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ORIGINAL RESEARCH

Market Inefficiency, Entry Order and Coordination

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Abstract

The causes of market inefficiency are many. We suggest an additional cause — buyers' random entry order. In a market where identical sellers compete for buyers of heterogeneous valuations, first come first served is the norm. Since all buyers choose the cheapest available good, a low-valuation buyer who enters the market late may find the remaining goods unaffordable, which causes markets not to clear. We therefore propose a coordination solution to the market inefficiency problem. We find that in a market where all the high-valuation buyers enter first and all the low-valuation buyers enter afterwards, the market clears effectively. Moreover, we find the inefficiency arising from buyers' entry order becomes less of a problem in larger economies and vanishes in the limit.

Keywords Market efficiency · Coordination · Random entry

JEL Classification D47 · L11

1 Introduction

The standard causes of market inefficiency are related to externalities or informational problems that manifest in the pay-off relevant private information of some market participants (see [1–6]). Adverse selection [7–9] and signalling [10, 11] are prominent examples. In this article, we point out an additional phenomenon that gives rise to inefficient outcomes. It pertains to the order in which market participants enter the market. When there are, say, buyers with heterogeneous valuations,

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and capacity-constrained sellers who price the goods before the buyers enter, the sellers typically use mixed strategies in pricing. The low pricing sellers trade with every buyer. Other sellers take some risk and target high-valuation buyers. The risk arises as buyers always choose the cheapest goods available. If high-valuation buyers enter first, there are only high-priced goods left to the low-valuation buyers, and not all possible trades are consummated.

Consider, as an example, an online dating network where women and men search for partners. Women post the selection criteria of desired men such as hobbies, job, education, height and age. Men read the posts and contact the women if they think they meet the criteria. As a matter of fact, the men regard the women with low or few criteria as more attractive than the selective women because the chance of a match is higher. The criteria can be looked upon as implicit prices. Women and men are in the roles of sellers and buyers. Of course, the dating markets are much more complicated but for this example we assume that the women are pretty much alike except for the criteria they post. We also assume that once a woman receives a contact, she leaves the dating network immediately so that there is no further search. Therefore, in this setting, the matches that form depend mainly on the price level (the posted selection criteria) and men's timing of entering the market (the time they start searching online). Entering the dating network too late results in a lower chance of finding a match since the less picky women are not anymore in the market.

This observation about the importance of entry order can be used to study the value of coordination in markets where prices and competition amongst the sellers do not solve the problem. We employ a setting where there are equal numbers of sellers and buyers. The sellers are identical and each of them has a unit of an indivisible good for sale. The buyers are of two types. Half of them are low-valuation buyers whose reservation price is $v < 1$, and the other half are high-valuation buyers with reservation price unity. Each buyer has unit demand for the good.

Our market setting is without any inherent frictions, like physically separated sellers, or private information of the market participants. We model the market as a two-stage game. In the first stage, the sellers set the prices to which they are committed to. In the second stage, the buyers arrive in the market in a random order, and each buyer chooses the cheapest remaining good if the price is below his valuation. This setting, without the possibility to revise prices, depicts the essence of the mechanism. It would still be present in a dynamic setting where prices could be adjusted if not all goods find a buyer. The high-valuation buyers would, of course, anticipate this, and they would not trade immediately. The dynamic setting would resemble the model of bargaining with one-sided incomplete information where the uninformed party makes the offers. This is covered in Fudenberg, Levine and Tirole.[1] It is clear that the equilibrium in our market setting would exhibit pricing in mixed strategies, and decreasing prices in time but they would depend on the number of goods sold each period. As this is random, it seems a complicated problem to figure the equilibrium of the dynamic setting.

[1] See Fudenberg et al. [17]. Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information. Game-theoretic models of bargaining, Cambridge University Press, West Nyack, NY, pp.73-98.

We analyze three different scenarios. In the benchmark which captures the problem of the entry order, the buyers enter the market in a random order, and in a symmetric equilibrium the sellers use mixed strategies in pricing. The remaining cases constitute the two polar ways of coordinating the buyers' order of entry to the markets. In one case, all the low-valuation buyers enter the market first, and get the low-priced goods before the high-valuation buyers. The equilibrium pricing is in mixed strategies, but the competition for the high-valuation buyers is more intense than in the benchmark case. As a result, the allocation is also more inefficient; in fact, this is the worst case from the efficiency point of view. In the other case, all the high-valuation buyers enter the market first. It turns out that the sellers' symmetric pricing strategy is a pure one where every seller asks price v. All the possible trades are consummated, and there are no inefficiencies.

Studying the polar cases is useful and interesting for several reasons. First, one is able to pinpoint how the competition for a fixed resource, i.e. the high-valuation buyers, affects the outcome in a market setting with price competition. Secondly, the first policy reaction to the resulting inefficiency of the benchmark case, where some low-valuation buyers remain unserved, is likely to be that low-valuation buyers should enter first. But this does not take into account the sellers' response in pricing, and consequently it is instructive to show what happens in this case. Finally, the other polar case leads to efficiency and as we show in Sect. 5 it is implementable in a decentralised fashion without any policy intervention.

What is notable is that in all three cases the sellers' expected pay-off is v; whatever the buyers do the sellers can respond in a way that retains their expected pay-off constant. For this reason, the value of coordination can be evaluated by considering the buyers only. We stress that the value of coordination does not arise from there being more resources available nor there being equilibria that can be Pareto-ranked; in all the cases, there is exactly one symmetric equilibrium. The inefficiencies arise because of the order in which the buyers enter the market, and for efficiency comparison, the ordering is the only thing we vary.

One would expect that if the markets are small strategic behaviour causes inefficiencies, and as the market grows it becomes less important. This turns out true in our model. When the number of agents in the economy grows in such a way that the resources per capita remain constant, strategic behaviour vanishes; it becomes more and more probable that the sellers choose price v. In the limit, the equilibrium is efficient regardless of the order of entry by the buyers.

We follow [12] and measure the market (in)efficiency using the ratio between the expected value created and the value created if the market clears. In the benchmark case, we attain explicit expressions for the inefficiency, and we show that the inefficiency vanishes quite quickly as the economy grows. The case where the low-valuation buyers enter the market first is more complicated in terms of explicit expressions but we provide an approximate solution and conduct numerical analysis. We find that the inefficiencies are much more substantive, and vanish much more slowly than in the benchmark case as the economy grows. In the case where the high-valuation buyers enter the market first, the market clears and the outcome is efficient.

We are not aware of literature that focuses directly on market performance where the order in which the participants enter the market is studied. Of course, the value of coordination is recognised in a multitude of settings. For instance, almost by definition, any model of congestion demonstrates the value of coordination. In traffic settings, taxes and tolls are proposed in [13] and [14].

There are many reasons why achieving coordination is difficult without an outside party but the number of participants is clearly one of the most important. Knez and Camerer [15] suggest that coordination succeeds in a two-player game but difficulties arise in a group of three or more players. Albrecht [16] uses a market setting to study a coordination problem. He studies it in a (imperfectly) competitive matching market with sunk investments which features Pareto-ranked equilibria. An outcome is deemed a coordination failure if the corresponding equilibrium is not Pareto-optimal. He shows that with sufficient heterogeneity of the participants all the equilibria are efficient once the solution concept is refined to trembling-hand perfectness.

