

Infinity norm bounds for the inverse of Nekrasov matrices

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Abstract

From the application point of view, it is important to have a good upper bound for the maximum norm of the inverse of a given matrix A . In this paper we will give two simple and practical upper bounds for the maximum norm of the inverse of a Nekrasov matrix.

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1 Introduction

Throughout this paper, we will use the following notations:

$\mathbb{C}^n(\mathbb{R}^n)$ for the complex (real) n dimensional vector space,

$\mathbb{C}^{n,n}(\mathbb{R}^{n,n})$ for collection of all $n \times n$ matrices with complex (real) entries,

$N := \{1, 2, \dots, n\}$ for the set of indices,

$$r_i(A) = \sum_{j \in N \setminus \{i\}}^n |a_{ij}|, \quad i \in N \text{ for the deleted } i\text{-th row sum.}$$

Also, we define $h_i(A)$ recursively:

$$h_1(A) := \sum_{j \neq 1} |a_{1j}|,$$

$$h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^n |a_{ij}|.$$

We always denote by $A = D - L - U$ the standard splitting of a matrix A into its diagonal (D), strict lower ($-L$) and strict upper ($-U$) triangular parts.

All classes of matrices, which are considered in this paper, are subclasses of H -matrices, more precisely nonsingular H -matrices. This class is well-known from many applications, and it can be defined by:

Definition 1 A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called an H -matrix if its comparison matrix $\langle A \rangle = [m_{ij}]$ defined by

$$\langle A \rangle = [m_{ij}] \in \mathbb{C}^{n,n}; \quad m_{ij} = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \neq j \end{cases}.$$

is an M -matrix, i.e., $\langle A \rangle^{-1} \geq 0$.

A well-known property of H -matrices is given by the following theorem (see [1]).

Theorem 1 Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be a nonsingular H -matrix. Then

$$|A^{-1}| \leq \langle A \rangle^{-1}.$$

The most important subclass of H -matrices is the class of strictly diagonally dominant (SDD) matrices, defined by:

Definition 2 A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called SDD matrix if, for each $i \in N$, it holds that

$$|a_{ii}| > r_i(A).$$

Finally, let us recall the famous result of Varah [8], which gives us an upper bound for SDD matrices:

$$\|A^{-1}\|_{\infty} \leq \frac{1}{\min_{i \in N} (|a_{ii}| - r_i(A))}. \quad (1)$$

Obviously, this upper bound works only for SDD matrices, and even then it is not always good enough. As we see in above written formula, the smaller $\min_{i \in N} (|a_{ii}| - r_i(A))$ is, the worse estimation will be. For that reason it can be

useful to obtain new upper bounds, for a wider class of matrices which, as we will see, will sometimes work better in the *SDD* case, too. The class which will be chosen for that purpose is the class of, so called, Nekrasov matrices.

Definition 3 *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ is called Nekrasov matrix if, for each $i \in N$, it holds that*

$$|a_{ii}| > h_i(A).$$

It is known, see, for example [2], [7], that the class of Nekrasov matrices contains *SDD* class, while, on the other hand, it is a subclass of *H*-matrices.

2 New estimations

We will present two possibilities to estimate the infinity norm of the inverse matrix of a Nekrasov matrix. Although they look very similar, numerical examples will show that each of them can be better than the other one.

Theorem 2 *Suppose that $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a Nekrasov matrix. Then,*

$$\|A^{-1}\|_{\infty} \leq \frac{\max_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}} \quad (2)$$

and

$$\|A^{-1}\|_{\infty} \leq \frac{\max_{i \in N} z_i(A)}{\min_{i \in N} (|a_{ii}| - h_i(A))}, \quad (3)$$

where

$$z_1(A) := 1, \quad z_i(A) := \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A) + 1, \quad i \in N \setminus \{1\}. \quad (4)$$

In order to prove these estimations, we will start with the following lemma, proved by Robert in [6].

Lemma 1 *Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, with $a_{ii} \neq 0$ for all $i \in N$, then*

$$h_i(A) = |a_{ii}| \left[(|D| - |L|)^{-1} |U| e \right]_i, \quad (5)$$

where $e \in \mathbb{C}^n$ is the vector with all components equal to 1.

An immediate corollary of this Lemma is the following characterization of Nekrasov matrices, given by Szulc in [7]:

Theorem 3 *A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ is a Nekrasov matrix if and only if*

$$(|D| - |L|)^{-1}|U|e < e, \quad (6)$$

i.e. if and only if $E - (|D| - |L|)^{-1}|U|$ is an SDD matrix, where E is the identity matrix..

