three-element bounded lattice

$$\mathbf{3} = \langle \{0, a, 1\}; \lor, \land, 0, 1 \rangle,$$

which has played a seminal role as an example in the theory of natural dualities.

We use the four compatible partial operations on **3** shown in Figure 2: the two unary operations f and g, and the two binary partial operations σ and h. The alter ego $\mathfrak{B} := \langle \{0, a, 1\}; f, g, \mathfrak{T} \rangle$ dualises **3**, and can be obtained from Priestley duality using general 'duality transfer' techniques (Davey [6]). The alter ego

$$\mathbb{B}_{\sigma} := \langle \{0, a, 1\}; f, g, \sigma, \mathfrak{T} \rangle$$

strongly dualises **3**, and can be obtained from Priestley duality using general 'strong duality transfer' techniques (Davey and Haviar [7]). Note that, since \Im_{σ} strongly dualises **3** at the finite level, it must be equivalent to the top alter ego of **3**; see Facts 3.9(4).

The first example of a finite-level full but not strong duality (given by Davey, Haviar and Willard [9]) was based on the alter ego

$$\mathfrak{B}_h := \langle \{0, a, 1\}; f, g, h, \mathfrak{T} \rangle.$$

This alter ego was not found using general techniques. Later in this paper, we shall give a general 'full duality transfer' technique that will allow us to obtain this alter ego in a natural way from \Im_{σ} ; see Example 8.10.

FIGURE 2. The compatible partial operations f, g, σ and h on **3**

In this section, we give a new proof that \mathfrak{Z}_h fully dualises **3** at the finite level. We will show how to transfer the finite-level full duality down from \mathfrak{Z}_σ to \mathfrak{Z}_h by using a basis for the universal Horn theory of \mathfrak{Z}_σ .

We want to apply the New-from-old Lemma 2.1. So we need a way to enrich each finite structure \mathbb{X} in $\mathfrak{X}_h := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathfrak{Z}_h)$ into a structure \mathbb{X}^{\sharp} in $\mathfrak{X}_{\sigma} := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathfrak{Z}_{\sigma})$. We will check membership of \mathfrak{X}_{σ} syntactically: we know that at the finite level \mathfrak{X}_{σ} is axiomatised by the universal Horn theory of \mathfrak{Z}_{σ} .

Definition 4.1. Let $\mathbb{X} = \langle X; f^{\mathbb{X}}, g^{\mathbb{X}}, h^{\mathbb{X}}, \mathfrak{I}^{\mathbb{X}} \rangle$ be a finite structure in \mathfrak{X}_h . We want to define a structure \mathbb{X}^{\sharp} of the same signature as \mathfrak{Z}_{σ} . The binary partial operation σ is described in \mathfrak{Z}_{σ} by the sentence

$$\forall uvw \left[\sigma(u,v) = w \leftrightarrow \left(f(w) = u \& g(w) = v \right) \right],$$

which is logically equivalent to a conjunction of uH-sentences. So we would like to define the partial operation $\sigma^{X^{\sharp}}$ on X by

$$\operatorname{graph}(\sigma^{\mathbb{X}^\sharp}) := \left\{ \, (x,y,z) \in X^3 \ \big| \ f^{\mathbb{X}}(z) = x \ \& \ g^{\mathbb{X}}(z) = y \, \right\}.$$

As the endomorphisms f and g separate the elements of **3**, the uH-sentence

$$\forall uv \left[\left(f(u) = f(v) \& g(u) = g(v) \right) \rightarrow u = v \right]$$

holds in \mathfrak{Z}_h and therefore in \mathbb{X} . This tells us that graph($\sigma^{\mathbb{X}^{\sharp}}$) really is the graph of a binary partial operation on X (possibly an empty operation). So we can define

$$\mathbb{X}^{\sharp} := \langle X; f^{\mathbb{X}}, g^{\mathbb{X}}, \sigma^{\mathbb{X}^{\sharp}}, \mathfrak{T}^{\mathbb{X}} \rangle,$$

and \mathbb{X}^{\sharp} is a (discrete) Boolean structure of the same signature as \mathfrak{Z}_{σ} .

Remark 4.2. The operation $\sigma^{\mathbb{X}^{\sharp}}$ defined above has a natural interpretation in the case that \mathbb{X} is a concrete structure in \mathfrak{X}_h . Assume that $\mathbb{X} \leq (\mathfrak{Z}_h)^k$, for some k > 0. Then we can impose the ternary relation graph(σ) coordinate-wise on the set X. The operation $\sigma^{\mathbb{X}^{\sharp}}$ is defined so that graph($\sigma^{\mathbb{X}^{\sharp}}$) = graph(σ)^X. Thus $\sigma^{\mathbb{X}^{\sharp}}$ is the maximum coordinate-wise extension of σ to X.

Lemma 4.3. Let \mathbb{X} be a finite structure in $\mathfrak{X}_h := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathfrak{F}_h)$. Then the structure \mathbb{X}^{\sharp} defined in 4.1 belongs to $\mathfrak{X}_{\sigma} := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathfrak{F}_{\sigma})$.

Proof. As the structure \mathbb{X}^{\sharp} is finite, we just have to check it is a model of the universal Horn theory of \mathfrak{Z}_{σ} . The basis for $\operatorname{Th}_{uH}(\mathfrak{Z}_{\sigma})$ given by Clark, Davey, Haviar, Pitkethly and Talukder [3, 3.6] can be reduced to the following set of sentences:

- (1) $\forall v | f(v) = f(f(v)) = g(f(v)) \& g(v) = f(g(v)) = g(g(v)) |;$
- (2) $\forall uvw [(f(w) = u \& g(w) = v) \leftrightarrow \sigma(u, v) = w];$
- (3) $\forall uv \left[\left(\sigma(u, v) = \sigma(u, v) \& \sigma(v, u) = \sigma(v, u) \right) \rightarrow u = v \right];$

(4) $\forall uvw [(\sigma(u,v) = \sigma(u,v) \& \sigma(v,w) = \sigma(v,w)) \rightarrow \sigma(u,w) = \sigma(u,w)].$

Since sentence (1) is in the language of f and g, it is also part of the uH-theory of \mathfrak{F}_h . So \mathbb{X}^{\sharp} satisfies (1), as $\mathbb{X} \in \mathfrak{X}_h$. Sentence (2) holds in \mathbb{X}^{\sharp} by construction. Sentence (3) can be transformed into a uH-sentence in the language of f and g, using sentence (2):

$$\forall uvxy \left[\left(f(x) = u \ \& \ g(x) = v \ \& \ f(y) = v \ \& \ g(y) = u \right) \to u = v \right].$$

So \mathbb{X}^{\sharp} satisfies (3), again as $\mathbb{X} \in \mathfrak{X}_h$.

When sentence (4) is translated into the language of f and g, it becomes

$$\begin{split} \varphi &:= \forall uvwxy \left[\left(f(x) = u \ \& \ g(x) = v \ \& \ f(y) = v \ \& \ g(y) = w \right) \rightarrow \\ & \exists z \big(f(z) = u \ \& \ g(z) = w \big) \right], \end{split}$$

which is not a uH-sentence. But we can overcome this problem using the partial operation h. It is easy to check that \mathfrak{I}_h satisfies the sentences

(5) $\forall xy [g(x) = f(y) \to h(x, y) = h(x, y)],$ (6) $\forall xy [h(x, y) = h(x, y) \to f(h(x, y)) = f(x)],$ (7) $\forall xy [h(x, y) = h(x, y) \to g(h(x, y)) = g(y)].$

It follows that \mathfrak{Z}_h satisfies

$$\begin{split} \psi := \forall uvwxy \left[\left(f(x) = u \ \& \ g(x) = v \ \& \ f(y) = v \ \& \ g(y) = w \right) \to \\ \left(f(h(x,y)) = u \ \& \ g(h(x,y)) = w \right) \right], \end{split}$$

which is logically equivalent to a conjunction of uH-sentences. Since $\psi \vdash \varphi$, it follows that $\mathbb{X} \models \varphi$ and therefore that \mathbb{X}^{\sharp} satisfies (4).

The original proof that \mathfrak{I}_h fully dualises **3** at the finite level piggybacked on Priestley duality. We obtain a more 'generalisable' proof by piggybacking on the strong duality given by \mathfrak{I}_{σ} .

Lemma 4.4 ([9]). The alter ego $\mathfrak{I}_h := \langle \{0, a, 1\}; f, g, h, \mathcal{T} \rangle$ fully dualises the bounded lattice **3** at the finite level.

Proof. We use the fact that **3** is dualised by $\Im := \langle \{0, a, 1\}; f, g, \mathcal{T} \rangle$ and strongly dualised by $\Im_{\sigma} := \langle \{0, a, 1\}; f, g, \sigma, \mathcal{T} \rangle$. We shall establish conditions (1) and (2) of the New-from-old Lemma 2.1, with $\mathbb{M}_1 = \Im_{\sigma}$ and $\mathbb{M}_2 = \Im_h$.

Since \Im dualises \Im , so does \Im_h . Now let \mathbb{X} be a finite structure in \mathfrak{X}_h , and construct the structure \mathbb{X}^{\sharp} as in Definition 4.1. We know that $\mathbb{X}^{\sharp} \in \mathfrak{X}_{\sigma}$, by the previous lemma. It remains to check that $\hom_{\mathfrak{X}_h}(\mathbb{X}, \Im_h) = \hom_{\mathfrak{X}_{\sigma}}(\mathbb{X}^{\sharp}, \Im_{\sigma})$.

Define $\mathfrak{X} := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathfrak{Z})$. Let \mathbb{X}_{\flat} denote the common reduct of \mathbb{X} and \mathbb{X}^{\sharp} to the language of \mathfrak{Z} ; thus $\mathbb{X}_{\flat} \in \mathfrak{X}$. Consider a morphism $\mu \colon \mathbb{X}_{\flat} \to \mathfrak{Z}$. The construction of \mathbb{X}^{\sharp} ensures that $\mu \colon \mathbb{X}^{\sharp} \to \mathfrak{Z}_{\sigma}$ is a morphism. Since $\operatorname{graph}(h) \in \operatorname{Rel}_{\operatorname{ca}}(\mathfrak{Z})$, via the sentence

$$\forall uvw \ | h(u,v) = w \leftrightarrow (f(u) = f(w) \& g(v) = g(w) \& g(u) = f(v)) |,$$

and since $\mathbb{X} \models \operatorname{Th}_{\mathrm{uH}}(\mathfrak{Z}_h)$, we also know that $\mu \colon \mathbb{X} \to \mathfrak{Z}_h$ is a morphism. Thus $\operatorname{hom}_{\mathfrak{X}_h}(\mathbb{X},\mathfrak{Z}_h) = \operatorname{hom}_{\mathfrak{X}}(\mathbb{X}_{\flat},\mathfrak{Z}) = \operatorname{hom}_{\mathfrak{X}_{\sigma}}(\mathbb{X}^{\sharp},\mathfrak{Z}_{\sigma})$, as required. \Box

Remark 4.5. We know that \mathfrak{I}_h does not fully dualise **3** [9]. So this proof must break down somewhere for infinite structures in \mathfrak{X}_h . For any structure $\mathbb{X} \in \mathfrak{X}_h$, we can construct \mathbb{X}^{\sharp} as in Definition 4.1, and we can show that \mathbb{X}^{\sharp} is a Boolean model of the uH-theory of \mathfrak{I}_{σ} . However, this does not imply that the structure \mathbb{X}^{\sharp} belongs to \mathfrak{X}_{σ} , because the alter ego \mathfrak{Z}_{σ} is not standard [3, 3.5]. The connection between full dualities and standardness is explored in Section 8.