The rest of the article is organised as follows. In Sect. 2, we develop a benchmark model where buyers enter the market in a random order. We derive the equilibrium strategy for sellers and analyze market efficiency. In Sect. 3, we study the coordination model where buyers of one type enter the market before the other type. In Sect. 4, we provide numerical solutions for the equilibrium strategies and measures for market efficiency. We compare the results of the benchmark and the coordination games and discuss how market size, buyers' valuation and coordinative actions affect market efficiency. In Sect. 5, we discuss the logic of coordination. In Sect. 6, we conclude the article.

2 Benchmark Model: Random Entry Order

We call the benchmark where buyers enter the market in a random order the Random Entry case and use the superscript R to represent it. We call the case where low-valuation (high-valuation) buyers enter the market first the Low-valuation (High-valuation) case and use the superscript L (superscript H) to represent it.

Consider an economy where there are $2n$ sellers (she), each with a unit of an indivisible good, serving $2n$ buyers (he).[2] The good is perishable so the unsold items cannot be put into the resale market with discounted prices.[3] The sellers are identical, while the buyers are of two types. Half of the buyers value the good at unity and the other half at $v < 1$.[4] All of this are common knowledge. The game is static and in two stages. In stage one, the sellers make the pricing decisions; once the prices are posted, the sellers do not adjust them. In stage two, the buyers enter the market in a random order. The choice set for each buyer is binary: he buys if his valuation is higher than the lowest available price, and otherwise he does not buy. All the sellers are in the same location so that the buyers can see all the prices and then choose the lowest-priced good.

[2] If there are more sellers than buyers, a Bertrand-outcome where the sellers reduce the price to zero ensues. If there are more buyers than sellers, sellers will set high prices to target the high-valuation buyers and leave some (or all) low-valuation buyers unserved. The profit will attract more sellers to enter the market. Equal numbers of buyers and sellers would be the outcome if we had an entry stage with entry cost $c < v$. We do not model this.

[3] One can also think that the discount rate is so high that static analysis is sufficient.

[4] One can make the buyers more valuable by increasing v or decreasing the proportion of low-valuation buyers; of these two 'free' variables, we fix the proportion of low-valuation buyers.

We impose the following assumption.

Assumption 1 The valuation v of the low-type buyers is less than 1/2.

If v is higher, the equilibrium is a pure strategy one where each seller asks price v. Assumption 1 guarantees that the pricing is in mixed strategies, and that there is something to analyze.

We study symmetric equilibrium which is as simple as it gets since only the sellers have a strategic decision to make. A buyer's optimal behaviour is to buy the cheapest good in the market as long as its price does not exceed his valuation. To derive the sellers' pricing strategies, we first show that there is no pure strategy equilibrium. Note that most of the technical or long proofs are relegated to the Appendix 1.

Lemma 1 *A symmetric pure strategy equilibrium does not exist in the Random Entry case.*

Proof We prove by contradiction. Suppose there is a pure strategy equilibrium such that the sellers set the price at p'. Assume first that $p' = v$. A seller who deviates and asks price $p = 1$ makes a trade if the last buyer to the market has a high valuation. This happens with probability one-half, and consequently the deviator's expected pay-off is $\frac{1}{2} > v$.

Next, we show that no $p' \in (v, 1]$ can be an equilibrium. In this price range, a seller makes a trade with probability one-half, and has expected pay-off $\frac{1}{2}p'$. Asking a price $p' - \epsilon$ results in a trade for certain and for small ϵ, this is clearly a profitable deviation.

Finally, it is clear that a price less than v or greater than unity cannot constitute an equilibrium. □

Next, we derive those features of the symmetric mixed strategy equilibrium we need in our analysis. It is clear that the sellers have a mass point $\rho^R_{(n)}$ at price v, and with probability $1 - \rho^R_{(n)}$ they use a continuous mixed pricing strategy G on some interval $[a, 1]$ where $v < a < 1$.[5] This means that in equilibrium the sellers target consumers of both types, or the probability $\rho^R_{(n)}$ cannot be zero. Otherwise, $2n$ sellers would compete for n high-valuation buyers and leave all the low-valuation buyers unserved. The outcome would be similar to Bertrand competition where the sellers reduce the price to v.

As we are interested in the performance of the market, we need not solve the mixed strategy but it is enough to determine $\rho^R_{(n)}$. It allows us to solve the probabilities for all the possible numbers of unsold items. We find the following result.

Lemma 2 *The expected number of unsold goods is $n(1 - \rho^R_{(n)})$.*

[5] The subindex refers to the size of the economy. We study three different scenarios with corresponding superindices, R,H and L.

Next, we determine $\rho^R_{(n)}$ the probability that the sellers ask price v. The prices are either at v or continuous in the range of $[a, 1]$ where $v < a < 1$. In equilibrium, a seller's pay-off from asking price v must equal the pay-off from asking price 1.

Denote a seller's expected pay-off by $U(p)$ where p is the price she asks.[6] If a seller asks price v, she trades for certain. Therefore, her expected pay-off is

$$U(v) = v \cdot 1 = v. \tag{1}$$

If a seller asks price $p = 1$, we need to consider two cases. If fewer than half of the sellers set their price at v, that is $j < n$, the seller does not trade regardless of the buyers' entry order. Unity would be too expensive for the low-valuation buyers and high-valuation buyers choose lower-priced goods. If at least half of the sellers set price at v, that is $j \geq n$, the buyers' entry order determines the trading probability. Let the indicators $\mathbb{1}_{\{j<n\}}$ and $\mathbb{1}_{\{j\geq n\}}$ represent the two cases discussed above. The expected pay-off from price $p = 1$ is

$$U(1) = \sum_{j=0}^{2n-1} \binom{2n-1}{j} \left(\rho^R_{(n)}\right)^j \left(1 - \rho^R_{(n)}\right)^{2n-1-j} \left[\mathbb{1}_{\{j<n\}} \cdot 0 + \mathbb{1}_{\{j\geq n\}} \cdot \frac{\binom{n}{j-n}\binom{n}{n}}{\binom{2n}{j}} \cdot 1\right]. \tag{2}$$

The index j in the binomial sum keeps track of remaining $n-1$ sellers who choose price $p = v$. The seller under study, with price $p = 1$, makes a sale only if all the n low-valuation buyers are amongst the first $j \geq n$ buyers to arrive in the market. The number of ways this can happen is given by the numerator in the second term in the brackets; the denominator is the total number of ways one can choose j buyers from $2n$ buyers. The pay-offs in Eqs. (1) and (2) should be equal in equilibrium. Imposing equality, and simplifying, we determine the probability that sellers ask price v.

$$\begin{aligned}
v &= \sum_{j=0}^{2n-1} \binom{2n-1}{j} \left(\rho^R_{(n)}\right)^j \left(1 - \rho^R_{(n)}\right)^{2n-1-j} \left[\mathbb{1}_{\{j<n\}} \cdot 0 + \mathbb{1}_{\{j\geq n\}} \cdot \frac{\binom{n}{j-n}\binom{n}{n}}{\binom{2n}{j}} \cdot 1\right] \\
&= \sum_{j=n}^{2n-1} \binom{2n-1}{j} \left(\rho^R_{(n)}\right)^j \left(1 - \rho^R_{(n)}\right)^{2n-1-j} \frac{\binom{n}{j-n}}{\binom{2n}{j}} \\
&= \frac{1}{2} \sum_{j=n}^{2n-1} \binom{n-1}{j-n} \left(\rho^R_{(n)}\right)^j \left(1 - \rho^R_{(n)}\right)^{2n-1-j} \\
&= \frac{1}{2} \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\rho^R_{(n)}\right)^n \left(\rho^R_{(n)}\right)^j \left(1 - \rho^R_{(n)}\right)^{n-1-j} \\
&= \frac{1}{2} \left(\rho^R_{(n)}\right)^n \\
\Rightarrow \rho^R_{(n)} &= (2v)^{1/n}.
\end{aligned} \tag{3}$$

[6] Of course, the pay-off depends on the prices of all the other sellers but we suppress this dependence.