Proof of Theorem 2: First of all, notice that elements of matrix $(|D| - |L|)^{-1}|U|$ are nonnegative, because matrix $(|D| - |L|)$ is an M -matrix. From inequality (6) we see that sum of all elements in each row is less than 1, therefore we can conclude that all diagonal elements are also less than 1.

Assume that matrix A is a Nekrasov matrix. Then, from Theorem 3, we know that $E - (|D| - |L|)^{-1}|U|$ is an SDD matrix, so we can apply Varah bound for estimation of the infinity norm of its inverse matrix:

$$\|C^{-1}\|_{\infty} \leq \max_{i \in N} \frac{1}{|c_{ii}| - r_i(C)},$$

where $C := E - (|D| - |L|)^{-1}|U|$.

According to the fact that all diagonal entries of matrix $(|D| - |L|)^{-1}|U|$ are less than 1, we have:

$$\begin{aligned} |c_{ii}| &= 1 - \left[(|D| - |L|)^{-1}|U| \right]_{ii}, \\ r_i(C) &= \sum_{j \neq i}^n \left[(|D| - |L|)^{-1}|U| \right]_{ij}, \end{aligned}$$

and, therefore:

$$\begin{aligned} |c_{ii}| - r_i(C) &= 1 - \sum_{j=1}^n \left[(|D| - |L|)^{-1}|U| \right]_{ij} = \\ &= 1 - \sum_{j=1}^n \left[(|D| - |L|)^{-1}|U|e \right]_i = 1 - \frac{h_i(A)}{|a_{ii}|}. \end{aligned}$$

Hence,

$$\|C^{-1}\|_{\infty} \leq \max_{i \in N} \frac{1}{1 - \frac{h_i(A)}{|a_{ii}|}} = \frac{1}{1 - \max_{i \in N} \frac{h_i(A)}{|a_{ii}|}}.$$

Now, in order to estimate $\|A^{-1}\|_{\infty}$, it only remains to find a link between matrices C^{-1} and A^{-1} .

Since

$$C = (|D| - |L|)^{-1}\langle A \rangle,$$

it holds that

$$\langle A \rangle = (|D| - |L|)C,$$

and, see Theorem 1,

$$\|A^{-1}\|_{\infty} \leq \|\langle A \rangle^{-1}\|_{\infty} \leq \|C^{-1}\|_{\infty} \|(|D| - |L|)^{-1}\|_{\infty}.$$

Finally, because $|D| - |L|$ is an M -matrix,

$$\|(|D| - |L|)^{-1}\|_{\infty} = \|(|D| - |L|)^{-1}e\|_{\infty},$$

and, if we denote by $y := (|D| - |L|)^{-1}e$, then $e = (|D| - |L|)y$, or, by components:

$$|a_{11}|y_1 = 1, \quad |a_{ii}|y_i = 1 + \sum_{j=1}^{i-1} |a_{ij}|y_j, \quad i \in N \setminus \{1\}.$$

Since $|a_{ii}|y_i = z_i(A)$, see (4), we get

$$\|(|D| - |L|)^{-1}\|_{\infty} = \|y\|_{\infty} = \max_{i \in N} \frac{z_i(A)}{|a_{ii}|},$$

and the first upper bound (2) is proved.

In order to prove the second upper bound, we will multiply matrix $E - (|D| - |L|)^{-1}|U|$ with diagonal matrix $|D|$ from the left side, and denote:

$$B := |D| - |D|(|D| - |L|)^{-1}|U|. \quad (7)$$

Clearly, this multiplication does not change SDD property, so matrix B is also an SDD matrix, and we can, again, use Varah bound for estimation of the infinity norm of its inverse matrix:

$$\|B^{-1}\|_{\infty} \leq \max_{i \in N} \frac{1}{|b_{ii}| - r_i(B)}.$$

Knowing that all diagonal entries of matrix $(|D| - |L|)^{-1}|U|$ are less than 1, we have:

$$\begin{aligned} |b_{ii}| &= |a_{ii}| - |a_{ii}| \left[(|D| - |L|)^{-1}|U| \right]_{ii}, \\ r_i(B) &= \sum_{j \neq i}^n |a_{ij}| \left[(|D| - |L|)^{-1}|U| \right]_{ij}, \end{aligned}$$

and, therefore:

$$\begin{aligned} |b_{ii}| - r_i(B) &= |a_{ii}| - \sum_{j=1}^n |a_{ij}| \left[(|D| - |L|)^{-1}|U| \right]_{ij} = \\ &= |a_{ii}| - |a_{ii}| \left[(|D| - |L|)^{-1}|U|e \right]_i = |a_{ii}| - h_i(A). \end{aligned}$$