5. Background uH-logic for the general case

The proof of our New-from-old Theorem 8.1 generalises the proof in the previous section: we transfer a [finite-level] full duality from one alter ego \mathbb{M}_1 to another alter ego \mathbb{M}_2 by using a basis for $\mathrm{Th}_{uH}(\mathbb{M}_1)$. In this section, we present the required background uH-logic, all of which is well known and elementary, except perhaps for the 'partial operations' twist.

Consider a uH-sentence $\forall \vec{v} \left[\left(\&_{i=1}^{\nu} \alpha_i(\vec{v}) \right) \rightarrow \gamma(\vec{v}) \right]$ in the language of (H, R). We call $\&_{i=1}^{\nu} \alpha_i(\vec{v})$ the premise or hypothesis of the sentence and $\gamma(\vec{v})$ the conclusion. We identify a particularly simple form of uH-sentence.

Definition 5.1.

(1) A formula α is hypothetically pure if it has one of the following forms: (a) $r(v_{i_1}, \ldots, v_{i_n})$, for some $r \in R$, or

(b) $h(v_{i_1}, \ldots, v_{i_n}) = v_{i_0}$, for some $h \in H$,

where $v_{i_0}, v_{i_1}, \ldots, v_{i_n}$ are variables (not necessarily distinct).

- (2) A formula γ is *conclusively pure* if it has one of the following forms:
 - (a) $r(v_{i_1},\ldots,v_{i_n})$, for some $r \in R$,
 - (b) $h(v_{i_1}, \ldots, v_{i_n}) = h(v_{i_1}, \ldots, v_{i_n})$, for some $h \in H$,
 - (c) u = v, or
 - (d) \perp ,

where $u, v, v_{i_1}, \ldots, v_{i_n}$ are variables (not necessarily distinct). (3) A uH-sentence $\forall \vec{v} \left[\left(\&_{i=1}^{\nu} \alpha_i(\vec{v}) \right) \to \gamma(\vec{v}) \right]$ is *pure* if

- - (a) each α_i in the premise is hypothetically pure, and
 - (b) the conclusion γ is conclusively pure.

Lemma 5.2. Every uH-sentence in a language allowing partial-operation symbols is logically equivalent to a conjunction of pure uH-sentences.

Proof. We show how to transform an impure uH-sentence σ into a finite number of new uH-sentences, each of which is nearer to being pure than σ (according to some appropriate well-founded measure). By recursively applying this process to each new sentence obtained, we ultimately obtain a finite set of pure uH-sentences whose conjunction is logically equivalent to σ .

For each transformation in the following list (other than the first transformation), we (a) state the relevant logical equivalence, and (b) display the new uH-sentence or sentences obtained from σ by using this equivalence and then applying standard prenex operations. Variables w, w' and $\vec{w} = (w_1, \ldots, w_n)$ appearing in the statements of the transformations are assumed not to occur in σ .

Given an impure uH-sentence $\sigma = \forall \vec{v} [(\&_i \alpha_i(\vec{v})) \rightarrow \gamma(\vec{v})], \text{ do the following.}$

- (0) If some α_k is of the form u = v, for variables u and v, then remove α_k from σ . If the variables u and v are distinct, then also remove $\forall v$ and replace v by uthroughout the resulting formula.
- (1) Else if some α_k is of the form $r(t_1, \ldots, t_n)$, with some t_ℓ not a variable:
 - (a) use $\alpha_k \equiv \exists \vec{w} [r(\vec{w}) \& (\&_{j=1}^n t_j = w_j)];$
 - (b) replace σ by $\forall \vec{v} \vec{w} \left[\left(\left(\&_{i \neq k} \alpha_i(\vec{v}) \right) \& \vec{r}(\vec{w}) \& \left(\&_{j=1}^n t_j = w_j \right) \right) \rightarrow \gamma(\vec{v}) \right].$

- (2) Else if some α_k is of the form s = t, with t not a variable:
 - (a) use $\alpha_k \equiv \exists w | s = w \& t = w |;$
 - (b) replace σ by $\forall \vec{v}w [((\&_{i \neq k} \alpha_i(\vec{v})) \& s = w \& t = w) \rightarrow \gamma(\vec{v})].$
- (3) Else if some α_k is of the form $h(t_1, \ldots, t_n) = v_m$, with some t_ℓ not a variable: (a) use $\alpha_k \equiv \exists \vec{w} [h(\vec{w}) = v_m \& (\bigotimes_{j=1}^n t_j = w_j)];$
 - (b) replace σ by the sentence

$$\forall \vec{v}\vec{w} \left[\left(\left(\&_{i \neq k} \alpha_i(\vec{v}) \right) \& h(\vec{w}) = v_m \& \left(\&_{j=1}^n t_j = w_j \right) \right) \to \gamma(\vec{v}) \right].$$

(4) Else if γ is of the form $r(t_1, \ldots, t_n)$, with some t_{ℓ} not a variable:

(a) use
$$\gamma \equiv \left(\&_{j=1}^{n} t_{j} = t_{j} \right) \& \forall \vec{w} \left[\left(\&_{j=1}^{n} t_{j} = w_{j} \right) \rightarrow r(\vec{w}) \right];$$

(b) replace σ by the n+1 sentences

$$\forall \vec{v} \left[\left(\&_i \alpha_i(\vec{v}) \right) \to t_j = t_j \right], \quad \text{for } j \in \{1, \dots, n\}, \text{ and} \\ \forall \vec{v} \vec{w} \left[\left(\left(\&_i \alpha_i(\vec{v}) \right) \& \left(\&_{j=1}^n t_j = w_j \right) \right) \to r(\vec{w}) \right].$$

(5) Else if γ is of the form s = t, where s and t are distinct terms, at least one of which is not a variable:

(a) use
$$\gamma \equiv (s = s \& t = t \& \forall ww' [(s = w \& t = w') \to w = w']);$$

(b) replace σ by the three sentences

$$\begin{aligned} &\forall \vec{v} \left[\left(\&_i \alpha_i(\vec{v}) \right) \to s = s \right], \\ &\forall \vec{v} \left[\left(\&_i \alpha_i(\vec{v}) \right) \to t = t \right], \\ &\forall \vec{v} ww' \left[\left(\left(\&_i \alpha_i(\vec{v}) \right) \& s = w \& t = w' \right) \to w = w' \right]. \end{aligned}$$

- (6) Else γ is of the form $h(t_1, \dots, t_n) = h(t_1, \dots, t_n)$, with some t_ℓ not a variable: (a) use $\gamma \equiv \left(\&_{j=1}^n t_j = t_j \right) \& \forall \vec{w} \left[\left(\&_{j=1}^n t_j = w_j \right) \to h(\vec{w}) = h(\vec{w}) \right];$
 - (b) replace σ by the n+1 sentences

$$\forall \vec{v} \left[\left(\&_i \alpha_i(\vec{v}) \right) \to t_j = t_j \right], \quad \text{for } j \in \{1, \dots, n\}, \text{ and} \\ \forall \vec{v} \vec{w} \left[\left(\left(\&_i \alpha_i(\vec{v}) \right) \& \left(\&_{j=1}^n t_j = w_j \right) \right) \to h(\vec{w}) = h(\vec{w}) \right].$$

Notation 5.3. Given a structure \mathbb{X} and an *n*-variable sentence σ of the form $\forall \vec{v} [\psi(\vec{v}) \rightarrow \gamma(\vec{v})]$ in the language of \mathbb{X} , we use $\operatorname{pr}_{\mathbb{X}}(\sigma)$ to denote the *n*-ary relation on X defined by the premise of σ , that is,

$$\operatorname{pr}_{\mathbb{X}}(\sigma) := \left\{ \left(x_1, \dots, x_n \right) \in X^n \mid \mathbb{X} \models \psi(x_1, \dots, x_n) \right\}.$$

Definition 5.4. Let r be a k-ary relation on M and let s be an ℓ -ary relation on M. We say that r is a *bijective projection* of s if there is a bijection $\rho: s \to r$ of the form $\rho(a_1, \ldots, a_\ell) = (a_{\theta(1)}, \ldots, a_{\theta(k)})$, for some map $\theta: \{1, \ldots, k\} \to \{1, \ldots, \ell\}$.

Remark 5.5. Let σ be a uH-sentence and let Φ be the logically equivalent set of pure uH-sentences obtained via the proof of the previous lemma. In Section 7, we will use the following two facts.

- (1) For all $\varphi \in \Phi$, the conclusion of φ is in the same language as the conclusion of the original uH-sentence σ . That is, any partial-operation or relation symbol occurring in the conclusion of φ also occurs in the conclusion of σ .
- (2) For all $\varphi \in \Phi$, the premise of the original uH-sentence σ is a 'bijective projection' of the premise of φ . That is, for each structure \mathbb{X} such that $\mathbb{X} \models \sigma$, the relation $\operatorname{pr}_{\mathbb{X}}(\sigma)$ is a bijective projection of the relation $\operatorname{pr}_{\mathbb{X}}(\varphi)$.

14

The New-from-old Theorem 8.1 will give conditions under which we can deduce that an alter ego \mathbb{M}_2 fully dualises **M** [at the finite level] if we know that another alter ego \mathbb{M}_1 fully dualises **M** [at the finite level]. In this section and the next, we set up and prove the theorem.

Assumptions 6.1. Fix a finite algebra M and define $\mathcal{A} := \mathsf{ISP}(M)$. Let

 $\mathbb{M}_1 = \langle M; H_1, R_1, \mathfrak{T} \rangle$ and $\mathbb{M}_2 = \langle M; H_2, R_2, \mathfrak{T} \rangle$

be two alter egos of **M**, and assume that

- (hm) every hom-minimal relation on \mathbf{M} belongs to $\operatorname{Rel}_{\operatorname{ca}}(\mathbb{M}_2)$, and
- (op) \mathbb{M}_2 is operationally rich at each relation in $\mathbb{R}_2 \cup \operatorname{dom}(\mathbb{H}_2)$.

To mimic the set-up for our motivating example from Section 4, take **M** to be the bounded lattice **3** and choose $\mathbb{M}_1 = \mathfrak{Z}_{\sigma}$ and $\mathbb{M}_2 = \mathfrak{Z}_h$.

Note that the two conditions (hm) and (op) are necessary for \mathbb{M}_2 to yield a finite-level full duality, by the Full Duality Lemma 3.7. Using the following easy lemma, the assumption (op) also ensures that \mathbb{M}_2 is operationally rich at each relation in graph(H_2).

Lemma 6.2. Let \mathbb{M} be an alter ego of a finite algebra \mathbf{M} . Let r and s be relations compatible with \mathbf{M} , and assume that there is a bijective projection $\rho: \mathbf{s} \to \mathbf{r}$. If \mathbb{M} is operationally rich at r, then \mathbb{M} is also operationally rich at s.

Proof. Say that r is m-ary and s is n-ary. The projection $\rho: \mathbf{s} \to \mathbf{r}$ is given by $\rho(a_1, \ldots, a_n) = (a_{\theta(1)}, \ldots, a_{\theta(m)})$, for some map $\theta: \{1, \ldots, m\} \to \{1, \ldots, n\}$.