This gives us what we need from the equilibrium strategy in a market where the buyers enter in a random order.

Proposition 1 *In the Random Entry case, the symmetric mixed strategy equilibrium is such that the sellers ask price v with probability $\rho^R_{(n)}$ given by*

$$\rho^R_{(n)} = (2v)^{1/n}.$$

□

Following [12], we measure the market efficiency by

$$M^R_{(n)} = \frac{T^R}{T} = \frac{\text{Value created by trades}}{\text{Value created under market clearing}} = \frac{n \cdot 1 + n \cdot \rho^R_{(n)} \cdot v}{n \cdot 1 + n \cdot v} = \frac{1 + (2v)^{1/n} v}{1 + v} \tag{4}$$

where T^R is the ex ante expected value created in equilibrium, and T is the total value created from trades that would be generated if the market cleared. $M^R_{(n)} = 1$ indicates the most efficient case and $M^R_{(n)} = 0$ the least efficient.[7] We find that $M^R_{(n)}$ increases in n and approaches 1 as n grows without bound.

$$\frac{\partial M^R_{(n)}}{\partial n} = -\frac{2^{1/n} \cdot v^{1+1/n} \log(2v)}{n^2(1+v)} > 0 \text{ and } \lim_{n\to\infty} M^R_{(n)} = \frac{1 + 1 \cdot v}{1+v} = 1.$$

This shows that in a large economy the inefficiency vanishes. Furthermore, fixing n, $M^R_{(n)}$ is U-shaped in v. The lowest value is at $\widetilde{v}$ where the derivative equals zero.

$$\frac{\partial M^R_{(n)}}{\partial v} = \frac{-n + 2^{1/n} v^{1/n}(1 + n + v)}{n(1+v)^2} = 0.$$

We depict the behaviour of $M^R_{(n)}$ numerically in Sect. 4. The interpretation is as follows. When v is close to zero, the only thing that matters for efficiency is trading with the high-valuation buyers. At the same time, competition for them is great as witnessed by the value of $\rho^R_{(n)}$ which is close to zero. The value of potential trades is also dominated by the high-valuation trades, and the economy is close to efficiency. When v grows also the low-valuation buyers increase in importance and the value created in equilibrium (T_R) increases.

$$\frac{\partial T^R}{\partial v} = n(2v)^{1/n}\left(1 + \frac{1}{n}\right) > 0.$$

When v is just below $\frac{1}{2}$, the sellers take hardly any risk of not trading in equilibrium, and once again the economy is close to efficiency. As T^R is convex and T linear in v, the efficiency measure reaches its minimum somewhere between zero and

[7] As the high-valuation buyers always trade, the minimum value in our model is $\frac{1}{1+v}$. An alternative would be to calculate which percentage of the potential gains are achieved, or $\frac{(2v)^{1/n}v}{v}$, which would result in more impressive values of inefficiency. Also, assuming that there is a production cost $\gamma < v$ that has to be paid whether a seller trades or not would make the inefficiency more pronounced.

one-half. At the left endpoint of the interval, competition for the high-valuation buyers is the greatest, and at the right endpoint of the interval, there is no competition at all for the high-valuation buyers. The graphs for T and T^R are in the Appendix 2.

We have established that the buyers' random entry to the markets associated with the sellers' capacity constraints lead to pricing in mixed strategies. This gives rise to inefficient outcomes where not every profitable trade is consummated. As Expression (4) shows, in a small economy, market efficiency largely depends on the low-type buyers' valuation for the good, while in a large economy the dependency as well as inefficiency vanishes.

3 Coordination

3.1 Low-Valuation First

In this section, we analyze the model where the buyers coordinate on the entry order. To start with, we consider the case where all the low-valuation buyers enter the market first and all the high-valuation buyers enter after them. This could be the first reaction to the inefficiency of the previous section where the low-valuation buyers cannot trade as the high-valuation buyers who enter before them grab the low-priced goods. But this turns out to be a bad solution, or the worst possible, as it ignores the sellers' reaction to the entry order. When the low-valuation buyers enter first, the competition for the high-valuation buyers is the highest, and consequently the inefficiencies are also at their highest level.

Lemma 3 *A symmetric pure strategy equilibrium does not exist in the Low-valuation case.*

Proof The argument is similar to that in Lemma 1. We prove by contradiction. Suppose there is a pure strategy such that the sellers choose the price p'. First, $p' = v$ cannot be an equilibrium. A seller who deviates to $p = 1$ makes a trade for certain as the last buyer to enter the market is a high-valuation one. The deviator gets the expected pay-off $1 > v$.

Next, $p' \in (v, 1]$ cannot be an equilibrium. At price p', sellers only get to trade with the high-valuation buyers and get expected pay-off $p'/2$. A seller who deviates to a slightly lower price $p = p' - \epsilon$ makes a trade with a high-valuation buyer for certain. For any small ϵ, the deviator gets a higher pay-off $p' - \epsilon$.

Clearly, any price less than v or higher than 1 cannot be an equilibrium. This completes the proof. □

We know that the equilibrium must be in a mixed strategy such that there is a mass point at price $p = v$, and a continuous part on $[b, 1]$, $v < b$. Denote the probability that the sellers ask price v by $\rho^L_{(n)}$. The pay-off of asking price v must be equal to that of asking price unity. In the former case, a seller makes a trade for certain. To determine the probability of trade in the latter case, we denote by j the number of the sellers who set price at v. If fewer than half of the sellers choose v, that is $j < n$, only j low-valuation buyers trade and the rest, $n - j$, remain unserved. The high-valuation buyers who enter the market choose the cheapest goods available. Therefore, the seller with price $p = 1$ does not trade.

If at least half of the sellers choose v, that is $j \geq n$, all buyers are served and the market clears. Thus, the indifference condition is given by

$$
\begin{aligned}
&U(1) = U(v) \\
&\Leftrightarrow \\
&\sum_{j=n}^{2n-1} \binom{2n-1}{j} \left(\rho_{(n)}^L\right)^j \left(1 - \rho_{(n)}^L\right)^{2n-1-j} = v,
\end{aligned} \tag{5}
$$

where the probability $\rho_{(n)}^L$ is the solution to the above equation.

Lemma 4 *For any $n > 0$ and $v \in (0, 1/2)$, the solution to the probability of asking price v, $\rho_{(n)}^L$, is unique.*

We analyze $\rho_{(n)}^L$ numerically in Sect. 4. Here we sketch the proof of the existence of the limit of $\rho_{(n)}^L$ when n grows without bound; a rigorous proof is in Appendix 1. We then provide an approximate solution to $\rho_{(n)}^L$ for large values of n.