Hence,

$$\|B^{-1}\|_{\infty} \leq \max_{i \in N} \frac{1}{|a_{ii}| - h_i(A)} = \frac{1}{\min_{i \in N} (|a_{ii}| - h_i(A))}.$$

Now, in order to estimate $\|A^{-1}\|_{\infty}$, it only remains to find a link between matrices B^{-1} and A^{-1} . Since

$$B = |D|(|D| - |L|)^{-1}\langle A \rangle,$$

it holds that

$$\langle A \rangle = (E - |L||D|^{-1})B,$$

and, see Theorem 1,

$$\|A^{-1}\|_{\infty} \leq \|\langle A \rangle^{-1}\|_{\infty} \leq \|B^{-1}\|_{\infty} \|(E - |L||D|^{-1})^{-1}\|_{\infty}.$$

Finally, in order to estimate $\|(E - |L||D|^{-1})^{-1}\|_{\infty}$, we start with

$$\|(E - |L||D|^{-1})^{-1}\|_{\infty} = \|(E - |L||D|^{-1})^{-1}e\|_{\infty},$$

which is true because $E - |L||D|^{-1}$ is an M matrix.

If we denote $z(A) := (E - |L||D|^{-1})^{-1}e$, then $e = (E - |L||D|^{-1})z(A)$, or, by components:

$$z_1(A) = 1, \quad z_i(A) = 1 + \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A), \quad i \in N \setminus \{1\},$$

which is, precisely, the definition of $z_i(A)$, given by (4).

Therefore,

$$\|(E - |L||D|^{-1})^{-1}\|_{\infty} = \|z(A)\|_{\infty} = \max_{i \in N} z_i(A),$$

and the second upper bound (3) is also proved. \square

Both estimations, given by (2) and (3), can be used for bounding the infinity norm of the inverse of a Gudkov matrix, defined as a matrix for which there exists a permutation matrix P , such that PAP^T is a Nekrasov matrix. However, before applying either (2) or (3), one can *find* the permutation matrix P , in order to ensure the first row of PAP^T to be strictly diagonally dominant.

3 Numerical examples

Keeping in mind that SDD matrices are subset of Nekrasov matrices, we will compare our two bounds with Varah's one, when it is applicable. For this purpose we will consider the following six matrices:

$$A_1 = \begin{bmatrix} -7 & 1 & -0.2 & 2 \\ 7 & 88 & 2 & -3 \\ 2 & 0.5 & 13 & -2 \\ 0.5 & 3 & 1 & 6 \end{bmatrix}, A_2 = \begin{bmatrix} 8 & 1 & -0.2 & 3.3 \\ 7 & 13 & 2 & -3 \\ -1.3 & 6.7 & 13 & -2 \\ 0.5 & 3 & 1 & 6 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 21 & -9.1 & -4.2 & -2.1 \\ -0.7 & 9.1 & -4.2 & -2.1 \\ -0.7 & -0.7 & 4.9 & -2.1 \\ -0.7 & -0.7 & -0.7 & 2.8 \end{bmatrix}, A_4 = \begin{bmatrix} 5 & 1 & 0.2 & 2 \\ 1 & 21 & 1 & -3 \\ 2 & 0.5 & 6.4 & -2 \\ 0.5 & -1 & 1 & 9 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 6 & -3 & -2 \\ -1 & 11 & -8 \\ -7 & -3 & 10 \end{bmatrix}, A_6 = \begin{bmatrix} 8 & -0.5 & -0.5 & -0.5 \\ -9 & 16 & -5 & -5 \\ -6 & -4 & 15 & -3 \\ -4.9 & -0.9 & -0.9 & 6 \end{bmatrix}.$$

The upper bounds for their inverse matrices are given in the following table:

Matrix	class	exact $\ A^{-1}\ _\infty$	Varah	(2)	(3)
A_1	SDD	0.1921	0.6667	0.3805	0.5263
A_2	SDD	0.2390	1	0.8848	0.6885
A_3	SDD	0.8759	1.4286	1.8076	0.9676
A_4	SDD	0.2707	0.5556	0.6200	0.7937
A_5	Nekrasov	1.1519	-	1.4909	2.4848
A_6	Nekrasov	0.4474	-	1.1557	0.5702

Obviously, each estimation (2) or (3) can work better than the other one. So, in general case, for Nekrasov matrices, one can take the smallest estimation of these two.

For SDD case, it can be, also, useful to invest more time and calculations in obtaining these new estimations, since they can be significantly better (for example, see matrix A_2).

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