Assume that \mathbb{M} is operationally rich at r. Let $h: \mathbf{s} \to \mathbf{M}$ be a partial operation compatible with \mathbf{M} . Then $h \circ \rho^{-1}: \mathbf{r} \to \mathbf{M}$ is also a partial operation compatible with \mathbf{M} . So there is an *m*-ary term $t(v_1, \ldots, v_m)$ in the language of \mathbb{M} such that $h \circ \rho^{-1}(a_1, \ldots, a_m) = t^{\mathbb{M}}(a_1, \ldots, a_m)$, for all $(a_1, \ldots, a_m) \in r$. Define the *n*-ary term $t_1(v_1, \ldots, v_n) = t(v_{\theta(1)}, \ldots, v_{\theta(m)})$. Then, for any $(a_1, \ldots, a_n) \in s$, we have

$$h(a_1, \dots, a_n) = h \circ \rho^{-1} \big(\rho(a_1, \dots, a_n) \big) = t^{\mathbb{M}}(a_{\theta(1)}, \dots, a_{\theta(m)}) = t_1^{\mathbb{M}}(a_1, \dots, a_n).$$

Thus $t_1^{\mathbb{M}}$ is an extension of h in $\operatorname{Clo}_{ep}(\mathbb{M})$.

Notation 6.3. We denote the top alter ego of **M** by $\mathbb{M}_{\Omega} = \langle M; H_{\Omega}, R_{\Omega}, \mathcal{T} \rangle$; see Section 3. Now, for each $k \in \{1, 2, \Omega\}$, let \mathbb{Z}_k denote the category of all Boolean structures of signature (H_k, R_k) , and define the two full subcategories

$$\mathfrak{X}_k := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_k) \text{ and } \mathfrak{Y}_k := \{ \mathbb{Y} \in \mathfrak{Z}_k \mid \mathbb{Y} \models \mathrm{Th}_{\mathrm{uH}}(\mathbb{M}_k) \}$$

within \mathfrak{Z}_k ; note that $\mathfrak{X}_k \subseteq \mathfrak{Y}_k$. For each $k \in \{1, 2\}$, let $F_k \colon \mathfrak{Z}_\Omega \to \mathfrak{Z}_k$ be the natural forgetful functor.

Our aim in this section is to set up a 'sharp' functor $S_2: \mathcal{Y}_2 \to \mathfrak{Z}_\Omega$ that enriches each Boolean model of $\operatorname{Th}_{uH}(\mathbb{M}_2)$ into a Boolean structure of signature (H_Ω, R_Ω) . This mimics our motivating example in Section 4, where we enriched each finite structure $\mathbb{X} \in \mathfrak{X}_h$ into a structure $\mathbb{X}^{\sharp} \in \mathfrak{Z}_{\sigma}$ by defining the graph of the partial operation $\sigma^{\mathbb{X}^{\sharp}}$ conjunct-atomically in the language of \mathfrak{Z}_h . In the general situation, not every compatible relation on \mathbf{M} is conjunct-atomic definable from \mathbb{M}_2 . But we now show that assumption 6.1(hm) ensures that every compatible relation on \mathbf{M} is primitive-positive definable from \mathbb{M}_2 .

Definition 6.4. For each *n*-ary compatible relation *r* on **M**, where $n \ge 0$, fix an enumeration f_1, \ldots, f_m of the hom-set $\hom_{\mathcal{A}}(\mathbf{r}, \mathbf{M})$ and define the (n+m)-ary compatible relation

$$\hat{r} := \{ (\vec{a}, f_1(\vec{a}), \dots, f_m(\vec{a})) \mid \vec{a} \in r \}$$

on \mathbf{M} .

In the definition above, the algebra $\mathbf{r} \leq \mathbf{M}^n$ is the isomorphic projection of the algebra $\hat{\mathbf{r}} \leq \mathbf{M}^{n+m}$ onto its first *n* coordinates. By construction, the relation \hat{r} is hom-minimal on \mathbf{M} . Therefore \hat{r} is conjunct-atomic definable from \mathbb{M}_2 , by assumption 6.1(hm). This justifies the next definition.

Definition 6.5. For each *n*-ary compatible relation r on \mathbf{M} , with \hat{r} the associated (n+m)-ary hom-minimal relation on \mathbf{M} ,

- (a) fix an (n + m)-variable conjunct-atomic formula $\hat{\beta}_r(\vec{v}, \vec{w})$ in the language of \mathbb{M}_2 that defines \hat{r} in \mathbb{M}_2 , and
- (b) define the primitive-positive formula $\beta_r(\vec{v}) := \exists \vec{w} \, \hat{\beta}_r(\vec{v}, \vec{w}).$

Lemma 6.6. Let r be an n-ary compatible relation on **M**, for some $n \ge 0$. Then the formula $\beta_r(\vec{v})$ defines the relation r in \mathbb{M}_2 .

Proof. For all $\vec{a} \in M^n$, we have the sequence of equivalences

$$\mathbb{M}_2 \models \beta_r(\vec{a}) \iff \exists \vec{c} \in M^m(\vec{a}, \vec{c}) \in \hat{r} \iff \vec{a} \in r,$$

as required.

Lemma 6.7. Let r be an n-ary compatible relation on \mathbf{M} , for some $n \ge 0$. Let $\mathbb{X} \in \mathcal{Y}_2$ and let $r_{\mathbb{X}}$ denote the n-ary relation defined in \mathbb{X} by the formula $\beta_r(\vec{v})$.

- (1) The relation $r_{\mathbb{X}}$ is topologically closed in \mathbb{X}^n .
- (2) If r is the graph of a partial operation on M, then $r_{\mathbb{X}}$ is the graph of a continuous partial operation on \mathbb{X} with a topologically closed domain.

Proof. (1): Let $\hat{r}_{\mathbb{X}}$ denote the (n + m)-ary relation defined in \mathbb{X} by the conjunctatomic formula $\hat{\beta}_r(\vec{v}, \vec{w})$. Then the relation $\hat{r}_{\mathbb{X}}$ is topologically closed in \mathbb{X}^{n+m} , since \mathbb{X} is a Boolean structure. But $r_{\mathbb{X}}$ is just the projection of $\hat{r}_{\mathbb{X}}$ onto its first ncoordinates. Since \mathbb{X}^{n+m} is compact and \mathbb{X}^n is Hausdorff, it follows that $r_{\mathbb{X}}$ is also topologically closed.

(2): Let r be the graph of an n-ary compatible partial operation on **M**, with corresponding (n + 1)-variable primitive-positive formula $\beta_r(\vec{v}, u)$. The sentence

$$\forall \vec{v} \, u u' \left[\left(\beta_r(\vec{v}, u) \& \beta_r(\vec{v}, u') \right) \to u = u' \right] \tag{\dagger}$$

is logically equivalent to a uH-sentence in the language of \mathbb{M}_2 . Since $\beta_r(\vec{v}, u)$ defines r in \mathbb{M}_2 (by Lemma 6.6), the sentence (\dagger) is true in \mathbb{M}_2 and therefore true in \mathcal{Y}_2 . Thus $r_{\mathbb{X}}$ is the graph of an *n*-ary partial operation h on X. It follows from part (1) that $r_{\mathbb{X}}$ is closed. Since the codomain of h is compact and Hausdorff and the graph of h is closed, it follows that h is continuous. The domain of h is closed as it is a projection of $r_{\mathbb{X}}$ from the compact space \mathbb{X}^{n+1} to the Hausdorff space \mathbb{X}^n . \Box

Definition 6.8. Define the sharp functor $S_2: \mathcal{Y}_2 \to \mathcal{Z}_{\Omega}$ as follows.

- (1) For each structure $\mathbb{X} \in \mathcal{Y}_2$, define $S_2(\mathbb{X})$ to be the Boolean structure of signature (H_Ω, R_Ω) such that:
 - $S_2(\mathbb{X})$ has the same underlying set and topology as \mathbb{X} ;

- for all $r \in R_{\Omega}$, the relation $r^{S_2(\mathbb{X})}$ is the relation $r_{\mathbb{X}}$ defined in \mathbb{X} by the formula $\beta_r(\vec{v})$;
- for all h ∈ H_Ω, the graph of the partial operation h^{S₂(X)} is the relation graph(h)_X defined in X by the formula β_{graph(h)}(v, u).
 (Note that S₂(X) ∈ Z_Ω, by the previous lemma.)
- (2) For each morphism $\mu: \mathbb{X} \to \mathbb{Y}$ in \mathcal{Y}_2 , the morphism $S_2(\mu): S_2(\mathbb{X}) \to S_2(\mathbb{Y})$ has the same underlying set-map as μ . (This works because morphisms are compatible with primitive-positive formulæ.)

Note 6.9. It follows at once from Lemma 6.6 that $S_2(\mathbb{M}_2) = \mathbb{M}_{\Omega}$.

Lemma 6.10. Let h be a compatible partial operation on M, and let $\mathbb{X} \in \mathcal{Y}_2$. Then $\operatorname{dom}(h^{S_2(\mathbb{X})}) = \operatorname{dom}(h)^{S_2(\mathbb{X})}$.

Proof. Consider the compatible relations r := dom(h) and s := graph(h) on \mathbf{M} . Let the fixed enumerations used in Definition 6.4 be f_1, \ldots, f_m for $\text{hom}_{\mathcal{A}}(\mathbf{r}, \mathbf{M})$ and g_1, \ldots, g_m for $\text{hom}_{\mathcal{A}}(\mathbf{s}, \mathbf{M})$. Note that the two hom-sets have the same size, since there is an isomorphism $\rho: \mathbf{s} \to \mathbf{r}$, given by $\rho(\vec{a}, h(\vec{a})) := \vec{a}$. Indeed, there is a permutation θ of $\{1, \ldots, m\}$ such that $g_i = f_{\theta(i)} \circ \rho$, for all $i \in \{1, \ldots, m\}$.

We now have

$$\hat{r} = \left\{ \left(\vec{a}, f_1(\vec{a}), \dots, f_m(\vec{a}) \right) \mid \vec{a} \in r \right\} \text{ and} \\ \hat{s} = \left\{ \left(\vec{a}, h(\vec{a}), f_{\theta(1)}(\vec{a}), \dots, f_{\theta(m)}(\vec{a}) \right) \mid \vec{a} \in r \right\}.$$

We can choose $j \in \{1, \ldots, m\}$ such that $h = f_j$. The sentence

$$\forall \vec{v}uw_1 \cdots w_m \left[\left(\widehat{\beta}_r(\vec{v}, w_1, \dots, w_m) \& u = w_j \right) \leftrightarrow \widehat{\beta}_s(\vec{v}, u, w_{\theta(1)}, \dots, w_{\theta(m)}) \right]$$

is equivalent to a conjunction of uH-sentences in the language of \mathbb{M}_2 . This sentence is true in \mathbb{M}_2 and therefore in \mathcal{Y}_2 , and logically implies the sentence

$$\forall \vec{v} | \beta_{\operatorname{dom}(h)}(\vec{v}) \leftrightarrow (\exists u) \beta_{\operatorname{graph}(h)}(\vec{v}, u) |.$$

Hence $\operatorname{dom}(h)^{S_2(\mathbb{X})} = \operatorname{dom}(h^{S_2(\mathbb{X})}).$

Lemma 6.11. Let $\mathbb{X} \in \mathcal{Y}_2$. Then

(1) $r^{S_2(\mathbb{X})} = r^{\mathbb{X}}$, for each $r \in R_2$, and (2) $h^{S_2(\mathbb{X})} = h^{\mathbb{X}}$, for each $h \in H_2$.