Lemma 5 *As n grows without bound $\rho_{(n)}^L$ approaches* 1/2.

Proof Think of $\{\rho_{(n)}^L\}_{n=1}^{\infty}$ as a sequence of real numbers satisfying Eq. (5) and denote the limit of the sequence by $\hat{\rho}$ if there exists any. Let $S_{2n-1} = Bin\left(2n-1, \rho_{(n)}^L\right)$ represent a sum of $2n-1$ Bernoulli-$\rho_{(n)}^L$ random variables X_j. We have

$$
\sum_{j=n}^{2n-1} \binom{2n-1}{j} \rho_n^j \left(1 - \rho_n\right)^{2n-1-j} = 1 - Pr\left(S_{2n-1} \leq n-1\right).
$$

Standardising by subtracting the mean and dividing by the standard deviation, the right-hand side of the above equation is equal to

$$
1 - Pr\left(\frac{S_{2n-1} - (2n-1)\rho_{(n)}^L}{\sqrt{(2n-1)\rho_{(n)}^L\left(1-\rho_n\right)}} \leq \frac{n-1-(2n-1)\rho_{(n)}^L}{\sqrt{(2n-1)\rho_{(n)}^L\left(1-\rho_n\right)}} \right).
$$

Denote the cumulative distribution function of $\frac{S_{2n-1}-(2n-1)\rho_{(n)}^L}{\sqrt{(2n-1)\rho_{(n)}^L(1-\rho_n)}}$ by F_{2n-1}. Then, the above expression is equivalent to

$$
1 - F_{2n-1}\left(a_n\right), \tag{6}
$$

where $a_n = \frac{n-1-(2n-1)\rho_{(n)}^L}{\sqrt{(2n-1)\rho_{(n)}^L(1-\rho_n)}}$. The central limit theorem implies that $F_{2n-1}\left(a_n\right) \approx \Phi\left(a_n\right)$ where Φ is the distribution function of the standardised normal distribution. If $\rho_{(n)}^L$

approaches anything but one-half, then a_n approaches either plus or minus infinity, Eq. (5) cannot hold, i.e. $\rho^L_{(n)} \approx 1/2$. □

Next, we define the probability $\rho^L_{(n)}$ of choosing the low price $p = v$ in terms of quantity $b_{(n)}$ that can be determined from the distribution function of the standardised normal distribution, and which is used in the numerical analysis.

Lemma 6 *For large values of n, the probability of asking price v is approximately given by*

$$\rho^L_{(n)} = \frac{\sqrt{2} + b_{(n)} - \sqrt{2 + \left(b_{(n)}\right)^2}}{2b_{(n)}}$$

where

$$b_{(n)} = \frac{y}{\sqrt{n}}$$

and y is determined by $\Phi(y) = 1 - v$.

Note that Lemma 6 can be used to verify the result in Lemma 5. Equation (15) shows that $b_{(n)}$ decreases in n and converges to zero. The approximation of $\rho^L_{(n)}$, as shown in Eq. (16), thus converges. The limit is[8]

[8] As $\rho^L_{(n)}$ approaches one-half, it may seem that Condition $X(\rho^L_{(n)}) = 0$ cannot hold for large n because v can be anything less than one-half. The resolution is that $\rho^L_{(n)}$ grows slowly with n so that the condition remains valid. We check the rate of convergence. Assuming, for instance, that $v = 0.1$, we have

$$1 - \Phi(y) = 0.1 \text{ or } \Phi(y) = 0.9$$

from which we get

$$y \approx 1.28 \text{ or } b_{(n)} \approx \frac{1.28}{\sqrt{n}}.$$

Note that the Z-table is required for the value of y. Therefore we have

$$\begin{aligned}
\rho^L_{(n)} &= \frac{\sqrt{2} + b_{(n)} - \sqrt{2 + b^2_{(n)}}}{2b_{(n)}} \\
&= \frac{\sqrt{2} + \frac{1.28}{\sqrt{n}} - \sqrt{2 + (\frac{1.28}{\sqrt{n}})^2}}{2\frac{1.28}{\sqrt{n}}} \\
&\Rightarrow \lim_{n\to\infty} \frac{|\rho_{n+1} - 1/2|}{|\rho_n - 1/2|} = 1 \text{ and } \lim_{n\to\infty} \frac{|\rho_{n+2} - \rho_{n+1}|}{|\rho_{n+1} - \rho_n|} = 1.
\end{aligned}$$

The sequence $\rho^L_{(n)}$ converges to 1/2 sublinearly and logarithmically. The rate of convergence is 1.

$$\rho^L_{(n)} \to 1/2 := \hat{\rho}.$$

Based on the analysis above, we get

Proposition 2 *In the Low-valuation case, the symmetric mixed strategy equilibrium is such that the sellers ask price v with probability* $\rho^L_{(n)}$ *which is the solution to the indifference condition*

$$\sum_{j=n}^{2n-1} \binom{2n-1}{j} \left(\rho^L_{(n)}\right)^j \left(1-\rho^L_{(n)}\right)^{2n-1-j} = v.$$

The probability $\rho^L_{(n)}$ *approaches* 1/2 *when the economy grows without bound.*

The total ex-ante value created in equilibrium is given by

$$\sum_{j=0}^{n-1} \binom{2n}{j} \left(\rho^L_{(n)}\right)^j \left(1-\rho^L_{(n)}\right)^{2n-j} (j \cdot v + n \cdot 1) + \sum_{j=n}^{2n} \binom{2n}{j} \left(\rho^L_{(n)}\right)^j \left(1-\rho^L_{(n)}\right)^{2n-j} (n \cdot v + n \cdot 1).$$

The first sum represents the value created if less than half of the sellers asks price v, or $j < n$. In this case, j low-valuation and n high-valuation buyers are served, leaving $n - j$ goods unconsumed. The second sum represents the value created if at least half of the sellers ask price v, or $j \geq n$. In this case, all the buyers are served. Thus, the measure of market efficiency becomes

$$\begin{aligned} M^L_{(n)} &= \frac{T^L}{T} = \frac{\text{Value created by trades}}{\text{Value created under market clearing}} \\ &= \frac{\sum_{j=0}^{n-1} \binom{2n}{j} \left(\rho^L_{(n)}\right)^j \left(1-\rho^L_{(n)}\right)^{2n-j} (j \cdot v + n \cdot 1) + \sum_{j=n}^{2n} \binom{2n}{j} \left(\rho^L_{(n)}\right)^j \left(1-\rho^L_{(n)}\right)^{2n-j} (n \cdot v + n \cdot 1)}{n \cdot 1 + n \cdot v}. \end{aligned} \tag{7}$$

3.2 High-Valuation First

Next we cover the other polar case where the high-valuation buyers enter the market first and the low-valuation buyers after them. The high-valuation buyers buy the lowest-priced goods, which may leave some (or all) of the low-valuation buyers unserved depending on how many low-priced goods remain by the time they enter the market. This, of course, depends on the sellers' pricing strategy.