Proof. In cases (1) and (2), respectively, let *s* be the compatible relation *r* or graph(*h*) and let $\alpha(\vec{v})$ be the atomic formula $r(v_1, \ldots, v_n)$ or $h(v_1, \ldots, v_{n-1}) = v_n$. Then the sentence $\forall \vec{v} \left[\beta_s(\vec{v}) \leftrightarrow \alpha(\vec{v})\right]$ is true in \mathbb{M}_2 (by Lemma 6.6), and it suffices to prove that this sentence is true in \mathcal{Y}_2 . Since the implication $\forall \vec{v} \left[\beta_s(\vec{v}) \rightarrow \alpha(\vec{v})\right]$ is logically equivalent to a uH-sentence and is true in \mathbb{M}_2 , it is true in \mathcal{Y}_2 . So it remains to consider the converse implication.

By assumption 6.1(op), the alter ego \mathbb{M}_2 is operationally rich at each relation in $R_2 \cup \operatorname{dom}(H_2)$. By Lemma 6.2, it follows that \mathbb{M}_2 is operationally rich at each relation in graph (H_2) . So \mathbb{M}_2 is operationally rich at s. Let f_1, \ldots, f_m be the fixed enumeration of $\operatorname{hom}_{\mathcal{A}}(\mathbf{s}, \mathbf{M})$ used in Definition 6.4. Then f_1, \ldots, f_m have extensions g_1, \ldots, g_m in $\operatorname{Clo}_{ep}(\mathbb{M}_2)$ by operational richness. Thus

$$\widehat{s} = \left\{ \left(\overrightarrow{a}, f_1(\overrightarrow{a}), \dots, f_m(\overrightarrow{a}) \right) \mid \overrightarrow{a} \in s \right\} = \left\{ \left(\overrightarrow{a}, g_1(\overrightarrow{a}), \dots, g_m(\overrightarrow{a}) \right) \mid \overrightarrow{a} \in s \right\}.$$

Let t_1, \ldots, t_m be terms in the language of \mathbb{M}_2 that yield the partial operations g_1, \ldots, g_m . Then the sentence

$$\forall \vec{v} \left[\alpha(\vec{v}) \to \widehat{\beta}_s \left(\vec{v}, t_1(\vec{v}), \dots, t_m(\vec{v}) \right) \right] \tag{\dagger}$$

is equivalent to a conjunction of uH-sentences in the language of \mathbb{M}_2 . The sentence (†) holds in \mathbb{M}_2 and thus in \mathfrak{Y}_2 . But (†) logically implies $\forall \vec{v} \left[\alpha(\vec{v}) \rightarrow \beta_s(\vec{v}) \right]$, as required.

Lemma 6.12.

- (1) For each $\mathbb{X} \in \mathfrak{Y}_2$, we have $\mathbb{X} = F_2 S_2(\mathbb{X})$.
- (2) For each $\mathbb{X} \in \mathcal{Y}_{\Omega}$, we have $\mathbb{X} = S_2 F_2(\mathbb{X})$.

Proof. Part (1) follows directly from the previous lemma. To prove part (2), first note that the top alter ego \mathbb{M}_{Ω} satisfies the assumptions 6.1(hm) and 6.1(op) (i.e., with \mathbb{M}_{Ω} replacing \mathbb{M}_2). Consider the functor $S_{\Omega} : \mathcal{Y}_{\Omega} \to \mathcal{Z}_{\Omega}$ obtained by applying Definition 6.8 with \mathbb{M}_{Ω} as \mathbb{M}_2 , but using the same formulæ β_r as for \mathbb{M}_2 . Then $S_{\Omega} = S_2 F_2$. The previous lemma with \mathbb{M}_{Ω} as \mathbb{M}_2 yields $S_{\Omega} = \mathrm{id}_{\mathcal{Y}_{\Omega}}$.

7. PROOF OF THE NEW-FROM-OLD THEOREM: THE TRANSFER FUNCTOR

Throughout this section, the assumptions 6.1(hm) and 6.1(op) remain in force. In the previous section, we defined the sharp functor $S_2: \mathcal{Y}_2 \to \mathcal{Z}_{\Omega}$. In this section, we aim to show that the transfer functor

$$T_{21} := F_1 S_2 \colon \mathcal{Y}_2 \to \mathcal{Y}_1$$

is well defined. As in the motivating example from Section 4, we will use a basis Σ_1 for the uH-theory of \mathbb{M}_1 . By Lemma 5.2, we can assume that all the sentences in Σ_1 are pure. We will need to strengthen our assumptions on \mathbb{M}_2 , but to do this we require some definitions.

Definition 7.1. Let $\varphi = \forall \vec{v} \left[\left(\&_{i=1}^{\nu} \alpha_i(\vec{v}) \right) \to \gamma(\vec{v}) \right]$ be a pure uH-sentence in the language of \mathbb{M}_{Ω} . Define φ^{\natural} to be the sentence in the language of \mathbb{M}_2 constructed from φ as follows.

- (1) First, simultaneously make the following replacements:
 - (a) replace each $r(v_{i_1}, \ldots, v_{i_n})$ in φ with $\beta_r(v_{i_1}, \ldots, v_{i_n})$;
 - (b) replace each $h(v_{i_1}, \ldots, v_{i_n}) = v_{i_0}$ in φ with $\beta_{\operatorname{graph}(h)}(v_{i_1}, \ldots, v_{i_n}, v_{i_0})$; (c) if the conclusion $\gamma(\vec{v})$ is $h(v_{i_1}, \ldots, v_{i_n}) = h(v_{i_1}, \ldots, v_{i_n})$, replace it with $\beta_{\operatorname{dom}(h)}(v_{i_1}, \ldots, v_{i_n})$.
- (2) Then convert the new existential quantifiers in the premise into universal quantifiers out the front.

Let the new sentence so constructed be

$$\varphi^{\natural} = \forall \vec{v} \vec{w}_1 \dots \vec{w}_{\nu} \left[\left(\bigotimes_{i=1}^{\nu} \alpha_i^{\natural}(\vec{v}, \vec{w}_i) \right) \to \gamma^{\natural}(\vec{v}) \right].$$

Note that each α_i^{\natural} in the premise is of the form $\widehat{\beta}_r(v_{i_1}, \ldots, v_{i_n}, \vec{w}_i)$, for some $r \in R_{\Omega}$, and therefore is a conjunct-atomic formula in the language of \mathbb{M}_2 . The conclusion γ^{\natural} is either a primitive-positive formula in the language of \mathbb{M}_2 or else \perp .

Lemma 7.2. Let $\mathbb{X} \in \mathcal{Y}_2$ and let φ be a pure uH-sentence in the language of \mathbb{M}_{Ω} . Then $S_2(\mathbb{X}) \models \varphi$ if and only if $\mathbb{X} \models \varphi^{\natural}$. *Proof.* This follows from the definitions of $S_2(\mathbb{X})$ and φ^{\natural} . The only complication is replacement 7.1(1)(c). But Lemma 6.10 tells us that, for all $h \in H_{\Omega}$, we have

$$S_2(\mathbb{X}) \models h(\vec{a}) = h(\vec{a}) \iff \vec{a} \in \operatorname{dom}(h^{S_2(\mathbb{X})}) \iff \vec{a} \in \operatorname{dom}(h)^{S_2(\mathbb{X})}$$
$$\iff \mathbb{X} \models \beta_{\operatorname{dom}(h)}(\vec{a}).$$

So the result holds.

Recall that the notation $\operatorname{pr}_{\mathbb{X}}(\sigma)$ was introduced in 5.3.

Lemma 7.3. Let $\mathbb{X} \in \mathcal{Y}_2$ and let φ be a pure uH-sentence such that $\mathbb{M}_{\Omega} \models \varphi$. Then $S_2(\mathbb{X}) \models \varphi$ provided either

- (1) the conclusion of φ is in the language of \mathbb{M}_2 , or
- (2) \mathbb{M}_2 is operationally rich at the relation $\mathrm{pr}_{\mathbb{M}_2}(\varphi^{\natural})$.

Proof. Assume that (1) or (2) holds. We want to show that $S_2(\mathbb{X}) \models \varphi$. By Lemma 7.2, it suffices to show that $\mathbb{X} \models \varphi^{\natural}$. Using Note 6.9, we have $S_2(\mathbb{M}_2) = \mathbb{M}_{\Omega} \models \varphi$. So it follows by Lemma 7.2 that $\mathbb{M}_2 \models \varphi^{\natural}$.

Say that $\varphi = \forall \vec{v} \left[\left(\&_{i=1}^{\nu} \alpha_i(\vec{v}) \right) \to \gamma(\vec{v}) \right]$ and let $\gamma^{\natural}(\vec{v})$ be the conclusion of φ^{\natural} ; see Definition 7.1. Since the uH-sentence φ is pure, we know that its conclusion $\gamma(\vec{v})$ must take one of the following four forms:

- (a) $r(v_{i_1},\ldots,v_{i_n})$, for some $r \in R_{\Omega}$, in which case $\gamma^{\natural}(\vec{v})$ is $\beta_r(v_{i_1},\ldots,v_{i_n})$;
- (b) $h(v_{i_1}, \ldots, v_{i_n}) = h(v_{i_1}, \ldots, v_{i_n})$, for some $h \in H_{\Omega}$, in which case $\gamma^{\natural}(\vec{v})$ is $\beta_r(v_{i_1}, \ldots, v_{i_n})$, where $r := \operatorname{dom}(h)$;
- (c) $v_{i_1} = v_{i_2}$, in which case $\gamma^{\natural}(\vec{v})$ is also $v_{i_1} = v_{i_2}$;
- (d) \perp , in which case $\gamma^{\natural}(\vec{v})$ is also \perp .

If $\gamma(\vec{v})$ is of type (c) or (d), then φ^{\natural} is a uH-sentence true in \mathbb{M}_2 and thus in X. So we can now assume that $\gamma(\vec{v})$ is of type (a) or (b).

Case (1): the conclusion of φ is in the language of \mathbb{M}_2 . We can construct a uH-sentence ψ in the language of \mathbb{M}_2 from φ^{\natural} by changing the conclusion $\gamma^{\natural}(\vec{v})$ back to $\gamma(\vec{v})$. The conclusion $\gamma^{\natural}(\vec{v})$ is $\beta_r(v_{i_1}, \ldots, v_{i_n})$, for some $r \in R_2 \cup \operatorname{dom}(H_2)$. We know that β_r defines the interpretation of r in \mathbb{M}_2 (by Lemma 6.6) and also in X (by Lemmas 6.10 and 6.11). Thus $\varphi^{\natural} \leftrightarrow \psi$ is true in both \mathbb{M}_2 and X. Since $\mathbb{M}_2 \models \varphi^{\natural}$ and ψ is a uH-sentence, it follows that $X \models \varphi^{\natural}$.

Case (2): \mathbb{M}_2 is operationally rich at the relation $\operatorname{pr}_{\mathbb{M}_2}(\varphi^{\natural})$. To show that $\mathbb{X} \models \varphi^{\natural}$, it is enough to find a set Σ of uH-sentences in the language of \mathbb{M}_2 such that $\mathbb{M}_2 \models \Sigma$ and $\Sigma \vdash \varphi^{\natural}$.