Consider a pure strategy $p = v$ that generates expected profit v. There is no profitable deviation from this strategy. A deviation to price $p' > v$ leads to no trade and zero profit. A deviation to price $p' < v$ ensures trade with one of the high-valuation buyers but generates a profit less than v. There is also not a symmetric mixed strategy equilibrium as in the previous section as the highest pricing seller would not trade at all. We state this reasoning as

Proposition 3 *In the High-valuation case, the symmetric equilibrium is a pure pricing strategy $p = v$.*

Proof The reasoning above shows that $p = v$ constitutes a pure strategy equilibrium. It is clear that there are no other pure strategy equilibria. Neither are there symmetric mixed strategy equilibria. The lowest price in the support is not less than v, and let the highest price be $\bar{p} > v$. It is clear that there is not a mass point at $\bar{p}$, and consequently any seller asking $\bar{p}$ would trade with probability zero. □

In equilibrium, all the sellers get v, and buyers of both types are served clearing the market. The value of the efficiency measure is therefore $M^H = T^H/T = 1$ regardless of the market size which is indexed by n.

4 Comparison

In this section, we conduct numerical analysis, and compare the models of Sects. 2 and 3.1. We discuss how market size, the valuation of low-type buyers and the buyers' entry order affect market efficiency.

Table 1 reports the numerical solutions for the probabilities $\rho^R_{(n)}$, as solved explicitly in Eq. (3), and $\rho^L_{(n)}$, as shown in Eq. (5), or Lemma (6). The market size is $4n$

Table 1 The probabilities of asking price v in the benchmark and the coordination cases

Market size indexed by n[a]	Valuation (v)							
	0.1		0.2		0.3		0.4	
	$\rho^L_{(n)}$	$\rho^R_{(n)}$	$\rho^L_{(n)}$	$\rho^R_{(n)}$	$\rho^L_{(n)}$	$\rho^R_{(n)}$	$\rho^L_{(n)}$	$\rho^R_{(n)}$
2	0.1958	0.4472	0.2871	0.6325	0.3633	0.7746	0.4329	0.8944
3	0.2466	0.5848	0.3266	0.7368	0.3898	0.8434	0.4463	0.9283
4	0.2786	0.6687	0.3501	0.7953	0.4052	0.8801	0.4539	0.9457
5	0.3010	0.7248	0.3661	0.8326	0.4156	0.9029	0.4590	0.9564
6	0.3177	0.7647	0.3779	0.8584	0.4232	0.9184	0.4627	0.9635
7	0.3309	0.7946	0.3870	0.8773	0.4290	0.9296	0.4656	0.9686
8	0.3415	0.8178	0.3944	0.8918	0.4337	0.9381	0.4679	0.9725
9	0.3504	0.8363	0.4004	0.9032	0.4376	0.9448	0.4698	0.9755
10	0.3579	0.8513	0.4056	0.9124	0.4408	0.9502	0.4713	0.9779
50	0.4360	0.9683	0.4579	0.9818	0.4737	0.9898	0.4873	0.9955
100	0.4547	0.9840	0.4702	0.9909	0.4814	0.9949	0.4910	0.9978
1000	0.4857	0.9984	0.4906	0.9991	0.4941	0.9995	0.4972	0.9998
2000	0.4899	0.9992	0.4933	0.9995	0.4959	0.9997	0.4980	0.9999

[a]The first column of the table shows the values of n. The number of sellers and buyers are both $2n$ which make the size of the economy $4n$

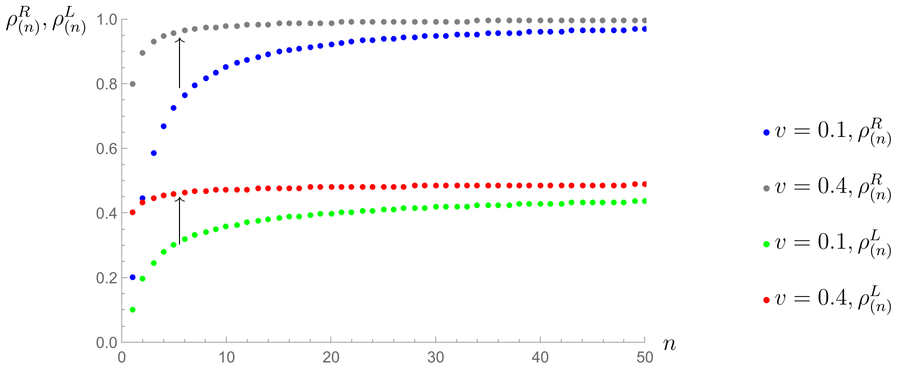

Fig. 1 The comparison of the probabilities

where the values of n are chosen from 2 to 2000. The low-valuation types are given four different valuations $v \in \{0.1, 0.2, 0.3, 0.4\}$. As n grows, the probability $\rho^R_{(n)}$ in the Random Entry case approaches 1 while the probability $\rho^L_{(n)}$ in the Low-valuation case approaches 1/2 for each v. It is also clear that $\rho^R_{(n)}$ and $\rho^L_{(n)}$ increase in v, holding n constant.

Figure 1 illustrates how the valuation v affects the equilibrium probabilities, especially in small economies. The values of $\rho^L_{(n)}$ at $v = 0.1$ (green dots) are much lower than at $v = 0.4$ (red dots) for small values of n. When the economy grows, dots in both colours move upwards, as the arrow points. The green curve increases quickly and gets very close to the red one at $n = 30$. For sufficiently large n, say $n \geq 45$, the values of $\rho^L_{(n)}$ are close to 1/2 and become much less dependent on the selection of v. The trend for $\rho^R_{(n)}$ is very similar, as illustrated in blue (for $v = 0.1$) and grey dots (for $v = 0.4$).

We report the measures of market efficiency in Table 2. The columns with $M^R_{(n)}$ measure the performance in the Random Entry case given in Eq. (4), and the columns with $M^L_{(n)}$ in the Low-valuation case given in Eq. (7). The market size is given by $4n$ where $n \in \{5, 10, 50, 100\}$. The low-valuation buyers' valuation, v, ranges from 0.1 to 0.475. We find that the values of market efficiency in both models vary in the same way but differ in levels. In each column, n is constant and the valuation v varies. We find that both $M^R_{(n)}$ and $M^L_{(n)}$ decrease first and then increase. The cut-off points are highlighted in the blue cells. For each pair of n and v, $M^R_{(n)}$ is always larger than $M^L_{(n)}$. It shows that when the buyers enter the market randomly, market inefficiency is, by comparison, a smaller problem. Buyers' coordination in the Low-valuation case results in a greater level of inefficiency. In each row, v is constant and n increases. We find that the higher the valuation v, the higher $M^R_{(n)}$ and $M^L_{(n)}$ are. The values of market efficiency approach 1 as v approaches 0.5.