The conclusion $\gamma^{\natural}(\vec{v})$ is $\beta_r(v_{i_1}, \ldots, v_{i_n})$, for some $r \in R_{\Omega} \cup \text{dom}(H_{\Omega})$. Define the compatible relation $p := \text{pr}_{\mathbb{M}_2}(\varphi^{\natural})$ on **M**. Since $\mathbb{M}_2 \models \varphi^{\natural}$ and since β_r defines r in \mathbb{M}_2 (by Lemma 6.6), we have

$$(\vec{a}, \vec{c}_1, \dots, \vec{c}_{\nu}) \in p \iff \mathbb{M}_2 \models \&_{i=1}^{\nu} \alpha_i^{\natural}(\vec{a}, \vec{c}_i) \implies \mathbb{M}_2 \models \gamma^{\natural}(\vec{a}) \implies \mathbb{M}_2 \models \beta_r(a_{i_1}, \dots, a_{i_n}) \implies (a_{i_1}, \dots, a_{i_n}) \in r.$$

Let f_1, \ldots, f_m be the fixed enumeration of $\hom_{\mathcal{A}}(\mathbf{r}, \mathbf{M})$ used in Definition 6.4. Then, for all $j \in \{1, \ldots, m\}$, we can define $g_j : \mathbf{p} \to \mathbf{M}$ by

$$g_j(\vec{a}, \vec{c}_1, \dots, \vec{c}_\nu) := f_j(a_{i_1}, \dots, a_{i_n}).$$

Each g_j is a compatible partial operation on **M** with domain p. We are assuming that \mathbb{M}_2 is operationally rich at $\operatorname{pr}_{\mathbb{M}_2}(\varphi^{\natural}) = p$. Thus there are terms t_1, \ldots, t_m in

the language of \mathbb{M}_2 that define extensions of g_1, \ldots, g_m in \mathbb{M}_2 . Define the sentence

$$\psi := \forall \vec{v} \vec{w}_1 \dots \vec{w}_{\nu} \left[\left(\bigotimes_{i=1}^{\nu} \alpha_i^{\natural}(\vec{v}, \vec{w}_i) \right) \rightarrow \widehat{\beta}_r \left(v_{i_1}, \dots, v_{i_n}, t_1(\vec{v}, \vec{w}_1, \dots, \vec{w}_{\nu}), \dots, t_m(\vec{v}, \vec{w}_1, \dots, \vec{w}_{\nu}) \right) \right].$$

Then ψ is equivalent to a conjunction of uH-sentences in the language of \mathbb{M}_2 , with $\mathbb{M}_2 \models \psi$ and $\psi \vdash \varphi^{\natural}$. Hence it follows that $\mathbb{X} \models \varphi^{\natural}$, as required. \Box

The next lemma will be used later to simplify the checking of condition 7.3(2).

Lemma 7.4. Let φ be a pure uH-sentence in the language of \mathbb{M}_{Ω} , and define the sentence φ^{\natural} as in 7.1. If \mathbb{M}_2 is operationally rich at the relation $\mathrm{pr}_{\mathbb{M}_{\Omega}}(\varphi)$, then \mathbb{M}_2 is also operationally rich at $\mathrm{pr}_{\mathbb{M}_2}(\varphi^{\natural})$.

Proof. By Lemma 6.2, it is enough to show that $\operatorname{pr}_{\mathbb{M}_{\Omega}}(\varphi)$ is a bijective projection of $\operatorname{pr}_{\mathbb{M}_{2}}(\varphi^{\natural})$. Referring to the notation of Definition 7.1, first note that each $\alpha_{j}^{\natural}(\vec{v}, \vec{w}_{j})$ in the premise of φ^{\natural} is of the form $\widehat{\beta}_{r_{j}}(\vec{v}_{j}, \vec{w}_{j})$, for some compatible relation r_{j} on \mathbf{M} , some tuple $\vec{v}_{j} = (v_{i_{j,1}}, \ldots, v_{i_{j,n_{j}}})$ of variables from \vec{v} , and some tuple of new variables \vec{w}_{j} of length $|\hom_{\mathcal{A}}(\mathbf{r}_{j}, \mathbf{M})|$. Let \vec{f}_{j} be the fixed enumeration of $\hom_{\mathcal{A}}(\mathbf{r}_{j}, \mathbf{M})$ used in Definition 6.4. Then

$$(\vec{a}, \vec{c}_1, \dots, \vec{c}_{\nu}) \in \operatorname{pr}_{\mathbb{M}_2}(\varphi^{\natural}) \iff \mathbb{M}_2 \models \bigotimes_{j=1}^{\mathcal{K}} \alpha_j^{\natural}(\vec{a}, \vec{c}_j)$$
$$\iff \mathbb{M}_2 \models \bigotimes_{j=1}^{\nu} \widehat{\beta}_{r_j}(\vec{a}_j, \vec{c}_j)$$
$$\iff (\forall j \in \{1, \dots, \nu\}) \ (\vec{a}_j \in r_j \& \vec{c}_j = \vec{f}_j(\vec{a}_j))$$
$$\iff \vec{a} \in \operatorname{pr}_{\mathbb{M}_{\Omega}}(\varphi) \& \ (\forall j \in \{1, \dots, \nu\}) \ \vec{c}_j = \vec{f}_j(\vec{a}_j).$$

It now follows that $\rho: \operatorname{pr}_{\mathbb{M}_2}(\varphi^{\natural}) \to \operatorname{pr}_{\mathbb{M}_\Omega}(\varphi)$, given by $(\vec{a}, \vec{c}_1, \ldots, \vec{c}_{\nu}) \mapsto \vec{a}$, is a bijective projection.

We now add to our initial assumptions, 6.1, in order to ensure that the transfer functor $T_{21} := F_1 S_2 : \mathcal{Y}_2 \to \mathcal{Y}_1$ is well defined.

Assumptions 7.5. Choose a basis Σ_1 for the universal Horn theory of \mathbb{M}_1 such that each uH-sentence in Σ_1 is pure. Assume that

(ax) for each $\varphi \in \Sigma_1$, if the conclusion of φ is not in the language of \mathbb{M}_2 , then \mathbb{M}_2 is operationally rich at the relation $\operatorname{pr}_{\mathbb{M}_2}(\varphi^{\natural})$.

Note 7.6. Each relation $\operatorname{pr}_{\mathbb{M}_2}(\varphi^{\natural})$ is conjunct-atomic definable from hom-minimal relations on **M**; see Definition 7.1. So assumption 7.5(ax) is necessary for \mathbb{M}_2 to yield a finite-level full duality, by the Full Duality Lemma 3.7.

Lemma 7.7. The transfer functor $T_{21} := F_1S_2 : \mathcal{Y}_2 \to \mathcal{Y}_1$ is well defined. That is, if $\mathbb{X} \in \mathcal{Y}_2$, then $F_1S_2(\mathbb{X}) \in \mathcal{Y}_1$.

Proof. Let $\mathbb{X} \in \mathcal{Y}_2$ and $\varphi \in \Sigma_1$. Then $S_2(\mathbb{X}) \models \varphi$, using 7.5(ax) and Lemma 7.3. So $F_1S_2(\mathbb{X}) \models \varphi$. It follows that $F_1S_2(\mathbb{X}) \models \Sigma_1$ and therefore $F_1S_2(\mathbb{X}) \in \mathcal{Y}_1$. \Box

Remark 7.8. Suppose that, in addition to our assumptions 6.1 and 7.5 on \mathbb{M}_2 , we assume that \mathbb{M}_1 fully dualises **M** at the finite level. Then \mathbb{M}_1 also satisfies conditions 6.1(hm) and 6.1(op) (i.e., with \mathbb{M}_1 replacing \mathbb{M}_2) by the Full Duality

Lemma 3.7. This means that we can use the method of Section 6 to define a sharp functor $S_1: \mathfrak{Y}_1 \to \mathfrak{Z}_{\Omega}$ based on \mathbb{M}_1 . As in Definition 6.5, for each $r \in R_{\Omega}$, we will need to choose some conjunct-atomic formula $\hat{\delta}_r(\vec{v}, \vec{w})$ in the language of \mathbb{M}_1 that defines \hat{r} in \mathbb{M}_1 , and this formula may well be different from the one $\hat{\beta}_r(\vec{v}, \vec{w})$ chosen for \mathbb{M}_2 . The alter ego \mathbb{M}_1 also satisfies condition 7.5(ax), for any pure basis Σ_2 for $\mathrm{Th}_{\mathrm{uH}}(\mathbb{M}_2)$, by Note 7.6. So we can follow the method of this section to establish that the transfer functor $T_{12} := F_2 S_1: \mathfrak{Y}_1 \to \mathfrak{Y}_2$ is well defined; see Lemma 7.7.

Lemma 7.9. Assume that \mathbb{M}_1 fully dualises \mathbf{M} at the finite level. Then the two transfer functors $T_{12} := F_2S_1 : \mathcal{Y}_1 \to \mathcal{Y}_2$ and $T_{21} := F_1S_2 : \mathcal{Y}_2 \to \mathcal{Y}_1$ are mutually inverse category isomorphisms.

Proof. By Lemma 7.7 and Remark 7.8, the two transfer functors are well defined. We just need to show that they are mutually inverse.

By the symmetry between the definitions of the functors T_{12} and T_{21} (see Remark 7.8), it is enough to show that $\mathbb{X} = T_{12}T_{21}(\mathbb{X})$, for some arbitrary $\mathbb{X} \in \mathcal{Y}_2$. Let $r \in R_2 \cup \text{graph}(H_2)$. We use \hat{r} to denote the associated hom-minimal relation; see Definition 6.4. We have a conjunct-atomic formula $\hat{\beta}_r(\vec{v}, \vec{w})$ in the language of \mathbb{M}_2 that defines \hat{r} in \mathbb{M}_2 , and a conjunct-atomic formula $\hat{\delta}_r(\vec{v}, \vec{w})$ in the language of \mathbb{M}_1 that defines \hat{r} in \mathbb{M}_1 ; see Definition 6.5 and Remark 7.8. Thus \mathbb{M}_Ω satisfies the sentence $\sigma := \forall \vec{v} \vec{w} [\hat{\beta}_r(\vec{v}, \vec{w}) \leftrightarrow \hat{\delta}_r(\vec{v}, \vec{w})]$.

By Lemma 6.11, the relation $r^{\mathbb{X}}$ is defined by the formula $\exists \vec{w} \, \hat{\beta}_r(\vec{v}, \vec{w})$ in $S_2(\mathbb{X})$. The relation $r^{T_{12}T_{21}(\mathbb{X})}$ is equal to the relation $r^{S_1F_1S_2(\mathbb{X})}$, which is described by the formula $\exists \vec{w} \, \hat{\delta}_r(\vec{v}, \vec{w})$ in $S_2(\mathbb{X})$. So we can show that $r^{\mathbb{X}} = r^{T_{12}T_{21}(\mathbb{X})}$ by checking that $S_2(\mathbb{X})$ satisfies the sentence σ .

First consider the backwards implication $\sigma_b := \forall \vec{v} \vec{w} [\hat{\delta}_r(\vec{v}, \vec{w}) \rightarrow \hat{\beta}_r(\vec{v}, \vec{w})]$. This is logically equivalent to a set Σ_b of uH-sentences, each of which holds in \mathbb{M}_{Ω} . Using Lemma 5.2, we can convert Σ_b into a logically equivalent set Φ_b of pure uH-sentences. The conclusion of each sentence in Φ_b is in the language of \mathbb{M}_2 ; see Remark 5.5(1). So $S_2(\mathbb{X}) \models \sigma_b$, by Lemma 7.3.

Now consider the forwards implication $\sigma_f := \forall \vec{v} \vec{w} [\hat{\beta}_r(\vec{v}, \vec{w}) \to \hat{\delta}_r(\vec{v}, \vec{w})]$. This is logically equivalent to a set Σ_f of uH-sentences, each of which holds in \mathbb{M}_{Ω} . Using Lemma 5.2 again, we can convert Σ_f into a logically equivalent set Φ_f of pure uH-sentences. Let $\varphi \in \Phi_f$. By Remark 5.5(2), since $\mathbb{M}_{\Omega} \models \sigma_f$, there is a bijective projection ρ : $\operatorname{pr}_{\mathbb{M}_{\Omega}}(\varphi) \to \operatorname{pr}_{\mathbb{M}_{\Omega}}(\sigma_f)$. The alter ego \mathbb{M}_2 is operationally rich at the relation $\hat{r} = \operatorname{pr}_{\mathbb{M}_{\Omega}}(\varphi)$, by Lemma 6.2. So $S_2(\mathbb{X}) \models \varphi$, by Lemmas 7.3 and 7.4. It follows that $S_2(\mathbb{X}) \models \sigma_f$, as required. \Box

We wrap up this section with the following result.

Lemma 7.10. Assume that \mathbb{M}_2 satisfies 6.1(hm), 6.1(op) and 7.5(ax). If \mathbb{M}_1 fully dualises **M** [at the finite level], then the following are equivalent:

- (1) \mathbb{M}_2 fully dualises **M** [at the finite level];
- (2) the transfer functor $T_{21} := F_1 S_2 : \mathcal{Y}_2 \to \mathcal{Y}_1$ sends each [finite] structure in \mathfrak{X}_2 into \mathfrak{X}_1 .

Proof. $(2) \Rightarrow (1)$: Assume that (2) holds. The alter ego \mathbb{M}_2 dualises \mathbf{M} at the finite level, by 6.1(hm) and the Duality Lemma 3.5. If \mathbb{M}_1 dualises \mathbf{M} , then so does \mathbb{M}_2 ;

see Remark 3.6. By the previous lemma, the transfer functor $T_{21}: \mathcal{Y}_2 \to \mathcal{Y}_1$ is a category isomorphism that preserves underlying sets and set-maps, and by Note 6.9 we have $T_{21}(\mathbb{M}_2) = \mathbb{M}_1$. It follows that \mathbb{M}_2 fully dualises **M** [at the finite level], using condition (2) and the New-from-old Lemma 2.1.

 $(1) \Rightarrow (2)$: Assume \mathbb{M}_2 fully dualises \mathbf{M} [at the finite level]. For $i \in \{1, 2, \Omega\}$, let $D_i: \mathcal{A} \to \mathfrak{X}_i$ and $E_i: \mathfrak{X}_i \to \mathcal{A}$ denote the hom-functors induced by \mathbf{M} and \mathbb{M}_i . Let \mathbb{X} be a [finite] structure in \mathfrak{X}_2 , and define the algebra $\mathbf{A} := E_2(\mathbb{X}) \in \mathcal{A}$. Then $\mathbb{X} \cong D_2 E_2(\mathbb{X}) = D_2(\mathbf{A})$. Since $D_i(\mathbf{A}) = F_i D_\Omega(\mathbf{A})$, for each $i \in \{1, 2\}$, using Lemma 6.12(2) yields

$$D_1(\mathbf{A}) = F_1 D_{\Omega}(\mathbf{A}) = F_1 S_2 F_2 D_{\Omega}(\mathbf{A}) = F_1 S_2 D_2(\mathbf{A}).$$

So we have $F_1S_2(\mathbb{X}) \cong F_1S_2D_2(\mathbf{A}) = D_1(\mathbf{A}) \in \mathfrak{X}_1$, as required.

8. The New-From-Old Theorem and its applications

We now have all the ingredients necessary to state and prove our main theorem. Recall that an alter ego \mathbb{M} is *standard* if the potential dual category $\mathfrak{X} = \mathsf{IS}_{c}\mathsf{P}^+(\mathbb{M})$ consists precisely of all Boolean models of $\mathrm{Th}_{uH}(\mathbb{M})$; see Definition 2.2.

New-from-old Theorem 8.1. Let \mathbf{M} be a finite algebra, and let \mathbb{M}_1 and \mathbb{M}_2 be alter egos of \mathbf{M} . Assume that \mathbb{M}_2 satisfies 6.1(hm), 6.1(op) and 7.5(ax).

- (1) If \mathbb{M}_1 fully dualises **M** at the finite level, then so does \mathbb{M}_2 .
- (2) If \mathbb{M}_1 is standard and fully dualises \mathbf{M} , then the same is true of \mathbb{M}_2 .

Proof. Part (1) follows directly from Lemma 7.10, because we automatically have $(\mathfrak{X}_1)_{\text{fin}} = (\mathfrak{Y}_1)_{\text{fin}}$; see Notation 6.3 and Definition 2.2.

To prove part (2), assume that \mathbb{M}_1 is standard and fully dualises \mathbf{M} . Since \mathbb{M}_1 is standard, we have $\mathbf{X}_1 = \mathbf{y}_1$. It follows by Lemma 7.10 that \mathbb{M}_2 fully dualises \mathbf{M} . To see that \mathbb{M}_2 is also standard, let $\mathbb{X} \in \mathbf{y}_2$. Then $T_{21}(\mathbb{X}) \in \mathbf{y}_1 = \mathbf{X}_1$. As we have shown that \mathbb{M}_2 fully dualises \mathbf{M} , we can use Lemma 7.10 (with the subscripts 1 and 2 swapped) to deduce that $T_{12}T_{21}(\mathbb{X}) \in \mathbf{X}_2$. Therefore Lemma 7.9 gives $\mathbb{X} = T_{12}T_{21}(\mathbb{X}) \in \mathbf{X}_2$. Thus \mathbb{M}_2 is standard.

Warning 8.2. In the signature of the alter ego $\mathbb{M}_1 = \langle M; H_1, R_1, T \rangle$, all operations are considered as partial operations. This means we are not privileging operations that happen to be total with the logical status of being total operations. To apply the New-from-old Theorem 8.1, the uH-basis chosen for \mathbb{M}_1 must imply all uH-sentences true in \mathbb{M}_1 , including those of the form $\forall v_1 \dots v_n [f(v_1, \dots, v_n) =$ $f(v_1, \dots, v_n)]$, where f is an n-ary total operation on M for some $n \ge 0$.

We now use this rather technical theorem to obtain a series of self-contained corollaries. First we use the theorem to give a new and very natural condition under which every finite-level full duality lifts to the infinite level.

Theorem 8.3. Let \mathbf{M} be a finite algebra. Assume that \mathbf{M} is fully dualised by a standard alter ego. If an alter ego \mathbb{M} fully dualises \mathbf{M} at the finite level, then \mathbb{M} is standard and fully dualises \mathbf{M} .

Proof. Let \mathbb{M}_1 be a standard alter ego that fully dualises \mathbf{M} . Assume that \mathbb{M} fully dualises \mathbf{M} at the finite level. Then we can take $\mathbb{M}_2 := \mathbb{M}$ and the assumptions of the New-from-old Theorem 8.1 are satisfied, by the Full Duality Lemma 3.7 and Note 7.6. Thus \mathbb{M} is standard and fully dualises \mathbf{M} .

Example 8.4. The previous theorem can be applied to quasi-primal algebras, that is, to finite algebras \mathbf{M} such that the ternary discriminator $t: M^3 \to M$ is a term function of \mathbf{M} , where

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y. \end{cases}$$

Davey and Werner [12, 2.7] have shown that every quasi-primal algebra has a standard, strongly dualising alter ego. So, for any quasi-primal algebra \mathbf{M} , the finite-level full dualities always lift to full dualities.

We can also use the New-from-old Theorem to refine the intrinsic description of finite-level full dualities given by the Full Duality Lemma 3.7.

Theorem 8.5. Let $\mathbb{M} = \langle M; H, R, T \rangle$ be an alter ego of a finite algebra **M**. Then the following are equivalent:

- (1) \mathbb{M} fully dualises \mathbf{M} at the finite level;
- (2) (a) every hom-minimal relation on **M** belongs to $\operatorname{Rel}_{\operatorname{ca}}(\mathbb{M})$, and
 - (b) \mathbb{M} is operationally rich at each relation in $\operatorname{Rel}_{\operatorname{ca}}(\mathbb{M})$;
- (3) (a) every hom-minimal relation on **M** belongs to $\operatorname{Rel}_{\operatorname{ca}}(\mathbb{M})$,
 - (b) \mathbb{M} is operationally rich at each relation in $R \cup \text{dom}(H)$, and
 - (c) M is operationally rich at each relation that is conjunct-atomic definable from hom-minimal relations.

Proof. Using the Full Duality Lemma 3.7, we only need to prove that $(3) \Rightarrow (1)$. So assume that (3) holds. We check that we can apply the New-from-old Theorem 8.1 with $\mathbb{M}_1 = \mathbb{M}_{\Omega}$ and $\mathbb{M}_2 = \mathbb{M}$. First note that the top alter ego \mathbb{M}_{Ω} must fully dualise **M** at the finite level, by the Full Duality Lemma 3.7. Conditions 6.1(hm) and 6.1(op) correspond to assumptions (3)(a) and (3)(b). Condition 7.5(ax) holds by assumption (3)(c), because each relation $\mathrm{pr}_{\mathbb{M}}(\varphi^{\natural})$ is conjunct-atomic definable from hom-minimal relations; see Note 7.6. □

From the previous result, we easily obtain a 'constructive' description of the smallest full-at-the-finite-level alter ego \mathbb{M}_{α} ; see Facts 3.9(2).

Theorem 8.6. Let M be a finite algebra. Define the sets

- R_{α} of all compatible relations on **M** that are conjunct-atomic definable from the hom-minimal relations on **M**, and
- H_{α} of all compatible partial operations on **M** with domain in R_{α} .

Then $\mathbb{M}_{\alpha} = \langle M; H_{\alpha}, R_{\alpha}, T \rangle$ is the smallest alter ego that fully dualises **M** at the finite level.

Proof. This follows from Theorem 8.5 (1)
$$\Leftrightarrow$$
 (3).

Using the transfer set-up from Section 7, we can give a new proof of the known characterisation of when a full duality is preserved under enriching the alter ego.

Theorem 8.7 ([11, 5.3]). Let $\mathbb{M}_1 = \langle M; H_1, R_1, T \rangle$ and $\mathbb{M}_2 = \langle M; H_2, R_2, T \rangle$ be alter egos of a finite algebra \mathbf{M} , with \mathbb{M}_1 a structural reduct of \mathbb{M}_2 . Assume that \mathbb{M}_1 fully dualises \mathbf{M} [at the finite level]. Then the following are equivalent:

- (1) \mathbb{M}_2 fully dualises **M** [at the finite level];
- (2) \mathbb{M}_2 is operationally rich at each relation in $(R_2 \setminus R_1) \cup \operatorname{dom}(H_2 \setminus H_1)$.

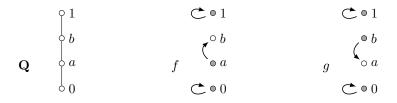


FIGURE 3. The partial automorphisms f and g of \mathbf{Q}

Proof. By the Full Duality Lemma 3.7, it suffices to prove $(2) \Rightarrow (1)$. Assume that (2) holds. Without loss of generality, we can assume \mathbb{M}_1 is a reduct of \mathbb{M}_2 .

Every hom-minimal relation on \mathbf{M} belongs to $\operatorname{Rel}_{\operatorname{ca}}(\mathbb{M}_1) \subseteq \operatorname{Rel}_{\operatorname{ca}}(\mathbb{M}_2)$, by the Duality Lemma 3.5. Thus 6.1(hm) holds. Now let $r \in R_2 \cup \operatorname{dom}(H_2)$. Using the Full Duality Lemma 3.7, if $r \in R_1 \cup \operatorname{dom}(H_1)$, then \mathbb{M}_1 is operationally rich at r, and so \mathbb{M}_2 is too. Otherwise, condition (2) ensures that \mathbb{M}_2 is operationally rich at r. Thus 6.1(op) holds. Since the language of \mathbb{M}_1 is contained in that of \mathbb{M}_2 , it follows immediately that 7.5(ax) holds.