Figure 2 illustrates how the efficiency measures $M^R_{(n)}$ and $M^L_{(n)}$ change in v when n gets values of 5 and 10. They are all U-shaped curves with different minimum points. Given $n = 5$, the value of $M^R_{(n)}$ (magenta dots) drops slightly until it reaches the lowest

Table 2 The measures of the market efficiencies

Valuation (v)	n [a]							
	5		10		50		100	
	$M^R_{(n)}$	$M^L_{(n)}$	$M^R_{(n)}$	$M^L_{(n)}$	$M^R_{(n)}$	$M^L_{(n)}$	$M^R_{(n)}$	$M^L_{(n)}$
0.1	0.97498	0.96271	0.98649	0.97330	0.99712	0.98795	0.99855	0.99147
0.125	0.96271	0.95815	0.98562	0.97014	0.99696	0.98655	0.99847	0.99048
0.15	0.97209	0.95465	0.98521	0.96771	0.99690	0.98549	0.99844	0.98973
0.175	0.97179	0.95203	0.98516	0.96590	0.99691	0.98469	0.99845	0.98917
0.2	0.97209	0.95016	0.98541	0.96461	0.99697	0.98412	0.99848	0.98877
0.225	0.97289	0.94891	0.98590	0.96375	0.99709	0.98375	0.99854	0.98850
0.25	0.97411	0.94821	0.98661	0.96327	0.99725	0.98354	0.99862	0.98836
0.275	0.97569	0.94798	0.98748	0.96311	0.99744	0.98347	0.99871	0.98830
0.3	0.97759	0.94816	0.98851	0.96323	0.99765	0.98352	0.99882	0.98834
0.325	0.97975	0.94869	0.98965	0.96359	0.99790	0.98367	0.99895	0.98845
0.35	0.98215	0.94952	0.99092	0.96416	0.99816	0.98392	0.99908	0.98862
0.375	0.98475	0.95062	0.99227	0.96491	0.99844	0.98424	0.99922	0.98885
0.4	0.98753	0.95195	0.99369	0.96582	0.99873	0.98464	0.99936	0.98913
0.425	0.99046	0.95348	0.99519	0.96686	0.99903	0.98509	0.99952	0.98945
0.45	0.99353	0.95518	0.99675	0.96802	0.99935	0.98560	0.99967	0.98981
0.475	0.99671	0.95702	0.99835	0.96929	0.99967	0.98615	0.99983	0.99019

[a] The values of n are set to be 5, 10, 50 and 100. The number of sellers and buyers are both $2n$ which make the size of the economy $4n$.

point 0.971, after which it rises. Meanwhile, the value of $M^L_{(n)}$ (brown dots) is much smaller. It starts from 0.962 (at $v = 0.1$), moves downwards to the lowest point 0.947 (at $v = 0.275$) and increases thereafter. For $n = 10$, $M^R_{(n)}$ and $M^L_{(n)}$ follow a similar pattern, as illustrated in orange (for $M^R_{(n)}$) and cyan dots (for $M^L_{(n)}$). The cause of the shape for $M^R_{(n)}$ has been discussed in the benchmark case and the same argument applies to $M^L_{(n)}$. Recall that market efficiency is measured by the ratio between the total ex ante value created from trades that each equilibrium generates (T^R, T^L and T^H) and the total value created when the market clears (T). We provide the graphs for T^R and T in Appendix 2.

Note that the measure of market efficiency ($M^H_{(n)}$) is 1 in the High-valuation case, as described in Proposition 3. Therefore, such coordination effectively solves the market inefficiency problem.

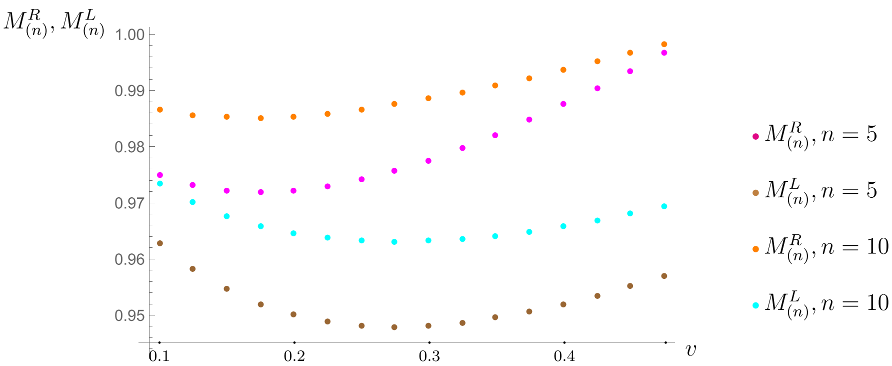

Fig. 2 The comparison of the market efficiencies

5 Discussion

A natural starting point to study the lack of coordination, and the resulting inefficiencies, is to assume that the buyers enter the market in a random order. This leads to pricing by mixed strategies, and indeed, to inefficient outcomes as some high pricing sellers are left with low-valuation buyers for whom the prices are too high.

We compare this to two polar cases, one in which the low-valuation buyers enter first, and the other in which the high-valuation buyers enter first. The former might be regarded as a natural solution to the problem that the early entering high-valuation buyers acquire the low-priced goods leaving low-valuation buyers without trading opportunities. This, however, ignores the sellers' reactions, and it turns out that fewer trades are consummated and more goods are wasted than in the benchmark.

When the low-valuation buyers arrive first, competition for the high-valuation buyers is the fiercest as the probability of asking the low price v is the smallest. In the model, competition manifests as high prices contrary to the standard market setting where competition lowers prices, and competition in our setting is harmful. That competition is not necessarily beneficial is no big news but the point is rarely made in a market setting. The underlying reason for the inefficiency is that the high-valuation buyers are a fixed and valuable resource, and competition for a fixed resource tends to be wasteful.

In the other case, the high-valuation buyers enter first. This turns out to be the most efficient order of entry; as the high-valuation buyers are the first in the market, there is no need to compete for them. Rather, the pricing, where all the sellers ask price v, reflects the fear of not being able to trade. The situation is very much like the sellers facing a downward sloping demand curve. This entry order is the choice of a social planner, and it can also be rationalised by the following informal argument.

Assume that the market is opened at a predetermined time and the buyers arrive in random order to a queue. This corresponds to our benchmark case. Consider the high-valuation buyers in the queue, and allow each of them to ask the person just before them whether the person would like to swap positions for a small sum of money. Each low-valuation buyer would be willing because they expect no more in the market than buying a good worth v at price v.

If this procedure is allowed to go on for sufficiently many rounds, the end result is that all the high-valuation buyers are before the low-valuation buyers in the queue. Notice that no high-valuation buyer is willing to pay enough to another high-valuation buyer to swap places. As far as the sellers understand what is going to happen, they revise their pricing strategy accordingly. This shows that in principle there is a decentralised solution to the potential inefficiencies and no policy measures are needed.

6 Conclusion

Efficiency considerations are of central interest in any market-mediated activity, and there are many causes of inefficiency. In this article, we study an issue that has not received much attention, that is, the buyers' entry order. We employ a simple game where identical sellers compete for the buyers of heterogeneous valuations. We derive the sellers' equilibrium pricing strategies, and find that in the benchmark model where the buyers enter the market in a random order, the sellers use mixed strategies in pricing. In equilibrium, not all the low-valuation buyers trade, which results in an inefficient outcome. We then compare the equilibrium results in the benchmark with those of two extreme cases of coordination where the same type buyers enter the market at the same time. We find that the sellers react to the buyers' new entry order by changing their pricing in a way that retains the level of equilibrium pay-off constant. In each of these three cases, the symmetric equilibrium is unique.

We then measure the performance of the market by the ratio between the ex ante expected value created and the total value created when the market clears. This comparison provides us with a measure of inefficiency, and also an understanding of how it vanishes as the economy grows. It is notable that in the limit the inefficiency vanishes regardless of the buyers' order of entry. The order only affects the speed of convergence to the efficient outcome.