We now apply Lemma 7.10 to show that \mathbb{M}_2 fully dualises \mathbf{M} [at the finite level]. Since \mathbb{M}_1 is a reduct of \mathbb{M}_2 , the transfer functor $T_{21} := F_1 S_2 : \mathcal{Y}_2 \to \mathcal{Y}_1$ is the forgetful functor, by Lemma 6.12(1). It follows that T_{21} sends each structure in \mathfrak{X}_2 into \mathfrak{X}_1 , as required.

We will now illustrate the general New-from-old Theorem using an important example from natural duality theory: the first known full-but-not-strong duality.

Example 8.8. Define the four-element lattice-based algebra

$$\mathbf{Q} := \langle \{0, a, b, 1\}; t, \lor, \land, 0, 1 \rangle$$

where 0 < a < b < 1 and the operation t is the ternary discriminator. Define two alter egos of **Q**:

$$\mathbb{Q}_0 := \langle \{0, a, b, 1\}; \operatorname{graph}(f), \mathfrak{T} \rangle \quad \text{and} \quad \mathbb{Q}_1 := \langle \{0, a, b, 1\}; f, g, \mathfrak{T} \rangle,$$

where the partial automorphisms f and g of \mathbf{Q} are shown in Figure 3.

By the Quasi-primal Strong Duality Theorem [2, 3.3.13], the alter ego \mathbb{Q}_1 strongly dualises \mathbf{Q} . Since \mathbb{Q}_0 and \mathbb{Q}_1 are clearly not structurally equivalent, the alter ego \mathbb{Q}_0 cannot strongly dualise \mathbf{Q} . Nevertheless, the alter ego \mathbb{Q}_0 fully dualises \mathbf{Q} : Clark, Davey and Willard [4] gave three different proofs to celebrate this discovery; we will now give yet another proof.

Since we know that \mathbb{Q}_1 dualises \mathbf{Q} , it follows easily that \mathbb{Q}_0 dualises \mathbf{Q} . Thus \mathbb{Q}_0 satisfies 6.1(hm), by the Duality Lemma 3.5. Since graph(f) is hom-minimal on \mathbf{Q} , it is trivial that \mathbb{Q}_0 satisfies 6.1(op).

As mentioned in Example 8.4, every quasi-primal algebra is strongly dualised by a standard alter ego. So \mathbb{Q}_1 is standard, by Theorem 8.3. It is easy to check that the following three uH-sentences form a basis for $\operatorname{Th}_{uH}(\mathbb{Q}_1)$:

- (1) $\forall uv [f(u) = v \rightarrow g(v) = u];$
- (2) $\forall uv \left[g(u) = v \rightarrow f(v) = u \right];$
- (3) $\forall uvw \left[\left(f(u) = v \& f(v) = w \right) \rightarrow u = v \right].$

Sentence (3) is pure, but sentences (1) and (2) are not. Sentence (1) converts into two pure uH-sentences:

(1a) $\forall uv [f(u) = v \rightarrow g(v) = g(v)];$

(1b) $\forall uvw \left[(f(u) = v \& g(v) = w) \to w = u \right].$

The purification of (2) is the pair of sentences (2a) and (2b) obtained from (1a) and (1b) by interchanging f and g. Of the five sentences (1a), (1b), (2a), (2b) and (3), only (1a) and (2a) have conclusions not in the language of \mathbb{Q}_0 . Since $\operatorname{pr}_{\mathbb{M}_{\Omega}}(1a) = \operatorname{graph}(f)$ and $\operatorname{pr}_{\mathbb{M}_{\Omega}}(2a) = \operatorname{graph}(g)$, both of which are hom-minimal, it follows from Lemma 7.4 that \mathbb{Q}_0 satisfies 7.5(ax) with respect to these five sentences. Thus \mathbb{Q}_0 is standard and fully dualises \mathbf{Q} , by the New-from-old Theorem 8.1.

To use the New-from-old Theorem directly, we need first to have come up with a candidate alter ego \mathbb{M}_2 that is going to fully dualise \mathbf{M} [at the finite level]. But we can easily adapt the New-from-old Theorem into an algorithm that can help you to find, for your favourite finite algebra \mathbf{M} , an alter ego of \mathbf{M} that is equivalent to the smallest full-at-the-finite-level alter ego \mathbb{M}_{α} .

Algorithm 8.9. Let M be a finite algebra. You need the following:

- (i) An alter ego $\mathbb{M}_0 = \langle M; H_0, R_0, \mathcal{T} \rangle$ of **M** such that
 - (a) \mathbb{M}_0 dualises **M** at the finite level,
 - (b) \mathbb{M}_0 is operationally rich at each relation in $R_0 \cup \text{dom}(H_0)$, and
 - (c) \mathbb{M}_0 is a reduct of \mathbb{M}_{α} .

(The easiest way to guarantee that (c) holds is to ensure that the signature of M_0 includes only total operations and hom-minimal relations.)

- (ii) An alter ego \mathbb{M}_1 that fully dualises **M** at the finite level.
- (iii) A finite basis Σ_1 for the uH-theory of \mathbb{M}_1 .

Start with $\mathbb{M}_2 := \mathbb{M}_0$. Then an alter ego equivalent to \mathbb{M}_α can be obtained by adding partial operations to the signature of \mathbb{M}_2 as follows.

For each uH-sentence $\psi \in \Sigma_1$ whose conclusion is not in the language of \mathbb{M}_0 , complete the following steps:

- (1) Convert ψ into a set of pure uH-sentences $\varphi_1, \ldots, \varphi_n$.
- (2) For each $i \in \{1, ..., n\}$ such that the conclusion of φ_i is not in the language of \mathbb{M}_0 , calculate the relation r_i on M as follows:
 - (a) if the premise of φ_i is in the language of \mathbb{M}_0 , then $r_i := \operatorname{pr}_{\mathbb{M}_0}(\varphi_i)$;

(b) if the premise of φ_i is not in the language of \mathbb{M}_0 , then $r_i := \operatorname{pr}_{\mathbb{M}_0}(\varphi_i^{\mathfrak{q}})$. (See 7.1 and 7.4.)

(3) For each relation r_i calculated in step (2), add all the compatible partial operations on **M** with domain r_i to the signature of \mathbb{M}_2 .

At the end of this process, you will have $\mathbb{M}_2 \equiv \mathbb{M}_{\alpha}$.

We finish this section by demonstrating this algorithm on the bounded lattice $\mathbf{3}$, whereby we shall 'rediscover' the partial operation h used in Section 4.

Example 8.10. Consider the bounded lattice $\mathbf{3} = \langle \{0, a, 1\}; \lor, \land, 0, 1 \rangle$, and define the two alter egos

 $\mathfrak{B}_0 := \langle \{0, a, 1\}; f, g, \mathfrak{T} \rangle \text{ and } \mathfrak{B}_1 := \langle \{0, a, 1\}; f, g, \sigma, \mathfrak{T} \rangle,$

as in Section 4. Then 3_0 and 3_1 dualise and strongly dualise 3, respectively.

A uH-axiomatisation for \Im_1 is given in the proof of Lemma 4.3:

- (1) $\forall v [f(v) = f(f(v)) = g(f(v)) \& g(v) = f(g(v)) = g(g(v))];$
- (2) $\forall uvw [(f(w) = u \& g(w) = v) \leftrightarrow \sigma(u, v) = w];$

(3)
$$\forall uv \left[\left(\sigma(u, v) = \sigma(u, v) \& \sigma(v, u) = \sigma(v, u) \right) \rightarrow u = v \right];$$

(4)
$$\forall uvw \mid (\sigma(u,v) = \sigma(u,v) \& \sigma(v,w) = \sigma(v,w)) \to \sigma(u,w) = \sigma(u,w)$$

We only need to consider (4) and the forward direction of (2).

The forward direction of (2) converts into a pair of pure uH-sentences, of which we need only consider (2a):

- (2a) $\forall uvw [(f(w) = u \& g(w) = v) \rightarrow \sigma(u, v) = \sigma(u, v)];$
- (2b) $\forall uvwx \left[\left(f(w) = u \& g(w) = v \& \sigma(u, v) = x \right) \rightarrow x = w \right].$

The premise of (2a) defines the ternary relation graph(σ) = {000, 01a, 111}. This relation is hom-minimal on **3**, so every compatible partial operation with domain graph(σ) already has an extension in Clo_{ep}(\mathfrak{Z}_0).

Using (2), we can rewrite (4) as

$$\forall uvwxy \left[\left(f(x) = u \& g(x) = v \& f(y) = v \& g(y) = w \right) \rightarrow \sigma(u, w) = \sigma(u, w) \right].$$

$$(4)'$$

Note that the premise of (4)' is in the language of \mathfrak{Z}_0 , so step (2)(a) of Algorithm 8.9 applies. The premise of (4)' defines the 5-ary relation

 $r := \operatorname{pr}_{\mathbb{M}_0}((4)') = \{00000, 0010a, 011a1, 11111\}.$

This relation forms a four-element chain, and so there are six homomorphisms from \mathbf{r} to $\mathbf{3}$. Thus there is only one compatible partial operation on $\mathbf{3}$ with domain \mathbf{r} that is not the restriction of a projection. We could just add this 5-ary partial operation to the signature of \mathfrak{Z}_0 , and we would be done.

But instead, we note from the premise of (4)' that the 5-ary relation r is isomorphic (via a projection) to the binary relation defined by g(x) = f(y), which is dom(h) = {00, 0a, a1, 11}. The missing partial operation with domain **r** is a restriction of $h(\pi_4, \pi_5)$. Since dom(h) is in Rel_{ca}(\mathfrak{Z}_0) and since every compatible partial operation with domain dom(h) is generated from the projections by f, g, h, we can add h to \mathfrak{Z}_0 to obtain the familiar alter ego $\mathfrak{Z}_2 := \langle \{0, a, 1\}; f, g, h, \mathfrak{T} \rangle \equiv \mathfrak{Z}_{\alpha}$.

9. Distinguishing full dualities

In this final section, we clarify the precise sense in which there can be two 'different' full dualities based on the same algebra \mathbf{M} . We first recall the categorical description of structural equivalence; see Davey, Haviar and Willard [10, p. 404].

Lemma 9.1. Let \mathbb{M}_1 and \mathbb{M}_2 be alter egos of a finite algebra \mathbf{M} . For $i \in \{1, 2\}$, define $\mathbf{X}_i := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_i)$. Then the following are equivalent:

- (1) \mathbb{M}_1 and \mathbb{M}_2 are structurally equivalent;
- (2) there is a concrete category isomorphism $F: \mathfrak{X}_2 \to \mathfrak{X}_1$ such that
 - (a) $F(\mathbb{M}_2) = \mathbb{M}_1$, and
 - (b) both F and F^{-1} preserve structural embeddings.

Moreover, we can take F to be the natural 'forgetful' functor.

Using our transfer set-up from Section 7, we obtain the following similar result. As mentioned in the introduction, if two alter egos \mathbb{M}_1 and \mathbb{M}_2 both fully dualise \mathbf{M} , then the categories $\mathsf{IS}_c\mathsf{P}^+(\mathbb{M}_1)$ and $\mathsf{IS}_c\mathsf{P}^+(\mathbb{M}_2)$ are equivalent, as they are both dually equivalent to $\mathcal{A} = \mathsf{ISP}(\mathbf{M})$. We now show that these two categories are in fact concretely isomorphic.

26

Lemma 9.2. Let \mathbb{M}_1 and \mathbb{M}_2 be alter egos of a finite algebra \mathbf{M} . For $i \in \{1, 2\}$, define $\mathbf{X}_i := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{M}_i)$. Assume that \mathbb{M}_1 fully dualises \mathbf{M} . Then the following are equivalent:

- (1) \mathbb{M}_2 fully dualises **M**;
- (2) there is a concrete category isomorphism F: X₂ → X₁ such that
 (a) F(M₂) = M₁, and
 - (b) F preserves structural embeddings of the form $\mathbb{X} \stackrel{\text{incl}}{\hookrightarrow} (\mathbb{M}_2)^S$, where X is closed under all compatible partial operations on **M**.

Moreover, if \mathbb{M}_1 is a structural reduct of \mathbb{M}_2 , then we can take F to be the natural 'forgetful' functor, and if \mathbb{M}_2 is a structural reduct of \mathbb{M}_1 , then we can take F^{-1} to be the natural 'forgetful' functor.

Proof. (2) \Rightarrow (1): Assume that we have $F: \mathfrak{X}_2 \to \mathfrak{X}_1$ as in (2). Let $\mathcal{A} := \mathsf{ISP}(\mathbf{M})$ and, for $i \in \{1, 2\}$, let $D_i: \mathcal{A} \to \mathfrak{X}_i$ and $E_i: \mathfrak{X}_i \to \mathcal{A}$ denote the hom-functors induced by \mathbf{M} and \mathbb{M}_i .

We first show that \mathbb{M}_2 dualises **M**. Let $\mathbf{A} \in \mathcal{A}$. The functor F preserves the structural embedding $D_2(\mathbf{A}) \stackrel{\text{incl}}{\hookrightarrow} (\mathbb{M}_2)^A$, by (2)(b), and so $FD_2(\mathbf{A})$ is an induced substructure of $F((\mathbb{M}_2)^A)$. We have $F(\mathbb{M}_2) = \mathbb{M}_1$, by (2)(a). Therefore

$$F((\mathbb{M}_2)^A) = (F(\mathbb{M}_2))^A = (\mathbb{M}_1)^A,$$

as the concrete category isomorphism $F: \mathfrak{X}_2 \to \mathfrak{X}_1$ preserves concrete products. The underlying set of $FD_2(\mathbf{A})$ is $\hom_{\mathcal{A}}(\mathbf{A}, \mathbf{M})$, as F also preserves underlying sets. Hence $FD_2(\mathbf{A}) = D_1(\mathbf{A})$, and therefore

$$\hom_{\mathfrak{X}_2}(D_2(\mathbf{A}), \mathbb{M}_2) = \hom_{\mathfrak{X}_1}(FD_2(\mathbf{A}), F(\mathbb{M}_2)) = \hom_{\mathfrak{X}_1}(D_1(\mathbf{A}), \mathbb{M}_1),$$

and so $E_2D_2(\mathbf{A}) = E_1D_1(\mathbf{A})$. Since \mathbb{M}_1 dualises \mathbf{M} , it follows that \mathbb{M}_2 does too.

For each structure $\mathbb{X} \in \mathfrak{X}_2$, we have $\hom_{\mathfrak{X}_2}(\mathbb{X}, \mathbb{M}_2) = \hom_{\mathfrak{X}_1}(F(\mathbb{X}), \mathbb{M}_1)$. Hence \mathbb{M}_2 fully dualises \mathbf{M} , by the New-from-old Lemma 2.1.

 $(1) \Rightarrow (2)$: Assume that \mathbb{M}_2 fully dualises **M**. Since \mathbb{M}_1 also fully dualises **M**, the transfer functors $T_{21}: \mathfrak{X}_2 \to \mathfrak{X}_1$ and $T_{12}: \mathfrak{X}_1 \to \mathfrak{X}_2$ are well defined, using Lemma 7.10 twice. Thus $T_{21}: \mathfrak{X}_2 \to \mathfrak{X}_1$ is a concrete category isomorphism, by Lemma 7.9. We have $T_{21}(\mathbb{M}_2) = \mathbb{M}_1$, by Note 6.9, and so T_{21} satisfies (2)(a).

Now let \mathbb{X} be an induced substructure of $(\mathbb{M}_2)^S$ with the property that X is closed under all compatible partial operations on **M**. Then $\mathbb{X} = F_2(\mathbb{X}^{\sharp})$, where $\mathbb{X}^{\sharp} \leq (\mathbb{M}_{\Omega})^S$. Using Lemma 6.12(2), we have

$$T_{21}(\mathbb{X}) = T_{21}F_2(\mathbb{X}^\sharp) = F_1S_2F_2(\mathbb{X}^\sharp) = F_1(\mathbb{X}^\sharp) \leqslant (\mathbb{M}_1)^S.$$

Note that $T_{21}((\mathbb{M}_2)^S) = (T_{21}(\mathbb{M}_2))^S = (\mathbb{M}_1)^S$. Hence T_{21} satisfies (2)(b).

If \mathbb{M}_1 is a structural reduct of \mathbb{M}_2 , then the transfer functor T_{21} is the natural 'forgetful' functor, by Lemma 6.12(1). Similarly, if \mathbb{M}_2 is a structural reduct of \mathbb{M}_1 , then the inverse transfer functor T_{12} is the natural 'forgetful' functor. \Box

Remark 9.3. We now demonstrate that the notion of 'structural embedding' we are using is not always categorically expressible in the concrete dual category arising from a full duality.

Our example is based on the quasi-primal algebra \mathbf{Q} from Example 8.8. We know that the two alter egos

$$\mathbb{Q}_0 = \langle \{0, a, b, 1\}; \operatorname{graph}(f), \mathfrak{T} \rangle \quad \text{and} \quad \mathbb{Q}_1 = \langle \{0, a, b, 1\}; f, g, \mathfrak{T} \rangle$$

fully dualise **Q**. For each $i \in \{0, 1\}$, define the dual category $\mathfrak{X}_i := \mathsf{IS}_{\mathsf{c}}\mathsf{P}^+(\mathbb{Q}_i)$. By Lemma 9.2, the natural 'forgetful' functor $F : \mathfrak{X}_1 \to \mathfrak{X}_0$ is a category isomorphism. The inverse category isomorphism $F^{-1} : \mathfrak{X}_0 \to \mathfrak{X}_1$ preserves underlying sets and set-maps, but does not preserve structural embeddings.

For example, consider the induced substructure \mathbb{X} of \mathbb{Q}_0 with $X := \{a\}$. The inclusion $i: \mathbb{X} \to \mathbb{Q}_0$ is a structural embedding. But the one-to-one morphism $F^{-1}(i): F^{-1}(\mathbb{X}) \to \mathbb{Q}_1$ is not a structural embedding: its image $\{a\}$ does not form an induced substructure of \mathbb{Q}_1 , as it is not closed under f. (The morphism $F^{-1}(i)$ is an embedding in \mathfrak{X}_1 in the concrete category-theoretic sense; see Adámek, Herrlich and Strecker [1, Definition 8.6].)

In the strong dual category \mathfrak{X}_1 , the structural embeddings correspond exactly to surjections in the quasivariety $\mathcal{A} = \mathsf{ISP}(\mathbf{Q})$. The dual category \mathfrak{X}_0 has more structural embeddings.

Comparing Lemmas 9.1 and 9.2, we see that it is the non-categorical nature of structural embeddings that allows a finite algebra to have truly different full dualities.

References

- J. Adámek, H. Herrlich and G. E. Strecker, Abstract and Concrete Categories: The Joy of Cats, Wiley, 1990. Republished in Reprints in Theory and Applications of Categories, no. 17 (2006), 1–507.
- [2] D. M. Clark and B. A. Davey, Natural Dualities for the Working Algebraist, Cambridge University Press, 1998.
- [3] D. M. Clark, B. A. Davey, M. Haviar, J. G. Pitkethly and M. R. Talukder, Standard topological quasi-varieties, *Houston J. Math.* 29 (2003), 859–887.
- [4] D. M. Clark, B. A. Davey and R. Willard, Not every full duality is strong!, Algebra Universalis 57 (2007), 375–381.
- [5] D. M. Clark and P. H. Krauss, Topological quasi varieties, Acta Sci. Math. (Szeged) 47 (1984), 3–39.
- [6] B. A. Davey, Dualisability in general and endodualisability in particular, Logic and Algebra (Pontignano, 1994) (A. Ursini and P. Agliano, eds), Lecture Notes in Pure and Applied Mathematics 180, Marcel Dekker, 1996, pp. 437–455.
- [7] B. A. Davey and M. Haviar, A schizophrenic operation which aids the efficient transfer of strong dualities, *Houston J. Math.* 26 (2000), 215–222.
- [8] B. A. Davey, M. Haviar and J. G. Pitkethly, Full dualisability is independent of the generating algebra, Algebra Universalis 67 (2012), 257–272.
- [9] B. A. Davey, M. Haviar and R. Willard, Full does not imply strong, does it?, Algebra Universalis 54 (2005), 1–22.
- [10] B. A. Davey, M. Haviar and R. Willard, Structural entailment, Algebra Universalis 54 (2005), 397–416.
- [11] B. A. Davey, J. G. Pitkethly and R. Willard, The lattice of alter egos, Internat. J. Algebra Comput. 22 (2012), 1250007, 36 pp.
- [12] B. A. Davey and H. Werner, Dualities and equivalences for varieties of algebras, *Contributions to Lattice Theory (Szeged, 1980)* (A. P. Huhn and E. T. Schmidt, eds), Colloquia Mathematica Societatis János Bolyai **33**, North-Holland, 1983, pp. 101–275.
- [13] B. A. Davey and R. Willard, The dualisability of a quasi-variety is independent of the generating algebra, Algebra Universalis 45 (2001), 103–106.
- [14] K. H. Hofmann, M. Mislove and A. Stralka, The Pontryagin Duality of Compact 0-Dimensional Semilattices and its Applications, Lecture Notes in Mathematics 396, Springer, 1974.
- [15] J. Isbell, General functorial semantics. I, Amer. J. Math. 94 (1972), 535-596.
- [16] P. T. Johnstone, Stone Spaces, Cambridge University Press, 1982.
- [17] J. Lambek and P. J. Scott, Introducton to Higher Order Categorical Logic, Cambridge University Press, 1986.

- [18] H.-E. Porst and W. Tholen, Concrete categories, *Category Theory at Work* (H. Herrlich and H.-E. Porst, eds), Heldermann, 1991, pp. 111–136.
- [19] H. A. Priestley, Representation of distributive lattices by means of ordered Stone spaces, Bull. London Math. Soc. 2 (1970), 186–190.
- [20] M. J. Saramago, Some remarks on dualisability and endodualisability, Algebra Universalis 43 (2000), 197–212.
- [21] M. H. Stone, The theory of representations for Boolean algebras, Trans. Amer. Math. Soc. 40 (1936), 37–111.
- [22] A. Stralka, A partially ordered space which is not a Priestley space, Semigroup Forum 20 (1980), 293–297.

Department of Mathematics and Statistics, La Trobe University, Victoria 3086, Australia

E-mail address: B.Davey@latrobe.edu.au

Department of Mathematics and Statistics, La Trobe University, Victoria 3086, Australia

 $E\text{-}mail\ address:$ J.Pitkethly@latrobe.edu.au

Department of Pure Mathematics, University of Waterloo, Ontario N2L 3G1, Canada E-mail address: rdwillar@uwaterloo.ca