Some real-life markets where the entry order may be important include markets for goods that do not keep very well such as fish. The resale markets for tickets to concerts or comparable events are such that high pricing sellers often do not manage to sell their tickets. Also, small labour markets, say for a particular occupation, with firms in the role of sellers and job seekers in the role of buyers may fit our theoretical framework.

Appendix 1. Proof of Lemmas

Proof of Lemma 2

Proof Denote the number of sellers who ask price v by j. We first note that if fewer than half of the sellers ask price v, that is, if $j < n$, the number of unsold goods varies between $n - j$ (if the low-valuation buyers happen to enter the market first) and n (if the high-valuation buyers happen to enter the market first). The probability that the number of unsold items is $n - j + k$, $k \in \{0, 1, ..., j\}$, is given by $Pr(n - j + k) = \frac{\binom{n}{j-k}\binom{n}{k}}{\binom{2n}{j}}$. In the denominator, there is the number of ways to choose j from a total of $2n$ buyers in the market. In the numerator, there is the product of choosing $j - k$ low-valuation buyers and k high-valuation buyers amongst the first j buyers.

Analogously, when $j \geq n$, the number of unsold goods varies between zero, if the low-valuation buyers happen to enter the market first, and $2n - j$, if the high-valuation buyers enter first. The probabilities of unsold goods are given by $Pr(k) = \frac{\binom{n}{j-(n-k)}\binom{n}{n-k}}{\binom{2n}{j}}$, $k \in \{0, 1, ..., 2n - j\}$.

The expected number of unsold goods, when $j < n$, is given by

$$\sum_{k=0}^{j} \frac{\binom{n}{j-k}\binom{n}{k}}{\binom{2n}{j}}(n-j+k). \tag{8}$$

This event happens with probability $\binom{2n}{j}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}$. Consequently, the ex ante expectation of the number of unsold goods, conditional on $j < n$, is given by

$$\begin{aligned}
&\sum_{k=0}^{j} \frac{\binom{n}{j-k}\binom{n}{k}}{\binom{2n}{j}}(n-j+k)\binom{2n}{j}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}\\
&=\sum_{k=0}^{j}\binom{n}{j-k}\binom{n}{k}(n-j+k)\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}\\
&=\sum_{k=0}^{j} n\binom{n-1}{j-k}\binom{n}{k}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}
\end{aligned}$$

where $\rho^R_{(n)}$ is the probability that sellers set the price at v. By Vandermonde's identity, we have $\sum_{k=0}^{j}\binom{n-1}{j-k}\binom{n}{k}=\binom{2n-1}{j}$. Therefore, the last equality is simplified to

$$\binom{2n-1}{j} n\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}. \tag{9}$$

When $j \geq n$, the expected number of unsold goods is given by

$$\sum_{k=1}^{2n-j} \frac{\binom{n}{j-(n-k)}\binom{n}{n-k}}{\binom{2n}{j}}k. \tag{10}$$

This event happens with probability $\binom{2n}{j}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}$. Consequently, the ex ante expectation of the number of unsold goods, conditional on $j \geq n$, is given by

$$\begin{aligned}
&\sum_{k=1}^{2n-j} \frac{\binom{n}{j-(n-k)}\binom{n}{n-k}}{\binom{2n}{j}}k\binom{2n}{j}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}\\
&=\sum_{k=1}^{2n-j}\binom{n}{j-(n-k)}\binom{n}{n-k}k\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}\\
&=\sum_{k=1}^{2n-j} n\binom{n}{2n-j-k}\binom{n-1}{k-1}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j} \quad \text{by } \binom{n}{j-(n-k)}=\binom{n}{2n-j-k}\\
&=\sum_{k=0}^{2n-j-1} n\binom{n}{2n-j-1-k}\binom{n-1}{k}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}.
\end{aligned}$$

Again, by Vandermonde's identity, we have $\sum_{k=0}^{2n-j-1}\binom{n}{2n-j-1-k}\binom{n-1}{k}=\binom{2n-1}{2n-1-j}$. Therefore, the last equation is equivalent to

$$n\binom{2n-1}{2n-1-j}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}. \tag{11}$$

From Expressions (9) and (11), the expected number of unsold goods for any j is given by

$$\sum_{j=0}^{n-1} n\binom{2n-1}{j}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j} + \sum_{j=n}^{2n-1} n\binom{2n-1}{2n-j-1}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-j}$$
$$= n\left(1-\rho^R_{(n)}\right)\sum_{j=0}^{n-1}\binom{2n-1}{j}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-1-j}$$
$$+ n\left(1-\rho^R_{(n)}\right)\sum_{j=n}^{2n-1}\binom{2n-1}{j}\left(\rho^R_{(n)}\right)^j\left(1-\rho^R_{(n)}\right)^{2n-1-j}.$$

Denote the distribution function of a random variable with density $Bin(p, n)$ by $F(p, n, x)$. The above expression is equivalent to

$$n\left(1-\rho^R_{(n)}\right)\left[F\left(\rho^R_{(n)}, 2n-1, n-1\right) + \left(1 - F\left(\rho^R_{(n)}, 2n-1, n-1\right)\right)\right]$$

which reduces to $n\left(1-\rho^R_{(n)}\right)$. This completes the proof of Lemma 2. □

Proof of Lemma 4

Proof Inspired by Eq. (5), we define the function

$$X(\rho^L_{(n)}) = \sum_{j=n}^{2n-1}\binom{2n-1}{j}\left(\rho^L_{(n)}\right)^j\left(1-\rho^L_{(n)}\right)^{2n-1-j} - v. \tag{12}$$

We show that for any $n > 0$ and $v \in (0, 1/2)$, there exists a unique solution $\rho^L_{(n)} \in (0, 1)$ satisfying $X(\rho^L_{(n)}) = 0$. $X(\rho^L_{(n)})$ is clearly continuous. $X(0) = -v < 0$ and $X(1) = 1 - v > 0$. By the intermediate value theorem, there exists at least one ρ such that $X(\rho) = 0$. Also by $X'(\rho^L_{(n)}) > 0$ on the interval (0, 1),[9] the function $X(\rho^L_{(n)})$ is monotonically increasing. Therefore for each pair (n, v), the solution to Equation (5) is unique in the range of (0, 1). □

Proof of Lemma 5

By Barry-Esseen theorem

[9] The formula of $X'(\rho^L_{(n)})$ is available upon request.

$$|F_{2n-1}(a_n) - \Phi(a_n)| \le \frac{3\,\mathbb{E}\left(|X_1|^3\right)}{\sigma^3\sqrt{2n-1}} = \frac{3\rho^L_{(n)}}{\left(\rho^L_{(n)}\left(1-\rho^L_{(n)}\right)\right)^{3/2}}\frac{1}{\sqrt{2n-1}}$$

Since $F_{2n-1}(a_n) = 1 - v$ for all n, we get

$$|1 - v - \Phi(a_n)| \le \frac{3}{\left(1-\rho^L_{(n)}\right)^{3/2}}\frac{1}{\sqrt{(2n-1)\rho^L_{(n)}}} \tag{13}$$

Next, we show that $\rho^L_{(n)}$ does not converge to zero or unity, and to simplify we denote it by ρ_n. First, we have

$$\begin{aligned} v &= \sum_{k=n}^{2n-1}\binom{2n-1}{k}\rho_n^k\left(1-\rho_n\right)^{2n-1-k} \le \rho_n^n\sum_{k=n}^{2n-1}\binom{2n-1}{k} \\ &\le \rho_n^n\sum_{k=0}^{2n-1}\binom{2n-1}{k} = \rho_n^n 2^{2n-1} = \left(4\rho_n\right)^n\frac{1}{2} \end{aligned}$$

Consequently, $4\rho_n \ge (2v)^{\frac{1}{n}}$, and $\lim_{n\to\infty}\rho_n \ge \frac{1}{4}$.
By the same reasoning

$$\begin{aligned} 1 - v &= \sum_{k=0}^{n-1}\binom{2n-1}{k}\rho_n^k\left(1-\rho_n\right)^{2n-1-k} \le \left(1-\rho_n\right)^n\sum_{k=0}^{n-1}\binom{2n-1}{k} \\ &\le \left(1-\rho_n\right)^n 2^{2n-1} = \left(4\left(1-\rho_n\right)\right)^n\frac{1}{2} \end{aligned}$$

and from this we get $4\left(1-\rho_n\right) \ge (2(1-v))^{\frac{1}{n}}$. Taking limits, we get $\lim_{n\to\infty}\left(1-\rho_n\right) \ge \frac{1}{4}\lim_{n\to\infty}(2(1-v))^{\frac{1}{n}} = \frac{1}{4}$.

Since the sequence $\left\{\rho^L_{(n)}\right\}$ belongs to a compact set, to show convergence to $\frac{1}{2}$, it is enough to show that every convergent subsequence converges to one-half. From the reasoning above, we see that for large enough n we have $\frac{1}{8} < \rho^L_{(n)} < \frac{7}{8}$. This means that the right-hand side of (13) goes to zero when n grows indefinitely. Consequently, we have $\Phi(a_n)\lim_{n\to\infty} 1 - v \in \left(\frac{1}{2}, 1\right)$.

Note that $a_n = \frac{\sqrt{n}\left(1-2\rho^L_{(n)}\right)}{\sqrt{(2-1/n)\rho^L_{(n)}\left(1-\rho^L_{(n)}\right)}} - \frac{1-\rho^L_{(n)}}{\sqrt{(2n-1)\rho^L_{(n)}\left(1-\rho^L_{(n)}\right)}}$ where the denominator of the first term remains bounded and the second term converges to zero as n grows without limit.

Let us next assume that the subsequence $\left\{\rho_{n_k}\right\}_{k=1}^{\infty} \to c$. If $c > 1/2$ then $a_{n_k} \to -\infty$, and $\Phi\left(a_{n_k}\right) \to 0$ which is a contradiction. Analogously, if $c < 1/2$ then $a_{n_k} \to \infty$, and $\Phi\left(a_{n_k}\right) \to 1$ which is a contradiction.

Since every convergent subsequence converges to $\frac{1}{2}$, by compactness, the whole sequence converges to 1/2. □

Proof of Lemma 6

Proof For large n, normal distribution is a good approximation for binomial distribution. We rewrite expression $\sum_{j=n}^{2n-1} \binom{2n-1}{j} \left(\rho_{(n)}^L\right)^j \left(1-\rho_{(n)}^L\right)^{2n-1-j}$ as $Pr\left(S_{2n-1} \geq n\right)$, where S_{2n-1} is a binomial random variable with the success probability $\rho_{(n)}^L$. When n is large, the expression is about

$$Pr\left(Z \geq \frac{n-\rho_{(n)}^L(2n-1)}{\rho_{(n)}^L\left(1-\rho_{(n)}^L\right)\sqrt{2n-1}}\right) \approx Pr\left(Z \geq \frac{\sqrt{n}\left(1-2\rho_{(n)}^L\right)}{\rho_{(n)}^L\left(1-\rho_{(n)}^L\right)\sqrt{2}}\right),$$

where Z is the standardised normal distribution. Let

$$b_{(n)} = \frac{1-2\rho_{(n)}^L}{\rho_{(n)}^L(1-\rho_{(n)}^L)\sqrt{2}}. \tag{14}$$

The probability can be expressed as

$$Pr\left(Z \geq \sqrt{n}b_{(n)}\right) = 1-\Phi\left(\sqrt{n}b_{(n)}\right).$$

Given any $v \in \left(0, \frac{1}{2}\right)$, we solve y from $\Phi(y) = 1-v$ using the Z-table, and we get $b_{(n)}$ by

$$b_{(n)} = \frac{y}{\sqrt{n}}. \tag{15}$$

Rewriting Expression (14), we get

$$\rho_{(n)}^L = \frac{\sqrt{2}+b_{(n)}-\sqrt{2+b_{(n)}^2}}{2b_{(n)}}. \tag{16}$$

□

Appendix 2. The Values Created in Equilibrium and Under Market Clearing

The figure shows how the valuation of low-type buyers (v) affects the total ex ante value created from trades that equilibrium generates in the benchmark model (T^R) and the total ex ante value created under market clearing (T), when n is set to be 2 (see Fig. 3). For large ns, the two curves are very close to each other; therefore, we cannot see their difference properly. T^R is convex and T is linear. The difference in the

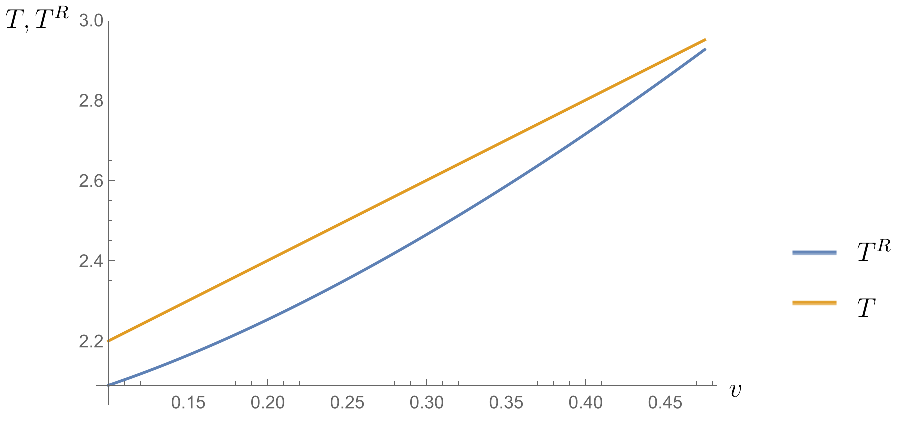

Fig. 3 The values created in equilibrium and under market clearing

rates of increase for T^R and T is the cause for the trend of market efficiency measures, as stated after Proposition 1. The efficiency $M^R_{(n)}$ evaluated by the ratio between T^R and T has the trend such that for a given n, the efficiency decreases for small vs, and after some cutoff point, increases. $M^L_{(n)}$ follows a similar trend, as illustrated in Fig. 2.

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Data Availability We confirm that all the data has been presented in the article. There will be no supplementary data. The data is produced by Mathematica and the codes that generate the data are available upon request.

Declarations

Ethical Approval We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

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Conflict of Interest The authors declare no competing interests.

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